Introduction to

Modern Economic Growth

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Introduction to Modern Economic Growth: Parts 1-5

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Preface

This book is intended to serve two purposes:

(1) First and foremost, this is a book about economic growth and long-run economic development. The process of economic growth and the sources of differences in economic performance across nations are some of the most interesting, important and challenging areas in modern social science. The primary purpose of this book is to introduce graduate students to these major questions and to the theoretical tools necessary for studying them. The book therefore strives to provide students with a strong background in dynamic economic analysis, since only such a background will enable a serious study of economic growth and economic development. It also tries to provide a clear discussion of the broad empirical patterns and historical processes underlying the current state of the world economy. This is motivated by my belief that to understand why some countries grow and some fail to do so, economists have to move beyond the mechanics of models and pose questions about the fundamental causes of economic growth.

(2) In a somewhat different capacity, this book is also a graduate-level introduction to modern macroeconomics and dynamic economic analysis. It is sometimes commented that, unlike basic microeconomic theory, there is no core of current macroeconomic theory that is shared by all economists. This is not entirely true. While there is disagreement among macroeconomists about how to approach short-run macroeconomic phenomena and what the boundaries of macroeconomics should be, there is broad agreement about the workhorse models of dynamic macroeconomic analysis. These include the Solow growth model, the neoclassical growth model, the overlapping-generations model and models of technological change and technology adoption. Since these are all models of economic growth, a thorough treatment of modern economic growth can also provide (and perhaps should provide) an introduction to this core material of modern macroeconomics. Although there are several good graduate-level macroeconomic textbooks, they typically spend relatively little time on the basic core material and do not develop the links between modern macroeconomic analysis and economic dynamics on the one hand and general equilibrium theory on the other. In contrast, the current book does not cover any of the short-run topics in macroeconomics, but provides a thorough and rigorous introduction to what I view to be the core of macroeconomics. Therefore, the second purpose of the book is to provide a first graduate-level course in modern macroeconomics.

The selection of topics is designed to strike a balance between the two purposes of the book. Chapters 1, 3 and 4 introduce many of the salient features of the process of economic growth and the sources of cross-country differences in economic performance. Even though these chapters cannot do justice to the large literature on economic growth empirics, they provide a sufficient background for students to appreciate the set of issues that are central to the study of economic growth and also a platform for a further study of this large literature.

Chapters 5-7 provide the conceptual and mathematical foundations of modern macroeconomic analysis. Chapter 5 provides the microfoundations for much of the rest of the book (and for much of modern macroeconomics), while Chapters 6 and 7 provide a quick but relatively rigorous introduction to dynamic optimization. Most books on macroeconomics or economic growth use either continuous time or discrete time exclusively. I believe that a serious study of both economic growth and modern macroeconomics requires the student (and the researcher) to be able to go between discrete and continuous time and choose whichever one is more convenient or appropriate for the set of questions at hand. Therefore, I have deviated from this standard practice and included both continuous time and discrete time material throughout the book.

Chapters 2, 8, 9 and 10 introduce the basic workhorse models of modern macroeconomics and traditional economic growth, while Chapter 11 presents the first generation models of sustained (endogenous) economic growth. Chapters 12-15 cover models of technological progress, which are an essential part of any modern economic growth course.

Chapter 16 generalizes the tools introduced in Chapter 6 to stochastic environments. Using these tools, Chapter 17 presents a number of models of stochastic growth, most notably, the neoclassical growth model under uncertainty, which is the foundation of much of modern macroeconomics (though it is often left out of economic growth courses). The canonical Real Business Cycle model is presented as an application. This chapter also covers another major workhorse model of modern macroeconomics, the incomplete markets model of Bewley. Finally, this chapter also presents a number of other approaches to modeling the interaction between incomplete markets and economic growth and shows how models of stochastic growth can be useful in understanding how economies transition from stagnation or slow growth to an equilibrium with sustained growth.

Chapters 18-21 cover a range of topics that are sometimes left out of economic growth textbooks. These include models of technology adoption, technology diffusion, the interaction between international trade and technology, the process of structural change, the demographic transition, the possibility of poverty traps, the effects of inequality on economic growth and the interaction between financial and economic development. These topics are important for creating a bridge between the empirical patterns we observe in practice and the theory. Most traditional growth models consider a single economy in isolation and often after it has already embarked upon a process of steady economic growth. A study of models that incorporate cross-country interdependences, structural change and the possibility of takeoffs will enable us to link core topics of development economics, such as structural change, poverty traps or the demographic transition, to the theory of economic growth.

Finally, Chapters 22 and 23 consider another topic often omitted from macroeconomics and economic growth textbooks; political economy. This is motivated by the belief that the study of economic growth would be seriously hampered if we failed to ask questions about the fundamental causes of why countries differ in their economic performances. These questions invariably bring us to differences in economic policies and institutions across nations. Political economy enables us to develop models to understand why economic policies and institutions differ across countries and must therefore be an integral part of the study of economic growth.

A few words on the philosophy and organization of the book might also be useful for students and teachers. The underlying philosophy of the book is that all the results that are stated should be proved or at least explained in detail. This implies a somewhat different organization than existing books. Most textbooks in economics do not provide proofs for many of the results that are stated or invoked, and mathematical tools that are essential for the analysis are often taken for granted or developed in appendices. In contrast, I have strived to provide simple proofs of almost all results stated in this book. It turns out that once unnecessary generality is removed, most results can be stated and proved in a way that is easily accessible to graduate students. In fact, I believe that even somewhat long proofs are much easier to understand than general statements made without proof, which leave the reader wondering about why these statements are true.

I hope that the style I have chosen not only makes the book self-contained, but also gives the students an opportunity to develop a thorough understanding of the material. In addition, I present the basic mathematical tools necessary for analysis within the main body of the text. My own experience suggests that a "linear" progression, where the necessary mathematical tools are introduced when needed, makes it easier for the students to follow and appreciate the material. Consequently, analysis of stability of dynamical systems, dynamic programming in discrete time and optimal control in continuous time are all introduced within the main body of the text. This should both help the students appreciate the foundations of the theory of economic growth and also provide them with an introduction to the main tools of dynamic economic analysis, which are increasingly used in every subdiscipline of economics. Throughout, when some material is technically more difficult and can be skipped without loss of continuity, it is clearly marked with a "*". Only material that is tangentially related to the main results in the text or those that should be familiar to most graduate students are left for the Mathematical Appendices.

I have also included a large number of exercises. Students can only gain a thorough understanding of the material by working through the exercises. The exercises that are somewhat more difficult are also marked with a "*".

This book can be used in a number of different ways. First, it can be used in a one-quarter or one-semester course on economic growth. Such a course might start with Chapters 1-4, then depending on the nature of the course, use Chapters 5-7 either for a thorough study of the general equilibrium and dynamic optimization foundations of growth theory or only for reference. Chapters 8-11 cover the traditional growth theory and Chapters 12-15 provide the basics of endogenous growth theory. Depending on time and interest, any selection of Chapters 16-23 can be used for the last part of such a course.

Second, the book can be used for a one-quarter first-year graduate-level course in macroeconomics. In this case, Chapter 1 is optional. Chapters 3, 5-7, 8-11 and 16 and 17 would be the core of such a course. The same material could also be covered in a one-semester course, but in this case, it could be supplemented either with some of the later chapters or with material from one of the leading graduate-level macroeconomic textbooks on short-run macroeconomics, fiscal policy, asset pricing, or other topics in dynamic macroeconomics.

Third, the book can be used for an advanced (second-year) course in economic growth or economic development. An advanced course on growth or development could use Chapters 1-11 as background and then focus on selected chapters from Chapters 12-23.

Finally, since the book is self-contained, I also hope that it can be used for self-study.

Acknowledgments. This book grew out of the first graduate-level introduction to macroeconomics course I have taught at MIT. Parts of the book have also been taught as part of a second-year graduate macroeconomics course. I would like to thank the students who have sat through these lectures and made comments that have improved the manuscript. I owe a special thanks to Monica Martinez-Bravo, Samuel Pienknagura, Lucia Tian Tian and especially Michael Peters and Alp Simsek for outstanding research assistance. In fact, without Michael and Alp's help this book would have taken me much longer and would have contain many more errors. I also thank Lauren Fahey for editorial suggestions and help with the references. I would also like to thank George-Marios Angeletos, Olivier Blanchard, Francesco Caselli, Melissa Dell, Peter Funk, Oded Galor, Hugo Hopenhayn, Simon Johnson, Chad Jones, Ismail Saglam, Jesse Zinn for useful suggestions and corrections on individual chapters, and especially Pol Antras, Kiminori Matsuyama, James Robinson, Jesus Fernandez-Villaverde and Pierre Yared for very valuable suggestions on multiple chapters.

Please note that this is a preliminary draft of the book manuscript. The draft certainly contains mistakes. Comments and suggestions for corrections are welcome.

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Part 1

Introduction

We start with a quick look at the stylized facts of economic growth and the most basic model of growth, the Solow growth model. The purpose is both to prepare us for the analysis of more modern models of economic growth with forward-looking behavior, explicit capital accumulation and endogenous technological progress, and also to give us a way of mapping the simplest model to data. We will also discuss differences between proximate and fundamental causes of economic growth and economic development.

CHAPTER 1

Economic Growth and Economic Development: The Questions

1.1. Cross-Country Income Differences

There are very large differences in income per capita and output per worker across countries today. Countries at the top of the world income distribution are more than thirty times as rich as those at the bottom. For example, in 2000, GDP (or income) per capita in the United States was over \$34000. In contrast, income per capita is much lower in many other countries: about \$8000 in Mexico, about \$4000 in China, just over \$2500 in India, only about \$1000 in Nigeria, and much much lower in some other sub-Saharan African countries such as Chad, Ethiopia, and Mali. These numbers are all in 2000 US dollars and are adjusted for purchasing power party (PPP) to allow for differences in relative prices of different goods across countries (all data from the Penn World tables compiled by Summers and Heston). The cross-country income gap is considerably larger when there is no PPP-adjustment. For example, without the PPP adjustment, GDP per capita in India and China in 2000 would be lower by a factor of four or so.

Figure 1.1 provides a first look at these differences. It plots estimates of the distribution of PPP-adjusted GDP per capita across the available set of countries in 1960, 1980 and 2000. A number of features are worth noting. First, the 1960 density shows that 15 years after the end of World War II, most countries had income per capita less than \$1500 (in 2000 US dollars); the mode of the distribution is around \$1250. The rightwards shift of the distributions for 1980 and for 2000 shows the growth of average income per capita for the next 40 years. In 2000, the mode is still slightly above \$3000, but now there is another concentration of countries between \$20,000 and \$30,000. The density estimate for the year 2000 shows the considerable inequality in income per capita today.

Part of the spreading out of the distribution in Figure 1.1 is because of the increase in average incomes. It may therefore be more informative to look at the logarithm (log) of income per capita. It is more natural to look at the log of variables, such as income per capita, that grow over time, especially when growth is approximately proportional as suggested by see Figure 1.8) (this is because when $x(t)$ grows at a proportional rate, $\log x(t)$ grows linearly, and more importantly, if $x_1(t)$ and $x_2(t)$ both grow by 10% over a certain period of

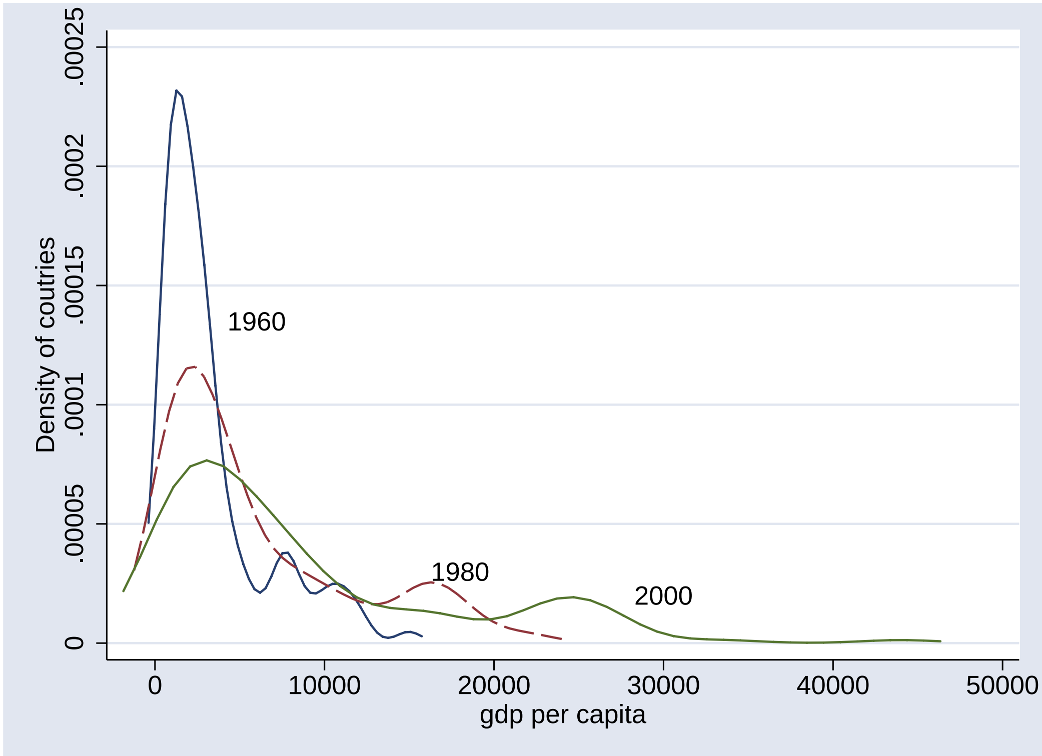

Figure 1.1. Estimates of the distribution of countries according to PPP-adjusted GDP per capita in 1960, 1980 and 2000.

time, $x_1(t) - x_2(t)$ will also grow, while $\log x_1(t) - \log x_2(t)$ will remain constant). Figure 1.2 shows a similar pattern, but now the spreading-out is more limited. This reflects the fact that while the absolute gap between rich and poor countries has increased considerably between 1960 and 2000, the proportional gap has increased much less. Nevertheless, it can be seen that the 2000 density for log GDP per capita is still more spread out than the 1960 density. In particular, both figures show that there has been a considerable increase in the density of relatively rich countries, while many countries still remain quite poor. This last pattern is sometimes referred to as the "stratification phenomenon", corresponding to the fact that some of the middle-income countries of the 1960s have joined the ranks of relatively high-income countries, while others have maintained their middle-income status or even experienced relative impoverishment.

Figures 1.1 and 1.2 demonstrate that there is somewhat greater inequality among nations. An equally relevant concept might be inequality among individuals in the world economy. Figures 1.1 and 1.2 are not directly informative on this, since they treat each country identically regardless of the size of its population. The alternative is presented in Figure 1.3, which shows the population-weighted distribution. In this case, countries such as China, India, the

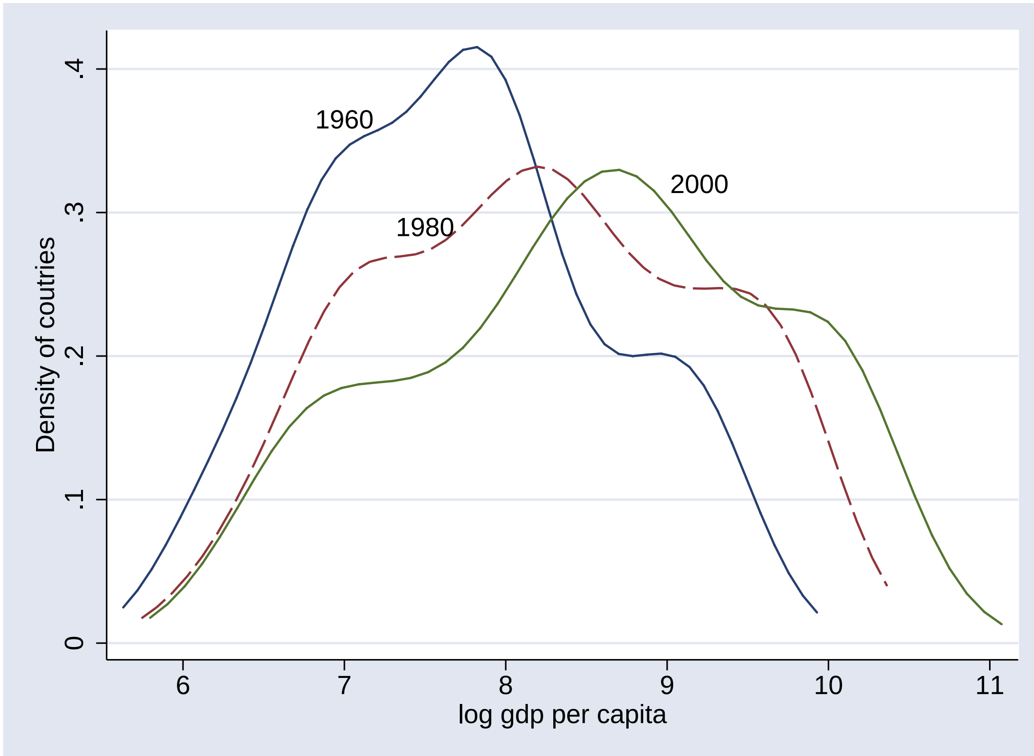

Figure 1.2. Estimates of the distribution of countries according to log GDP per capita (PPP-adjusted) in 1960, 1980 and 2000.

United States, and Russia receive greater weight because they have larger populations. The picture that emerges in this case is quite different. In fact, the 2000 distribution looks less spread out, with thinner left tail than the 1960 distribution. This reflects the fact that in 1960 China and India were among the poorest nations, whereas their relatively rapid growth in the 1990s puts them into the middle-poor category by 2000. Chinese and Indian growth has therefore created a powerful force towards relative equalization of income per capita among the inhabitants of the globe.

Figures 1.1, 1.2 and 1.3 look at the distribution of GDP per capita. While this measure is relevant for the welfare of the population, much of growth theory focuses on the productive capacity of countries. Theory is therefore easier to map to data when we look at output (GDP) per worker. Moreover, key sources of difference in economic performance across countries are national policies and institutions. So for the purpose of understanding the sources of differences in income and growth across countries (as opposed to assessing welfare questions), the unweighted distribution is more relevant than the population-weighted distribution. Consequently, Figure 1.4 looks at the unweighted distribution of countries according to (PPP-adjusted) GDP per worker. Since internationally comparable data on employment

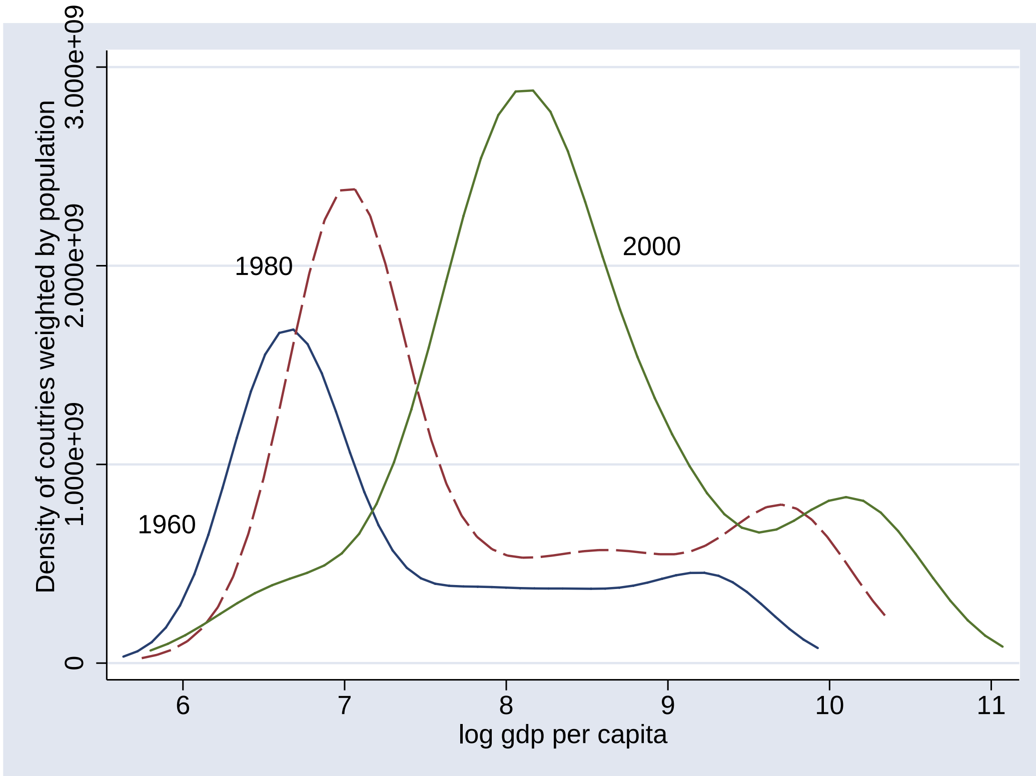

Figure 1.3. Estimates of the population-weighted distribution of countries according to log GDP per capita (PPP-adjusted) in 1960, 1980 and 2000.

are not available for a large number of countries, "workers" here refer to the total economically active population (according to the definition of the International Labour Organization). Figure 1.4 is very similar to Figure 1.2, and if anything, shows a greater concentration of countries in the relatively rich tail by 2000, with the poor tail remaining more or less the same as in Figure 1.2.

Overall, Figures 1.1-1.4 document two important facts: first, there is a large amount of inequality in income per capita and income per worker across countries as shown by the highly dispersed distributions. Second, there is a slight but noticeable increase in inequality across nations (though not necessarily across individuals in the world economy).

1.2. Income and Welfare

Should we care about cross-country income differences? The answer is *definitely yes*. High income levels reflect high standards of living. Economic growth might, at least over some range, increase pollution or it may raise individual aspirations, so that the same bundle of consumption may no longer make an individual as happy. But at the end of the day,

Figure 1.4. Estimates of the distribution of countries according to log GDP per worker (PPP-adjusted) in 1960, 1980 and 2000.

when one compares an advanced, rich country with a less-developed one, there are striking differences in the quality of life, standards of living and health.

Figures 1.5 and 1.6 give a glimpse of these differences and depict the relationship between income per capita in 2000 and consumption per capita and life expectancy at birth in the same year. Consumption data also come from the Penn World tables, while data on life expectancy at birth are available from the World Bank Development Indicators.

These figures document that income per capita differences are strongly associated with differences in consumption and differences in health as measured by life expectancy. Recall also that these numbers refer to PPP-adjusted quantities, thus differences in consumption do not (at least in principle) reflect the fact that the same bundle of consumption goods costs different amounts in different countries. The PPP adjustment corrects for these differences and attempts to measure the variation in real consumption. Therefore, the richest countries are not only producing more than thirty times as much as the poorest countries but are also consuming thirty times as much. Similarly, cross-country differences in health are quite remarkable; while life expectancy at birth is as high as 80 in the richest countries, it is only

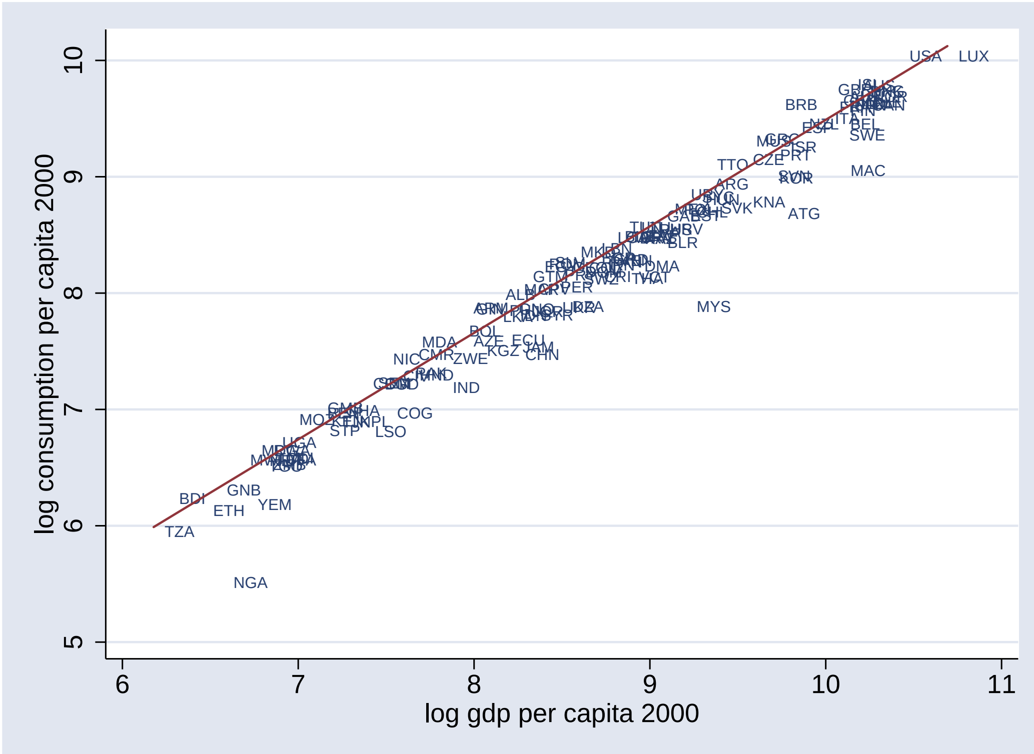

Figure 1.5. The association between income per capita and consumption per capita in 2000.

between 40 and 50 in many sub-Saharan African nations. These gaps represent huge welfare differences.

Understanding how some countries can be so rich while some others are so poor is one of the most important, perhaps *the* most important, challenges facing social science. It is important both because these income differences have major welfare consequences and because a study of these striking differences will shed light on how the economies of different nations function and sometimes how they fail to function.

The emphasis on income differences across countries implies neither that income per capita can be used as a "sufficient statistic" for the welfare of the average citizen nor that it is the only feature that we should care about. As we will discuss in detail later, the efficiency properties of the market economy (such as the celebrated *First Welfare Theorem* or Adam Smith's *invisible hand*) do not imply that there is no conflict among individuals or groups in society. Economic growth is generally good for welfare but it often creates "winners" and "losers." Joseph Schumpeter's famous notion of *creative destruction* emphasizes precisely this aspect of economic growth; productive relationships, firms and sometimes individual livelihoods will often be destroyed by the process of economic growth because growth is

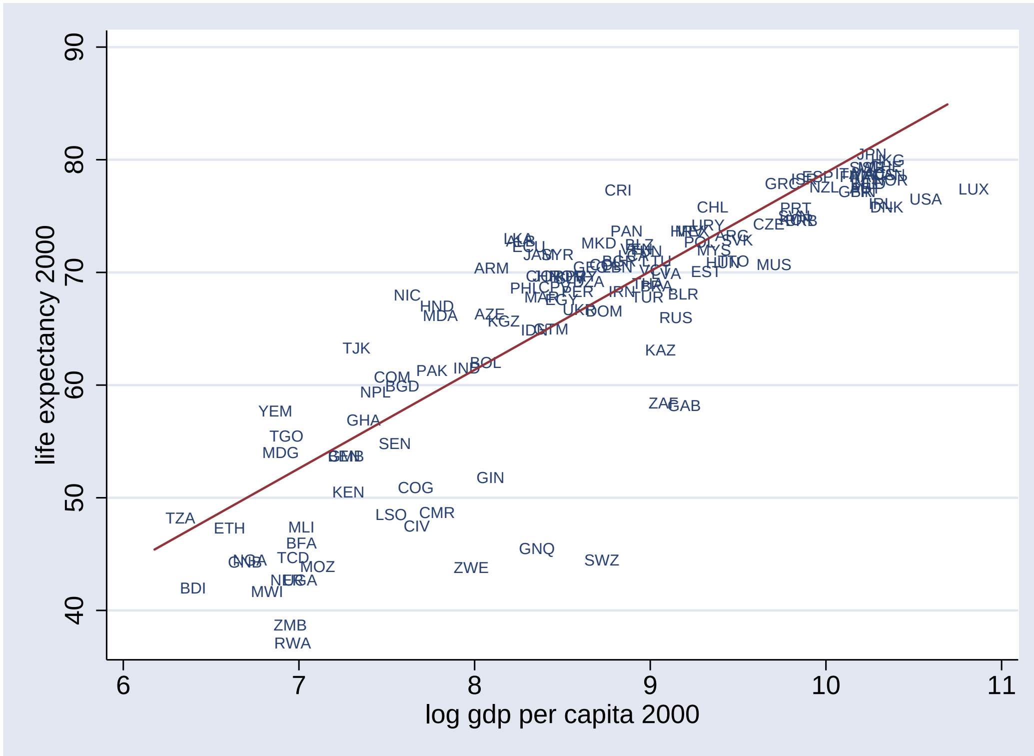

Figure 1.6. The association between income per capita and life expectancy at birth in 2000.

brought about by the introduction of new technologies and creation of new firms, which replaces existing firms and technologies. This creates a natural social tension, even in a growing society. Another source of social tension related to growth (and development) is that, as emphasized by Simon Kuznets and discussed in detail in Part 7 below, growth and development are often accompanied by sweeping structural transformations, also destroying certain established relationships and creating yet other winners and losers in the process. One of the important lessons of political economy analyses of economic growth, which will be discussed in the last part of the book, concerns how institutions and policies can be arranged so that those who lose out from the process of economic growth can be compensated or perhaps prevented from blocking economic progress.

A stark illustration of the fact that growth does not always mean an improvement in the living standards of all or even most citizens in a society comes from South Africa under Apartheid. Available data (from gold mining wages) illustrate that from the beginning of the 20th century until the fall of the Apartheid regime, GDP per capita grew considerably but the real wages of black South Africans, who make up the majority of the population, likely fell during this period. This of course does not imply that economic growth in South

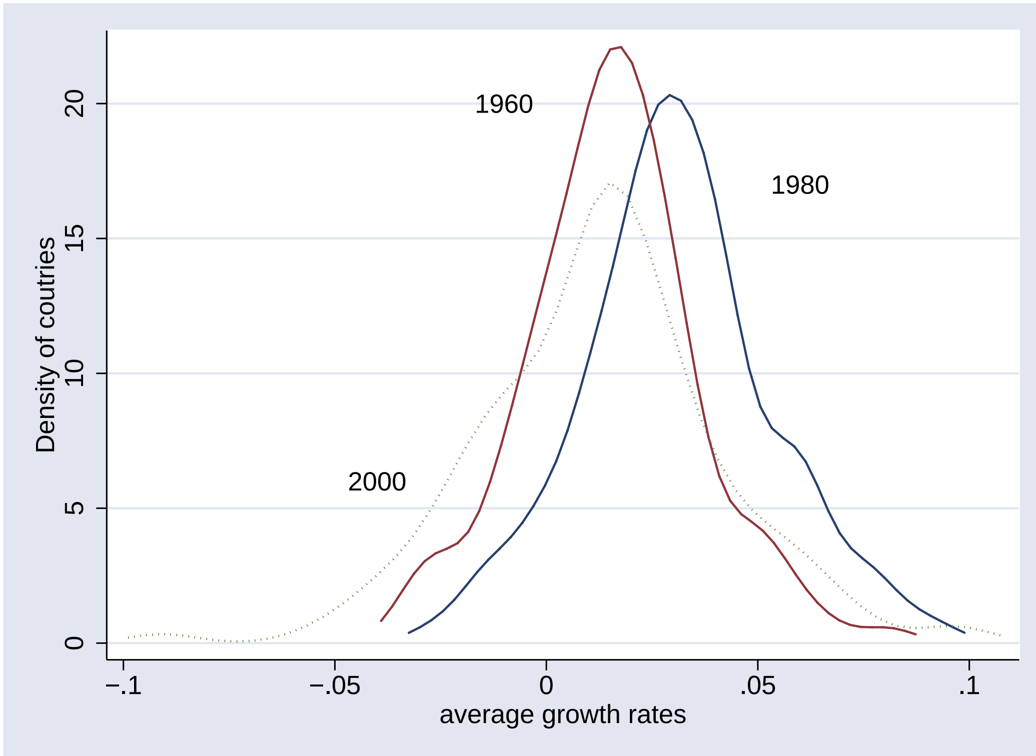

Figure 1.7. Estimates of the distribution of countries according to the growth rate of GDP per worker (PPP-adjusted) in 1960, 1980 and 2000.

Africa was not beneficial. South Africa is still one of the richest countries in sub-Saharan Africa. Nevertheless, this observation alerts us to other aspects of the economy and also underlines the potential conflicts inherent in the growth process. Similarly, most existing evidence suggests that during the early phases of the British Industrial Revolution, which started the process of modern economic growth, the living standards of most workers may have fallen or at best remained stagnant. This pattern of potential divergence between GDP per capita and the economic fortunes of large number of individuals and society is not only interesting in and of itself, but it may also inform us about why certain segments of the society may be in favor of policies and institutions that do not encourage growth.

1.3. Economic Growth and Income Differences

How could one country be more than thirty times richer than another? The answer lies in differences in growth rates. Take two countries, A and B, with the same initial level of income at some date. Imagine that country A has 0% growth per capita, so its income per capita remains constant, while country B grows at 2% per capita. In 200 years' time country B will be more than 52 times richer than country A. Therefore, the United States is considerably

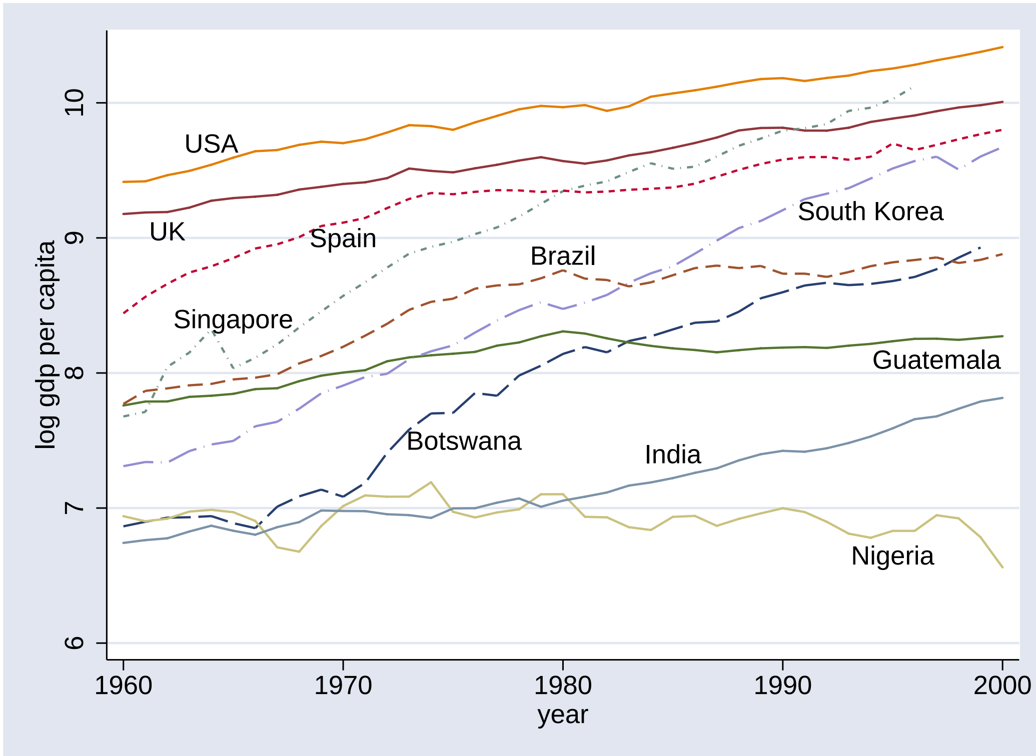

Figure 1.8. The evolution of income per capita in the United States, United Kingdom, Spain, Singapore, Brazil, Guatemala, South Korea, Botswana, Nigeria and India, 1960-2000.

richer than Nigeria because it has grown steadily over an extended period of time, while Nigeria has not (and we will see that there is a lot of truth to this simple calculation; see Figures 1.8, 1.10 and 1.12).

In fact, even in the historically-brief postwar era, we see tremendous differences in growth rates across countries. This is shown in Figure 1.7 for the postwar era, which plots the density of growth rates across countries in 1960, 1980 and 2000. The growth rate in 1960 refers to the (geometric) average of the growth rate between 1950 and 1969, the growth rate in 1980 refers to the average growth rate between 1970 and 1989 and 2000 refers to the average between 1990 and 2000 (in all cases subject to data availability; all data from Penn World tables). Figure 1.7 shows that in each time interval, there is considerable variability in growth rates; the cross-country distribution stretches from negative growth rates to average growth rates as high as 10% a year.

Figure 1.8 provides another look at these patterns by plotting log GDP per capita for a number of countries between 1960 and 2000 (in this case, we look at GDP per capita instead of GDP per worker both for data coverage and also to make the figures more comparable to the historical figures below). At the top of the figure, we see US and UK GDP per capita increasing at a steady pace, with a slightly faster growth in the United States, so that the log ("proportional") gap between the two countries is larger in 2000 than it is in 1960. Spain starts much poorer than the United States and the UK in 1960 but grows very rapidly between 1960 and the mid-1970s, thus closing the gap between itself and the United States and the UK. The three countries that show very rapid growth in this figure are Singapore, South Korea and Botswana. Singapore starts much poorer than the UK and Spain in 1960, but grows very rapidly and by the mid-1990s it has become richer than both. South Korea has a similar trajectory, though it starts out poorer than Singapore and grows slightly less rapidly, so that by the end of the sample it is still a little poorer than Spain. The other country that has grown very rapidly is the "African success story" Botswana, which was extremely poor at the beginning of the sample. Its rapid growth, especially after 1970, has taken Botswana to the ranks of the middle-income countries by 2000.

The two Latin American countries in this picture, Brazil and Guatemala, illustrate the often-discussed Latin American economic malaise of the postwar era. Brazil starts out richer than Singapore, South Korea and Botswana and has a relatively rapid growth rate between 1960 and 1980. But it experiences stagnation from 1980 onwards, so that by the end of the sample Singapore, South Korea and Botswana have become richer than Brazil. Guatemala's experience is similar but even more bleak. Contrary to Brazil, there is little growth in Guatemala between 1960 and 1980 and no growth between 1980 and 2000.

Finally, Nigeria and India start out at similar levels of income per capita as Botswana but experience little growth until the 1980s. Starting in 1980, the Indian economy experiences relatively rapid growth, though this has not been sufficient for its income per capita to catch up with the other nations in the figure. Finally, Nigeria, in a pattern that is unfortunately all-too-familiar in sub-Saharan Africa, experiences a contraction of its GDP per capita, so that in 2000 it is in fact poorer than it was in 1960.

The patterns shown in Figure 1.8 are what we would like to understand and explain. Why is the United States richer in 1960 than other nations and able to grow at a steady pace thereafter? How did Singapore, South Korea and Botswana manage to grow at a relatively rapid pace for 40 years? Why did Spain grow relatively rapidly for about 20 years, but then slow down? Why did Brazil and Guatemala stagnate during the 1980s? What is responsible for the disastrous growth performance of Nigeria?

1.4. Origins of Today's Income Differences and World Economic Growth

The growth rate differences shown in Figures 1.7 and 1.8 are interesting in their own right and could also be, in principle, responsible for the large differences in income per capita we observe today. But are they? The answer is *no*. Figure 1.8 shows that in 1960 there was already a very large gap between the United States on the one hand and India and Nigeria on the other.

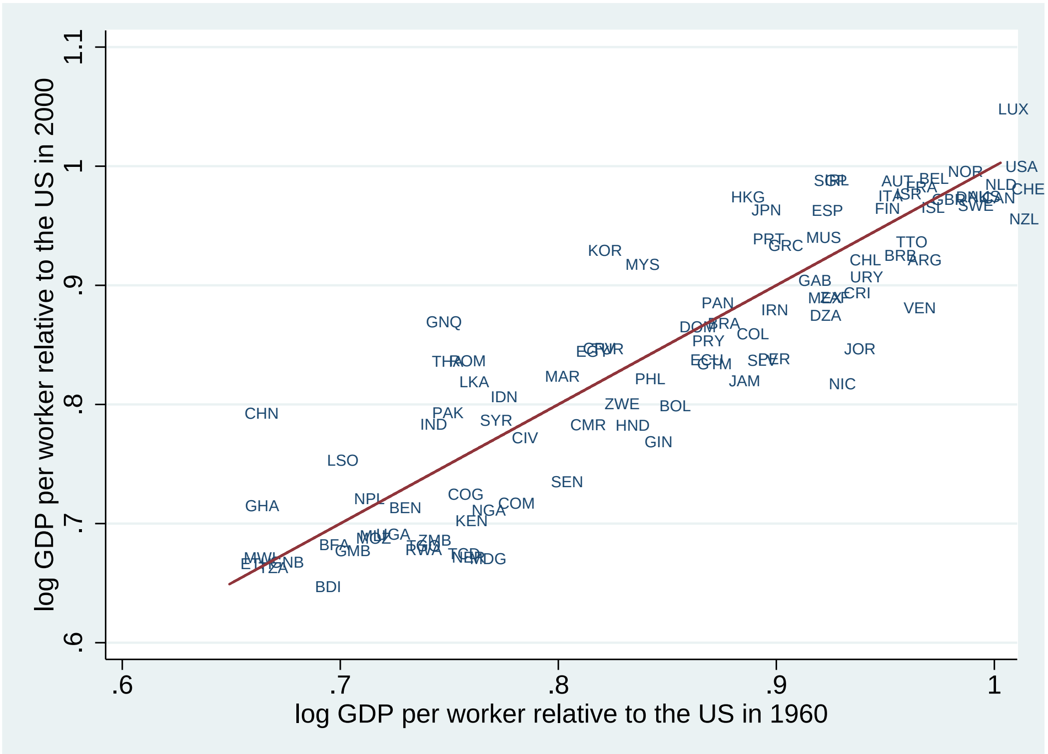

Figure 1.9. Log GDP per worker in 2000 versus log GDP per worker in 1960, together with the 45° line.

This can be seen more easily in Figure 1.9, which plots log GDP per worker in 2000 versus log GDP per capita in 1960 (in both cases relative to the US value) superimposed over the 45° line. Most observations are around the 45° line, indicating that the relative ranking of countries has changed little between 1960 and 2000. Thus the origins of the very large income differences across nations are not to be found in the postwar era. There are striking growth differences during the postwar era but the evidence presented so far suggests that the "world income distribution" has been more or less stable, with a slight tendency towards becoming more unequal.

If not in the postwar era, when did this growth gap emerge? The answer is that much of the divergence took place during the 19th and early 20th centuries. Figures 1.10 and 1.12 give a glimpse of these 19th-century developments by using the data compiled by Angus Maddison for GDP per capita differences across nations going back to 1820 (or sometimes earlier). These data are less reliable than Summers-Heston's Penn World tables, since they do not come from standardized national accounts. Moreover, the sample is more limited and

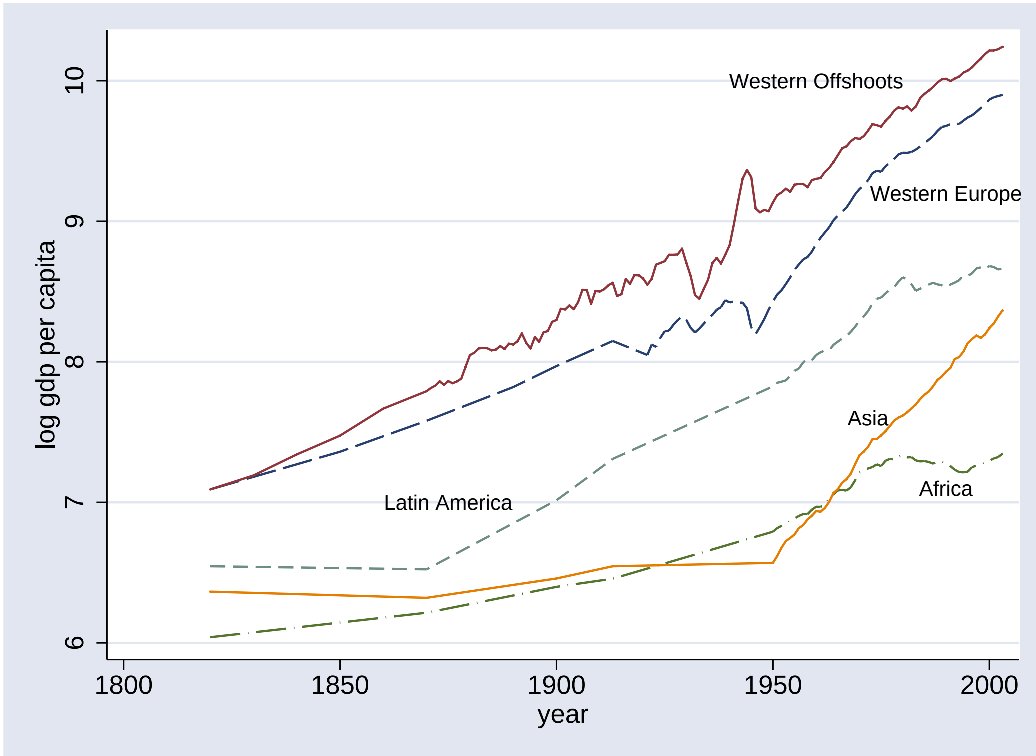

Figure 1.10. The evolution of average GDP per capita in Western Offshoots, Western Europe, Latin America, Asia and Africa, 1820-2000.

does not include observations for all countries going back to 1820. Finally, while these data include a correction for PPP, this is less reliable than the price comparisons used to construct the price indices in the Penn World tables. Nevertheless, these are the best available estimates for differences in prosperity across a large number of nations going back to the 19th century.

Figure 1.10 illustrates the divergence; it depicts the evolution of average income between five groups of countries, Western Offshoots of Europe (the United States, Canada, Australia and New Zealand), Western Europe, Latin America, Asia and Africa. It shows the relatively rapid growth of the Western Offshoots and West European countries during the 19th century, while Asia and Africa remained stagnant and Latin America showed little growth. The relatively small income gap in 1820 had become much larger by 1960.

Another major macroeconomic fact is visible in Figure 1.10: Western Offshoots and West European nations experience a noticeable dip in GDP per capita around 1929. This is because of the famous Great Depression. Western offshoots, in particular the United States, only recovered fully from this large recession in the wake of WWII. How an economy can experience such a sharp decline in output and how it recovers from such a shock are among the major questions of macroeconomics. While the Great Depression falls outside the scope of the current book, we will later discuss the relationship between economic crises and growth as well as potential sources of volatility in economic growth.

A variety of other evidence suggests that differences in income per capita were even smaller once we go back further than 1820. Maddison also has estimates for average income for the same groups of countries going back to 1000 AD or even earlier. We extend Figure 1.10 using these data; the results are shown in Figure 1.11. Although these numbers are based on scattered evidence and informed guesses, the general pattern is consistent with qualitative historical evidence and the fact that income per capita in any country cannot have been much less than $500 in terms of 2000 US dollars, since individuals could not survive with real incomes much less than this level. Figure 1.11 shows that as we go further back, the gap among countries becomes much smaller. This further emphasizes that the big divergence among countries has taken place over the past 200 years or so. Another noteworthy feature that becomes apparent from this figure is the remarkable nature of world economic growth. Much evidence suggests that there was only limited economic growth before the 18th century and certainly before the 15th century. While certain civilizations, including Ancient Greece, Rome, China and Venice, managed to grow, their growth was either not sustained (thus ending with collapses and crises) or progress at only at a slow pace. No society before 19th-century Western Europe and the United States achieved steady growth at comparable rates. In fact, Maddison's estimates show a slow but steady increase in West European GDP per capita even earlier, starting in 1000. This view is not shared by all economic historians, many of whom estimate that there was little increase in income per capita before 1500 or even before 1800. For our purposes this is not central, however. What is important is that, using Walter Rostow's terminology, Figure 1.11 shows a pattern of *takeoff* into sustained growth; the economic growth experience of Western Europe and Western Offshoots appears to have changed dramatically about 200 years or so ago. Economic historians debate whether there was a discontinuous change in economic activity to deserve the terms takeoff or Industrial Revolution. This debate is besides the point for our purposes. Whether or not the change was discontinuous, it was present and transformed the functioning of many economies. As a result of this transformation, the stagnant or slowly-growing economies of Europe embarked upon a path of sustained growth. The origins of today's riches and also of today's differences in prosperity are to be found in this pattern of takeoff during the 19th century. In the same time as much of Western Europe and its Offshoots grew rapidly, much of the rest of the world did not experience a comparable takeoff or did so much later. Therefore, an understanding of modern economic growth and current cross-country income differences ultimately necessitates an inquiry into the causes of why the takeoff occurred, why it did so about 200 years ago, and why it took place only in some areas and not in others.

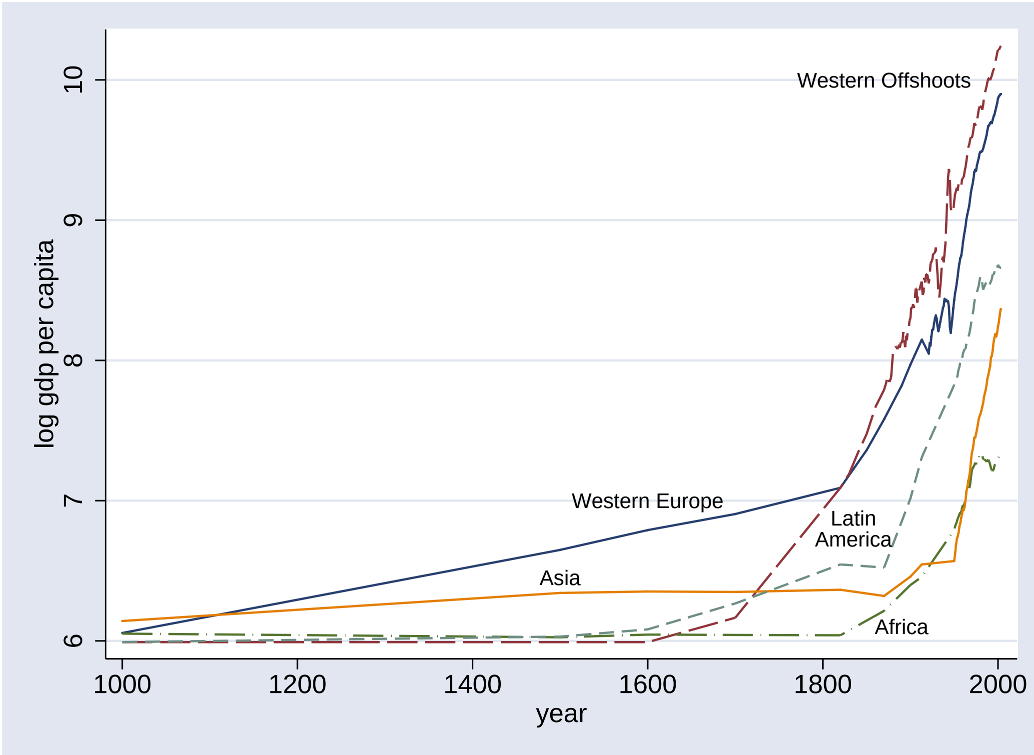

Figure 1.11. The evolution of average GDP per capita in Western Offshoots, Western Europe, Latin America, Asia and Africa, 1000-2000.

Figure 1.12 shows the evolution of income per capita for United States, Britain, Spain, Brazil, China, India and Ghana. This figure confirms the patterns shown in Figure 1.10 for averages, with the United States Britain and Spain growing much faster than India and Ghana throughout, and also much faster than Brazil and China except during the growth spurts experienced by these two countries.

Overall, on the basis of the available information we can conclude that the origins of the current cross-country differences in economic performance in income per capita lie during the 19th and early 20th centuries (or perhaps even during the late 18th century). This divergence took place at the same time as a number of countries in the world "took off" and achieved sustained economic growth. Therefore understanding modern economic growth is not only interesting and important in its own right but also holds the key to understanding the causes of cross-country differences in income per capita today.

1.5. Conditional Convergence

We have so far documented the large differences in income per capita across nations, the slight divergence in economic fortunes over the postwar era and the much larger divergence

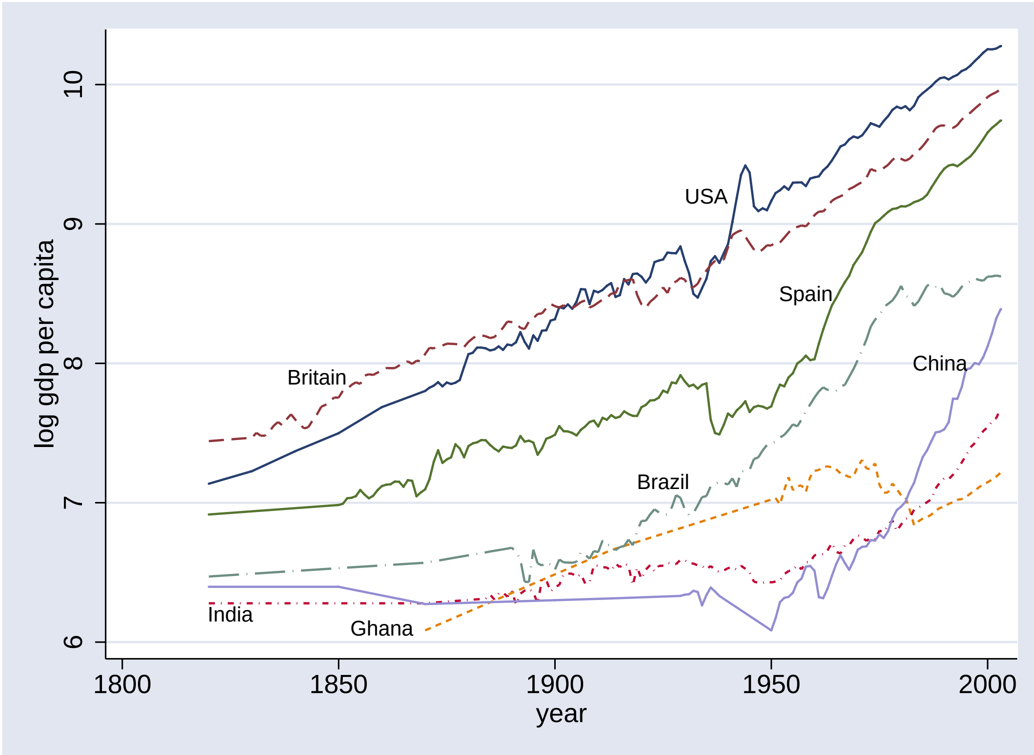

Figure 1.12. The evolution of income per capita in the United States, Britain, Spain, Brazil, China, India and Ghana, 1820-2000.

since the early 1800s. The analysis focused on the "unconditional" distribution of income per capita (or per worker). In particular, we looked at whether the income gap between two countries increases or decreases irrespective of these countries' "characteristics" (e.g., institutions, policies, technology or even investments). Alternatively, we can look at the "conditional" distribution (e.g., Barro and Sala-i-Martin, 1992). Here the question is whether the economic gap between two countries that are similar in observable characteristics is becoming narrower or wider over time. When we look at the conditional distribution of income per capita across countries the picture that emerges is one of conditional convergence: in the postwar period, the income gap between countries that share the same characteristics typically closes over time (though it does so quite slowly). This is important both for understanding the statistical properties of the world income distribution and also as an input into the types of theories that we would like to develop.

How do we capture conditional convergence? Consider a typical "Barro growth regression":

$$g_{t,t-1} = \beta \ln y_{t-1} + \mathbf{X}'_{t-1}\boldsymbol{\alpha} + \varepsilon_t \tag{1.1}$$

where $g_{t,t-1}$ is the *annual* growth rate between dates $t-1$ and t, y_{t-1} is output per worker (or income per capita) at date $t-1$, and $\mathbf{X}_{t-1}$ is a vector of variables that the regression is conditioning on with coefficient vector $\boldsymbol{\alpha}$. These variables are included because they are potential determinants of steady state income and/or growth. First note that without covariates equation (1.1) is quite similar to the relationship shown in Figure 1.9 above. In particular, since $g_{t,t-1} \simeq \ln y_t - \ln y_{t-1}$, equation (1.1) can be written as

$$\ln y_t \simeq (1+\beta) \ln y_{t-1} + \varepsilon_t.$$

Figure 1.9 showed that the relationship between log GDP per worker in 2000 and log GDP per worker in 1960 can be approximated by the 45° line, so that in terms of this equation, β should be approximately equal to 0. This is confirmed by Figure 1.13, which depicts the relationship between the (geometric) average growth rate between 1960 and 2000 and log GDP per worker in 1960. This figure reiterates that there is no "unconditional" convergence for the entire world over the postwar period.

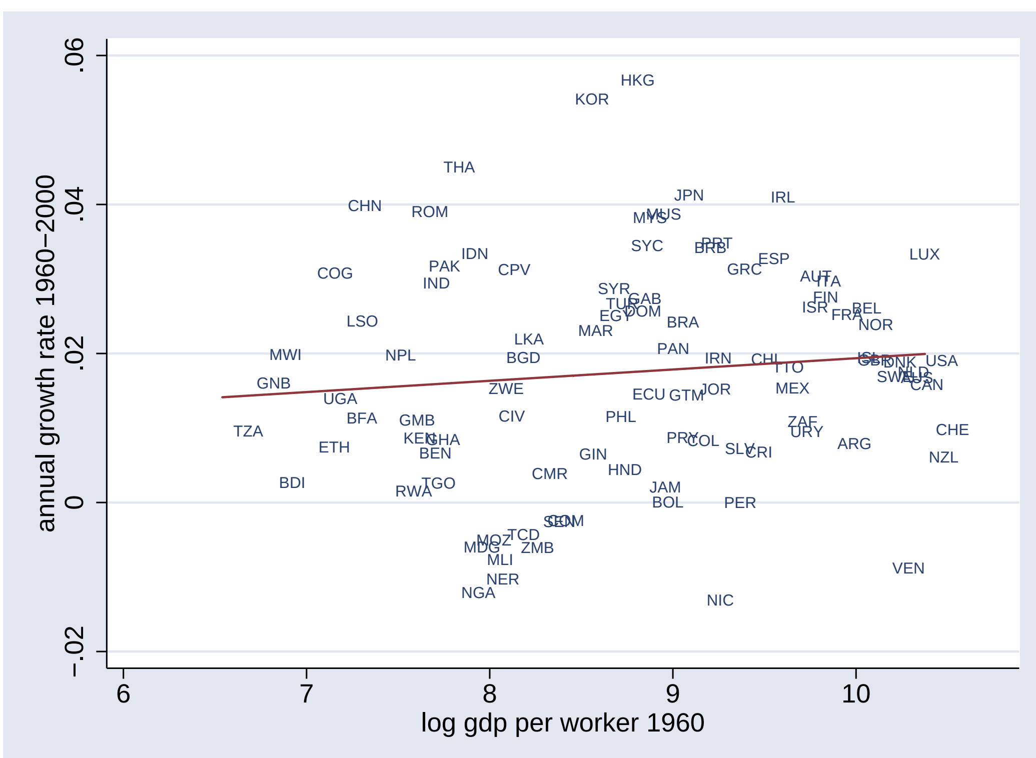

Figure 1.13. Annual growth rate of GDP per worker between 1960 and 2000 versus log GDP per worker in 1960 for the entire world.

While there is no convergence for the entire world, when we look among the "OECD" nations,[1] we see a different pattern. Figure 1.14 shows that there is a strong negative relationship between log GDP per worker in 1960 and the annual growth rate between 1960 and 2000 among the OECD countries. What distinguishes this sample from the entire world sample is the relative homogeneity of the OECD countries, which have much more similar institutions, policies and initial conditions than the entire world. This suggests that there might be a type of conditional convergence when we control for certain country characteristics potentially affecting economic growth.

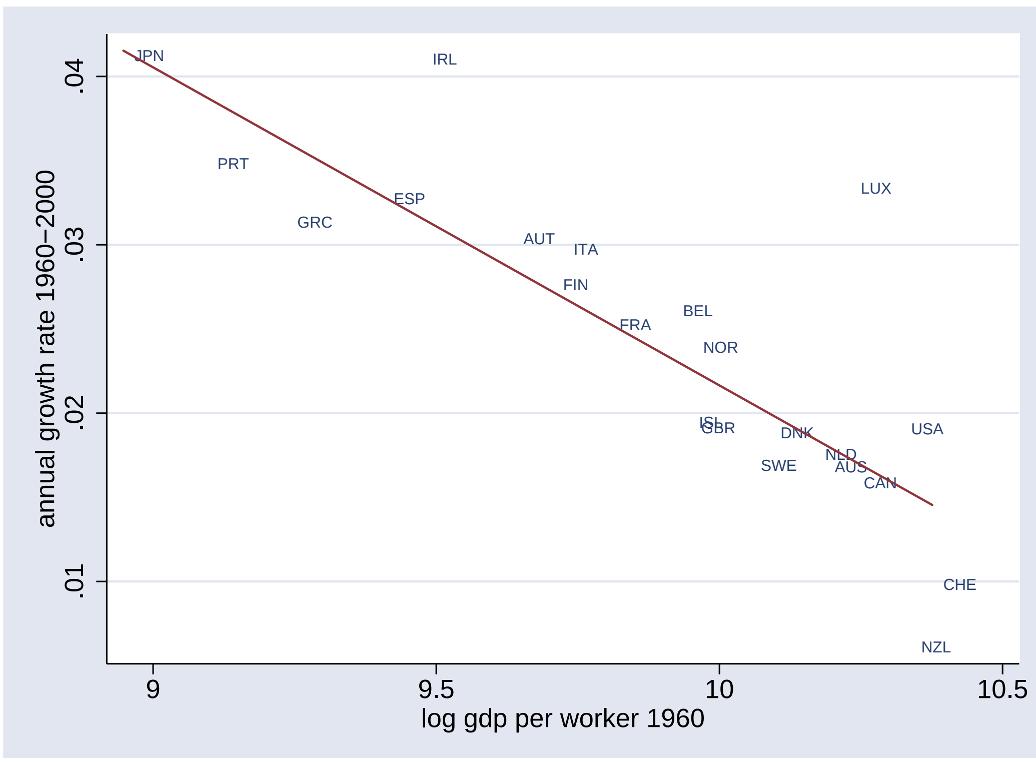

Figure 1.14. Annual growth rate of GDP per worker between 1960 and 2000 versus log GDP per worker in 1960 for core OECD countries.

This is what the vector $\mathbf{X}_{t-1}$ captures in equation (1.1). In particular, when this vector includes variables such as years of schooling or life expectancy, Barro and Sala-i-Martin estimate β to be approximately -0.02, indicating that the income gap between countries that have the same human capital endowment has been narrowing over the postwar period on average at about 2 percent a year.

Therefore, there is no evidence of (unconditional) convergence in the world income distribution over the postwar era (in fact, the evidence suggests some amount of divergence in

[1]That is, the initial members of the OECD club plotted in this picture, which excludes more recent OECD members such as Turkey, Mexico and Korea.

incomes across nations), there is some evidence for conditional convergence, meaning that the income gap between countries that are similar in observable characteristics appears to narrow over time. This last observation is relevant both for understanding among which countries the economic divergence has occurred and for determining what types of models we might want to consider for understanding the process of economic growth and differences in economic performance across nations. For example, we will see that many of the models we will study shortly, including the basic Solow and the neoclassical growth models, suggest that there should be "transitional dynamics" as economies below their steady-state (target) level of income per capita grow towards that level. Conditional convergence is consistent with this type of transitional dynamics.

1.6. Correlates of Economic Growth

The discussion of conditional convergence in the previous section emphasized the importance of certain country characteristics that might be related to the process of economic growth. What types of countries grow more rapidly? Ideally, we would like to answer this question at a "causal" level. In other words, we would like to know which specific characteristics of countries (including their policies and institutions) have a causal effect on growth. A causal effect here refers to the answer to the following counterfactual thought experiment: if, all else equal, a particular characteristic of the country were changed "exogenously" (i.e., not as part of equilibrium dynamics or in response to a change in other observable or unobservable variables), what would be the effect on equilibrium growth? Answering such causal questions is quite challenging, however, precisely because it is difficult to isolate changes in endogenous variables that are not driven by equilibrium dynamics or by some other potentially omitted factors.

For this reason, we start with the more modest question of what factors correlate with post-war economic growth. With an eye to the theories that will come in the next two chapters, the two obvious candidates to look at are investments in physical capital and in human capital.

Figure 1.15 shows a strong positive association between the average growth of investment to GDP ratio and economic growth. Figure 1.16 shows a positive correlation between average years of schooling and economic growth. These figures therefore suggest that the countries that have grown faster are typically those that have invested more in physical capital and those that started out the postwar era with greater human capital. It has to be stressed that these figures do not imply that physical or human capital investment are the causes of economic growth (even though we expect from basic economic theory that they should contribute to increasing output). So far these are simply correlations, and they are likely

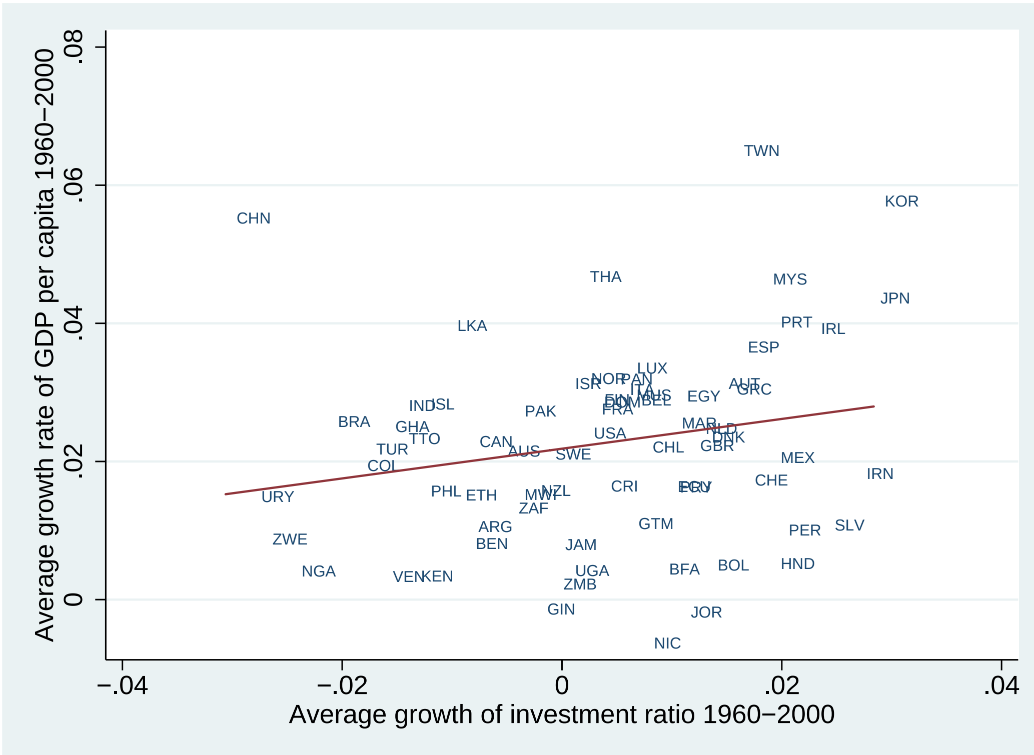

Figure 1.15. The relationship between average growth of GDP per capita and average growth of investments to GDP ratio, 1960-2000.

driven, at least in part, by omitted factors affecting both investment and schooling on the one hand and economic growth on the other.

We will investigate the role of physical and human capital in economic growth further in Chapter 3. One of the major points that will emerge from our analysis there is that focusing only on physical and human capital is not sufficient. Both to understand the process of sustained economic growth and to account for large cross-country differences in income, we also need to understand why societies differ in the efficiency with which they use their physical and human capital. We normally use the shorthand expression "technology" to capture factors other than physical and human capital affecting economic growth and performance (and we will do so throughout the book). It is therefore important to remember that technology differences across countries include both genuine differences in the techniques and in the quality of machines used in production, but also differences in productive efficiency resulting from differences in the organization of production, from differences in the way that markets are organized and from potential market failures (see in particular Chapter 21 on differences in productive efficiency resulting from the organization of markets and market failures). A detailed study of "technology" (broadly construed) is necessary for understanding both the

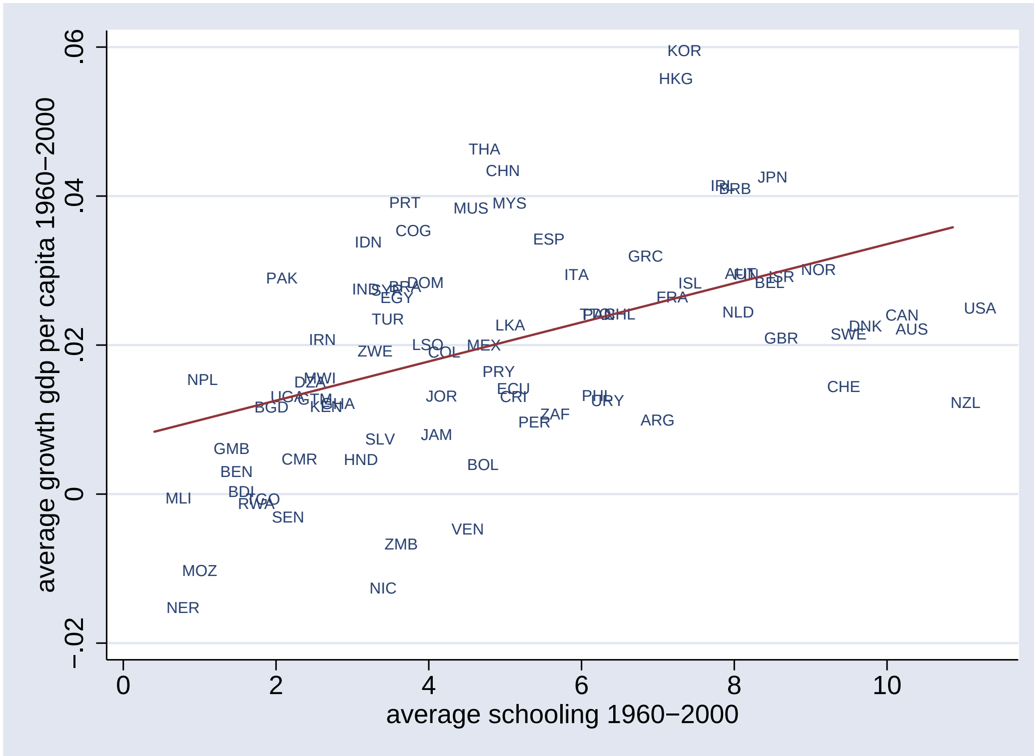

Figure 1.16

world-wide process of economic growth and cross-country differences. The role of technology in economic growth will be investigated in Chapter 3 and in later chapters.

1.7. From Correlates to Fundamental Causes

The correlates of economic growth, such as physical capital, human capital and technology, will be our first topic of study. But these are only *proximate causes* of economic growth and economic success (even if we convince ourselves that there is a causal in element the correlations shown above). It would not be entirely satisfactory to explain the process of economic growth and cross-country differences with technology, physical capital and human capital, since presumably there are reasons for why technology, physical capital and human capital differ across countries. In particular, if these factors are so important in generating large cross country income differences and causing the takeoff into modern economic growth, why do certain societies fail to improve their technologies, invest more in physical capital, and accumulate more human capital?

Let us return to Figure 1.8 to illustrate this point further. This figure shows that South Korea and Singapore have managed to grow at very rapid rates over the past 50 years, while Nigeria has failed to do so. We can try to explain the successful performance of South Korea and Singapore by looking at the correlates of economic growth—or at the proximate causes of economic growth. We can conclude, as many have done, that rapid capital accumulation has been a major cause of these growth miracles, and debate the role of human capital and technology. We can blame the failure of Nigeria to grow on its inability to accumulate capital and to improve its technology. These answers are undoubtedly informative for understanding the mechanics of economic successes and failures of the postwar era. But at some level they will also not have answered the central questions: how did South Korea and Singapore manage to grow, while Nigeria failed to take advantage of the growth opportunities? If physical capital accumulation is so important, why did Nigeria not invest more in physical capital? If education is so important, why are education levels in Nigeria still so low and why is existing human capital not being used more effectively? The answer to these questions is related to the *fundamental causes* of economic growth.

We will refer to potential factors affecting why societies end up with different technology and accumulation choices as the fundamental causes of economic growth. At some level, fundamental causes are the factors that enable us to link the questions of economic growth to the concerns of the rest of social sciences, and ask questions about the role of policies, institutions, culture and exogenous environmental factors. At the risk of oversimplifying complex phenomena, we can think of the following list of potential fundamental causes: (i) luck (or multiple equilibria) that lead to divergent paths among societies with identical opportunities, preferences and market structures; (ii) geographic differences that affect the environment in which individuals live and that influence the productivity of agriculture, the availability of natural resources, certain constraints on individual behavior, or even individual attitudes; (iii) institutional differences that affect the laws and regulations under which individuals and firms function and thus shape the incentives they have for accumulation, investment and trade; and (iv) cultural differences that determine individuals' values, preferences and beliefs. Chapter 4 will present a detailed discussion of the distinction between proximate and fundamental causes and what types of fundamental causes are more promising in explaining the process of economic growth and cross-country income differences.

For now, it is useful to briefly return to South Korea and Singapore versus Nigeria, and ask the questions (even if we are not in a position to fully answer them yet): can we say that South Korea and Singapore owe their rapid growth to luck, while Nigeria was unlucky? Can we relate the rapid growth of South Korea and Singapore to geographic factors? Can we relate them to institutions and policies? Can we find a major role for culture? Most detailed accounts of post-war economics and politics in these countries emphasize the role of growth-promoting policies in South Korea and Singapore—including the relative security of property rights and investment incentives provided to firms. In contrast, Nigeria's postwar history is one of civil war, military coups, extreme corruption and an overall environment failing to provide incentives to businesses to invest and upgrade their technologies. It therefore seems necessary to look for fundamental causes of economic growth that make contact with these facts and then provide coherent explanations for the divergent paths of these countries. Jumping ahead a little, it will already appear implausible that luck can be the major explanation. There were already significant differences between South Korea, Singapore and Nigeria at the beginning of the postwar era. It is also equally implausible to link the divergent fortunes of these countries to geographic factors. After all, their geographies did not change, but the growth spurts of South Korea and Singapore started in the postwar era. Moreover, even if we can say that Singapore benefited from being an island, without hindsight one might have concluded that Nigeria had the best environment for growth, because of its rich oil reserves.[2] Cultural differences across countries are likely to be important in many respects, and the rapid growth of many Asian countries is often linked to certain "Asian values". Nevertheless, cultural explanations are also unlikely to provide the whole story when it comes to fundamental causes, since South Korean or Singaporean culture did not change much after the end of WWII, while their rapid growth performances are distinctly post-war phenomena. Moreover, while South Korea grew rapidly, North Korea, whose inhabitants share the same culture and Asian values, had one of the most disastrous economic performances of the past 50 years.

This admittedly quick (and perhaps partial) account suggests that we have to look at the fundamental causes of economic growth in institutions and policies that affect incentives to accumulate physical and human capital and improve technology. Institutions and policies were favorable to economic growth in South Korea and Singapore, but not in Nigeria. Understanding the fundamental causes of economic growth is, in large part, about understanding the impact of these institutions and policies on economic incentives and why, for example, they have been growth-enhancing in the former two countries, but not in Nigeria. The intimate link between fundamental causes and institutions highlighted by this discussion motivates the last part of the book, which is devoted to the political economy of growth, that is, to the study of how institutions affect growth and why they differ across countries.

An important caveat should be noted at this point. Discussions of geography, institutions and culture can sometimes be carried out without explicit reference to growth models or even to growth empirics. After all, this is what many non-economist social scientists do. However, fundamental causes can only have a big impact on economic growth if they affect parameters and policies that have a first-order influence on physical and human capital and

[2]One can then turn this around and argue that Nigeria is poor because of a "natural resource curse," i.e., precisely because it has abundant and valuable natural resources. But this is not an entirely compelling empirical argument, since there are other countries, such as Botswana, with abundant natural resources that have grown rapidly over the past 50 years. More important, the only plausible channel through which abundance of natural resources may lead to worse economic outcomes is related to institutional and political economy factors. This then takes us to the realm of institutional fundamental causes.

technology. Therefore, an understanding of the mechanics of economic growth is essential for evaluating whether candidate fundamental causes of economic growth could indeed play the role that they are sometimes ascribed. Growth empirics plays an equally important role in distinguishing among competing fundamental causes of cross-country income differences. It is only by formulating parsimonious models of economic growth and confronting them with data that we can gain a better understanding of both the proximate and the fundamental causes of economic growth.

1.8. The Agenda

This discussion points to the following set of facts and questions that are central to an investigation of the determinants of long-run differences in income levels and growth. The three major questions that have emerged from our brief discussion are:

(1) Why are there such large differences in income per capita and worker productivity across countries?
(2) Why do some countries grow rapidly while other countries stagnate?
(3) What sustains economic growth over long periods of time and why did sustained growth start 200 years or so ago?

- In each case, a satisfactory answer requires a set of well-formulated models that illustrate the mechanics of economic growth and cross-country income differences, together with an investigation of the fundamental causes of the different trajectories which these nations have embarked upon. In other words, in each case we need a combination of theoretical models and empirical work.
- The traditional growth models—in particular, the basic Solow and the neoclassical models—provide a good starting point, and the emphasis they place on investment and human capital seems consistent with the patterns shown in Figures 1.15 and 1.16. However, we will also see that technological differences across countries (either because of their differential access to technological opportunities or because of differences in the efficiency of production) are equally important. Traditional models treat technology (market structure) as given or at best as evolving exogenously like a blackbox. But if technology is so important, we ought to understand why and how it progresses and why it differs across countries. This motivates our detailed study of models of endogenous technological progress and technology adoption. Specifically, we will try to understand how differences in technology may arise, persist and contribute to differences in income per capita. Models of technological change will also be useful in thinking about the sources of sustained growth of the world economy over the past 200 years and why the growth process took off 200 years or so ago and has proceeded relatively steadily since then.

- Some of the other patterns we encountered in this chapter will inform us about the types of models that have the most promise in explaining economic growth and cross-country differences in income. For example, we have seen that cross-country income differences can only be accounted for by understanding why some countries have grown rapidly over the past 200 years, while others have not. Therefore, we need models that can explain how some countries can go through periods of sustained growth, while others stagnate.

  Nevertheless, we have also seen that the postwar world income distribution is relatively stable (at most spreading out slightly from 1960 to 2000). This pattern has suggested to many economists that we should focus on models that generate large "permanent" cross-country differences in income per capita, but not necessarily large "permanent" differences in growth rates (at least not in the recent decades). This is based on the following reasoning: with substantially different long-run growth rates (as in models of endogenous growth, where countries that invest at different rates grow at different rates), we should expect significant divergence. We saw above that despite some widening between the top and the bottom, the cross-country distribution of income across the world is relatively stable.

  Combining the post-war patterns with the origins of income differences related to the economic growth over the past two centuries suggests that we should look for models that can account both for long periods of significant growth differences and also for a "stationary" world income distribution, with large differences across countries. The latter is particularly challenging in view of the nature of the global economy today, which allows for free-flow of technologies and large flows of money and commodities across borders. We therefore need to understand how the poor countries fell behind and what prevents them today from adopting and imitating the technologies and organizations (and importing the capital) of the richer nations.
- And as our discussion in the previous section suggests, all of these questions can be (and perhaps should be) answered at two levels. First, we can use the models we develop in order to provide explanations based on the mechanics of economic growth. Such answers will typically explain differences in income per capita in terms of differences in physical capital, human capital and technology, and these in turn will be related to some other variables such as preferences, technology, market structure, openness to international trade and perhaps some distortions or policy variables. These will be our answers regarding the proximate causes of economic growth.

  We will next look at the fundamental causes underlying these proximate factors, and try to understand why some societies are organized differently than others. Why do they have different market structures? Why do some societies adopt policies that encourage economic growth while others put up barriers against technological change? These questions are central to a study of economic growth, and can only be answered by developing systematic models of the political economy of development and looking at the historical process of economic growth to generate data that can shed light on these fundamental causes.

Our next task is to systematically develop a series of models to understand the mechanics of economic growth. In this process, we will encounter models that underpin the way economists think about the process of capital accumulation, technological progress, and productivity growth. Only by understanding these mechanics can we have a framework for thinking about the causes of why some countries are growing and some others are not, and why some countries are rich and others are not.

Therefore, the approach of the book will be two-pronged: on the one hand, it will present a detailed exposition of the mathematical structure of a number of dynamic general equilibrium models useful for thinking about economic growth and macroeconomic phenomena; on the other, we will try to uncover what these models imply about which key parameters or key economic processes are different across countries and why. Using this information, we will then attempt to understand the potential fundamental causes of differences in economic growth.

1.9. References and Literature

The empirical material presented in this chapter is largely standard and parts of it can be found in many books, though interpretations and exact emphases differ. Excellent introductions, with slightly different emphases, are provided in Jones's (1998, Chapter 1) and Weil's (2005, Chapter 1) undergraduate economic growth textbooks. Barro and Sala-i-Martin (2004) also present a brief discussion of the stylized facts of economic growth, though their focus is on postwar growth and conditional convergence rather than the very large cross-country income differences and the long-run perspective emphasized here. An excellent and very readable account of the key questions of economic growth, with a similar perspective to the one here, is provided in Helpman (2005).

Much of the data used in this chapter comes from Summers-Heston's Penn World tables (latest version, Summers, Heston and Aten, 2005). These tables are the result of a very careful study by Robert Summers and Alan Heston to construct internationally comparable price indices and internationally comparable estimates of income per capita and consumption. PPP adjustment is made possible by these data. Summers and Heston (1991) give a very lucid discussion of the methodology for PPP adjustment and its use in the Penn World tables. PPP adjustment enables us to construct measures of income per capita that are comparable across countries. Without PPP adjustment, differences in income per capita across countries can be computed using the current exchange rate or some fundamental exchange-rate. There are many problems with such exchange-rate-based measures. The most important one is that they do not make an allowance for the fact that relative prices and even the overall price level differ markedly across countries. PPP-adjustment brings us much closer to differences in "real income" and "real consumption". Information on "workers" (active population), consumption and investment are also from this dataset. GDP, consumption and investment data from the Penn World tables are expressed in 1996 constant US dollars. Life expectancy data are from the World Bank's World Development Indicators CD-ROM, and refer to the average life expectancy of males and females at birth. This dataset also contains a range of other useful information. Schooling data are from Barro and Lee's (2002) dataset, which contains internationally comparable information on years of schooling.

In all figures and regressions, growth rates are computed as geometric averages. In particular, the geometric average growth rate of output per capita y between date t and $t+T$ is

$$g_{t,t+T} \equiv \left(\frac{y_{t+T}}{y_t}\right)^{1/T} - 1.$$

The geometric average growth rate is more appropriate to use in the context of income per capita than the arithmetic average, since the growth rate refers to "proportional growth". It can be easily verified from this formula that if $y_{t+1} = (1+g)\, y_t$ for all t, then $g_{t+T} = g$.

Historical data are from various works by Angus Maddison (2001, 2005). While these data are not as reliable as the estimates from the Penn World tables, the general patterns they show are typically consistent with evidence from a variety of different sources. Nevertheless, there are points of contention. For example, as Figure 1.11 shows, Maddison's estimates show a slow but relatively steady growth of income per capita in Western Europe starting in 1000. This is disputed by some historians and economic historians. A relatively readable account, which strongly disagrees with this conclusion, is provided in Pomeranz (2001), who argues that income per capita in Western Europe and the Yangtze Valley in China were broadly comparable as late as 1800. This view also receives support from recent research by Allen (2004), which documents that the levels of agricultural productivity in 1800 were comparable in Western Europe and China. Acemoglu, Johnson and Robinson (2002 and 2005) use urbanization rates as a proxy for income per capita and obtain results that are intermediate between those of Maddison and Pomeranz. The data in Acemoglu, Johnson and Robinson (2002) also confirms the fact that there were very limited income differences across countries as late as the 1500s, and that the process of rapid economic growth started sometime in the 19th century (or perhaps in the late 18th century). Recent research by Broadberry and Gupta (2006) also disputes Pomeranz's arguments and gives more support to a pattern in which there was already an income gap between Western Europe and China by the end of the 18th century.

The term *takeoff* I used in Section 1.4 is introduced in Walter Rostow's famous book *Stages of Economic Growth* (1960) and has a broader connotation than the term "Industrial Revolution," which economic historians typically use to refer to the process that started in Britain at the end of the 18th century (e.g., Ashton, 1968). Mokyr (1990) contains an excellent discussion of the debate on whether the beginning of industrial growth was due to a continuous or discontinuous change and, consistent with my emphasis here, concludes that this is secondary to the more important fact that the modern process of growth *did* start around this time.

There is a large literature on the "correlates of economic growth," starting with Barro (1991), which is surveyed in Barro and Sala-i-Martin (2004) and Barro (1999). Much of this literature, however, interprets these correlations as causal effects, even when this is not warranted (see the further discussion in Chapters 3 and 4). Note that while Figure 1.15 looks at the relationship between the average growth of investment to GDP ratio and economic growth, Figure 1.16 shows the relationship between average schooling (not its growth) and economic growth. There is a much weaker relationship between growth of schooling and economic growth, which may be for a number of reasons; first, there is considerable measurement error in schooling estimates (see Krueger and Lindahl, 2000); second, as shown in in some of the models that will be discussed later, the main role of human capital may be to facilitate technology adoption, thus we may expect a stronger relationship between the level of schooling and economic growth than the change in schooling and economic growth (see Chapter 10); finally, the relationship between the level of schooling and economic growth may be partly spurious, in the sense that it may be capturing the influence of some other omitted factors also correlated with the level of schooling; if this is the case, these omitted factors may be removed when we look at changes. While we cannot reach a firm conclusion on these alternative explanations, the strong correlation between the level of average schooling and economic growth documented in Figure 1.16 is interesting in itself.

The narrowing of income per capita differences in the world economy when countries are weighted by population is explored in Sala-i-Martin (2005). Deaton (2005) contains a critique of Sala-i-Martin's approach. The point that incomes must have been relatively equal around 1800 or before, because there is a lower bound on real incomes necessary for the survival of an individual, was first made by Maddison (1992) and Pritchett (1996). Maddison's estimates of GDP per capita and Acemoglu, Johnson and Robinson's estimates based on urbanization confirm this conclusion.

The estimates of the density of income per capita reported above are similar to those used by Quah (1994, 1995) and Jones (1996). These estimates use a nonparametric Gaussian kernel. The specific details of the kernel estimates do not change the general shape of the densities. Quah was also the first to emphasize the stratification in the world income distribution and the possible shift towards a "bi-modal" distribution, which is visible in Figure 1.3. He dubbed this the "Twin Peaks" phenomenon (see also Durlauf and Quah, 1994). Barro (1991) and Barro and Sala-i-Martin (1992) emphasize the presence and importance of conditional convergence, and argue against the relevance of the stratification pattern emphasized by Quah and others. The first chapter of Barro and Sala-i-Martin's (2004) textbook contains a detailed discussion from this viewpoint.

The first economist to emphasize the importance of conditional convergence and conduct a cross-country study of convergence was Baumol (1986), but he was using lower quality data than the Summers-Heston data. This also made him conduct his empirical analysis on a selected sample of countries, potentially biasing his results (see De Long, 1991). Barro's (1991) and Barro and Sala-i-Martin's (1992) work using the Summers-Heston data has been instrumental in generating renewed interest in cross-country growth regressions.

The data on GDP growth and black real wages in South Africa are from Wilson (1972). Wages refer to real wages in gold mines. Feinstein (2004) provides an excellent economic history of South Africa. The implications of the British Industrial Revolution for real wages and living standards of workers are discussed in Mokyr (1993). Another example of rapid economic growth with falling real wages is provided by the experience of the Mexican economy in the early 20th century (see Gómez-Galvarriato, 1998). There is also evidence that during this period, the average height of the population might be declining as well, which is often associated with falling living standards, see López Alonso and Porras Condy (2003).

There is a major debate on the role of technology and capital accumulation in the growth experiences of East Asian nations, particularly South Korea and Singapore. See Young (1994) for the argument that increases in physical capital and labor inputs explain almost all of the rapid growth in these two countries. See Klenow and Rodriguez-Clare (1996) and Hsieh (2001) for the opposite point of view.

The difference between proximate and fundamental causes will be discussed further in later chapters. This distinction is emphasized in a different context by Diamond (1996), though it is implicitly present in North and Thomas's (1973) classic book. It is discussed in detail in the context of long-run economic development and economic growth in Acemoglu, Johnson and Robinson (2006). We will revisit these issues in greater detail in Chapter 4.

CHAPTER 2

The Solow Growth Model

The previous chapter introduced a number of basic facts and posed the main questions concerning the sources of economic growth over time and the causes of differences in economic performance across countries. These questions are central not only for growth theory but also for macroeconomics and social sciences more generally. Our next task is to develop a simple framework that can help us think about the *proximate* causes and the mechanics of the process of economic growth and cross-country income differences. We will use this framework both to study potential sources of economic growth and also to perform simple comparative statics to gain an understanding of what features of societies are conducive to higher levels of income per capita and more rapid economic growth.

Our starting point will be the so-called Solow-Swan model named after Robert (Bob) Solow and Trevor Swan, or simply the *Solow model* for the more famous of the two economists. These two economists published two pathbreaking articles in the same year, 1956 (Solow, 1956, and Swan, 1956) introducing the Solow model. Bob Solow later developed many implications and applications of this model and was awarded the Nobel prize in economics for these contributions. This model has shaped the way we approach not only economic growth but the entire field of macroeconomics. Consequently, a byproduct of our analysis of this chapter will be a detailed exposition of the workhorse model of much of macroeconomics.

The Solow model is remarkable in its simplicity. Looking at it today, one may fail to appreciate how much of an intellectual breakthrough it was relative to what came before. Before the advent of the Solow growth model, the most common approach to economic growth built on the model developed by Roy Harrod and Evsey Domar (Harrod, 1939, Domar, 1946). The Harrod-Domar model emphasized potential dysfunctional aspects of economic growth, for example, how economic growth could go hand-in-hand with increasing unemployment (see Exercise 2.16 on this model). The Solow model demonstrated why the Harrod-Domar model was not an attractive place to start. At the center of the Solow growth model, distinguishing it from the Harrod-Domar model, is the *neoclassical* aggregate production function. This function not only enables the Solow model to make contact with microeconomics, but it also serves as a bridge between the model and the data as we will see in the next chapter.

An important feature of the Solow model, which will be shared by many models we will see in this book, is that it is a simple and *abstract* representation of a complex economy.

At first, it may appear too simple or too abstract. After all, to do justice to the process of growth or macroeconomic equilibrium, we have to think of many different individuals with different tastes, abilities, incomes and roles in society, many different sectors and multiple social interactions. Instead, the Solow model cuts through these complications by constructing a simple one-good economy, with little reference to individual decisions. Therefore, for us the Solow model will be both a starting point and a springboard for richer models.

Despite its mathematical simplicity, the Solow model can be best appreciated by going back to the microeconomic foundations of general equilibrium theory, and this is where we begin. Since the Solow model is the workhorse model of macroeconomics in general, a good grasp of its workings and foundations is not only useful in our investigations of economic growth, but also essential for modern macroeconomic analysis. In this chapter, I present the basic Solow model. The closely related neoclassical growth model will be presented in Chapter 8.

2.1. The Economic Environment of the Basic Solow Model

Economic growth and development are dynamic processes, focusing on how and why output, capital, consumption and population change over time. The study of economic growth and development therefore necessitates dynamic models. Despite its simplicity, the Solow growth model is a dynamic general equilibrium model (though many key features of dynamic general equilibrium models emphasized in Chapter 5, such as preferences and dynamic optimization are missing in this model).

The Solow model can be formulated either in discrete or in continuous time. We start with the discrete time version, both because it is conceptually simpler and it is more commonly used in macroeconomic applications. However, many growth models are formulated in continuous time and we will also provide a detailed exposition of the continuous-time version of the Solow model and show that it is often more tractable.

2.1.1. Households and Production. Consider a closed economy, with a unique final good. The economy is in discrete time running to an infinite horizon, so that time is indexed by $t = 0, 1, 2, ...$ Time periods here can correspond to days, weeks, or years. So far we do not need to take a position on this.

The economy is inhabited by a large number of households, and for now we are going to make relatively few assumptions on households because in this baseline model, they will not be optimizing. This is the main difference between the Solow model and the *neoclassical growth model*. The latter is the Solow model plus dynamic consumer (household) optimization. To fix ideas, you may want to assume that all households are identical, so that the economy admits *a representative household*—meaning that the demand and labor supply side of the economy can be represented as if it resulted from the behavior of a single household. We will return to what the representative household assumption entails in Chapter 5 and see that it is not totally innocuous. But that is for later.

What do we need to know about households in this economy? The answer is: not much. We do not yet endow households with preferences (utility functions). Instead, for now, we simply assume that they save a constant exogenous fraction s of their disposable income—irrespective of what else is happening in the economy. This is the same assumption used in basic Keynesian models and in the Harrod-Domar model mentioned above. It is also at odds with reality. Individuals do not save a constant fraction of their incomes; for example, if they did, then the announcement by the government that there will be a large tax increase next year should have no effect on their saving decisions, which seems both unreasonable and empirically incorrect. Nevertheless, the exogenous constant saving rate is a convenient starting point and we will spend a lot of time in the rest of the book analyzing how consumers behave and make intertemporal choices.

The other key agents in the economy are firms. Firms, like consumers, are highly heterogeneous in practice. Even within a narrowly-defined sector of an economy (such as sports shoes manufacturing), no two firms are identical. But again for simplicity, we start with an assumption similar to the representative household assumption, but now applied to firms. We assume that all firms in this economy have access to the same production function for the final good, or in other words, we assume that the economy admits a *representative firm*, with a representative (or aggregate) production function. Moreover, we also assume that this aggregate production function exhibits *constant returns to scale* (see below for a definition). More explicitly, the aggregate production function for the unique final good is

$$Y(t) = F[K(t), L(t), A(t)] \tag{2.1}$$

where $Y(t)$ is the total amount of production of the final good at time t, $K(t)$ is the capital stock, $L(t)$ is total employment, and $A(t)$ is technology at time t. Employment can be measured in different ways. For example, we may want to think of $L(t)$ as corresponding to hours of employment or number of employees. The capital stock $K(t)$ corresponds to the quantity of "machines" (or more explicitly, equipment and structures) used in production, and it is typically measured in terms of the value of the machines. There are multiple ways of thinking of capital (and equally many ways of specifying how capital comes into existence). Since our objective here is to start out with a simple workable model, we make the rather sharp simplifying assumption that capital is the same as the final good of the economy. However, instead of being consumed, capital is used in the production process of more goods. To take a concrete example, think of the final good as "corn". Corn can be used both for consumption and as an input, as "seed", for the production of more corn tomorrow. Capital then corresponds to the amount of corn used as seeds for further production.

Technology, on the other hand, has no natural unit. This means that $A(t)$, for us, is a *shifter* of the production function (2.1). For mathematical convenience, we will often represent $A(t)$ in terms of a number, but it is useful to bear in mind that, at the end of the day, it is a representation of a more abstract concept. Later we will discuss models in which $A(t)$ can be multidimensional, so that we can analyze economies with different types of technologies. As noted in Chapter 1, we may often want to think of a broad notion of technology, incorporating the effects of the organization of production and of markets on the efficiency with which the factors of production are utilized. In the current model, $A(t)$ represents all these effects.

A major assumption of the Solow growth model (and of the neoclassical growth model we will study in Chapter 8) is that technology is *free*; it is publicly available as a non-excludable, non-rival good. Recall that a good is *non-rival* if its consumption or use by others does not preclude my consumption or use. It is *non-excludable*, if it is impossible to prevent the person from using it or from consuming it. Technology is a good candidate for a non-excludable, non–rival good, since once the society has some knowledge useful for increasing the efficiency of production, this knowledge can be used by any firm without impinging on the use of it by others. Moreover, it is typically difficult to prevent firms from using this knowledge (at least once it is in the public domain and it is not protected by patents). For example, once the society knows how to make wheels, everybody can use that knowledge to make wheels without diminishing the ability of others to do the same (making the knowledge to produce wheels non-rival). Moreover, unless somebody has a well-enforced patent on wheels, anybody can decide to produce wheels (making the know-how to produce wheels non-excludable). The implication of the assumptions that technology is non-rival and non-excludable is that $A(t)$ is freely available to all potential firms in the economy and firms do not have to pay for making use of this technology. Departing from models in which technology is freely available will be a major step towards developing models of endogenous technological progress in Part 4 and towards understanding why there may be significant technology differences across countries in Part 6 below.

As an aside, you might want to note that some authors use x_t or K_t when working with discrete time and reserve the notation $x(t)$ or $K(t)$ for continuous time. Since we will go back and forth between continuous time and discrete time, we use the latter notation throughout. When there is no risk of confusion, we will drop time arguments, but whenever there is the slightest risk of confusion, we will err on the side of caution and include the time arguments.

Now we impose some standard assumptions on the production function.

Assumption 1. ***(Continuity, Differentiability, Positive and Diminishing Marginal Products, and Constant Returns to Scale)*** *The production function $F : \mathbb{R}^3_+ \to \mathbb{R}_+$ is twice continuously differentiable in K and L, and satisfies*

$$F_K(K, L, A) \equiv \frac{\partial F(K, L, A)}{\partial K} > 0, \quad F_L(K, L, A) \equiv \frac{\partial F(K, L, A)}{\partial L} > 0,$$
$$F_{KK}(K, L, A) \equiv \frac{\partial^2 F(K, L, A)}{\partial K^2} < 0, \quad F_{LL}(K, L, A) \equiv \frac{\partial^2 F(K, L, A)}{\partial L^2} < 0.$$

Moreover, F exhibits constant returns to scale in K and L.

All of the components of Assumption 1 are important. First, the notation $F : \mathbb{R}^3_+ \to \mathbb{R}_+$ implies that the production function takes nonnegative arguments (i.e., $K, L \in \mathbb{R}_+$) and maps to nonnegative levels of output ($Y \in \mathbb{R}_+$). It is natural that the level of capital and the level of employment should be positive. Since A has no natural units, it could have been negative. But there is no loss of generality in restricting it to be positive. The second important aspect of Assumption 1 is that F is a continuous function in its arguments and is also differentiable. There are many interesting production functions which are not differentiable and some interesting ones that are not even continuous. But working with continuously differentiable functions makes it possible for us to use differential calculus, and the loss of some generality is a small price to pay for this convenience. Assumption 1 also specifies that marginal products are positive (so that the level of production increases with the amount of inputs); this also rules out some potential production functions and can be relaxed without much complication (see Exercise 2.6). More importantly, Assumption 1 imposes that the marginal product of both capital and labor are diminishing, i.e., $F_{KK} < 0$ and $F_{LL} < 0$, so that more capital, holding everything else constant, increases output by less and less, and the same applies to labor. This property is sometimes also referred to as "diminishing returns" to capital and labor. We will see below that the degree of diminishing returns to capital will play a very important role in many of the results of the basic growth model. In fact, these features distinguish the Solow growth model from its antecedent, the Harrod-Domar model (see Exercise 2.16).

The other important assumption is that of constant returns to scale. Recall that F exhibits *constant returns to scale* in K and L if it is *linearly homogeneous* (homogeneous of degree 1) in these two variables. More specifically:

Definition 2.1. *Let K be an integer. The function $g : \mathbb{R}^{K+2} \to \mathbb{R}$ is homogeneous of degree m in $x \in \mathbb{R}$ and $y \in \mathbb{R}$ if and only if*

$$g(\lambda x, \lambda y, z) = \lambda^m g(x, y, z) \text{ for all } \lambda \in \mathbb{R}_+ \text{ and } z \in \mathbb{R}^K.$$

It can be easily verified that linear homogeneity implies that the production function F is concave, though not strictly so (see Exercise 2.1).

Linearly homogeneous (constant returns to scale) production functions are particularly useful because of the following theorem:

Theorem 2.1. ***(Euler's Theorem)*** *Suppose that* $g : \mathbb{R}^{K+2} \to \mathbb{R}$ *is continuously differentiable in* $x \in \mathbb{R}$ *and* $y \in \mathbb{R}$*, with partial derivatives denoted by* g_x *and* g_y *and is homogeneous of degree* m *in* x *and* y*. Then*

$$mg(x, y, z) = g_x(x, y, z)\, x + g_y(x, y, z)\, y \text{ for all } x \in \mathbb{R},\ y \in \mathbb{R} \text{ and } z \in \mathbb{R}^K.$$

Moreover, $g_x(x, y, z)$ *and* $g_y(x, y, z)$ *are themselves homogeneous of degree* $m - 1$ *in* x *and* y*.*

Proof. We have that g is continuously differentiable and

$$\lambda^m g(x, y, z) = g(\lambda x, \lambda y, z)\,. \tag{2.2}$$

Differentiate both sides of equation (2.2) with respect to λ, which gives

$$m\lambda^{m-1} g(x, y, z) = g_x(\lambda x, \lambda y, z)\, x + g_y(\lambda x, \lambda y, z)\, y$$

for any λ. Setting $\lambda = 1$ yields the first result. To obtain the second result, differentiate both sides of equation (2.2) with respect to x:

$$\lambda g_x(\lambda x, \lambda y, z) = \lambda^m g_x(x, y, z)\,.$$

Dividing both sides by λ establishes the desired result. □

2.1.2. Market Structure, Endowments and Market Clearing. For most of the book, we will assume that all factor markets are competitive. This is yet another assumption that is not totally innocuous. Both labor markets and capital markets have imperfections that have important implications for economic growth. But it is only by starting out with the competitive benchmark that we can best appreciate the implications of these imperfections for economic growth. Furthermore, until we come to models of endogenous technological change, we will assume that product markets are also competitive, so ours will be a prototypical *competitive general equilibrium model*.

As in standard competitive general equilibrium models, the next step is to specify endowments, that is, what the economy starts with in terms of labor and capital and who owns these endowments. Let us imagine that all factors of production are owned by households. In particular, households own all of the labor, which they supply inelastically. Inelastic supply means that there is some endowment of labor in the economy, for example equal to the population, $\bar{L}(t)$, and all of this will be supplied regardless of the price (as long as it is nonnegative). The *labor market clearing* condition can then be expressed as:

$$L(t) = \bar{L}(t) \tag{2.3}$$

for all t, where $L(t)$ denotes the demand for labor (and also the level of employment). More generally, this equation should be written in complementary slackness form. In particular, let the *wage rate* (or the rental price of labor) at time t be $w(t)$, then the labor market clearing condition takes the form $L(t) \leq \bar{L}(t)$, $w(t) \geq 0$ and $\left(L(t) - \bar{L}(t)\right) w(t) = 0$. The complementary slackness formulation makes sure that labor market clearing does not happen at a negative wage—or that if labor demand happens to be low enough, employment could be below $\bar{L}(t)$ at zero wage. However, this will not be an issue in most of the models studied in this book (in particular, Assumption 1 and competitive labor markets make sure that wages have to be strictly positive), thus we will use the simpler condition (2.3) throughout.

The households also own the capital stock of the economy and rent it to firms. We denote the *rental price of capital* at time t be $R(t)$. The capital market clearing condition is similar to (2.3) and requires the demand for capital by firms to be equal to the supply of capital by households: $K^s(t) = K^d(t)$, where $K^s(t)$ is the supply of capital by households and $K^d(t)$ is the demand by firms. Capital market clearing is straightforward to impose in the class of models analyzed in this book by imposing that the amount of capital $K(t)$ used in production at time t is consistent with household behavior and firms' optimization.

We take households' initial holdings of capital, $K(0)$, as given (as part of the description of the environment), and this will determine the initial condition of the dynamical system we will be analyzing. For now how this initial capital stock is distributed among the households is not important, since households optimization decisions are not modeled explicitly and the economy is simply assumed to save a fraction s of its income. When we turn to models with household optimization below, an important part of the description of the environment will be to specify the preferences and the budget constraints of households.

At this point, we could also introduce $P(t)$ as the price of the final good at time t. But we do not need to do this, since we have a choice of a numeraire commodity in this economy, whose price will be normalized to 1. In particular, as we will discuss in greater detail in Chapter 5, Walras' Law implies that we should choose the price of one of the commodities as numeraire. In fact, throughout we will do something stronger. We will normalize the price of the final good to 1 *in all periods*. Ordinarily, one cannot choose more than one numeraire—otherwise, one would be fixing the relative price between the two numeraires. But again as will be explained in Chapter 5, we can build on an insight by Kenneth Arrow (Arrow, 1964) that it is sufficient to price *securities* (assets) that transfer one unit of consumption from one date (or state of the world) to another. In the context of dynamic economies, this implies that we need to keep track of an *interest rate* across periods, denoted by $r(t)$, and this will enable us to normalize the price of the final good to 1 in every period (and naturally, we will keep track of the wage rate $w(t)$, which will determine the intertemporal price of labor relative to final goods at any date t).

This discussion should already alert you to a central fact: you should think of all of the models we discuss in this book as *general equilibrium economies*, where different commodities correspond to the same good at different dates. Recall from basic general equilibrium theory that the same good at different dates (or in different states or localities) is a different commodity. Therefore, in almost all of the models that we will study in this book, there will be *an infinite number of commodities*, since time runs to infinity. This raises a number of special issues, which we will discuss as we go along.

Now returning to our treatment of the basic model, the next assumption is that capital depreciates, meaning that machines that are used in production lose some of their value because of wear and tear. In terms of our corn example above, some of the corn that is used as seeds is no longer available for consumption or for use as seeds in the following period. We assume that this depreciation takes an "exponential form," which is mathematically very tractable. This means that capital depreciates (exponentially) at the rate δ, so that out of 1 unit of capital this period, only $1-\delta$ is left for next period. As noted above, depreciation here stands for the wear and tear of the machinery, but it can also represent the replacement of old machines by new machines in more realistic models (see Chapter 14). For now it is treated as a black box, and it is another one of the black boxes that will be opened later in the book.

The loss of part of the capital stock affects the interest rate (rate of return to savings) faced by the household. Given the assumption of exponential depreciation at the rate δ and the normalization of the price of the final goods to 1, this implies that the *interest rate* faced by the household will be $r(t) = R(t) - \delta$. Recall that a unit of final good can be consumed now or used as capital and rented to firms. In the latter case, the household will receive $R(t)$ units of good in the next period as the rental price, but will lose δ units of the capital, since δ fraction of capital depreciates over time. This implies that the individual has given up one unit of commodity dated $t-1$ for $r(t)$ units of commodity dated t. The relationship between $r(t)$ and $R(t)$ explains the similarity between the symbols for the interest rate and the rental rate of capital. The interest rate faced by households will play a central role when we model the dynamic optimization decisions of households. In the Solow model, this interest rate does not directly affect the allocation of resources.

2.1.3. Firm Optimization. We are now in a position to look at the optimization problem of firms. Throughout the book we assume that the only objective of firms is to maximize profits. Since we have assumed the existence of an aggregate production function, we only need to consider the problem of a *representative firm*. Throughout, unless otherwise stated, we assume that capital markets are functioning, so firms can rent capital in spot markets. This implies that the maximization problem of the (representative) firm can be written as a static program

$$\max_{L(t)\geq 0, K(t)\geq 0} F[K(t), L(t), A(t)] - w(t) L(t) - R(t) K(t). \tag{2.4}$$

When there are irreversible investments or costs of adjustments, as discussed in Section 7.8 in Chapter 7, we would need to consider the dynamic optimization problems of firms. But in the absence of these features, the production side can be represented as a static maximization problem.

A couple of additional feature are worth noting:

(1) The maximization problem is set up in terms of aggregate variables. This is without loss of any generality given the representative firm.
(2) There is nothing multiplying the F term, since the price of the final good has been normalized to 1. Thus the first term in (2.4) is the revenues of the representative firm (or the revenues of all of the firms in the economy).
(3) This way of writing the problem already imposes competitive factor markets, since the firm is taking as given the rental prices of labor and capital, $w(t)$ and $R(t)$ (which are in terms of the numeraire, the final good).
(4) This is a concave problem, since F is concave (see Exercise 2.1).

Since F is differentiable from Assumption 1, the first-order necessary conditions of the maximization problem (2.4) imply the important and well-known result that the competitive rental rates are equal to marginal products:

$$w(t) = F_L[K(t), L(t), A(t)], \tag{2.5}$$

and

$$R(t) = F_K[K(t), L(t), A(t)]. \tag{2.6}$$

Note also that in (2.5) and (2.6), we used the symbols $K(t)$ and $L(t)$. These represent the amount of capital and labor used by firms. In fact, solving for $K(t)$ and $L(t)$, we can derive the capital and labor demands of firms in this economy at rental prices $R(t)$ and $w(t)$—thus we could have used $K^d(t)$ instead of $K(t)$, but this additional notation is not necessary.

This is where Euler's Theorem, Theorem 2.1, becomes useful. Combined with competitive factor markets, this theorem implies:

Proposition 2.1. *Suppose Assumption 1 holds. Then in the equilibrium of the Solow growth model, firms make no profits, and in particular,*

$$Y(t) = w(t) L(t) + R(t) K(t).$$

Proof. This follows immediately from Theorem 2.1 for the case of $m = 1$, i.e., constant returns to scale. □

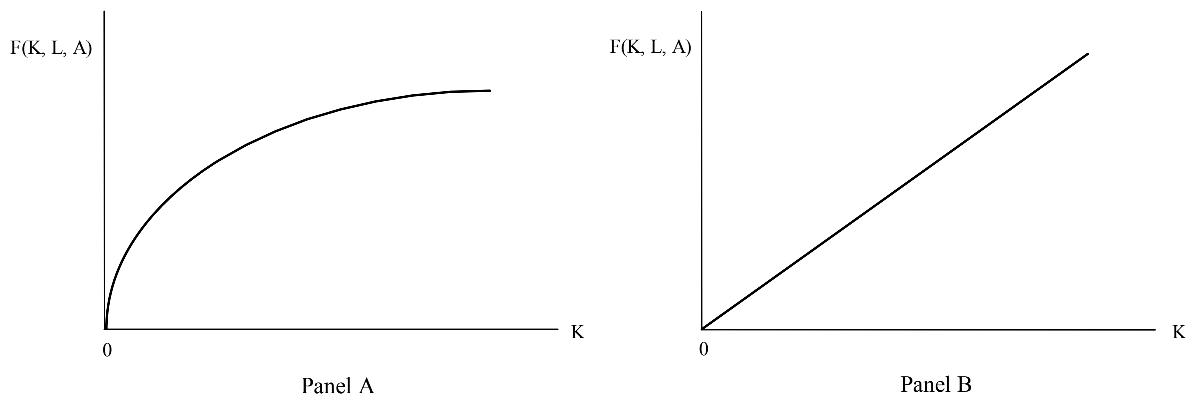

Figure 2.1. Production functions and the marginal product of capital. The example in Panel A satisfies the Inada conditions in Assumption 2, while the example in Panel B does not.

This result is both important and convenient; it implies that firms make no profits, so in contrast to the basic general equilibrium theory with strictly convex production sets, the ownership of firms does not need to be specified. All we need to know is that firms are profit-maximizing entities.

In addition to these standard assumptions on the production function, in macroeconomics and growth theory we often impose the following additional boundary conditions, referred to as Inada conditions.

Assumption 2. ***(Inada conditions)*** *F satisfies the Inada conditions*

$$\begin{aligned} \lim_{K\to 0} F_K(K,L,A) &= \infty \text{ and } \lim_{K\to\infty} F_K(K,L,A)=0 \text{ for all } L>0 \text{ and all } A \\ \lim_{L\to 0} F_L(K,L,A) &= \infty \text{ and } \lim_{L\to\infty} F_L(K,L,A)=0 \text{ for all } K>0 \text{ and all } A. \end{aligned}$$

The role of these conditions—especially in ensuring the existence of *interior equilibria*—will become clear in a little. They imply that the "first units" of capital and labor are highly productive and that when capital or labor are sufficiently abundant, their marginal products are close to zero. Figure 2.1 draws the production function $F(K,L,A)$ as a function of K, for given L and A, in two different cases; in Panel A, the Inada conditions are satisfied, while in Panel B, they are not.

We will refer to Assumptions 1 and 2 throughout much of the book.

2.2. The Solow Model in Discrete Time

We now start with the analysis of the dynamics of economic growth in the discrete time Solow model.

2.2.1. Fundamental Law of Motion of the Solow Model. Recall that K depreciates exponentially at the rate δ, so that the law of motion of the capital stock is given by

$$K(t+1) = (1-\delta) K(t) + I(t), \tag{2.7}$$

where $I(t)$ is investment at time t.

From national income accounting for a closed economy, we have that the total amount of final goods in the economy must be either consumed or invested, thus

$$Y(t) = C(t) + I(t), \tag{2.8}$$

where $C(t)$ is consumption.[1] Using (2.1), (2.7) and (2.8), any *feasible* dynamic allocation in this economy must satisfy

$$K(t+1) \leq F[K(t), L(t), A(t)] + (1-\delta) K(t) - C(t)$$

for $t = 0, 1, \ldots$. The question now is to determine the equilibrium dynamic allocation among the set of feasible dynamic allocations. Here the *behavioral rule* of the constant saving rate simplifies the structure of equilibrium considerably. It is important to notice that the constant saving rate is a behavioral rule—it is not derived from the maximization of a well-defined utility function. This means that any welfare comparisons based on the Solow model have to be taken with a grain of salt, since we do not know what the preferences of the individuals are.

Since the economy is closed (and there is no government spending), aggregate investment is equal to savings,

$$S(t) = I(t) = Y(t) - C(t).$$

Individuals are assumed to save a constant fraction s of their income,

$$S(t) = sY(t), \tag{2.9}$$

while they consume the remaining $1-s$ fraction of their income:

$$C(t) = (1-s) Y(t) \tag{2.10}$$

In terms of capital market clearing, this implies that the supply of capital resulting from households' behavior can be expressed as $K^s(t) = (1-\delta)K(t) + S(t) = (1-\delta)K(t) + sY(t)$. Setting supply and demand equal to each other, this implies $K^s(t) = K(t)$. Moreover, from (2.3), we have $L(t) = \bar{L}(t)$. Combining these market clearing conditions with (2.1) and (2.7), we obtain *the fundamental law of motion* the Solow growth model:

$$K(t+1) = sF[K(t), L(t), A(t)] + (1-\delta) K(t). \tag{2.11}$$

This is a nonlinear *difference equation*. The equilibrium of the Solow growth model is described by this equation together with laws of motion for $L(t)$ (or $\bar{L}(t)$) and $A(t)$.

[1] In addition, we can introduce government spending $G(t)$ on the right-hand side of (2.8). Government spending does not play a major role in the Solow growth model, thus we set it equal to 0 (see Exercise 2.5).

2.2.2. Definition of Equilibrium. The Solow model is a mixture of an old-style Keynesian model and a modern dynamic macroeconomic model. Households do not optimize when it comes to their savings/consumption decisions. Instead, their behavior is captured by a *behavioral rule*. Nevertheless, firms still maximize and factor markets clear. Thus it is useful to start defining equilibria in the way that is customary in modern dynamic macro models. Since $L(t) = \bar{L}(t)$ from (2.3), throughout we write the exogenous evolution of labor endowments in terms of $L(t)$ to simplify notation.

Definition 2.2. *In the basic Solow model for a given sequence of* $\{L(t), A(t)\}_{t=0}^{\infty}$ *and an initial capital stock* $K(0)$*, an equilibrium path is a sequence of capital stocks, output levels, consumption levels, wages and rental rates* $\{K(t), Y(t), C(t), w(t), R(t)\}_{t=0}^{\infty}$ *such that* $K(t)$ *satisfies (2.11),* $Y(t)$ *is given by (2.1),* $C(t)$ *is given by (2.10), and* $w(t)$ *and* $R(t)$ *are given by (2.5) and (2.6).*

The most important point to note about Definition 2.2 is that an equilibrium is defined as an entire path of allocations and prices. An economic equilibrium does *not* refer to a static object; it specifies the entire path of behavior of the economy.

2.2.3. Equilibrium Without Population Growth and Technological Progress. We can make more progress towards characterizing the equilibria by exploiting the constant returns to scale nature of the production function. To do this, let us make some further assumptions, which will be relaxed later in this chapter:

(1) There is no population growth; total population is constant at some level $L > 0$. Moreover, since individuals supply labor inelastically, this implies $L(t) = L$.
(2) There is no technological progress, so that $A(t) = A$.

Let us define the capital-labor ratio of the economy as

$$k(t) \equiv \frac{K(t)}{L}, \tag{2.12}$$

which is a key object for the analysis. Now using the constant returns to scale assumption, we can express output (income) per capita, $y(t) \equiv Y(t)/L$, as

$$\begin{aligned} y(t) &= F\left[\frac{K(t)}{L}, 1, A\right] \\ &\equiv f(k(t)). \end{aligned} \tag{2.13}$$

In other words, with constant returns to scale output per capita is simply a function of the capital-labor ratio. Note that $f(k)$ here depends on A, so I could have written $f(k, A)$; I do not do this to simplify the notation and also because until 2.6, there will be no technological progress thus A is constant and can be normalized to $A = 1$.[2] From Theorem 2.1, we can also express the marginal products of capital and labor (and thus their rental prices) as

$$\begin{aligned} R(t) &= f'(k(t)) > 0 \text{ and} \\ w(t) &= f(k(t)) - k(t) f'(k(t)) > 0. \end{aligned} \tag{2.14}$$

The fact that both of these factor prices are positive follows from Assumption 1, which imposed that the first derivatives of F with respect to capital and labor are always positive.

Example 2.1. **(The Cobb-Douglas Production Function)** Let us consider the most common example of production function used in macroeconomics, the Cobb-Douglas production function—and already add the caveat that even though the Cobb-Douglas production function is convenient and widely used, it is a very special production function and many interesting phenomena are ruled out by this production function as we will discuss later in this book. The Cobb-Douglas production function can be written as

$$\begin{aligned} Y(t) &= F[K(t), L(t), A(t)] \\ &= AK(t)^{\alpha} L(t)^{1-\alpha}, \; 0 < \alpha < 1. \end{aligned} \tag{2.15}$$

It can easily be verified that this production function satisfies Assumptions 1 and 2, including the constant returns to scale feature imposed in Assumption 1. Dividing both sides by $L(t)$, we have the representation of the production function in per capita terms as in (2.13):

$$y(t) = Ak(t)^{\alpha},$$

with $y(t)$ as output per worker and $k(t)$ capital-labor ratio as defined in (2.12). The representation of factor prices as in (2.14) can also be verified. From the per capita production function representation, in particular equation (2.14), the rental price of capital can be expressed as

$$\begin{aligned} R(t) &= \frac{\partial Ak(t)^{\alpha}}{\partial k(t)}, \\ &= \alpha Ak(t)^{-(1-\alpha)}. \end{aligned}$$

Alternatively, in terms of the original production function (2.15), the rental price of capital in (2.6) is given by

$$\begin{aligned} R(t) &= \alpha AK(t)^{\alpha-1} L(t)^{1-\alpha} \\ &= \alpha Ak(t)^{-(1-\alpha)}, \end{aligned}$$

[2] Later, when the focus on specific (labor-augmenting) technological change, the A can also be taken out and the per capita production function can be written as $y = Af(k)$, with a slightly different definition of k as effective capital-labor ratio (see, for example, equation 2.45 in Section 2.6.

which is equal to the previous expression and thus verifies the form of the marginal product given in equation (2.14). Similarly, from (2.14),

$$\begin{aligned} w(t) &= Ak(t)^{\alpha} - \alpha Ak(t)^{-(1-\alpha)} \times k(t) \\ &= (1-\alpha) AK(t)^{\alpha} L(t)^{-\alpha}, \end{aligned}$$

which verifies the alternative expression for the wage rate in (2.5).

Returning to the analysis with the general production function, the per capita representation of the aggregate production function enables us to divide both sides of (2.11) by L to obtain the following simple difference equation for the evolution of the capital-labor ratio:

$$k(t+1) = sf(k(t)) + (1-\delta) k(t). \tag{2.16}$$

Since this difference equation is derived from (2.11), it also can be referred to as the *equilibrium difference equation* of the Solow model, in that it describes the equilibrium behavior of the key object of the model, the capital-labor ratio. The other equilibrium quantities can be obtained from the capital-labor ratio $k(t)$.

At this point, we can also define a *steady-state equilibrium* for this model.

Definition 2.3. *A steady-state equilibrium without technological progress and population growth is an equilibrium path in which $k(t) = k^*$ for all t.*

In a steady-state equilibrium the capital-labor ratio remains constant. Since there is no population growth, this implies that the level of the capital stock will also remain constant. Mathematically, a "steady-state equilibrium" corresponds to a "stationary point" of the equilibrium difference equation (2.16). Most of the models we will analyze in this book will admit a steady-state equilibrium, and typically the economy will tend to this steady state equilibrium over time (but often never reach it in finite time). This is also the case for this simple model.

This can be seen by plotting the difference equation that governs the equilibrium behavior of this economy, (2.16), which is done in Figure 2.2. The thick curve represents (2.16) and the dashed line corresponds to the 45° line. Their (positive) intersection gives the steady-state value of the capital-labor ratio k^*, which satisfies

$$\frac{f(k^*)}{k^*} = \frac{\delta}{s}. \tag{2.17}$$

Notice that in Figure 2.2 there is another intersection between (2.16) and the 45^0 line at $k = 0$. This is because the figure assumes that $f(0) = 0$, thus there is no production without capital, and if there is no production, there is no savings, and the system remains at $k = 0$, making $k = 0$ a steady-state equilibrium. We will ignore this intersection throughout. This is for a number of reasons. First, $k = 0$ is a steady-state equilibrium only when $f(0) = 0$,

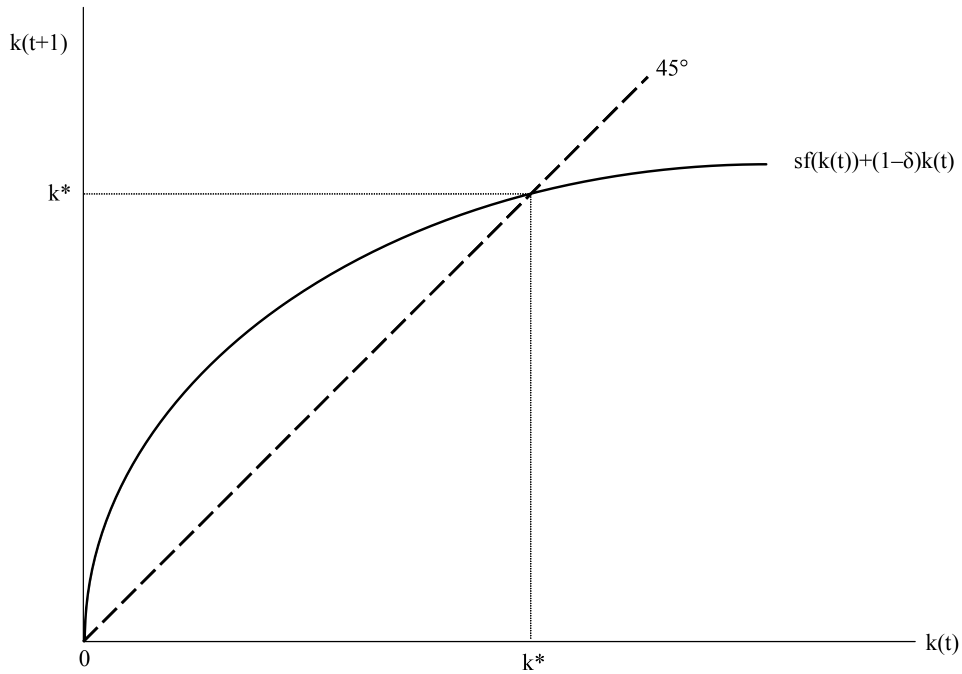

Figure 2.2. Determination of the steady-state capital-labor ratio in the Solow model without population growth and technological change.

which corresponds to the case where capital is an essential factor, meaning that if $K(t) = 0$, then output is equal to zero irrespective of the amount of labor and the level of technology. However, if capital is not essential, $f(0)$ will be positive and $k = 0$ will cease to be a steady state equilibrium (a stationary point of the difference equation (2.16)). This is illustrated in Figure 2.3, which draws (2.16) for the case where $f(0) = \varepsilon$ for any $\varepsilon > 0$. Second, as we will see below, this intersection, even when it exists, is an *unstable point*, thus the economy would never travel towards this point starting with $K(0) > 0$. Third, Hakenes and Irmen (2006) show that even with $f(0) = 0$, the Inada conditions imply that in the continuous time version of the Solow model $k = 0$ may not be a steady-state equilibrium. Finally and most importantly, this intersection has no economic interest for us.

An alternative visual representation of the steady state is to view it as the intersection between a ray through the origin with slope δ (representing the function δk) and the function $sf(k)$. Figure 2.4 shows this picture, which is also useful for two other purposes. First, it depicts the levels of consumption and investment in a single figure. The vertical distance between the horizontal axis and the δk line at the steady-state equilibrium gives the amount of investment per capita (equal to δk^*), while the vertical distance between the function $f(k)$ and the δk line at k^* gives the level of consumption per capita. Clearly, the sum of

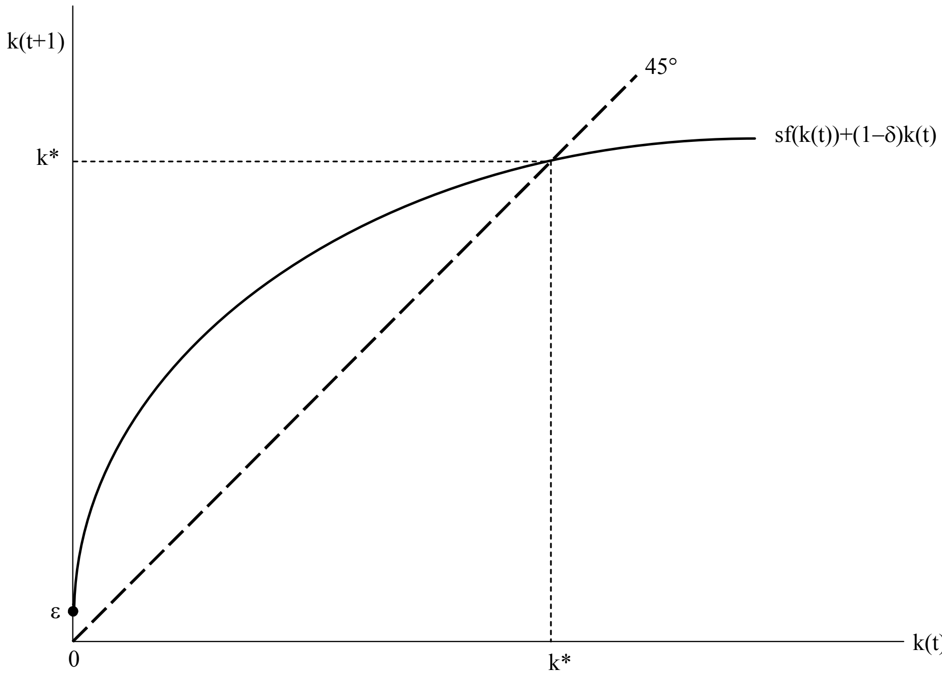

Figure 2.3. Unique steady state in the basic Solow model when $f(0) = \varepsilon > 0$.

these two terms make up $f(k^*)$. Second, Figure 2.4 also emphasizes that the steady-state equilibrium in the Solow model essentially sets investment, $sf(k)$, equal to the amount of capital that needs to be "replenished", δk. This interpretation will be particularly useful when we incorporate population growth and technological change below.

This analysis therefore leads to the following proposition (with the convention that the intersection at $k = 0$ is being ignored even when $f(0) = 0$):

Proposition 2.2. *Consider the basic Solow growth model and suppose that Assumptions 1 and 2 hold. Then there exists a unique steady state equilibrium where the capital-labor ratio* $k^* \in (0, \infty)$ *is given by (2.17), per capita output is given by*

$$y^* = f(k^*) \tag{2.18}$$

and per capita consumption is given by

$$c^* = (1 - s) f(k^*) . \tag{2.19}$$

Proof. The preceding argument establishes that any k^* that satisfies (2.17) is a steady state. To establish existence, note that from Assumption 2 (and from L'Hospital's rule, see Theorem A.19 in Appendix Chapter A), $\lim_{k \to 0} f(k)/k = \infty$ and $\lim_{k \to \infty} f(k)/k = 0$. Moreover, $f(k)/k$ is continuous from Assumption 1, so by the Intermediate Value Theorem

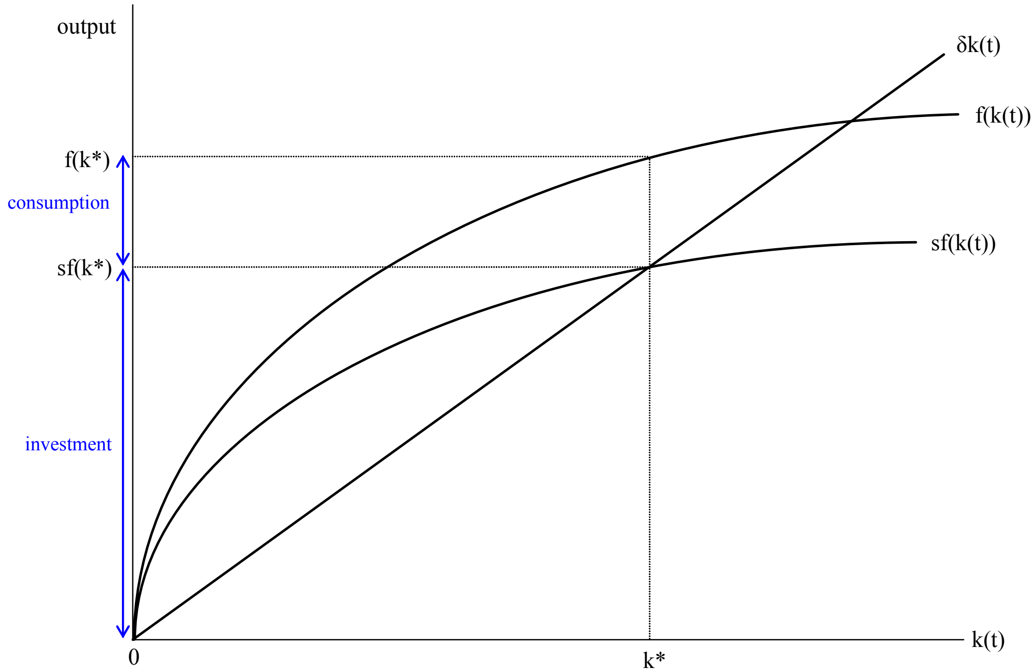

Figure 2.4. Investment and consumption in the steady-state equilibrium.

(see Theorem A.3 in Appendix Chapter A) there exists k^* such that (2.17) is satisfied. To see uniqueness, differentiate $f(k)/k$ with respect to k, which gives

$$\frac{\partial [f(k)/k]}{\partial k} = \frac{f'(k)k - f(k)}{k^2} = -\frac{w}{k^2} < 0, \tag{2.20}$$

where the last equality uses (2.14). Since $f(k)/k$ is everywhere (strictly) decreasing, there can only exist a unique value k^* that satisfies (2.17).

Equations (2.18) and (2.19) then follow by definition. □

Figure 2.5 shows through a series of examples why Assumptions 1 and 2 cannot be dispensed with for the existence and uniqueness results in Proposition 2.2. In the first two panels, the failure of Assumption 2 leads to a situation in which there is no steady state equilibrium with positive activity, while in the third panel, the failure of Assumption 1 leads to non-uniqueness of steady states.

So far the model is very parsimonious: it does not have many parameters and abstracts from many features of the real world in order to focus on the question of interest. Recall that an understanding of how cross-country differences in certain parameters translate into differences in growth rates or output levels is essential for our focus. This will be done in the next proposition. But before doing so, let us generalize the production function in one

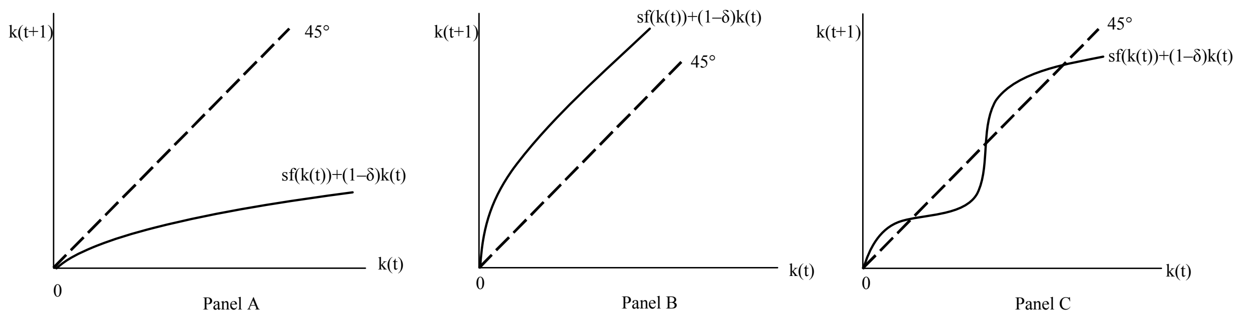

FIGURE 2.5. Examples of nonexistence and nonuniqueness of interior steady states when Assumptions 1 and 2 are not satisfied.

simple way, and assume that

$$f(k) = a\tilde{f}(k),$$

where $a > 0$, so that a is a shift parameter, with greater values corresponding to greater productivity of factors. This type of productivity is referred to as "Hicks-neutral" as we will see below, but for now it is just a convenient way of looking at the impact of productivity differences across countries. Since $f(k)$ satisfies the regularity conditions imposed above, so does $\tilde{f}(k)$.

PROPOSITION 2.3. *Suppose Assumptions 1 and 2 hold and $f(k) = a\tilde{f}(k)$. Denote the steady-state level of the capital-labor ratio by $k^*(a, s, \delta)$ and the steady-state level of output by $y^*(a, s, \delta)$ when the underlying parameters are a, s and δ. Then we have*

$$\begin{aligned} \frac{\partial k^*(a,s,\delta)}{\partial a} &> 0, \; \frac{\partial k^*(a,s,\delta)}{\partial s} > 0 \text{ and } \frac{\partial k^*(a,s,\delta)}{\partial \delta} < 0 \\ \frac{\partial y^*(a,s,\delta)}{\partial a} &> 0, \; \frac{\partial y^*(a,s,\delta)}{\partial s} > 0 \text{ and } \frac{\partial y^*(a,s,\delta)}{\partial \delta} < 0. \end{aligned}$$

PROOF. The proof follows immediately by writing

$$\frac{\tilde{f}(k^*)}{k^*} = \frac{\delta}{as},$$

which holds for an open set of values of k^*. Now apply the implicit function theorem to obtain the results. For example,

$$\frac{\partial k^*}{\partial s} = \frac{\delta (k^*)^2}{as^2 w^*} > 0$$

where $w^* = f(k^*) - k^* f'(k^*) > 0$. The other results follow similarly. □

Therefore, countries with higher saving rates and better technologies will have higher capital-labor ratios and will be richer. Those with greater (technological) depreciation, will tend to have lower capital-labor ratios and will be poorer. All of the results in Proposition

2.3 are intuitive, and start giving us a sense of some of the potential determinants of the capital-labor ratios and output levels across countries.

The same comparative statics with respect to a and δ immediately apply to c^* as well. However, it is straightforward to see that c^* will not be monotone in the saving rate (think, for example, of the extreme case where $s = 1$), and in fact, there will exist a specific level of the saving rate, s_{gold}, referred to as the "golden rule" saving rate, which maximizes the steady-state level of consumption. Since we are treating the saving rate as an exogenous parameter and have not specified the objective function of households yet, we cannot say whether the golden rule saving rate is "better" than some other saving rate. It is nevertheless interesting to characterize what this golden rule saving rate corresponds to.

To do this, let us first write the steady state relationship between c^* and s and suppress the other parameters:

$$\begin{aligned} c^*(s) &= (1-s) f(k^*(s)), \\ &= f(k^*(s)) - \delta k^*(s), \end{aligned}$$

where the second equality exploits the fact that in steady state $sf(k) = \delta k$. Now differentiating this second line with respect to s (again using the implicit function theorem), we have

$$\frac{\partial c^*(s)}{\partial s} = \left[f'(k^*(s)) - \delta\right] \frac{\partial k^*}{\partial s}. \tag{2.21}$$

We define the golden rule saving rate s_{gold} to be such that $\partial c^*(s_{gold})/\partial s = 0$. The corresponding steady-state golden rule capital stock is defined as k^*_{gold}. These quantities and the relationship between consumption and the saving rate are plotted in Figure 2.6.

The next proposition shows that s_{gold} and k^*_{gold} are uniquely defined and the latter satisfies (2.22).

Proposition 2.4. *In the basic Solow growth model, the highest level of steady-state consumption is reached for s_{gold}, with the corresponding steady state capital level k^*_{gold} such that*

$$f'\left(k^*_{gold}\right) = \delta. \tag{2.22}$$

Proof. By definition $\partial c^*(s_{gold})/\partial s = 0$. From Proposition 2.3, $\partial k^*/\partial s > 0$, thus (2.21) can be equal to zero only when $f'(k^*(s_{gold})) = \delta$. Moreover, when $f'(k^*(s_{gold})) = \delta$, it can be verified that $\partial^2 c^*(s_{gold})/\partial s^2 < 0$, so $f'(k^*(s_{gold})) = \delta$ indeed corresponds a local maximum. That $f'(k^*(s_{gold})) = \delta$ also yields the global maximum is a consequence of the following observations: $\forall\ s \in [0,1]$ we have $\partial k^*/\partial s > 0$ and moreover, when $s < s_{gold}$, $f'(k^*(s)) - \delta > 0$ by the concavity of f, so $\partial c^*(s)/\partial s > 0$ for all $s < s_{gold}$, and by the converse argument, $\partial c^*(s)/\partial s < 0$ for all $s > s_{gold}$. Therefore, only s_{gold} satisfies $f'(k^*(s)) = \delta$ and gives the unique global maximum of consumption per capita. □

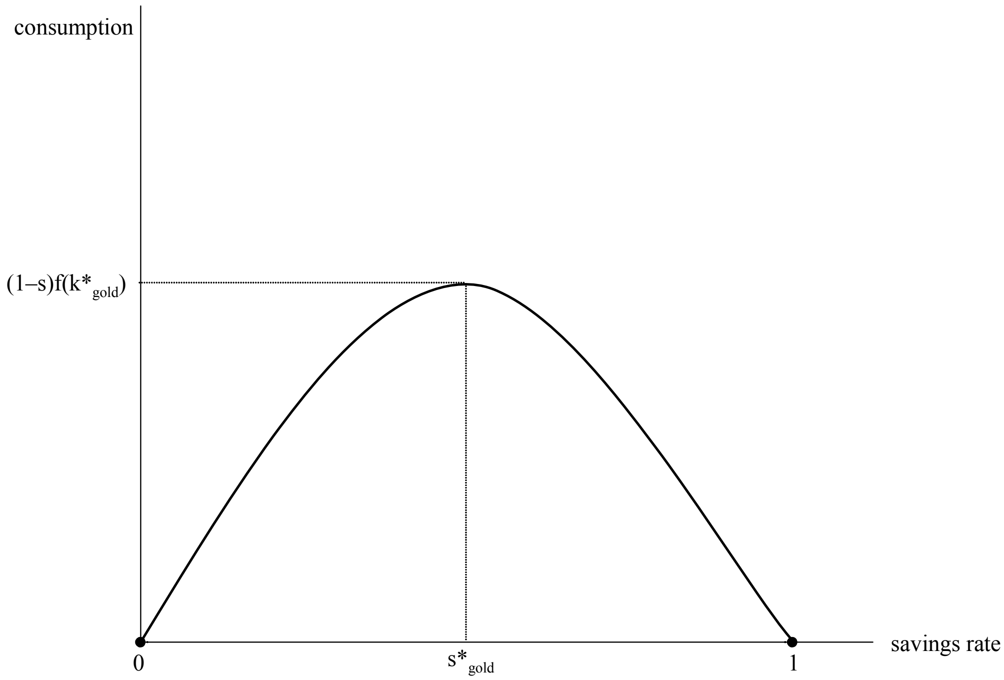

Figure 2.6. The "golden rule" level of savings rate, which maximizes steady-state consumption.

In other words, there exists a unique saving rate, s_{gold}, and also unique corresponding capital-labor ratio, k^*_{gold}, which maximize the level of steady-state consumption. When the economy is below k^*_{gold}, the higher saving rate will increase consumption, whereas when the economy is above k^*_{gold}, steady-state consumption can be increased by saving less. In the latter case, lower savings translate into higher consumption because the capital-labor ratio of the economy is too high so that individuals are investing too much and not consuming enough. This is the essence of what is referred to as *dynamic inefficiency*, which we will encounter in greater detail in models of overlapping generations in Chapter 9. However, recall that there is no explicit utility function here, so statements about "inefficiency" have to be considered with caution. In fact, the reason why such dynamic inefficiency will not arise once we endogenize consumption-saving decisions of individuals will be apparent to many of you already.

2.3. Transitional Dynamics in the Discrete Time Solow Model

Proposition 2.2 establishes the existence of a unique steady-state equilibrium (with positive activity). Recall, however, that an *equilibrium path* does not refer simply to the steady state, but to the entire path of capital stock, output, consumption and factor prices. This is an important point to bear in mind, especially since the term "equilibrium" is used differently in economics than in physical sciences. Typically, in engineering and physical sciences, an equilibrium refers to a point of rest of a dynamical system, thus to what we have so far referred to as *the steady state equilibrium.* One may then be tempted to say that the system is in "disequilibrium" when it is away from the steady state. However, in economics, the non-steady-state behavior of an economy is also governed by optimizing behavior of households and firms and market clearing. Most economies spend much of their time in non-steady-state situations. Thus we are typically interested in the entire dynamic equilibrium path of the economy, not just its steady state.

To determine what the equilibrium path of our simple economy looks like we need to study the "transitional dynamics" of the equilibrium difference equation (2.16) starting from an arbitrary capital-labor ratio, $k(0) > 0$. Of special interest is the answer to the question of whether the economy will tend to this steady state starting from an arbitrary capital-labor ratio, and how it will behave along the transition path. It is important to consider an arbitrary capital-labor ratio, since, as noted above, the total amount of capital at the beginning of the economy, $K(0)$, is taken as a state variable, while for now, the supply of labor L is fixed. Therefore, at time $t = 0$, the economy starts with $k(0) = K(0)/L$ as its initial value and then follows the law of motion given by the difference equation (2.16). Thus the question is whether the difference equation (2.16) will take us to the unique steady state starting from an arbitrary initial capital-labor ratio.

Before doing this, recall some definitions and key results from the theory of dynamical systems. Consider the nonlinear system of autonomous difference equations,

$$\mathbf{x}(t+1) = \mathbf{G}(\mathbf{x}(t)), \tag{2.23}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ and $\mathbf{G} : \mathbb{R}^n \to \mathbb{R}^n$. Let $\mathbf{x}^*$ be a fixed point of the mapping $\mathbf{G}(\cdot)$, i.e.,

$$\mathbf{x}^* = \mathbf{G}(\mathbf{x}^*).$$

Such a $\mathbf{x}^*$ is sometimes referred to as "an equilibrium point" of the difference equation (2.23). Since in economics, equilibrium has a different meaning, we will refer to $\mathbf{x}^*$ as a stationary point or a *steady state* of (2.23). We will often make use of the stability properties of the steady states of systems of difference equations. The relevant notion of stability is introduced in the next definition.

Definition 2.4. *A steady state* $\mathbf{x}^*$ *is (locally)* asymptotically stable *if there exists an open set* $B(\mathbf{x}^*) \ni \mathbf{x}^*$ *such that for any solution* $\{\mathbf{x}(t)\}_{t=0}^{\infty}$ *to (2.23) with* $\mathbf{x}(0) \in B(\mathbf{x}^*)$*, we have* $\mathbf{x}(t) \to \mathbf{x}^*$*. Moreover,* $\mathbf{x}^*$ *is* globally asymptotically stable *if for all* $\mathbf{x}(0) \in \mathbb{R}^n$*, for any solution* $\{\mathbf{x}(t)\}_{t=0}^{\infty}$*, we have* $\mathbf{x}(t) \to \mathbf{x}^*$*.*

The next theorem provides the main results on the stability properties of systems of linear difference equations. The Chapter B in the Appendix contains an overview of eigenvalues and some other properties of differential equations, which are also relevant for difference equations. Definitions and certain elementary results the matrix of partial derivatives (the Jacobian), which we will use below, are provided in Appendix Chapter A. The following theorems are special cases of the results presented in Appendix Chapter B.

Theorem 2.2. *Consider the following linear difference equation system*

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{b} \tag{2.24}$$

with initial value $\mathbf{x}(0)$, *where* $\mathbf{x}(t) \in \mathbb{R}^n$ *for all* t, $\mathbf{A}$ *is an* $n \times n$ *matrix and* $\mathbf{b}$ *is a* $n \times 1$ *column vector. Let* $\mathbf{x}^*$ *be the steady state of the difference equation given by* $\mathbf{A}\mathbf{x}^* + \mathbf{b} = \mathbf{x}^*$. *Suppose that all of the eigenvalues of* $\mathbf{A}$ *are strictly inside the unit circle in the complex plane. Then the steady state of the difference equation (2.24),* $\mathbf{x}^*$, *is* globally asymptotically stable, *in the sense that starting from any* $\mathbf{x}(0) \in \mathbb{R}^n$, *the unique solution* $\{\mathbf{x}(t)\}_{t=0}^{\infty}$ *satisfies* $\mathbf{x}(t) \to \mathbf{x}^*$.

Next let us return to the nonlinear autonomous system (2.23). Unfortunately, much less can be said about nonlinear systems, but the following is a standard *local* stability result (see Appendix Chapter B).

Theorem 2.3. *Consider the following nonlinear autonomous system*

$$\mathbf{x}(t+1) = \mathbf{G}[\mathbf{x}(t)] \tag{2.25}$$

with initial value $\mathbf{x}(0)$, *where* $\mathbf{G}:\mathbb{R}^n \to \mathbb{R}^n$. *Let* $\mathbf{x}^*$ *be a steady state of this system, i.e.,* $\mathbf{G}(\mathbf{x}^*) = \mathbf{x}^*$, *and suppose that* $\mathbf{G}$ *is continuously differentiable at* $\mathbf{x}^*$. *Define*

$$\mathbf{A} \equiv D\mathbf{G}(\mathbf{x}^*),$$

where $D\mathbf{G}$ *denotes the matrix of partial derivatives (Jacobian) of* $\mathbf{G}$. *Suppose that all of the eigenvalues of* $\mathbf{A}$ *are strictly inside the unit circle. Then the steady state of the difference equation (2.25)* $\mathbf{x}^*$ *is locally* asymptotically stable, *in the sense that there exists an open neighborhood of* $\mathbf{x}^*$, $\mathbf{B}(\mathbf{x}^*) \subset \mathbb{R}^n$ *such that starting from any* $\mathbf{x}(0) \in \mathbf{B}(\mathbf{x}^*)$, *we have* $\mathbf{x}(t) \to \mathbf{x}^*$.

An immediate corollary of Theorem 2.3 the following useful result:

Corollary 2.1. *Let* $x(t), a, b \in \mathbb{R}$, *then the unique steady state of the linear difference equation* $x(t+1) = ax(t) + b$ *is globally asymptotically stable (in the sense that* $x(t) \to x^* = b/(1-a)$*) if* $|a| < 1$.

Let $g : \mathbb{R} \to \mathbb{R}$ *be a continuous function, differentiable at the steady state* x^*, *defined by* $g(x^*) = x^*$. *Then, the steady state of the nonlinear difference equation* $x(t+1) = g(x(t))$,

x^*, is locally asymptotically stable if $|g'(x^*)| < 1$. Moreover, if $|g'(x)| < 1$ for all $x \in \mathbb{R}$, then x^* is globally asymptotically stable.

Proof. The first part follows immediately from Theorem 2.2. The local stability of g in the second part follows from Theorem 2.3. Global stability follows since

$$\begin{aligned} |x(t+1) - x^*| &= |g(x(t)) - g(x^*)| \\ &= \left| \int_{x^*}^{x(t)} g'(x)dx \right| \\ &< |x(t) - x^*|, \end{aligned}$$

where the last inequality follows from the hypothesis that $|g'(x)| < 1$ for all $x \in \mathbb{R}$. □

We can now apply Corollary 2.1 to the equilibrium difference equation of the Solow model, (2.16):

Proposition 2.5. *Suppose that Assumptions 1 and 2 hold, then the steady-state equilibrium of the Solow growth model described by the difference equation (2.16) is globally asymptotically stable, and starting from any $k(0) > 0$, $k(t)$ monotonically converges to k^*.*

Proof. Let $g(k) \equiv sf(k) + (1-\delta)k$. First observe that $g'(k)$ exists and is always strictly positive, i.e., $g'(k) > 0$ for all k. Next, from (2.16), we have

$$k(t+1) = g(k(t)), \tag{2.26}$$

with a unique steady state at k^*. From (2.17), the steady-state capital k^* satisfies $\delta k^* = sf(k^*)$, or

$$k^* = g(k^*). \tag{2.27}$$

Now recall that $f(\cdot)$ is concave and differentiable from Assumption 1 and satisfies $f(0) \geq 0$ from Assumption 2. For any strictly concave differentiable function, we have

$$f(k) > f(0) + kf'(k) \geq kf'(k), \tag{2.28}$$

where the second inequality uses the fact that $f(0) \geq 0$. Since (2.28) implies that $\delta = sf(k^*)/k^* > sf'(k^*)$, we have $g'(k^*) = sf'(k^*) + 1 - \delta < 1$. Therefore,

$$g'(k^*) \in (0,1).$$

Corollary 2.1 then establishes local asymptotic stability.

To prove global stability, note that for all $k(t) \in (0, k^*)$,

$$\begin{aligned} k(t+1) - k^* &= g(k(t)) - g(k^*) \\ &= -\int_{k(t)}^{k^*} g'(k)\,dk, \\ &< 0 \end{aligned}$$

where the first line follows by subtracting (2.27) from (2.26), the second line uses the fundamental theorem of calculus, and the third line follows from the observation that $g'(k) > 0$ for all k. Next, (2.16) also implies

$$\begin{aligned}\frac{k(t+1)-k(t)}{k(t)} &= s\frac{f(k(t))}{k(t)}-\delta \\ &> s\frac{f(k^*)}{k^*}-\delta \\ &= 0,\end{aligned}$$

where the second line uses the fact that $f(k)/k$ is decreasing in k (from (2.28) above) and the last line uses the definition of k^*. These two arguments together establish that for all $k(t) \in (0, k^*)$, $k(t+1) \in (k(t), k^*)$. An identical argument implies that for all $k(t) > k^*$, $k(t+1) \in (k^*, k(t))$. Therefore, $\{k(t)\}_{t=0}^{\infty}$ monotonically converges to k^* and is globally stable. □

This stability result can be seen diagrammatically in Figure 2.7. Starting from initial capital stock $k(0)$, which is below the steady-state level k^*, the economy grows towards k^* and the economy experiences *capital deepening*—meaning that the capital-labor ratio will increase. Together with capital deepening comes growth of per capita income. If, instead, the economy were to start with $k'(0) > k^*$, it would reach the steady state by decumulating capital and contracting (i.e., negative growth).

The following proposition is an immediate corollary of Proposition 2.5:

Proposition 2.6. *Suppose that Assumptions 1 and 2 hold, and $k(0) < k^*$, then $\{w(t)\}_{t=0}^{\infty}$ is an increasing sequence and $\{R(t)\}_{t=0}^{\infty}$ is a decreasing sequence. If $k(0) > k^*$, the opposite results apply.*

Proof. See Exercise 2.7. □

Recall that when the economy starts with too little capital relative to its labor supply, the capital-labor ratio will increase. This implies that the marginal product of capital will fall due to diminishing returns to capital, and the wage rate will increase. Conversely, if it starts with too much capital, it will decumulate capital, and in the process the wage rate will decline and the rate of return to capital will increase.

The analysis has established that the Solow growth model has a number of nice properties; unique steady state, asymptotic stability, and finally, simple and intuitive comparative statics. Yet, so far, it has no growth. The steady state is the point at which there is no growth in the capital-labor ratio, no more capital deepening and no growth in output per capita. Consequently, the basic Solow model (without technological progress) can only generate economic growth along the transition path when the economy starts with $k(0) < k^*$. This growth is not sustained, however: it slows down over time and eventually comes to

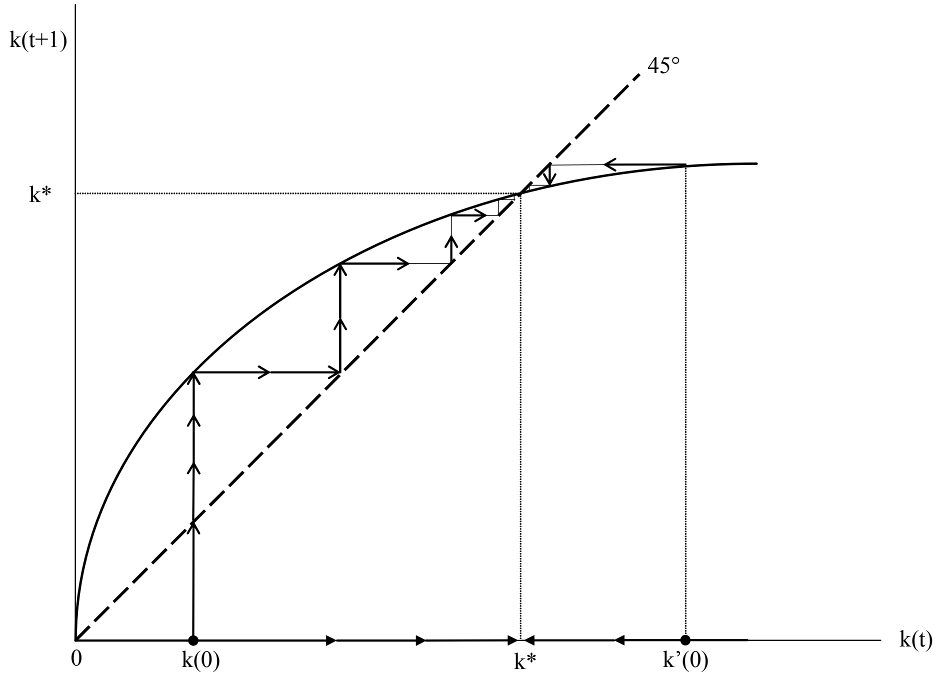

Figure 2.7. Transitional dynamics in the basic Solow model.

an end. We will see in Section 2.6 that the Solow model can incorporate economic growth by allowing *exogenous* technological change. Before doing this, it is useful to look at the relationship between the discrete-time and continuous-time formulations.

2.4. The Solow Model in Continuous Time

2.4.1. From Difference to Differential Equations. Recall that the time periods could refer to days, weeks, months or years. In some sense, the time unit is not important. This suggests that perhaps it may be more convenient to look at dynamics by making the time unit as small as possible, i.e., by going to continuous time. While much of modern macroeconomics (outside of growth theory) uses discrete time models, many growth models are formulated in continuous time. The continuous time setup in general has a number of advantages, since some pathological results of discrete time disappear in continuous time (see Exercise 2.14). Moreover, continuous time models have more flexibility in the analysis of dynamics and allow explicit-form solutions in a wider set of circumstances. These considerations motivate a detailed study of both the discrete-time and the continuous-time versions of the basic models.

Let us start with a simple difference equation

$$x(t+1) - x(t) = g(x(t)). \tag{2.29}$$

This equation states that between time t and $t+1$, the absolute growth in x is given by $g(x(t))$. Let us now consider the following approximation

$$x(t+\Delta t) - x(t) \simeq \Delta t \cdot g(x(t)),$$

for any $\Delta t \in [0,1]$. When $\Delta t = 0$, this equation is just an identity. When $\Delta t = 1$, it gives (2.29). In-between it is a linear approximation, which should not be too bad if the distance between t and $t+1$ is not very large, so that $g(x) \simeq g(x(t))$ for all $x \in [x(t), x(t+1)]$ (however, you should also convince yourself that this approximation could in fact be quite bad if you take a very nonlinear function g, for which the behavior changes significantly between $x(t)$ and $x(t+1)$). Now divide both sides of this equation by Δt, and take limits to obtain

$$\lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} = \dot{x}(t) \simeq g(x(t)), \tag{2.30}$$

where throughout the book we use the "dot" notation

$$\dot{x}(t) \equiv \frac{\partial x(t)}{\partial t}$$

to denote time derivatives. Equation (2.30) is a differential equation representing the same dynamics as the difference equation (2.29) for the case in which the distance between t and $t+1$ is "small".

2.4.2. The Fundamental Equation of the Solow Model in Continuous Time. We can now repeat all of the analysis so far using the continuous time representation. Nothing has changed on the production side, so we continue to have (2.5) and (2.6) as the factor prices, but now these refer to instantaneous rental rates (i.e., $w(t)$ is the flow of wages that the worker receives for an instant etc.).

Savings are again given by

$$S(t) = sY(t),$$

while consumption is given by (2.10) above.

Let us now introduce population growth into this model for the first time, and assume that the labor force $L(t)$ grows proportionally, i.e.,

$$L(t) = \exp(nt) L(0). \tag{2.31}$$

The purpose of doing so is that in many of the classical analyses of economic growth, population growth plays an important role, so it is useful to see how it affects things here. We are not introducing technological progress yet, which will be done in the next section.

Recall that

$$k(t) \equiv \frac{K(t)}{L(t)},$$

which implies that

$$\frac{\dot{k}(t)}{k(t)} = \frac{\dot{K}(t)}{K(t)} - \frac{\dot{L}(t)}{L(t)},$$
$$= \frac{\dot{K}(t)}{K(t)} - n.$$

From the limiting argument leading to equation (2.30) in the previous subsection, the law of motion of the capital stock is given by

$$\dot{K}(t) = sF[K(t), L(t), A(t)] - \delta K(t).$$

Now using the definition of $k(t)$ as the capital-labor ratio and the constant returns to scale properties of the production function, we obtain the fundamental law of motion of the Solow model in continuous time for the capital-labor ratio as

$$\frac{\dot{k}(t)}{k(t)} = s\frac{f(k(t))}{k(t)} - (n+\delta), \tag{2.32}$$

where, following usual practice, we wrote the proportional change in the capital-labor ratio on the left-hand side by dividing both sides by $k(t)$.[3]

Definition 2.5. *In the basic Solow model in continuous time with population growth at the rate n, no technological progress and an initial capital stock $K(0)$, an equilibrium path is a sequence of capital stocks, labor, output levels, consumption levels, wages and rental rates $[K(t), L(t), Y(t), C(t), w(t), R(t)]_{t=0}^{\infty}$ such that $L(t)$ satisfies (2.31), $k(t) \equiv K(t)/L(t)$ satisfies (2.32), $Y(t)$ is given by (2.1), $C(t)$ is given by (2.10), and $w(t)$ and $R(t)$ are given by (2.5) and (2.6).*

As before, a *steady-state* equilibrium involves $k(t)$ remaining constant at some level k^*.

It is easy to verify that the equilibrium differential equation (2.32) has a unique steady state at k^*, which is given by a slight modification of (2.17) above to incorporate population growth:

$$\frac{f(k^*)}{k^*} = \frac{n+\delta}{s}. \tag{2.33}$$

In other words, going from discrete to continuous time has not changed any of the basic economic features of the model, and again the steady state can be plotted in diagram similar to the one used above (now with the population growth rate featuring in there as well). This is done in Figure 2.8, which also highlights that the logic of the steady state is the same with population growth as it was without population growth. The amount of investment, $sf(k)$, is used to replenish the capital-labor ratio, but now there are two reasons for replenishments.

[3]Throughout I adopt the notation $[x(t)]_{t=0}^{\infty}$ to denote the continuous time path of variable $x(t)$. An alternative notation often used in the literature is $(x(t); t \geq 0)$. I prefer the former both because it is slightly more compact and also because it is more similar to the discrete time notation for the time path of a variable, $\{x(t)\}_{t=0}^{\infty}$.

We still have a fraction δ of the capital stock depreciating. In addition, the capital stock of the economy also has to increase as population grows in order to maintain the capital-labor ratio constant. The amount of capital that needs to be replenished is therefore $(n + \delta) k$.

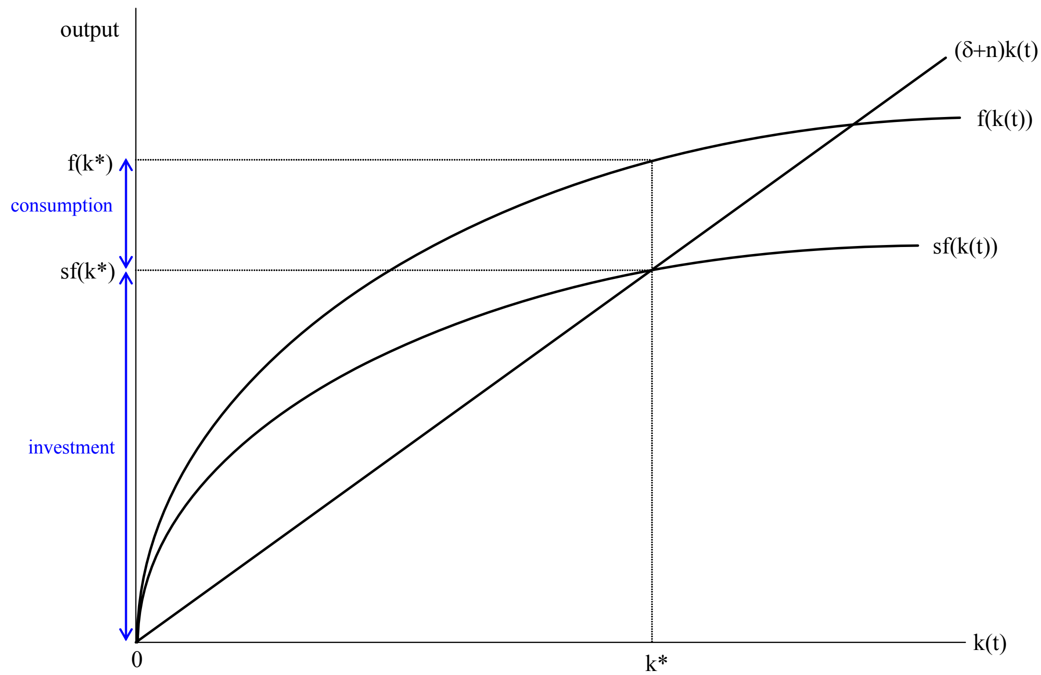

Figure 2.8. Investment and consumption in the state-state equilibrium with population growth.

Therefore we have:

Proposition 2.7. *Consider the basic Solow growth model in continuous time and suppose that Assumptions 1 and 2 hold. Then there exists a unique steady state equilibrium where the capital-labor ratio is equal to $k^* \in (0, \infty)$ and is given by (2.33), per capita output is given by*

$$y^* = f(k^*)$$

and per capita consumption is given by

$$c^* = (1 - s) f(k^*).$$

Proof. See Exercise 2.3. □

Moreover, again defining $f(k) = a\tilde{f}(k)$, we obtain:

Proposition 2.8. *Suppose Assumptions 1 and 2 hold and* $f(k) = a\tilde{f}(k)$. *Denote the steady-state equilibrium level of the capital-labor ratio by* $k^*(a, s, \delta, n)$ *and the steady-state level of output by* $y^*(a, s, \delta, n)$ *when the underlying parameters are given by* a, s *and* δ. *Then we have*

$$\frac{\partial k^*(a,s,\delta,n)}{\partial a} > 0,\ \frac{\partial k^*(a,s,\delta,n)}{\partial s} > 0,\ \frac{\partial k^*(a,s,\delta,n)}{\partial \delta} < 0 \text{ and } \frac{\partial k^*(a,s,\delta,n)}{\partial n} < 0$$
$$\frac{\partial y^*(a,s,\delta,n)}{\partial a} > 0,\ \frac{\partial y^*(a,s,\delta,n)}{\partial s} > 0,\ \frac{\partial y^*(a,s,\delta,n)}{\partial \delta} < 0 \text{and} \frac{\partial y^*(a,s,\delta,n)}{\partial n} < 0.$$

Proof. See Exercise 2.4. □

The new result relative to the earlier comparative static proposition is that now a higher population growth rate, n, also reduces the capital-labor ratio and output per capita. The reason for this is simple: a higher population growth rate means there is more labor to use the existing amount of capital, which only accumulates slowly, and consequently the equilibrium capital-labor ratio ends up lower. This result implies that countries with higher population growth rates will have lower incomes per person (or per worker).

2.5. Transitional Dynamics in the Continuous Time Solow Model

The analysis of transitional dynamics and stability with continuous time yields similar results to those in Section 2.3, but in many ways simpler. To do this in detail, we need to remember the equivalents of the above theorems for differential equations. The following theorems follow from the results presented in Appendix Chapter B.

Theorem 2.4. *Consider the following linear differential equation system*

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b} \tag{2.34}$$

with initial value $\mathbf{x}(0)$, *where* $\mathbf{x}(t) \in \mathbb{R}^n$ *for all* t, $\mathbf{A}$ *is an* $n\times n$ *matrix and* $\mathbf{b}$ *is a* $n\times 1$ *column vector. Let* $\mathbf{x}^*$ *be the steady state of the system given by* $\mathbf{A}\mathbf{x}^* + \mathbf{b} = 0$. *Suppose that all of the eigenvalues of* $\mathbf{A}$ *have negative real parts. Then the steady state of the differential equation (2.34)* $\mathbf{x}^*$ *is globally asymptotically stable, in the sense that starting from any* $\mathbf{x}(0) \in \mathbb{R}^n$, $\mathbf{x}(t) \to \mathbf{x}^*$.

Theorem 2.5. *Consider the following nonlinear autonomous differential equation*

$$\dot{\mathbf{x}}(t) = \mathbf{G}[\mathbf{x}(t)] \tag{2.35}$$

with initial value $\mathbf{x}(0)$, *where* $\mathbf{G} : \mathbb{R}^n \to \mathbb{R}^n$. *Let* $\mathbf{x}^*$ *be a steady state of this system, i.e.,* $\mathbf{G}(\mathbf{x}^*) = 0$, *and suppose that* $\mathbf{G}$ *is continuously differentiable at* $\mathbf{x}^*$. *Define*

$$\mathbf{A} \equiv D\mathbf{G}(\mathbf{x}^*),$$

and suppose that all of the eigenvalues of $\mathbf{A}$ *have negative real parts. Then the steady state of the differential equation (2.35)* $\mathbf{x}^*$ *is locally asymptotically stable, in the sense that there* *exists an open neighborhood of* $\mathbf{x}^*$, $\mathbf{B}(\mathbf{x}^*) \subset \mathbb{R}^n$ *such that starting from any* $\mathbf{x}(0) \in \mathbf{B}(\mathbf{x}^*)$, *we have* $\mathbf{x}(t) \to \mathbf{x}^*$.

Once again an immediate corollary is:

Corollary 2.2. *Let* $x(t) \in \mathbb{R}$, *then the steady state of the linear difference equation* $\dot{x}(t) = ax(t)$ *is globally asymptotically stable (in the sense that* $x(t) \to 0$*) if* $a < 0$.

Let $g : \mathbb{R} \to \mathbb{R}$ *be continuous and differentiable at* x^* *where* $g(x^*) = 0$. *Then, the steady state of the nonlinear differential equation* $\dot{x}(t) = g(x(t))$, x^*, *is locally asymptotically stable if* $g'(x^*) < 0$.

Proof. See Exercise 2.8. □

Finally, with continuous time, we also have another useful theorem:

Theorem 2.6. *Let* $g : \mathbb{R} \to \mathbb{R}$ *be a continuous function and suppose that there exists a unique* x^* *such that* $g(x^*) = 0$. *Moreover, suppose* $g(x) < 0$ *for all* $x > x^*$ *and* $g(x) > 0$ *for all* $x < x^*$. *Then the steady state of the nonlinear differential equation* $\dot{x}(t) = g(x(t))$, x^*, *is globally asymptotically stable, i.e., starting with any* $x(0)$, $x(t) \to x^*$.

Proof. The hypotheses of the theorem imply that for all $x > x^*$, $\dot{x} < 0$ and for all $x < x^*$, $\dot{x} > 0$. This establishes that $x(t) \to x^*$ starting from any $x(0) \in \mathbb{R}$. □

Notice that the equivalent of Theorem 2.6 is not true in discrete time, and this will be illustrated in Exercise 2.14.

In view of these results, Proposition 2.5 immediately generalizes:

Proposition 2.9. *Suppose that Assumptions 1 and 2 hold, then the basic Solow growth model in continuous time with constant population growth and no technological change is globally asymptotically stable, and starting from any* $k(0) > 0$, $k(t) \to k^*$.

Proof. The proof of stability is now simpler and follows immediately from Theorem 2.6 by noting that whenever $k < k^*$, $sf(k) - (n+\delta)k > 0$ and whenever $k > k^*$, $sf(k) - (n+\delta)k < 0$. □

Figure 2.9 shows the analysis of stability diagrammatically. The figure plots the right-hand side of (2.32) and makes it clear that whenever $k < k^*$, $\dot{k} > 0$ and whenever $k > k^*$, $\dot{k} < 0$, so that the capital-effective labor ratio monotonically converges to the steady-state value k^*.

Example 2.2. **(Dynamics with the Cobb-Douglas Production Function)** Let us return to the Cobb-Douglas production function introduced in Example 2.1

$$F[K, L, A] = AK^{\alpha}L^{1-\alpha} \text{ with } 0 < \alpha < 1.$$

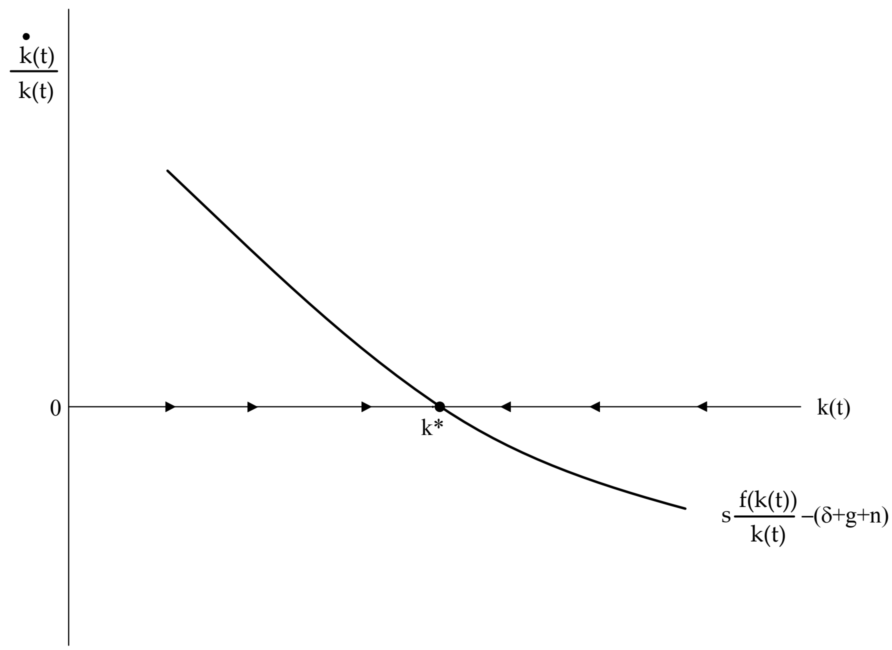

Figure 2.9. Dynamics of the capital-labor ratio in the basic Solow model.

As noted above, the Cobb-Douglas production function is special, mainly because it has an elasticity of substitution between capital and labor equal to 1. Recall that for a homothetic production function $F(K, L)$, the elasticity of substitution is defined by

$$\sigma \equiv -\left[\frac{\partial \ln (F_K/F_L)}{\partial \ln (K/L)}\right]^{-1}, \tag{2.36}$$

where F_K and F_L denote the marginal products of capital and labor. In addition, F is required to be homothetic, so that F_K/F_L is only a function of K/L. For the Cobb-Douglas production function $F_K/F_L = (\alpha/(1-\alpha)) \cdot (L/K)$, thus $\sigma = 1$. This feature implies that when the production function is Cobb-Douglas and factor markets are competitive, equilibrium factor shares will be constant irrespective of the capital-labor ratio. In particular:

$$\begin{aligned}
\alpha_K(t) &= \frac{R(t)\, K(t)}{Y(t)} \\
&= \frac{F_K(K(t), L(t))\, K(t)}{Y(t)} \\
&= \frac{\alpha A [K(t)]^{\alpha-1} [L(t)]^{1-\alpha} K(t)}{A [K(t)]^{\alpha} [L(t)]^{1-\alpha}} \\
&= \alpha.
\end{aligned}$$

Similarly, the share of labor is $\alpha_L(t) = 1 - \alpha$. The reason for this is that with an elasticity of substitution equal to 1, as capital increases, its marginal product decreases proportionally, leaving the capital share (the amount of capital times its marginal product) constant.

Recall that with the Cobb-Douglas technology, the per capita production function takes the form $f(k) = Ak^{\alpha}$, so the steady state is given again from (2.33) (with population growth at the rate n) as

$$A\left(k^*\right)^{\alpha-1} = \frac{n+\delta}{s}$$

or

$$k^* = \left(\frac{sA}{n+\delta}\right)^{\frac{1}{1-\alpha}},$$

which is a simple and interpretable expression for the steady-state capital-labor ratio. k^* is increasing in s and A and decreasing in n and δ (which is naturally consistent with the results in Proposition 2.8). In addition, k^* is increasing in α. This is because a higher α implies less diminishing returns to capital, thus a higher capital-labor ratio reduces the average return to capital to the level necessary for steady state as given in equation (2.33).

Transitional dynamics are also straightforward in this case. In particular, we have:

$$\dot{k}(t) = sA\left[k(t)\right]^{\alpha} - (n+\delta)\,k(t)$$

with initial condition $k(0)$. To solve this equation, let $x(t) \equiv k(t)^{1-\alpha}$, so the equilibrium law of motion of the capital labor ratio can be written in terms of $x(t)$ as

$$\dot{x}(t) = (1-\alpha)\,sA - (1-\alpha)(n+\delta)\,x(t),$$

which is a linear differential equation, with a general solution

$$x(t) = \frac{sA}{n+\delta} + \left[x(0) - \frac{sA}{n+\delta}\right]\exp\left(-(1-\alpha)(n+\delta)\,t\right).$$

(see, for example, the Appendix Chapter B, or Boyce and DiPrima, 1977, Simon and Bloom, 1994). Expressing this solution in terms of the capital-labor ratio

$$k(t) = \left\{\frac{sA}{n+\delta} + \left[\left[k(0)\right]^{1-\alpha} - \frac{sA}{\delta}\right]\exp\left(-(1-\alpha)(n+\delta)\,t\right)\right\}^{\frac{1}{1-\alpha}}.$$

This solution illustrates that starting from any $k(0)$, the equilibrium $k(t) \rightarrow k^* = \left(sA/(n+\delta)\right)^{1/(1-\alpha)}$, and in fact, the rate of adjustment is related to $(1-\alpha)(n+\delta)$, or more specifically, the gap between $k(0)$ and its steady-state value is closed at the exponential rate $(1-\alpha)(n+\delta)$. This is intuitive: a higher α implies less diminishing returns to capital, which slows down the rate at which the marginal and average product of capital declines as capital accumulates, and this reduces the rate of adjustment to steady state. Similarly, a smaller δ means less replacement of depreciated capital and a smaller n means slower population growth, both of those slowing down the adjustment of capital per worker and thus the rate of transitional dynamics.

Example 2.3. **(The Constant Elasticity of Substitution Production Function)** The previous example introduced the Cobb-Douglas production function, which featured an elasticity of substitution equal to 1. The Cobb-Douglas production function is a special case of the constant elasticity of substitution (CES) production function, first introduced by Arrow, Chenery, Minhas and Solow (1961). This production function imposes a constant elasticity, σ, not necessarily equal to 1. To write this function, consider a vector-valued index of technology $\mathbf{A}(t) = (A_H(t), A_K(t), A_L(t))$. Then the CES production function can be written as

$$\begin{aligned} Y(t) &= F[K(t), L(t), \mathbf{A}(t)] \\ &\equiv A_H(t)\left[\gamma (A_K(t)K(t))^{\frac{\sigma-1}{\sigma}} + (1-\gamma)(A_L(t)L(t))^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}, \end{aligned} \tag{2.37}$$

where $A_H(t) > 0$, $A_K(t) > 0$ and $A_L(t) > 0$ are three different types of technological change which will be discussed further in Section 2.6; $\gamma \in (0,1)$ is a distribution parameter, which determines how important labor and capital services are in determining the production of the final good and $\sigma \in [0, \infty]$ is the elasticity of substitution. To verify that σ is indeed the constant elasticity of substitution, let us use (2.36). In particular, it is easy to verify that the ratio of the marginal product of capital to the marginal productive labor, F_K/F_L, is given by

$$\frac{F_K}{F_L} = \frac{\gamma A_K(t)^{\frac{\sigma-1}{\sigma}} K(t)^{-\frac{1}{\sigma}}}{(1-\gamma) A_L(t)^{\frac{\sigma-1}{\sigma}} L(t)^{-\frac{1}{\sigma}}},$$

thus, we indeed have that

$$\sigma = -\left[\frac{\partial \ln(F_K/F_L)}{\partial \ln(K/L)}\right]^{-1}.$$

The CES production function is particularly useful because it is more general and flexible than the Cobb-Douglas form while still being tractable. As we take the limit $\sigma \to 1$, the CES production function (2.37) converges to the Cobb-Douglas function $Y(t) = A_H(t)(A_K(t))^{\gamma}(A_L(t))^{1-\gamma}(K(t))^{\gamma}(L(t))^{1-\gamma}$. As $\sigma \to \infty$, the CES production function becomes linear, i.e.

$$Y(t) = \gamma A_H(t) A_K(t) K(t) + (1-\gamma) A_H(t) A_L(t) L(t).$$

Finally, as $\sigma \to 0$, the CES production function converges to the Leontief production function with no substitution between factors,

$$Y(t) = A_H(t) \min\{\gamma A_K(t) K(t); (1-\gamma) A_L(t) L(t)\}.$$

The special feature of the Leontief production function is that if $\gamma A_K(t) K(t) \neq (1-\gamma) A_L(t) L(t)$, either capital or labor will be partially "idle" in the sense that a small reduction in capital or labor will have no effect on output or factor prices. Exercise 2.16 illustrates a number of the properties of the CES production function, while Exercise 2.17

provides an alternative derivation of this production function along the lines of the original article by Arrow, Chenery, Minhas and Solow (1961).

2.5.1. A First Look at Sustained Growth. Can the Solow model generate sustained growth *without* technological progress? The answer is yes, but only if we relax some of the assumptions we have imposed so far.

The Cobb-Douglas example above already showed that when α is close to 1, adjustment of the capital-labor ratio back to its steady-state level can be very slow. A very slow adjustment towards a steady-state has the flavor of "sustained growth" rather than the economy settling down to a stationary point quickly.

In fact, the simplest model of sustained growth essentially takes $\alpha = 1$ in terms of the Cobb-Douglas production function above. To do this, let us relax Assumptions 1 and 2 (which do not allow $\alpha = 1$), and suppose that

$$F\left[K\left(t\right), L\left(t\right), A\left(t\right)\right] = AK\left(t\right), \tag{2.38}$$

where $A > 0$ is a constant. This is the so-called "AK" model, and in its simplest form output does not even depend on labor. The results we would like to highlight apply with more general constant returns to scale production functions, for example,

$$F\left[K\left(t\right), L\left(t\right), A\left(t\right)\right] = AK\left(t\right) + BL\left(t\right), \tag{2.39}$$

but it is simpler to illustrate the main insights with (2.38), leaving the analysis of the richer production function (2.39) to Exercise 2.15.

Let us continue to assume that population grows at a constant rate n as before (cfr. equation (2.31)). Then, combining this with the production function (2.38), the fundamental law of motion of the capital stock becomes

$$\frac{\dot{k}\left(t\right)}{k\left(t\right)} = sA - \delta - n.$$

Therefore, if the parameters of the economy satisfy the inequality $sA - \delta - n > 0$, there will be sustained growth in the capital-labor ratio. From (2.38), this implies that there will be sustained growth in output per capita as well. This immediately establishes the following proposition:

Proposition 2.10. *Consider the Solow growth model with the production function (2.38) and suppose that $sA - \delta - n > 0$. Then in equilibrium, there is sustained growth of output per capita at the rate $sA - \delta - n$. In particular, starting with a capital-labor ratio $k(0) > 0$, the economy has*

$$k\left(t\right) = \exp\left(\left(sA - \delta - n\right)t\right)k\left(0\right)$$

and

$$y\left(t\right) = \exp\left(\left(sA - \delta - n\right)t\right)Ak\left(0\right).$$

This proposition not only establishes the possibility of endogenous growth, but also shows that in this simplest form, there are no transitional dynamics. The economy always grows at a constant rate $sA - \delta - n$, irrespective of what level of capital-labor ratio it starts from. Figure 2.10 shows this equilibrium diagrammatically.

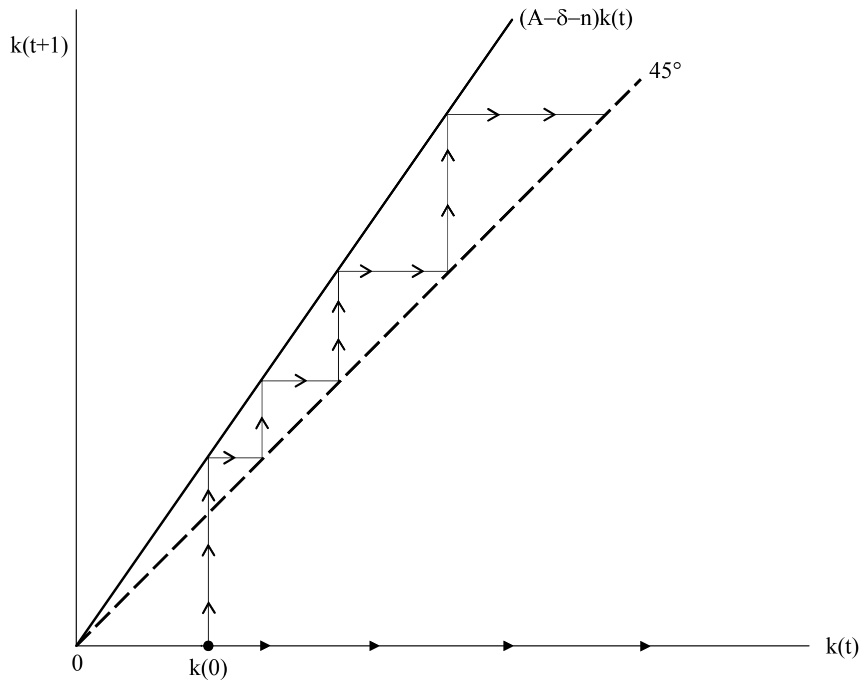

FIGURE 2.10. Sustained growth with the linear AK technology with $sA - \delta - n > 0$.

Does the AK model provide an appealing approach to explain sustained growth? While its simplicity is a plus, the model has a number of unattractive features. First, it is a somewhat knife-edge case, which does not satisfy Assumptions 1 and 2; in particular, it requires the production function to be ultimately linear in the capital stock. Second and relatedly, this feature implies that as time goes by the share of national income accruing to capital will increase towards 1. We will see in the next section that this does not seem to be borne out by the data. Finally and most importantly, we will see in the rest of the book that technological progress seems to be a major (perhaps the most major) factor in understanding the process of economic growth. A model of sustained growth without technological progress fails to capture this essential aspect of economic growth. Motivated by these considerations, we next turn to the task of introducing technological progress into the baseline Solow growth model.

2.6. Solow Model with Technological Progress

2.6.1. Balanced Growth. The models analyzed so far did not feature technological progress. We now introduce changes in $A(t)$ to capture improvements in the technological know-how of the economy. There is little doubt that today human societies know how to produce many more goods than before and they can do so much more efficiently than in the past. In other words, the productive knowledge of the human society has progressed tremendously over the past 200 years, and even more tremendously over the past 1,000 or 10,000 years. This suggests that an attractive way of introducing economic growth in the framework developed so far is to allow technological progress in the form of changes in $A(t)$. The question is how to do this. We will shortly see that the production function $F[K(t), L(t), A(t)]$ is too general to achieve our objective. In particular, with this general structure, we may not have *balanced growth*.

By balanced growth, we mean a path of the economy consistent with the *Kaldor facts* (Kaldor, 1963), that is, a path where, while output per capita increases, the capital-output ratio, the interest rate, and the distribution of income between capital and labor remain roughly constant. Figure 2.11, for example, shows the evolution of the shares of capital and labor in the US national income.

Despite fairly large fluctuations, there is no trend in these factors shares. Moreover, a range of evidence suggests that there is no apparent trend in interest rates over long time horizons and even in different societies (see, for example, Homer and Sylla, 1991). These facts and the relative constancy of capital-output ratios until the 1970s have made many economists prefer models with balanced growth to those without. It is not literally true that the share of capital in output and the capital-output ratio are exactly constant. For example, since the 1970s both the capital share and the capital-output ratio may have increased depending on how one measures them. Nevertheless, constant factor shares and a constant capital-output ratio are a good approximation to reality and a very useful starting point for our models.

Also for future reference, note that the capital share in national income is about 1/3, while the labor share is about 2/3. We are ignoring the share of land here as we did in the analysis so far: land is not a major factor of production. This is clearly not the case for the poor countries, where land is a major factor of production. It is useful to think about how incorporating land into this framework will change the implications of our analysis (see Exercise 2.10). For now, it suffices to note that this pattern of the factor distribution of income, combined with economists' desire to work with simple models, often makes them choose a Cobb-Douglas aggregate production function of the form $AK^{1/3}L^{2/3}$ as an approximation to reality (especially since it ensures that factor shares are constant by construction).

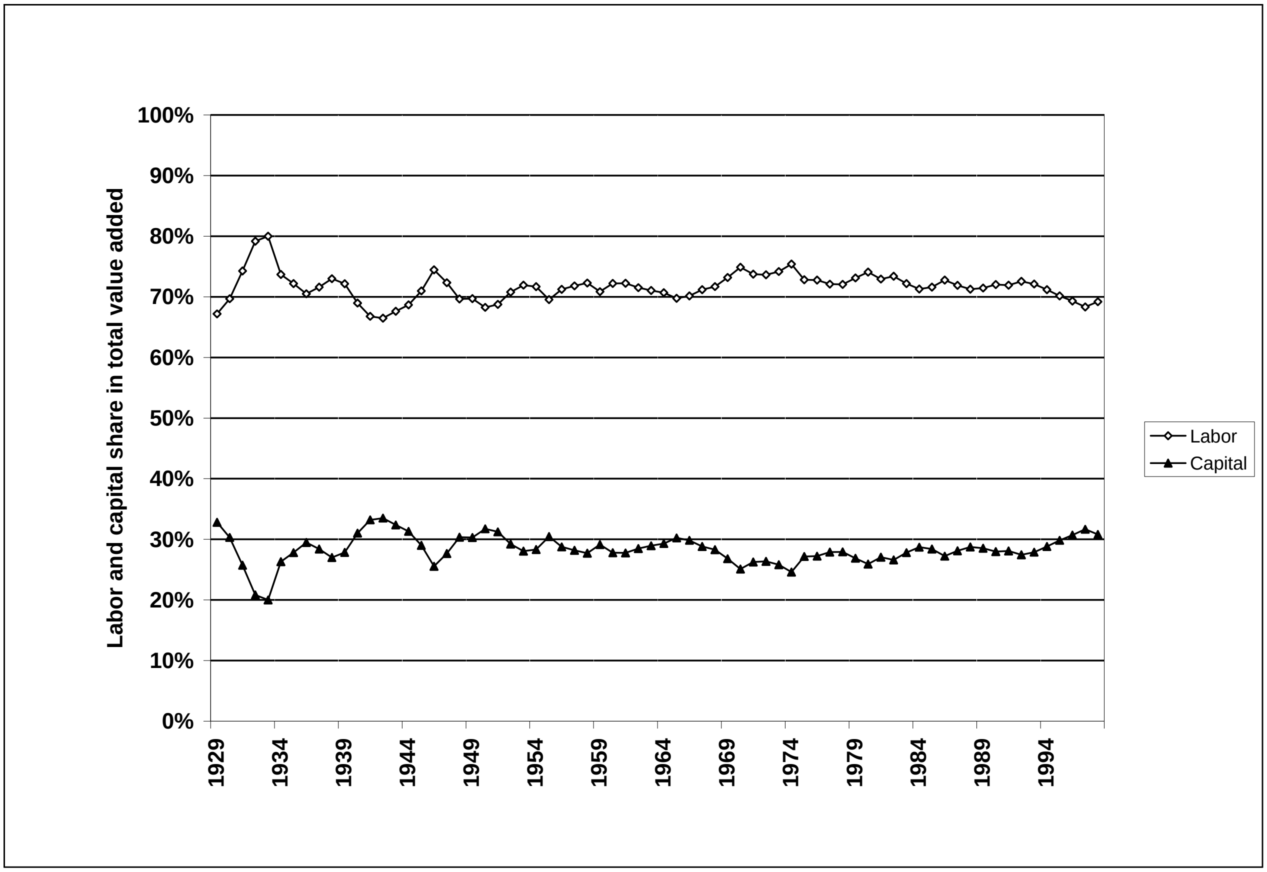

Figure 2.11. Capital and Labor Share in the U.S. GDP.

For us, the most important reason to start with balanced growth is that it is much easier to handle than non-balanced growth, since the equations describing the law of motion of the economy can be represented by difference or differential equations with well-defined steady states. Put more succinctly, the main advantage from our point of view is that balanced growth is the same as a steady-state in transformed variables—i.e., we will again have $\dot{k} = 0$, but the definition of k will change. This will enable us to use the same tools developed so far to analyze economies with sustained growth. It is nevertheless important to bear in mind that in reality, growth has many non-balanced features. For example, the share of different sectors changes systematically over the growth process, with agriculture shrinking, manufacturing first increasing and then shrinking. Ultimately, we would like to have models that combine certain quasi-balanced features with these types of structural transformations embedded in them. We will return to these issues in Part 7 of the book.

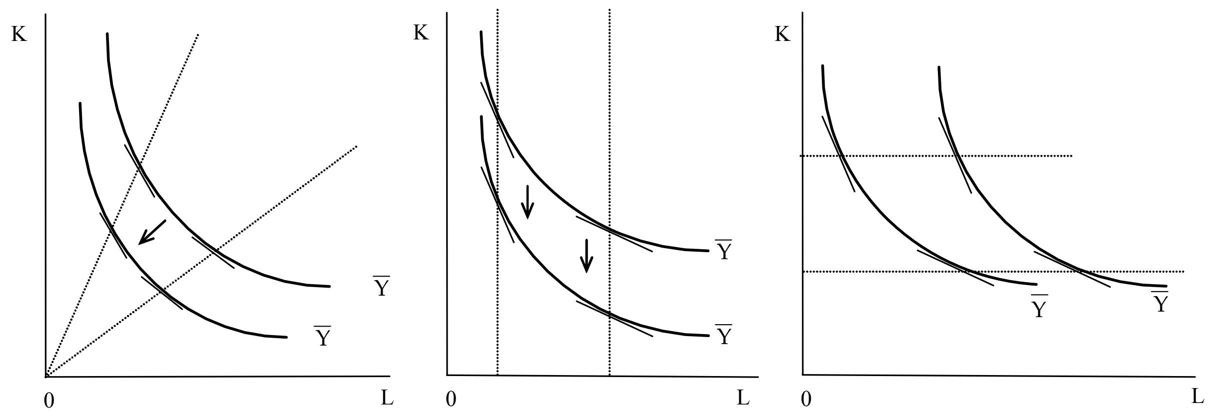

Figure 2.12. Hicks-neutral, Solow-neutral and Harrod-neutral shifts in isoquants.

2.6.2. Types of Neutral Technological Progress. What are some convenient special forms of the general production function $F[K(t), L(t), A(t)]$? First we could have

$$F[K(t), L(t), A(t)] = A(t)\tilde{F}[K(t), L(t)],$$

for some constant returns to scale function $\tilde{F}$. This functional form implies that the technology term $A(t)$ is simply a multiplicative constant in front of another (quasi-) production function $\tilde{F}$ and is referred to as *Hicks-neutral* after the famous British economist John Hicks. Intuitively, consider the isoquants of the function $F[K(t), L(t), A(t)]$ in the L-K space, which plot combinations of labor and capital for a given technology $A(t)$ such that the level of production is constant. This is shown in Figure 2.12. Hicks-neutral technological progress, in the first panel, corresponds to a relabeling of the isoquants (without any change in their shape).

Another alternative is to have capital-augmenting or *Solow-neutral* technological progress, in the form

$$F[K(t), L(t), A(t)] = \tilde{F}[A(t)K(t), L(t)].$$

This is also referred to as capital-augmenting progress, because a higher $A(t)$ is equivalent to the economy having more capital. This type of technological progress corresponds to the isoquants shifting with technological progress in a way that they have constant slope at a given labor-output ratio and is shown in the second panel of Figure 2.12.

Finally, we can have labor-augmenting or *Harrod-neutral* technological progress, named after an early influential growth theorist Roy Harrod, who we encountered above in the context of the Harrod-Domar model previously:

$$F[K(t), L(t), A(t)] = \tilde{F}[K(t), A(t)L(t)].$$

This functional form implies that an increase in technology $A(t)$ increases output as if the economy had more labor. Equivalently, the slope of the isoquants are constant along rays with constant capital-output ratio, and the approximate shape of the isoquants are plotted in the third panel of Figure 2.12.

Of course, in practice technological change can be a mixture of these, so we could have a vector valued index of technology $\mathbf{A}(t) = (A_H(t), A_K(t), A_L(t))$ and a production function that looks like

$$F[K(t), L(t), \mathbf{A}(t)] = A_H(t)\tilde{F}[A_K(t)K(t), A_L(t)L(t)], \tag{2.40}$$

which nests the constant elasticity of substitution production function introduced in Example 2.3 above. Nevertheless, even (2.40) is a restriction on the form of technological progress, since changes in technology, $A(t)$, could modify the entire production function.

It turns out that, although all of these forms of technological progress look equally plausible ex ante, our desire to focus on balanced growth forces us to one of these types of neutral technological progress. In particular, balanced growth necessitates that all technological progress be labor-augmenting or Harrod-neutral. This is a very surprising result and it is also somewhat troubling, since there is no ex ante compelling reason for why technological progress should take this form. We now state and prove the relevant theorem here and return to the discussion of why long-run technological change might be Harrod-neutral in Chapter 15.

2.6.3. The Steady-State Technological Progress Theorem. A version of the following theorem was first proved by the early growth economist Hirofumi Uzawa (1961). For simplicity and without loss of any generality, let us focus on continuous time models. The key elements of balanced growth, as suggested by the discussion above, are the constancy of factor shares and the constancy of the capital-output ratio, $K(t)/Y(t)$. Since there is only labor and capital in this model, by factor shares, we mean

$$\alpha_L(t) \equiv \frac{w(t)L(t)}{Y(t)} \text{ and } \alpha_K(t) \equiv \frac{R(t)K(t)}{Y(t)}.$$

By Assumption 1 and Theorem 2.1, we have that $\alpha_L(t) + \alpha_K(t) = 1$.

The following theorem is a stronger version of a result first stated and proved by Uzawa. Here we will present a proof along the lines of the more recent paper by Schlicht (2006). For this result, let us define an asymptotic path as a path of output, capital, consumption and labor as $t \to \infty$.

Theorem 2.7. ***(Uzawa)*** *Consider a growth model with a constant returns to scale aggregate production function*

$$Y(t) = F\left[K(t), L(t), \tilde{A}(t)\right],$$

with $\tilde{A}(t)$ *representing technology at time* t *and aggregate resource constraint*

$$\dot{K}(t) = Y(t) - C(t) - \delta K(t).$$

Suppose that there is a constant growth rate of population, i.e., $L(t) = \exp(nt) L(0)$ *and that there exists an asymptotic path where output, capital and consumption grow at constant rates, i.e.,* $\dot{Y}(t)/Y(t) = g_Y$, $\dot{K}(t)/K(t) = g_K$ *and* $\dot{C}(t)/C(t) = g_C$. *Suppose finally that* $g_K + \delta > 0$. *Then, along the growth path we have*

(1) $g_Y = g_K = g_C$; *and*

(2) *asymptotically, the aggregate production function can be represented as:*

$$Y(t) = \tilde{F}[K(t), A(t) L(t)],$$

where

$$\frac{\dot{A}(t)}{A(t)} = g = g_Y - n.$$

Proof. By hypothesis, as $t \to \infty$, we have $Y(t) = \exp(g_Y(t-\tau)) Y(\tau)$, $K(t) = \exp(g_K(t-\tau)) K(\tau)$ and $L(t) = \exp(n(t-\tau)) L(\tau)$ for some $\tau < \infty$. The aggregate resource constraint at time t implies

$$(g_K + \delta) K(t) = Y(t) - C(t).$$

Since the left-hand side is positive by hypothesis, we can divide both sides by $\exp(g_K(t-\tau))$ and write date t quantities in terms of date τ quantities to obtain

$$(g_K + \delta) K(\tau) = \exp((g_Y - g_K)(t-\tau)) Y(\tau) - \exp((g_C - g_K)(t-\tau)) C(\tau)$$

for all t. Differentiating with respect to time implies that

$$(g_Y - g_K) \exp((g_Y - g_K)(t-\tau)) Y(\tau) - (g_C - g_K) \exp((g_C - g_K)(t-\tau)) C(\tau) = 0$$

for all t. This equation can hold for all t either if $g_Y = g_K = g_C$ or if $g_Y = g_C$ and $Y(\tau) = C(\tau)$. However the latter condition is inconsistent with $g_K + \delta > 0$. Therefore, $g_Y = g_K = g_C$ as claimed in the first part of the theorem.

Next, the aggregate production function for time τ can be written as

$$\exp(-g_Y(t-\tau)) Y(t) = F\left[\exp(-g_K(t-\tau)) K(t), \exp(-n(t-\tau)) L(t), \tilde{A}(\tau)\right].$$

Multiplying both sides by $\exp(g_Y(t-\tau))$ and using the constant returns to scale property of F, we obtain

$$Y(t) = F\left[\exp((t-\tau)(g_Y - g_K)) K(t), \exp((t-\tau)(g_Y - n)) L(t), \tilde{A}(\tau)\right].$$

From part 1, $g_Y = g_K$, therefore

$$Y(t) = F\left[K(t), \exp((t-\tau)(g_Y - n)) L(t), \tilde{A}(\tau)\right].$$

Moreover, this equation is true for t irrespective of the initial τ, thus

$$\begin{aligned} Y(t) &= \tilde{F}\left[K(t), \exp\left((t-\tau)(g_Y - n)\right) L(t)\right], \\ &= \tilde{F}\left[K(t), A(t) L(t)\right], \end{aligned}$$

with

$$\frac{\dot{A}(t)}{A(t)} = g_Y - n$$

establishing the second part of the theorem. □

A remarkable feature of this result is that it was stated and proved without any reference to equilibrium behavior or market clearing. Also, contrary to Uzawa's original theorem, it is not stated for a balanced growth path (meaning an equilibrium path with constant factor shares), but only for an asymptotic path with constant rates of output, capital and consumption growth. The proposition only exploits the definition of asymptotic paths, the constant returns to scale nature of the aggregate production function and the resource constraint. Consequently, the result is a very powerful one.

Before providing a more economic intuition for this result, let us state an immediate implication of this theorem as a corollary, which will be useful both in the discussions below and for the intuition:

Corollary 2.3. *Under the assumptions of Theorem 2.7, if an economy has an asymptotic path with constant growth of output, capital and consumption, then asymptotically technological progress can be represented as Harrod neutral (purely labor-augmenting).*

The intuition for Theorem 2.7 and for the corollary is simple. We have assumed that the economy features capital accumulation in the sense that $g_K + \delta > 0$. From the aggregate resource constraint, this is only possible if output and capital grow at the same rate. Either this growth rate is equal to the rate of population growth, n, in which case, there is no technological change (i.e., the proposition applies with $g = 0$), or the economy exhibits growth of per capita income and capital-labor ratio. The latter case creates an asymmetry between capital and labor, in the sense that capital is accumulating faster than labor. Constancy of growth then requires technological change to make up for this asymmetry—that is, technology to take a labor-augmenting form.

This intuition does not provide a reason for why technology should take this labor-augmenting (Harrod-neutral) form, however. The proposition and its corollary simply state that if technology did not take this form, and asymptotic path with constant growth rates would not be possible. At some level, this is a distressing result, since it implies that balanced growth (in fact something weaker than balanced growth) is only possible under a very stringent assumption. It also provides no reason why technological change should take this form. Nevertheless, in Chapter 15, we will see that when technology is endogenous, the intuition in the previous paragraph also works to make technology endogenously more labor-augmenting than capital-augmenting.

Notice also that this proposition does not state that technological change has to be labor-augmenting all the time. Instead, it requires that technological change has to be labor-augmenting asymptotically, i.e., along the balanced growth path. This is exactly the pattern that certain classes of endogenous-technology models will generate.

Finally, it is important to emphasize that Theorem 2.7 does not require that $Y^*(t) = \tilde{F}[K^*(t), A(t)L(t)]$, but only that it has a representation of the form $Y^*(t) = \tilde{F}[K^*(t), A(t)L(t)]$. This allows one important exception to the statement that "asymptotically technological change has to be Harrod neutral". If the aggregate production function is Cobb-Douglas and takes the form

$$Y(t) = [A_K(t)K(t)]^{\alpha}[A_L(t)L(t)]^{1-\alpha},$$

then both $A_K(t)$ and $A_L(t)$ could grow asymptotically, while maintaining balanced growth. However, in this Cobb-Douglas example we can define $A(t) = [A_K(t)]^{\alpha/(1-\alpha)} A_L(t)$ and the production function can be represented as

$$Y(t) = [K(t)]^{\alpha}[A(t)L(t)]^{1-\alpha}.$$

In other words, technological change can be represented as purely labor-augmenting, which is what Theorem 2.7 requires. Intuitively, the differences between labor-augmenting and capital-augmenting (and other forms) of technological progress matter when the elasticity of substitution between capital and labor is not equal to 1. In the Cobb-Douglas case, as we have seen above, this elasticity of substitution is equal to 1, thus different forms of technological progress are simple transforms of each other.

Another important corollary of Theorem 2.7 is obtained when we also assume that factor markets are competitive.

Corollary 2.4. *Under the conditions of Theorem 2.7, if factor markets are competitive, then asymptotic factor shares are constant, i.e., as* $t \to \infty$, $\alpha_L(t) \to \alpha_L^*$ *and* $\alpha_K(t) \to \alpha_K^*$.

Proof. With competitive factor markets, we have that as $t \to \infty$

$$\begin{aligned}\alpha_K(t) &\equiv \frac{R(t)K(t)}{Y(t)} \\ &= \frac{K(t)}{Y(t)}\frac{\partial \tilde{F}[K(t), A(t)L(t)]}{\partial K(t)} \\ &= \alpha_K^*,\end{aligned}$$

where the second line uses the definition of the rental rate of capital in a competitive market and the third line uses the fact that as $t \to \infty$, $g_Y = g_K$ and $g_K = g + n$ from Theorem

2.7 and that $\tilde{F}$ exhibits constant returns to scale and thus its derivative is homogeneous of degree 0. □

This corollary, together with Theorem 2.7, implies that any asymptotic path with constant growth rates for output, capital and consumption must be a balanced growth path and can only be generated from an aggregate production function asymptotically featuring Harrod-neutral technological change.

In light of this corollary, we can provide further intuition for Theorem 2.7. Suppose the production function takes the special form $F[A_K(t)K(t), A_L(t)L(t)]$. The corollary implies that factor shares will be constant. Given constant returns to scale, this can only be the case when total capital inputs, $A_K(t)K(t)$, and total labor inputs, $A_L(t)L(t)$, grow at the same rate; otherwise, the share of either capital or labor will be increasing over time. The fact that the capital-output ratio is constant in steady state (or the fact that capital accumulates) implies that $K(t)$ must grow at the same rate as $A_L(t)L(t)$. Thus balanced growth can only be possible if $A_K(t)$ is asymptotically constant.

2.6.4. The Solow Growth Model with Technological Progress: Continuous Time. Now we are ready to analyze the Solow growth model with technological progress. We will only present the analysis for continuous time. The discrete time case can be analyzed analogously and we omit this to avoid repetition. From Theorem 2.7, we know that the production function must a representation of the form

$$Y(t) = F[K(t), A(t)L(t)],$$

with purely labor-augmenting technological progress asymptotically. For simplicity, let us assume that it takes this form throughout. Moreover, suppose that there is technological progress at the rate g, i.e.,

$$\frac{\dot{A}(t)}{A(t)} = g, \tag{2.41}$$

and population growth at the rate n,

$$\frac{\dot{L}(t)}{L(t)} = n.$$

Again using the constant saving rate we have

$$\dot{K}(t) = sF[K(t), A(t)L(t)] - \delta K(t). \tag{2.42}$$

The simplest way of analyzing this economy is to express everything in terms of a normalized variable. Since "effective" or efficiency units of labor are given by $A(t)L(t)$, and F exhibits constant returns to scale in its two arguments, we now define $k(t)$ as the *effective capital-labor* ratio, i.e., capital divided by efficiency units of labor,

$$k(t) \equiv \frac{K(t)}{A(t)L(t)}. \tag{2.43}$$

Differentiating this expression with respect to time, we obtain

$$\frac{\dot{k}(t)}{k(t)}=\frac{\dot{K}(t)}{K(t)}-g-n. \tag{2.44}$$

The quantity of output per unit of effective labor can be written as

$$\begin{aligned}\hat{y}(t) &\equiv \frac{Y(t)}{A(t)L(t)}\\ &= F\left[\frac{K(t)}{A(t)L(t)},1\right]\\ &\equiv f(k(t)).\end{aligned}$$

Income per capita is $y(t)\equiv Y(t)/L(t)$, i.e.,

$$\begin{aligned}y(t) &= A(t)\hat{y}(t)\\ &= A(t)f(k(t)).\end{aligned} \tag{2.45}$$

It should be clear that if $\hat{y}(t)$ is constant, income per capita, $y(t)$, will grow over time, since $A(t)$ is growing. This highlights that in this model, and more generally in models with technological progress, we should not look for "steady states" where income per capita is constant, but for *balanced growth paths*, where income per capita grows at a constant rate, while some transformed variables such as $\hat{y}(t)$ or $k(t)$ in (2.44) remain constant. Since these transformed variables remain constant, balanced growth paths can be thought of as steady states of a transformed model. Motivated by this, in models with technological change throughout we will use the terms "steady state" and balanced growth path interchangeably.

Substituting for $\dot{K}(t)$ from (2.42) into (2.44), we obtain:

$$\frac{\dot{k}(t)}{k(t)}=\frac{sF[K(t),A(t)L(t)]}{K(t)}-(\delta+g+n).$$

Now using (2.43),

$$\frac{\dot{k}(t)}{k(t)}=\frac{sf(k(t))}{k(t)}-(\delta+g+n), \tag{2.46}$$

which is very similar to the law of motion of the capital-labor ratio in the continuous time model, (2.32). The only difference is the presence of g, which reflects the fact that now k is no longer the capital-labor ratio but the *effective* capital-labor ratio. Precisely because it is the effective capital-labor ratio, k will remain constant in the balanced growth path of this economy.

An equilibrium in this model is defined similarly to before. A steady state or a balanced growth path is, in turn, defined as an equilibrium in which $k(t)$ is constant. Consequently, we have (proof omitted):

Proposition 2.11. *Consider the basic Solow growth model in continuous time, with Harrod-neutral technological progress at the rate g and population growth at the rate n. Suppose that Assumptions 1 and 2 hold, and define the effective capital-labor ratio as in (2.43). Then there exists a unique steady state (balanced growth path) equilibrium where the effective capital-labor ratio is equal to $k^* \in (0, \infty)$ and is given by*

$$\frac{f(k^*)}{k^*} = \frac{\delta + g + n}{s}. \tag{2.47}$$

Per capita output and consumption grow at the rate g.

Equation (2.47), which determines the steady-state level of effective capital-labor ratio, emphasizes that now total savings, $sf(k)$, are used for replenishing the capital stock for three distinct reasons. The first is again the depreciation at the rate δ. The second is population growth at the rate n, which reduces capital per worker. The third is Harrod-neutral technological progress at the rate g. Recall that we now need to keep the effective capital-labor ratio, k, given by (2.43) constant. Even if K/L is constant, k will now decline because of the growth of A. Thus the replenishment of the effective capital-labor ratio requires investments to be equal to $(\delta + g + n) k$, which is the intuitive explanation for equation (2.47).

The comparative static results are also similar to before, with the additional comparative static with respect to the initial level of the labor-augmenting technology, $A(0)$ (since the level of technology at all points in time, $A(t)$, is completely determined by $A(0)$ given the assumption in (2.41)). We therefore have:

Proposition 2.12. *Suppose Assumptions 1 and 2 hold and let $A(0)$ be the initial level of technology. Denote the balanced growth path level of effective capital-labor ratio by $k^*(A(0), s, \delta, n)$ and the level of output per capita by $y^*(A(0), s, \delta, n, t)$ (the latter is a function of time since it is growing over time). Then we have*

$$\begin{aligned}
\frac{\partial k^*(A(0), s, \delta, n)}{\partial A(0)} &= 0, \ \frac{\partial k^*(A(0), s, \delta, n)}{\partial s} > 0, \\
\frac{\partial k^*(A(0), s, \delta, n)}{\partial n} &< 0 \text{ and } \frac{\partial k^*(A(0), s, \delta, n)}{\partial \delta} < 0,
\end{aligned}$$

and also

$$\begin{aligned}
\frac{\partial y^*(A(0), s, \delta, n, t)}{\partial A(0)} &> 0, \ \frac{\partial y^*(A(0), s, \delta, n, t)}{\partial s} > 0, \\
\frac{\partial y^*(A(0), s, \delta, n, t)}{\partial n} &< 0 \text{ and } \frac{\partial y^*(A(0), s, \delta, n, t)}{\partial \delta} < 0,
\end{aligned}$$

for each t.

Proof. See Exercise 2.18. □

Finally, we also have very similar transitional dynamics.

Proposition 2.13. *Suppose that Assumptions 1 and 2 hold, then the Solow growth model with Harrod-neutral technological progress and population growth in continuous time is asymptotically stable, i.e., starting from any $k(0) > 0$, the effective capital-labor ratio converges to a steady-state value k^* ($k(t) \to k^*$).*

Proof. See Exercise 2.19. □

Therefore, the comparative statics and dynamics are very similar to the model without technological progress. The major difference is that now the model generates growth in output per capita, so can be mapped to the data much better. However, the disadvantage is that growth is driven entirely *exogenously*. The growth rate of the economy is exactly the same as the exogenous growth rate of the technology stock. The model specifies neither where this technology stock comes from nor how fast it grows.

2.7. Comparative Dynamics

In this section, we briefly undertake some simple "comparative dynamic" exercises. By comparative dynamics, we refer to the analysis of the dynamic response of an economy to a change in its parameters or to shocks. Comparative dynamics are different from comparative statics in Propositions 2.3, 2.8 or 2.12 in that we are interested in the entire path of adjustment of the economy following the shock or changing parameter. The basic Solow model is particularly well suited to such an analysis because of its simplicity. Such an exercise is also useful because the basic Solow model, and its neoclassical cousin, are often used for analysis of policy changes, medium-run shocks and business cycle dynamics, so understanding of how the basic model response to various shocks is useful for a range of applications. We will see in Chapter 8 that comparative dynamics are more interesting in the neoclassical growth model than the basic Solow model. Consequently, the analysis here will be brief and limited to a diagrammatic exposition. Moreover, for brevity we will focus on the continuous time economy.

Recall that the law of motion of the effective capital-labor ratio in the continuous time Solow model is given by (2.46) $\dot{k}(t)/k(t) = sf(k(t))/k(t) - (\delta + g + n)$. The right-hand side of this equation is plotted in Figure 2.13. The intersection with the horizontal axis gives the steady state (balanced growth) equilibrium, k^*. This figure is sufficient for us to carry out comparative dynamic exercises. Consider, for example, a one-time, unanticipated, permanent increase in the saving rate from s to s'. This shifts to curve to the right as shown by the dotted line, with a new intersection with the horizontal axis, k^{**}. The arrows on the horizontal axis show how the effective capital-labor ratio adjusts gradually to the new balanced growth effective capital-labor ratio, k^{**}. Immediately, when the increase in the saving rate is realized, the capital stock remains unchanged (since it is a *state* variable). After this point, it follows the dashed arrows on the horizontal axis.

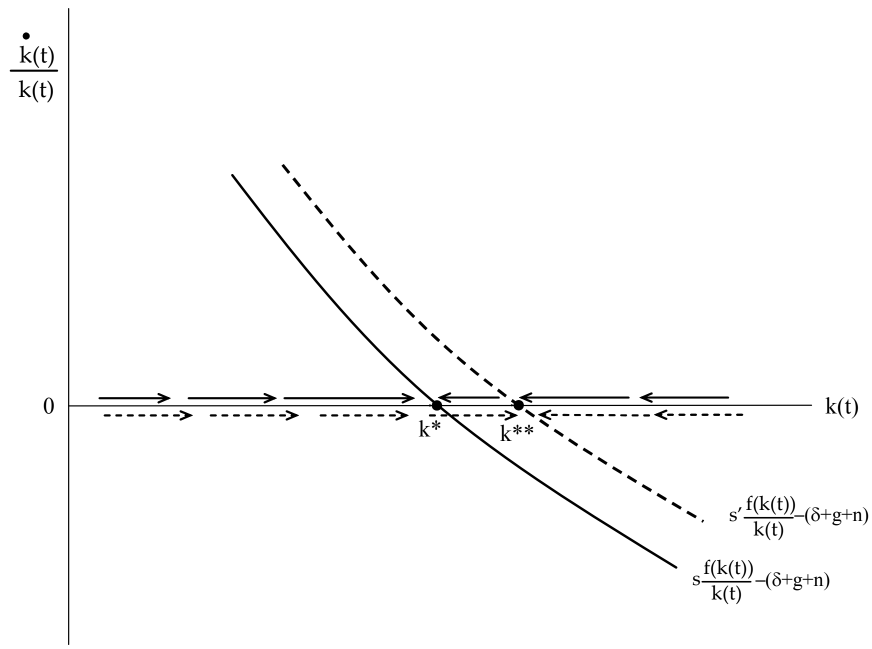

Figure 2.13. Dynamics following an increase in the savings rate from s to s'. The solid arrows show the dynamics for the initial steady state, while the dashed arrows show the dynamics for the new steady state.

The comparative dynamics following a one-time, unanticipated, permanent decrease in δ or n are identical.

We can also use the diagrammatic analysis to look at the effect of an unanticipated, but transitory change in parameters. For example, imagine that s changes in unanticipated manner at $t = t'$, but this change will be reversed and the saving rate will return back to its original value at some known future date $t = t'' > t'$. In this case, starting at t', the economy follows the rightwards arrows until t'. After t'', the original steady state of the differential equation applies and together with this the leftwards arrows become effective. Thus from t'' onwards, the economy gradually returns back to its original balanced growth equilibrium, k^*.

We will see that similar comparative dynamics can be carried out in the neoclassical growth model as well, but the response of the economy to some of these changes will be more complex.

2.8. Taking Stock

What have we learned from the Solow model? At some level, a lot. We now have a simple and tractable framework, which allows us to discuss capital accumulation and the implications of technological progress. As we will see in the next chapter, this framework is already quite useful in helping us think about the data.

However, at some other level, we have learned relatively little. The questions that Chapter 1 posed are related to why some countries are rich while others are poor, to why some countries grow while others stagnate, and to why the world economy embarked upon the process of steady growth over the past few centuries. The Solow model shows us that if there is no technological progress, and as long as we are not in the AK world, ruled out by Assumption 2, there will be no sustained growth. In this case, we can talk about cross-country output differences, but *not* about growth of countries or growth of the world economy.

The Solow model does generate per capita output growth, but only by introducing exogenous technological progress. But in this case, everything is being driven by technological progress, and technological progress itself is exogenous, just a blackbox, outside the model and outside the influence of economic incentives. If technological progress is "where it's at", then we have to study and understand which factors generate technological progress, what makes some firms and societies invent better technologies, and what induces firms and societies to adopt and use these superior technologies.

Even on the question of capital accumulation, the Solow model is not entirely satisfactory. The rate of capital accumulation is determined by the saving rate, the depreciation rate and the rate of population growth. All of these are taken as exogenous.

In this light, the Solow growth model is most useful for us as a framework laying out the general issues and questions. It emphasizes that to understand growth, we have to understand physical capital accumulation (and human capital accumulation, which will be discussed in the next chapter) and perhaps most importantly, technological progress. All of these are black boxes in the Solow growth model. Therefore, much of what we will do in the rest of the book will be to dig deeper and understand what lies in these black boxes. We start by introducing consumer optimization in Chapter 8, so that we can talk about capital accumulation more systematically. Then we will turn to models in which human capital accumulation and technological progress are endogenous. A framework in which the rate of accumulation of factors of production and technology are endogenous gives us a framework for posing and answering questions related to the fundamental causes of economic growth.

Nevertheless, we will also see that even in its bare-bones form the Solow model is useful in helping us think about the world and bringing useful perspectives, especially related to the proximate causes of economic growth. This is the topic of the next chapter.

2.9. References and Literature

The model analyzed in this chapter was first developed in Solow (1956) and Swan (1956). Solow (1970) gives a nice and accessible treatment, with historical references. Barro and Sala-i-Martin's (2004, Chapter 1) textbook presents a more up-to-date treatment of the basic Solow model at the graduate level, while Jones (1998, Chapter 2) presents an excellent undergraduate treatment.

The treatment in the chapter made frequent references to basic consumer and general equilibrium theory. These are prerequisites for an adequate understanding of the theory of economic growth. Mas-Colell, Whinston and Green's (1995) graduate microeconomics theory textbook contains an excellent treatment of all of the necessary material, including basic producer theory and an accessible presentation of the basic notions of general equilibrium theory, including a discussion of Arrow securities and the definition of Arrow-Debreu commodities. A good understanding of basic general equilibrium is essential for the study of both the material in this book and of macroeconomics more generally. Some of the important results from general equilibrium theory will be discussed in Chapter 5.

Properties of homogeneous functions and Euler's Theorem can be found, for example, in Simon and Blume (1994, Chapter 20). The reader should be familiar with the implicit function theorem and properties of concave and convex functions, which will be used throughout the book. A review is given in the Appendix Chapter A. The reader may also want to consult Simon and Blume (1994) and Rudin (1976).

Appendix Chapter B provides an overview of solutions to differential equations and difference equations and a discussion of stability. Theorems 2.2, 2.3, 2.4 and 2.5 follow from the results presented there. In addition, the reader may want to consult Simon and Blume (1994), Luenberger (1979) or Boyce and DiPrima (1977) for various results on difference and differential equations. Throughout these will feature frequently. Knowledge of solutions to simple differential equations and stability properties of difference and differential equations at the level of Appendix Chapter B will be useful. The material in Luenberger (1979) is particularly useful since it contains a unified treatment of difference and differential equations. Galor (2005) gives an introduction to difference equations and discrete time dynamical systems for economists.

The golden rule saving rate was introduced by Edmund Phelps (1961). It is called the "golden rule" rate with reference to the biblical golden rule "do unto others as you would have them do unto you" applied in an intergenerational setting—i.e., thinking that those living and consuming it each day to form a different generation. While the golden rule saving rate is of historical interest and useful for discussions of dynamic efficiency it has no intrinsic optimality property, since it is not derived from well-defined preferences. Optimal saving policies will be discussed in greater detail in Chapter 8.

The balanced growth facts were first noted by Kaldor (1963). Figure 2.11 uses data from Piketty and Saez (2004). Homer and Sylla (1991) discuss the history of interest rates over many centuries and across different societies; they show that there is no notable upward or downward trend in interest rate. Nevertheless, not all aspects of the economic growth process are "balanced", and the non-balanced nature of growth will be discussed in detail in Part 7 of the book, which also contains references to changes in the sectoral composition of output in the course of the growth process.

The steady state theorem, Theorem 2.7, was first proved by Uzawa (1961). Many different proofs are available. The proof given here is adapted from Schlicht (2006), which is also discussed in Jones and Scrimgeour (2006). A similar proof also appears in Wan (1971). Barro and Sala-i-Martin's (2004, Chapter 1) also suggest a proof. Nevertheless, their argument is incomplete, since it assumes that technological change *must be* a combination of Harrod and Solow-neutral technological change, which is rather restrictive and not necessary for the proof. The proposition and the proof provided here are therefore more general and complete.

As noted in the text, the CES production function was first introduced by Arrow, Chenery, Minhas and Solow (1961).

Finally, the interested reader should look at the paper by Hakenes and Irmen (2006) for why Inada conditions can remove the steady state at $k = 0$ in continuous time even when $f(0) = 0$. Here it suffices to say that whether this steady state exists or not is a matter of the order in which limits are taken. In any case, as noted in the text, the steady state at $k = 0$ has no economic content and will be ignored throughout the book.

2.10. Exercises

Exercise 2.1. Prove that Assumption 1 implies that $F(A, K, L)$ is concave in K and L, but not strictly so.

Exercise 2.2. Consider the Solow growth model in continuous time with the following per capita production function

$$f(k) = k^4 - 6k^3 + 11k^2 - 6k.$$

(1) Which parts of Assumptions 1 and 2 does the underlying production function $F(K, L)$ violate?
(2) Show that with this production function, there exist three steady-state equilibria.
(3) Prove that two of these steady-state equilibria are locally stable, while one of them is locally unstable. Can any of these steady-state equilibria be globally stable?

Exercise 2.3. Prove Proposition 2.7.

Exercise 2.4. Prove Proposition 2.8.

Exercise 2.5. Let us introduce government spending in the basic Solow model. Consider the basic model without technological change. In particular, suppose that (2.8) takes the form

$$Y(t) = C(t) + I(t) + G(t),$$

with $G(t)$ denoting government spending at time t. Imagine that government spending is given by $G(t) = \sigma Y(t)$.

(1) Discuss how the relationship between income and consumption should be changed? Is it reasonable to assume that $C(t) = sY(t)$?
(2) Suppose that government spending partly comes out of private consumption, so that $C(t) = (s - \lambda\sigma) Y(t)$, where $\lambda \in [0, 1]$. What is the effect of higher government spending (in the form of higher σ) on the equilibrium of the Solow model?
(3) Now suppose that part of government spending is invested in the capital stock of the economy. In particular, let a fraction ϕ of $G(t)$ be invested in the capital stock, so that total investment at time t is given by

$$I(t) = (1 - s - (1 - \lambda)\sigma + \phi\sigma) Y(t).$$

Show that if ϕ is sufficiently high, the steady-state level of capital-labor ratio will increase as a result of higher government spending (corresponding to higher σ). Is this reasonable? How would you alternatively introduce public investments in this model?

Exercise 2.6. Suppose that $F(A, K, L)$ is (weakly) concave in K and L and satisfies Assumption 2. Prove Propositions 2.2 and 2.5. How do we need to modify Proposition 2.6?

Exercise 2.7. Prove Proposition 2.6.

Exercise 2.8. Prove Corollary 2.2.

Exercise 2.9. Recall the definition of the elasticity of substitution σ in (2.36). Suppose labor markets are competitive and the wage rate is equal to w. Prove that if the aggregate production function $F(K, L, A)$ exhibits constant returns to scale in K and L, then

$$\varepsilon_{y,w} \equiv \frac{\partial y / \partial w}{y / w} = \sigma,$$

where, as usual, $y \equiv F(K, L, A)/L$.

Exercise 2.10. Consider a modified version of the continuous time Solow growth model where the aggregate production function is

$$F(K, L, Z) = L^{\beta} K^{\alpha} Z^{1-\alpha-\beta},$$

where Z is land, available in fixed inelastic supply. Assume that $\alpha + \beta < 1$, capital depreciates at the rate δ, and there is an exogenous saving rate of s.

(1) First suppose that there is no population growth. Find the steady-state capital-labor ratio and output level. Prove that the steady state is unique and globally stable.
(2) Now suppose that there is population growth at the rate n, i.e., $\dot{L}/L = n$. What happens to the capital-labor ratio and output level as $t \to \infty$? What happens to return to land and the wage rate as $t \to \infty$?
(3) Would you expect the population growth rate n or the saving rate s to change over time in this economy? If so, how?

EXERCISE 2.11. Consider the continuous time Solow model without technological progress and with constant rate of population growth equal to n. Suppose that the production function satisfies Assumptions 1 and 2. Assume that capital is owned by capitalists and labor is supplied by a different set of agents, the workers. Following a suggestion by Kaldor (1957), suppose that capitalists save a fraction s_K of their income, while workers consume all of their income.

(1) Define and characterize the steady-state equilibrium of this economy and study its stability.
(2) What is the relationship between the steady-state capital-labor ratio in this economy k^* and the golden rule capital stock k^*_{gold} defined above?

EXERCISE 2.12. Consider the Solow growth model with constant saving rate s and depreciation rate of capital equal to δ. Assume that population is constant and the aggregate production function is given by the constant returns to scale production function

$$F\left[A_K\left(t\right)K\left(t\right), A_L\left(t\right)L\left(t\right)\right]$$

where $\dot{A}_L(t)/A_L(t) = g_L > 0$ and $\dot{A}_K(t)/A_K(t) = g_K > 0$.

(1) Suppose that F is Cobb-Douglas. Determine the steady-state growth rate and the adjustment of the economy to the steady state.
(2) Suppose that F is not Cobb-Douglas. Prove that there does not exist a steady state. Explain why this is.
(3) For the case in which F is not Cobb-Douglas, determine what happens to the capital-labor ratio and output per capita as $t \to \infty$.

EXERCISE 2.13. Consider the Solow model with non-competitive labor markets. In particular, suppose that there is no population growth and no technological progress and output is given by $F(K, L)$. The saving rate is equal to s and the depreciation rate is given by δ.

(1) First suppose that there is a minimum wage $\bar{w}$, such that workers are not allowed to be paid less than $\bar{w}$. If labor demand at this wage falls short of L, employment is equal to the amount of labor demanded by firms, L^d. Assume that $\bar{w} > f(k^*) - k^* f'(k^*)$, where k^* is the steady-state capital-labor ratio of the basic

Solow model given by $f(k^*)/k^* = \delta/s$. Characterize the dynamic equilibrium path of this economy starting withsome amount of physical capital $K(0) > 0$.

(2) Next consider a different form of labor market imperfection, whereby workers receive a fraction $\beta > 0$ of output in each firm as their wage income. Characterize a dynamic equilibrium path in this case. [Hint: recall that the saving rate is still equal to s].

Exercise 2.14. Consider the discrete-time Solow growth model with constant population growth at the rate n, no technological change, constant depreciation rate of δ and a constant saving rate s. Assume that the per capita production function is given by the following continuous but non-neoclassical function:

$$f(k) = Ak + b,$$

where $A, b > 0$.

(1) Explain why this production function is non-neoclassical (i.e., why does it violate Assumptions 1 and 2 above?).
(2) Show that if $A - n - \delta = 1$, then for any $k(0) \neq b/2$, the economy settles into an asymptotic cycle and continuously fluctuates between $k(0)$ and $b - k(0)$.
(3) Now consider a more general continuous production function $f(k)$ that does not satisfy Assumptions 1 and 2, such that there exist $k_1, k_2 \in \mathbb{R}_+$ with $k_1 \neq k_2$ and

$$\begin{aligned} k_2 &= f(k_1) - (n+\delta) k_1 \\ k_1 &= f(k_2) - (n+\delta) k_2. \end{aligned}$$

Show that when such (k_1, k_2) exist, there may also exist a stable steady state.
(4) Prove that such cycles are not possible in the continuous-time Solow growth model for any (possibly non-neoclassical) continuous production function $f(k)$. [Hint: consider the equivalent of Figure 2.9 above].
(5) What does the result in parts 2 and 3 imply for the approximations of discrete time by continuous time suggested in Section 2.4?
(6) In light of your answer to part 6, what do you think of the cycles in parts 2 and 3?

Exercise 2.15. Characterize the asymptotic equilibrium of the modified Solow/AK model mentioned above, with a constant saving rate s, depreciation rate δ, no population growth and an aggregate production function of the form

$$F[K(t), L(t)] = A_K K(t) + A_L L(t).$$

Exercise 2.16. Consider the basic Solow growth model with a constant saving rate s, constant population growth at the rate n, aggregate production function given by (2.37), and no technological change.

(1) Determine conditions under which this production function satisfies Assumptions 1 and 2.
(2) Characterize the unique steady-state equilibrium when Assumptions 1 and 2 hold.
(3) Now suppose that σ is sufficiently high so that Assumption 2 does not hold. Show that in this case equilibrium behavior can be similar to that in Exercise 2.15 with sustained growth in the long run. Interpret this result.
(4) Now suppose that $\sigma \to 0$, so that the production function becomes Leontief,

$$Y(t) = \min \{\gamma A_K(t) K(t); (1-\gamma) A_L(t) L(t)\}.$$

The model is then identical to the classical Harrod-Domar growth model developed by Roy Harrod and Evsey Domar (Harrod, 1939, Domar, 1946). Show that in this case there is typically no steady-state equilibrium with full employment and no idle capital. What happens to factor prices in these cases? Explain why this case is "pathological," giving at least two reasons why we may expect equilibria with idle capital or idle labor not to apply in practice.

Exercise 2.17. * We now derive the CES production function following the method in the original article by Arrow, Chenery, Minhas and Solow (1961). These authors noted that a good empirical approximation to the relationship between income per capita and the wage rate was provided by an equation of the form

$$y = \alpha w^{-\sigma},$$

where $y = f(k)$ is again output per capita and w is the wage rate. With competitive markets, recall that $w = f(k) - kf'(k)$. Thus the above equation can be written as

$$y = \alpha \left(y - ky'\right)^{-\sigma},$$

where $y = y(k) \equiv f(k)$ and y' denotes $f'(k)$. This is a nonlinear first-order differential equation.

(1) Using separation of variables (see Appendix Chapter B), show that the solution to this equation satisfies

$$y(k) = \left[\alpha^{1/\sigma} + c_0 k^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}},$$

where c_0 is a constant of integration.
(2) How would you put more structure on α and c_0 and derive the exact form of the CES production function in (2.37)?

Exercise 2.18. Prove Proposition 2.12.

Exercise 2.19. Prove Proposition 2.13.

Exercise 2.20. In this exercise, we work through an alternative conception of technology, which will be useful in the next chapter. Consider the basic Solow model in continuous time and suppose that $A(t) = A$, so that there is no technological progress of the usual kind. However, assume that the relationship between investment and capital accumulation modified to

$$K(t+1) = (1-\delta) K(t) + q(t) I(t),$$

where $[q(t)]_{t=0}^{\infty}$ is an exogenously given time-varying process. Intuitively, when $q(t)$ is high, the same investment expenditure translates into a greater increase in the capital stock. Therefore, we can think of $q(t)$ as the inverse of the relative prices of machinery to output. When $q(t)$ is high, machinery is relatively cheaper. Gordon (1990) documented that the relative prices of durable machinery has been declining relative to output throughout the postwar era. This is quite plausible, especially given our recent experience with the decline in the relative price of computer hardware and software. Thus we may want to suppose that $\dot{q}(t) > 0$. This exercise asks you to work through a model with this feature based on Greenwood, Hercowitz and Krusell (1997).

(1) Suppose that $\dot{q}(t)/q(t) = \gamma_K > 0$. Show that for a general production function, $F(K, L)$, there exists no steady-state equilibrium.
(2) Now suppose that the production function is Cobb-Douglas, $F(K, L) = L^{1-\alpha} K^{\alpha}$, and characterize the unique steady-state equilibrium.
(3) Show that this steady-state equilibrium does not satisfy the Kaldor fact of constant K/Y. Is this a problem? [Hint: how is "K" measured in practice? How is it measured in this model?].

CHAPTER 3

The Solow Model and the Data

In this chapter, we will see how the Solow model or its simple extensions can be used to interpret both economic growth over time and cross-country output differences. Our focus is on *proximate causes* of economic growth, that is, the factors such as investment or capital accumulation highlighted by the basic Solow model, as well as technology and human capital differences. What lies underneath these proximate causes is the topic of the next chapter.

There are multiple ways of using the basic Solow model to look at the data. Here we start with the growth accounting framework, which is most commonly applied for decomposing the sources of growth over time. After briefly discussing the theory of growth accounting and some of its uses, we turn to applications of the Solow model to cross-country output differences. In this context, we introduce the augmented Solow model with human capital, and show how various different regression-based approaches can be motivated from this framework. We will then see how the growth accounting framework can be modified to a "development accounting framework" to form another bridge between the Solow model and the data. A constant theme that emerges from many of these approaches concerns the importance of productivity differences over time and across countries. The chapter ends with a brief discussion of various other approaches to estimating cross-country productivity differences.

3.1. Growth Accounting

As discussed in the previous chapter, at the center of the Solow model is the aggregate production function, (2.1), which we rewrite here in its general form:

$$Y(t) = F[K(t), L(t), A(t)].$$

Another major contribution of Bob Solow to the study of economic growth was the observation that this production function, combined with competitive factor markets, also gives us a framework for accounting for the sources of economic growth. In particular, Solow (1957) developed what has become one of the most common tools in macroeconomics, the *growth accounting framework.*

For our purposes, it is sufficient to expose the simplest version of this framework. Consider a continuous-time economy and differentiate the production function (2.1) with respect to time. Dropping time dependence and denoting the partial derivatives of F with respect to its arguments by F_A, F_K and F_L, this yields

$$\frac{\dot{Y}}{Y} = \frac{F_A A}{Y}\frac{\dot{A}}{A} + \frac{F_K K}{Y}\frac{\dot{K}}{K} + \frac{F_L L}{Y}\frac{\dot{L}}{L}. \tag{3.1}$$

Now denote the growth rates of output, capital stock and labor by $g \equiv \dot{Y}/Y$, $g_K \equiv \dot{K}/K$ and $g_L \equiv \dot{L}/L$, and also define

$$x \equiv \frac{F_A A}{Y}\frac{\dot{A}}{A}$$

as the contribution of technology to growth. Next, recall from the previous chapter that with competitive factor markets, we have $w = F_L$ and $R = F_K$ (equations (2.5) and (2.6)) and define the factor shares as $\alpha_K \equiv RK/Y$ and $\alpha_L \equiv wL/Y$. Putting all these together, (3.1) can be written as

$$x = g - \alpha_K g_K - \alpha_L g_L. \tag{3.2}$$

This is the *fundamental growth accounting* equation. At some level it is no more than an identity. However, it also allows us to estimate the contribution of technological progress to economic growth using data on factor shares, output growth, labor force growth and capital stock growth. This contribution from technological progress is typically referred to as *Total Factor Productivity* (TFP) or sometimes as Multi Factor Productivity.

In particular, denoting an estimate by "^", we have the estimate of TFP growth at time t as:

$$\hat{x}(t) = g(t) - \alpha_K(t)\, g_K(t) - \alpha_L(t)\, g_L(t). \tag{3.3}$$

we only put the "^" on x, but one may want to take into account that all of the terms on the right-hand side are also "estimates" obtained with a range of assumptions from national accounts and other data sources.

If we are interested in $\dot{A}/A$ rather than x, we would need to make further assumptions. For example, if we assume that the production function takes the standard labor-augmenting form

$$Y(t) = \tilde{F}[K(t), A(t)L(t)],$$

then we would have

$$\frac{\dot{A}}{A} = \frac{1}{\alpha_L}[g - \alpha_K g_K - \alpha_L g_L],$$

but this equation is not particularly useful, since $\dot{A}/A$ is not something we are inherently interested in. The economically interesting object is in fact $\hat{x}$ in (3.3), since it measures the effect of technological progress on output growth directly.

In continuous time, equation (3.3) is exact because it is defined in terms of instantaneous changes (derivatives). In practice, instead of instantaneous changes, we look at changes over discrete time intervals, for example over a year (or sometimes with better data, over a quarter or a month). With discrete time intervals, there is a potential problem in using (3.3); over the time horizon in question, factor shares can change; should we use beginning-of-period or end-of-period values of α_K and α_L? It can be shown that the use of either beginning-of-period or end-of-period values might lead to seriously biased estimates of the contribution of TFP to output growth, $\hat{x}$. This is particularly likely when the distance between the two time periods is large (see Exercise 3.1). The best way of avoiding such biases is to use as high-frequency data as possible.

For now, taking the available data as given, let us look at how one could use the growth accounting framework with data over discrete intervals. The most common way of dealing with the problems pointed out above is to use factor shares calculated as the average of the beginning of period and end of period values. Therefore in discrete time, for a change between times t and $t+1$, the analog of equation (3.3) becomes

$$\hat{x}_{t,t+1} = g_{t,t+1} - \bar{\alpha}_{K,t,t+1} g_{K,t,t+1} - \bar{\alpha}_{L,t,t+1} g_{L,t,t+1}, \tag{3.4}$$

where $g_{t,t+1}$ is the growth rate of output between t and $t+1$, and other growth rates are defined analogously. Moreover,

$$\bar{\alpha}_{K,t,t+1} \equiv \frac{\alpha_K(t) + \alpha_K(t+1)}{2} \text{ and } \bar{\alpha}_{L,t,t+1} \equiv \frac{\alpha_L(t) + \alpha_L(t+1)}{2}$$

are average factor shares between t and $t+1$. Equation (3.4) would be a fairly good approximation to (3.3) when the difference between t and $t+1$ is small and the capital-labor ratio does not change much during this time interval.

Solow's (1957) article not only developed this growth accounting framework but also applied it to US data for a preliminary assessment of the "sources of growth" during the early 20th century. The question Bob Solow asked was this: how much of the growth of the US economy can be attributed to increased labor and capital inputs, and how much of it is due to the residual, "technological progress"? Solow's conclusion was quite striking: a large part of the growth was due to technological progress.

This has been a landmark finding, emphasizing the importance of technological progress as the driver of economic growth not only in theory (as we saw in the previous chapter), but also in practice. It focused the attention of economists on sources of technology differences over time, across nations, industries and firms.

From early days, however, it was recognized that calculating the contribution of technological progress to economic growth in this manner has a number of pitfalls. Moses Abramovitz (1956), famously, dubbed the $\hat{x}$ term "the measure of our ignorance"—after all, it was the residual that we could not explain and we decided to call it "technology".

In its extreme form, this criticism is unfair, since $\hat{x}$ does correspond to technology according to equation (3.3); thus the growth accounting framework is an example of using theory to inform measurement. Yet at another level, the criticism has validity. If we mismeasure the growth rates of labor and capital inputs, g_L and g_K, we will arrive at inflated estimates of $\hat{x}$. And in fact there are good reasons for suspecting that Solow's estimates and even the higher-quality estimates that came later may be mismeasuring the growth of inputs. The most obvious reason for this is that what matters is not labor hours, but effective labor hours, so it is important—though difficult—to make adjustments for changes in the *human capital* of workers. We will discuss issues related to human capital in Section 3.3 below and then in greater detail in Chapter 10. Similarly, measurement of capital inputs is not straightforward. In the theoretical model, capital corresponds to the final good used as input to produce more goods. But in practice, capital is machinery, and in measuring the amount of capital used in production one has to make assumptions about how relative prices of machinery change over time. The typical assumption, adopted for a long time in national accounts and also naturally in applications of the growth accounting framework, was to use capital expenditures. However, if the same machines are much cheaper today than they had been in the past (as has been the case for computers, for example), then this methodology would severely underestimate g_K (recall Exercise 2.20 in the previous chapter). Therefore, there is indeed a danger in applying equation (3.3), since underestimating g_K will naturally inflate our estimates of the role of technology as a source of economic growth.

What the best way of making adjustments to labor and capital inputs in order to arrive to the best estimate of technology is still a hotly debated area. Dale Jorgensen, for example, has shown that the "residual" technology can be reduced very substantially (perhaps almost to 0) by making adjustments for changes in the quality of labor and capital (see, for example, Jorgensen, Gollop and Fraumeni, 1987, or Jorgensen, 2005). These issues will also become relevant when we think of applying similar ideas to decomposing cross-country output differences. Before doing this, however, we turn to applications of the Solow model to data using regression analysis.

3.2. Solow Model and Regression Analyses

Another popular approach of taking the Solow model to data is to use *growth regressions*, which involve estimating regression models with country growth rates on the left-hand side. These growth regressions have been used extensively following the work by Barro (1991). To see how these regressions are motivated and what their shortcomings are, let us return to the basic Solow model with constant population growth and labor-augmenting technological change in continuous time. Recall that, in this model, the equilibrium of an economy is described by the following equations:

$$y(t) = A(t) f(k(t)), \tag{3.5}$$

and

$$\frac{\dot{k}(t)}{k(t)} = \frac{sf(k(t))}{k(t)} - \delta - g - n, \tag{3.6}$$

where $A(t)$ is the labor-augmenting (Harrod-neutral) technology term, $k(t) \equiv K(t)/(A(t)L(t))$ is the effective capital labor ratio and $f(\cdot)$ is the per capita production function. Equation (3.6) follows from the constant technological progress and constant population growth assumptions, i.e., $\dot{A}(t)/A(t) = g$ and $\dot{L}(t)/L(t) = n$. Now differentiating (3.5) with respect to time and dividing both sides by $y(t)$, we obtain

$$\frac{\dot{y}(t)}{y(t)} = g + \varepsilon_f(k(t))\frac{\dot{k}(t)}{k(t)}, \tag{3.7}$$

where

$$\varepsilon_f(k(t)) \equiv \frac{f'(k(t))\,k(t)}{f(k(t))} \in (0,1)$$

is the elasticity of the $f(\cdot)$ function. The fact that it is between 0 and 1 follows from Assumption 1. For example, with the Cobb-Douglas technology from Example 2.1 in the previous chapter, we would have $\varepsilon_f(k(t)) = \alpha$, that is, it is a constant independent of $k(t)$ (see Example 3.1 below). However, generally, this elasticity is a function of $k(t)$.

Now let us consider a first-order Taylor expansion of (3.6) with respect to $\log k(t)$ around the steady-state value k^* (and recall that $\partial y/\partial \log x = (\partial y/\partial x)\cdot x$). This expansion implies that for $k(t)$ in the neighborhood of k^*, we have

$$\begin{aligned}\frac{\dot{k}(t)}{k(t)} &\simeq \left(\frac{sf(k^*)}{k^*} - \delta - g - n\right) + \left(\frac{f'(k^*)\,k^*}{f(k^*)} - 1\right)s\frac{f(k^*)}{k^*}(\log k(t) - \log k^*).\\ &\simeq (\varepsilon_f(k^*) - 1)(\delta + g + n)(\log k(t) - \log k^*).\end{aligned}$$

The use of the symbol "$\simeq$" here is to emphasize that this is an approximation, ignoring second-order terms. In particular, the first line follows simply by differentiating $\dot{k}(t)/k(t)$ with respect to $\log k(t)$ and evaluating the derivatives at k^* (and ignoring second-order terms). The second line uses the fact that the first term in the first line is equal to zero by definition of the steady-state value k^* (recall that from equation (2.47) in the previous chapter, $sf(k^*)/k^* = \delta + g + n$), the definition of the elasticity of the f function, $\varepsilon_f(k(t))$, and again the fact that $sf(k^*)/k^* = \delta + g + n$. Now substituting this into (3.7), we obtain

$$\frac{\dot{y}(t)}{y(t)} \simeq g - \varepsilon_f(k^*)(1 - \varepsilon_f(k^*))(\delta + g + n)(\log k(t) - \log k^*).$$

Let us define $y^*(t) \equiv A(t)f(k^*)$ as the level of per capita output that would apply if the effective capital-labor ratio were at its steady-state value and technology were at its time t level. We therefore refer to $y^*(t)$ as the "steady-state level of output per capita" even though it is not constant. Now taking first-order Taylor expansions of $\log y(t)$ with respect to $\log k(t)$ around $\log k^*(t)$ gives

$$\log y(t) - \log y^*(t) \simeq \varepsilon_f(k^*)(\log k(t) - \log k^*).$$

Combining this with the previous equation, we obtain the following "convergence equation":

$$\frac{\dot{y}(t)}{y(t)} \simeq g - (1 - \varepsilon_f(k^*))(\delta + g + n)(\log y(t) - \log y^*(t)). \tag{3.8}$$

Equation (3.8) makes it clear that, in the Solow model, there are two sources of growth in output per capita: the first is g, the rate of technological progress, and the second is "convergence". This latter source of growth results from the negative impact of the gap between the current level of output per capita and the steady-state level of output per capita on the rate of capital accumulation (recall that $0 < \varepsilon_f(k^*) < 1$). Intuitively, the further below is a country from its steady state capital-labor ratio, the more capital it will accumulate and the faster it will grow. This pattern is in fact visible in Figure 2.7 from the previous chapter. The reason is also clear from the analysis in the previous chapter. The lower is $y(t)$ relative to $y^*(t)$, and thus the lower is $k(t)$ relative to k^*, the greater is the average product of capital $f(k^*)/k^*$, and this leads to faster growth in the effective capital-labor ratio.

Another noteworthy feature is that the speed of convergence in equation (3.8), measured by the term $(1 - \varepsilon_f(k^*))(\delta + g + n)$ multiplying the gap between $\log y(t)$ and $\log y^*(t)$, depends on $\delta + g + n$ and the elasticity of the production function $\varepsilon_f(k^*)$. Both of these capture intuitive effects. As discussed in the previous chapter, the term $\delta + g + n$ determines the rate at which effective capital-labor ratio needs to be replenished. The higher is this rate of replenishment, the larger is the amount of investment in the economy (recall Figure 2.7 in the previous chapter) and thus there is room for faster adjustment. On the other hand, when $\varepsilon_f(k^*)$ is high, we are close to a linear—AK—production function, and as demonstrated in the previous chapter, in this case convergence should be slow. In the extreme case where $\varepsilon_f(k^*)$ is equal to 1, we will be in the AK economy and there will be no convergence.

Example 3.1. **(Cobb-Douglas production function and convergence)** Consider briefly the Cobb-Douglas production function from Example 2.1 in the previous chapter, where $Y(t) = A(t) K(t)^{\alpha} L(t)^{1-\alpha}$. This implies that $y(t) = A(t) k(t)^{\alpha}$. Consequently, as noted above, $\varepsilon_f(k(t)) = \alpha$. Therefore, (3.8) becomes

$$\frac{\dot{y}(t)}{y(t)} \simeq g - (1 - \alpha)(\delta + g + n)(\log y(t) - \log y^*(t)).$$

This equation also enables us to "calibrate" the speed of convergence in practice—meaning to obtain a back-of-the-envelope estimate of the speed of convergence by using plausible values of parameters. Let us focus on advanced economies. In that case, plausible values for these parameters might be $g \simeq 0.02$ for approximately 2% per year output per capita growth, $n \simeq 0.01$ for approximately 1% population growth and $\delta \simeq 0.05$ for about 5% per year depreciation. Recall also from the previous chapter that the share of capital in national income is about 1/3, so with the Cobb-Douglas production function we should have $\alpha \simeq 1/3$. Consequently, we may expect the convergence coefficient in front of $\log y(t) - \log y^*(t)$ to be around 0.054 ($\simeq 0.67 \times 0.08$). This is a very rapid rate of convergence and would imply that income gaps between two similar countries that have the same technology, the same depreciation rate and the same rate of population growth should narrow rather quickly. For example, it can be computed that with these numbers, the gap of income between two similar countries should be halved in little more than 10 years (see Exercise 3.4). This is clearly at odds with the patterns we saw in Chapter 1.

Using equation (3.8), we can obtain a growth regression similar to those estimated by Barro (1991). In particular, using discrete time approximations, equation (3.8) yields the regression equation:

$$g_{i,t,t-1} = b^0 + b^1 \log y_{i,t-1} + \varepsilon_{i,t}, \tag{3.9}$$

where $g_{i,t,t-1}$ is the growth rate of country i between dates $t-1$ and t, $\log y_{i,t-1}$ is the "initial" (i.e., time $t-1$) log output per capita of this country, and $\varepsilon_{i,t}$ is a stochastic term capturing all omitted influences. Regressions on this form have been estimated by, among others, Baumol (1986), Barro (1991) and Barro and Sala-i-Martin (1992). If such an equation is estimated in the sample of core OECD countries, b^1 is indeed estimated to be negative; countries like Greece, Spain and Portugal that were relatively poor at the end of World War II have grown faster than the rest as shown in Figure 1.14 in Chapter 1.

Yet, Figure 1.13 in Chapter 1 shows, when we look at the whole world, there is no evidence for a negative b^1. Instead, this figure makes it clear that, if anything, b^1 would be positive. In other words, there is no evidence of world-wide convergence.

Barro and Sala-i-Martin refer to this type of convergence as "unconditional convergence," meaning the convergence of countries regardless of differences in characteristics and policies. However, this notion of unconditional convergence may be too demanding. It requires that there should be a tendency for the income gap between any two countries to decline, irrespective of what types of technological opportunities, investment behavior, policies and institutions these countries have. If countries do differ with respect to these factors, the Solow model would *not* predict that they should converge in income level. Instead, each should converge to their own level of steady-state income per capita or balanced growth path. Thus in a world where countries differ according to their characteristics, a more appropriate regression equation may take the form:

$$g_{i,t,t-1} = b_i^0 + b^1 \log y_{i,t-1} + \varepsilon_{i,t}, \tag{3.10}$$

where the key difference is that now the constant term, b_i^0, is country specific. (In principle, the slope term, measuring the speed of convergence, b^1, should also be country specific, but in empirical work, this is generally taken to be a constant, and we assume the same here to simplify the exposition). One may then model b_i^0 as a function of certain country characteristics, such as institutional factors, human capital (see next section), or even the investment rate.

If the true equation is (3.10), in the sense that the Solow model applies but certain determinants of economic growth differ across countries, equation (3.9) would not be a good fit to the data. Put differently, there is no guarantee that the estimates of b^1 resulting from this equation will be negative. In particular, it is natural to expect that $Cov\left(b_i^0, \log y_{i,t-1}\right) < 0$ (where Cov refers to the population covariance), since economies with certain growth-reducing characteristics will have low levels of output. This implies a negative bias in the estimate of b^1 in equation (3.9), when the more appropriate equation is (3.10).

With this motivation, Barro (1991) and Barro and Sala-i-Martin (2004) favor the notion of "conditional convergence," which means that the convergence effects emphasized by the Solow model should lead to negative estimates of b^1 once b_i^0 is allowed to vary across countries. To implement this idea of conditional convergence empirically, Barro (1991) and Barro and Sala-i-Martin (2004) estimate models where b_i^0 is assumed to be a function of, among other things, the male schooling rate, the female schooling rate, the fertility rate, the investment rate, the government-consumption ratio, the inflation rate, changes in terms of trades, openness and institutional variables such as rule of law and democracy. In regression form, this can be written as

$$g_{i,t,t-1} = \mathbf{X}_{i,t}'\boldsymbol{\beta} + b^1 \log y_{i,t-1} + \varepsilon_{i,t}, \tag{3.11}$$

where $\mathbf{X}_{i,t}$ is a (column) vector including the variables mentioned above (as well as a constant), with a vector of coefficients $\boldsymbol{\beta}$. In other words, this specification imposes that b_i^0 in equation (3.10) can be approximated by $\mathbf{X}_{i,t}'\boldsymbol{\beta}$. Consistent with the emphasis on conditional convergence, regressions of equation (3.11) tend to show a negative estimate of b^1, but the magnitude of this estimate is much lower than that suggested by the computations in Example 3.1.

Regressions similar to (3.11) have not only been used to support "conditional convergence," that is, the presence of transitional dynamics similar to those implied by the Solow growth model, but they have also been used to estimate the "determinants of economic growth". In particular, it may appear natural to presume that the estimates of the coefficient vector $\boldsymbol{\beta}$ will contain information about the *causal effects* of various variables on economic growth. For example, the fact that the schooling variables enter with positive coefficients in the estimates of regression (3.11) is interpreted as evidence that "schooling causes growth". The simplicity of the regression equations of the form (3.11) and the fact that they create an attractive bridge between theory and data have made them very popular over the past two decades.

Nevertheless, there are several problematic features with regressions of this form. These include:

(1) Most, if not all, of the variables in $\mathbf{X}_{i,t}$ as well as $\log y_{i,t-1}$, are *econometrically endogenous* in the sense that they are jointly determined with the rate of economic growth between dates $t-1$ and t. For example, the same factors that make the country relatively poor in 1950, thus reducing $\log y_{i,t-1}$, should also affect its growth rate after 1950. Or the same factors that make a country invest little in physical and human capital could have a direct effect on its growth rate (through other channels such as its technology or the efficiency with which the factors of production are being utilized). This creates an obvious source of bias (and lack of econometric consistency) in the regression estimates of the coefficients. This bias makes it unlikely that the effects captured in the coefficient vector $\boldsymbol{\beta}$ correspond to causal effects of these characteristics on the growth potential of economies. One may argue that the convergence coefficient b^1 is of interest, even if it does not have a "causal interpretation". This argument is not entirely compelling, however. A basic result in econometrics is that if $\mathbf{X}_{i,t}$ is econometrically endogenous, so that the parameter vector $\boldsymbol{\beta}$ is estimated inconsistently, the estimate of the parameter b^1 will also be inconsistent unless $\mathbf{X}_{i,t}$ is independent from $\log y_{i,t-1}$.[1] This makes the estimates of the convergence coefficient, b^1, hard to interpret.

(2) Even if $\mathbf{X}_{i,t}$'s were econometrically exogenous, a negative coefficient estimate for b^1 could be caused by other econometric problems, such as measurement error or other transitory shocks to $y_{i,t}$. For example, because national accounts data are always measured with error, suppose that our available data on $\log y_{i,t}$ contains measurement error. In particular, suppose that we only observe estimates of output per capita $\tilde{y}_{i,t} = y_{i,t} \exp(u_{i,t})$, where $y_{i,t}$ is the true output per capita and $u_{i,t}$ is a random and serially uncorrelated error term. When we use the variable $\log \tilde{y}_{i,t}$ measured with error in our regressions, the error term $u_{i,t-1}$ will appear both on the left-hand side and the right-hand side of (3.11). In particular, note that

$$\log \tilde{y}_{i,t} - \log \tilde{y}_{i,t-1} = \log y_{i,t} - \log y_{i,t-1} + u_{i,t} - u_{i,t-1}.$$

Since the measured growth is $\tilde{g}_{i,t,t-1} \approx \log \tilde{y}_{i,t} - \log \tilde{y}_{i,t-1} = \log y_{i,t} - \log y_{i,t-1} + u_{i,t} - u_{i,t-1}$, when we look at the growth regression

$$\tilde{g}_{i,t,t-1} = \mathbf{X}'_{i,t}\boldsymbol{\beta} + b^1 \log \tilde{y}_{i,t-1} + \varepsilon_{i,t},$$

the measurement error $u_{i,t-1}$ will be part of both the error term $\varepsilon_{i,t}$ and $\log \tilde{y}_{i,t-1} = \log y_{i,t-1} + u_{i,t-1}$, leading to a negative bias in the estimation of b^1. Therefore,

[1] An example of the endogeneity of these variables will be given in Section 3.4 below.

we can end up with a negative estimate of b^1, *even when* there is no conditional convergence.

(3) The interpretation of regression equations like (3.11) is not always straightforward either. Many of the regressions used in the literature include the investment rate as part of the vector $\mathbf{X}_{i,t}$ (and all of them include the schooling rate). However, in the Solow model, differences in investment rates are *the* channel via which convergence will take place—in the sense that economies below their steady state will grow faster by having higher investment rates. Therefore, strictly speaking, conditional on the investment rate, there should be no further effect of the gap between the current level of output and the steady-state level of output. The same concern applies to interpreting the effect of the variables in the vector $\mathbf{X}_{i,t}$ (such as institutions or openness), which are typically included as potential determinants of economic growth. However, many of these variables would affect growth primarily by affecting the investment rate (or the schooling rate). Therefore, once we condition on the investment rate and the schooling rate, the coefficients on these variables no longer measure their impact on economic growth. Consequently, estimates of (3.11) with investment-like variables on the right-hand side are difficult to link to theory.

(4) Regressions of the form (3.11) are often thought to be appealing because they model the "process of economic growth" and may give us information about the potential determinants of economic growth (to the extent that these are included in the vector $\mathbf{X}_{i,t}$). Nevertheless, there is a sense in which growth regressions are not much different than "levels regressions"—similar to those discussed in Section 3.4 below and further in Chapter 4. In particular, again noting that $g_{i,t,t-1} \approx \log y_{i,t} - \log y_{i,t-1}$, equation (3.11) can be rewritten as

$$\log y_{i,t} \approx \mathbf{X}'_{i,t}\boldsymbol{\beta} + \left(1 + b^1\right) \log y_{i,t-1} + \varepsilon_{i,t}.$$

Therefore, essentially the level of output is being regressed on $\mathbf{X}_{i,t}$, so that regression on the form (3.11) will typically uncover the correlations between the variables in the vector $\mathbf{X}_{i,t}$ and output per capita. Thus growth regressions are often informative about which economic or social processes go hand-in-hand with high levels of output. A specific example is life expectancy. When life expectancy is included in the vector $\mathbf{X}_{i,t}$ in a growth regression, it is often highly significant. But this simply reflects the high correlation between income per capita and life expectancy we have already seen in Figure 1.6 in Chapter 1. It does not imply that life expectancy has a causal effect on economic growth, and more likely reflects the joint determination of life expectancy and income per capita and the impact of prosperity on life expectancy.

(5) Finally, the motivating equation for the growth regression, (3.8), is derived for a closed Solow economy. When we look at cross-country income differences or growth experiences, the use of this equation imposes the assumption that "each country is an island". In other words, we are representing the world as a collection of non-interacting closed economies. In practice, countries trade goods, exchange ideas and borrow and lend in international financial markets. This implies that the behavior of different countries will not be given by equation (3.8), but by a system of equations characterizing the joint world equilibrium. Even though a world equilibrium is typically a better way of representing differences in income per capita and their evolution, in the first part of the book we will follow the approach of the growth regressions and often use the "each country is an island" assumption. We will return to models of world equilibrium in Chapter 19.

This discussion suggests that growth regressions need to be used with caution and ought to be supplemented with other methods of mapping the basic Solow model to data. What other methods will be useful in empirical analyses of economic growth? A number of answers emerge from our discussion above. To start with, for many of the questions we are interested in, regressions that control for fixed country characteristics might be equally useful as, or more useful than, growth regressions. For example, a more natural regression framework for investigating the economic (or statistical) relationship between the variables in the vector $\mathbf{X}_{i,t}$ and economic growth might be

$$\log y_{i,t} = \alpha \log y_{i,t-1} + \mathbf{X}'_{i,t}\boldsymbol{\beta} + \delta_i + \mu_t + \varepsilon_{i,t}, \tag{3.12}$$

where δ_i's denote a full set of country fixed effects and μ_t's denote a full set of year effects. This regression framework differs from the growth regressions in a number of respects. First, the regression equation is specified in levels rather than with the growth rate on the left-hand side. But as we have seen above, this is not important and corresponds to a transformation of the left-hand side variable. Second, although we have included the lagged dependent variable, $\log y_{i,t-1}$, on the right-hand side of (3.12), models with fixed effects and lagged dependent variables are difficult to estimate, thus it is often more convenient to omit this term. Third and most important, by including the country fixed effects, this regression equation takes out fixed country characteristics that might be simultaneously affecting economic growth (or the level of income per capita) and the right-hand side variables of interest. For instance, in terms of the example discussed in (4) above, if life expectancy and income per capita are correlated because of some omitted factors, the country fixed effects, δ_i's, will remove the influence of these factors. Therefore, panel data regressions as in (3.12) may be more informative about the relationship between a range of factors and income per capita. Nevertheless, it is important to emphasize that including country fixed effects is not a panacea against all omitted variable biases and econometric endogeneity problems. Simultaneity bias often results from time-varying influences, which cannot be removed by including fixed effects. Moreover, to the extent that some of the variables in the vector $\mathbf{X}_{i,t}$ are slowly-varying themselves, the inclusion of country fixed effects will make it difficult to uncover the statistical relationship between these variables and income per capita.

This discussion highlights that econometric models similar to (3.12) are often useful and a good complement to (or substitute for) growth regressions. But they are not a good substitute for specifying the structural economic relationships fully and for estimating the causal relationships of interest. In the next chapter, we will see how some progress can be made in this regard by looking at the historical determinants of long-run economic growth and using specific historical episodes to generate potential sources of exogenous variation in the variables of interest that can allow an instrumental-variables strategy.

In the remainder of this chapter, we will see how the structure of the Solow model can be further exploited to look at the data. Before doing this, we will present an augmented version of the Solow model incorporating human capital, which will be useful in these empirical exercises.

3.3. The Solow Model with Human Capital

Before discussing further applications of the Solow model to the data, let us enrich the model by including human capital. Human capital is a term we use to represent the stock of skills, education, competencies and other productivity-enhancing characteristics embedded in labor. Put differently, human capital represents the efficiency units of labor embedded in raw labor hours.

The notion of human capital will be discussed in greater detail in Chapter 10 below, where we will study models in which individuals invest in their human capital in order to increase their earnings. In fact, the notion and the name "human capital" comes from the observation that individuals will invest in their skills and competencies in the same way as firms invest in their physical capital—to increase their productivity. The seminal work by Ted Schultz, Jacob Mincer and Gary Becker brought the notion of human capital to the forefront of economics. For now, all we need to know is that labor hours supplied by different individuals do not contain the same efficiency units; a highly trained carpenter can produce a chair in a few hours, while an amateur would spend many more hours to perform the same task. We capture this notion by thinking that the trained carpenter has more efficiency units of labor embedded in the labor hours he supplies, or alternatively he has more human capital. The theory of human capital is very rich and some of the important notions will be discussed in Chapter 10. For now, our objective is more modest, to investigate how including human capital makes the Solow model a better fit to the data. The inclusion of human capital will enable us to embed all three of the main *proximate sources* of income differences; physical capital, human capital and technology.

For the purposes of this section, let us focus on the continuous time economy and suppose that the aggregate production function of the economy is given by a variant of equation (2.1):

$$Y = F(K, H, AL), \tag{3.13}$$

where H denotes "human capital". How this is measured in the data will be discussed below. As usual, we assume throughout (often implicitly) that $A > 0$.

Let us also modify Assumption 1 as follows

Assumption 1′: The production function $F : \mathbb{R}_+^3 \to \mathbb{R}_+$ in (3.13) is twice continuously differentiable in K, H and L, and satisfies

$$\frac{\partial F(K,H,AL)}{\partial K} > 0, \quad \frac{\partial F(K,H,AL)}{\partial H} > 0, \quad \frac{\partial F(K,H,AL)}{\partial L} > 0$$
$$\frac{\partial^2 F(K,H,AL)}{\partial K^2} < 0, \quad \frac{\partial^2 F(K,H,AL)}{\partial H^2} < 0, \quad \frac{\partial^2 F(K,H,AL)}{\partial L^2} < 0,$$

Moreover, F exhibits constant returns to scale in its three arguments.

We also replace Assumption 2 with the following:

Assumption 2′: F satisfies the Inada conditions

$$\begin{aligned}
\lim_{K\to 0} \frac{\partial F(K,H,AL)}{\partial K} &= \infty \text{ and } \lim_{K\to\infty} \frac{\partial F(K,H,AL)}{\partial K} = 0 \text{ for all } H > 0 \text{ and } AL > 0, \\
\lim_{H\to 0} \frac{\partial F(K,H,AL)}{\partial H} &= \infty \text{ and } \lim_{H\to\infty} \frac{\partial F(K,H,AL)}{\partial H} = 0 \text{ for all } K > 0 \text{ and } AL > 0, \\
\lim_{L\to 0} \frac{\partial F(K,H,AL)}{\partial L} &= \infty \text{ and } \lim_{L\to\infty} \frac{\partial F(K,H,AL)}{\partial L} = 0 \text{ for all } K,\ H, A > 0.
\end{aligned}$$

Moreover, we assume that investments in human capital take a similar form to investments in physical capital; households save a fraction s_k of their income to invest in physical capital and a fraction s_h to invest in human capital. Human capital also depreciates in the same way as physical capital, and we denote the depreciation rates of physical and human capital by δ_k and δ_h, respectively.

We continue to assume that there is constant population growth and a constant rate of labor-augmenting technological progress, i.e.,

$$\frac{\dot{L}(t)}{L(t)} = n \text{ and } \frac{\dot{A}(t)}{A(t)} = g.$$

Now defining effective human and physical capital ratios as

$$k(t) \equiv \frac{K(t)}{A(t)L(t)} \text{ and } h(t) \equiv \frac{H(t)}{A(t)L(t)},$$

and using the constant returns to scale feature in Assumption 1′, output per effective unit of labor can be written as

$$\begin{aligned}\hat{y}(t) &\equiv \frac{Y(t)}{A(t)L(t)} \\ &= F\left(\frac{K(t)}{A(t)L(t)}, \frac{H(t)}{A(t)L(t)}, 1\right) \\ &\equiv f(k(t), h(t)).\end{aligned}$$

With the same steps as in Chapter 2, the law of motion of $k(t)$ and $h(t)$ can then be obtained as:

$$\begin{aligned}\dot{k}(t) &= s_k f(k(t), h(t)) - (\delta_k + g + n) k(t), \\ \dot{h}(t) &= s_h f(k(t), h(t)) - (\delta_h + g + n) h(t).\end{aligned}$$

A steady-state equilibrium is now defined not only in terms of effective capital-labor ratio, but effective human and physical capital ratios, (k^*, h^*), which satisfies the following two equations:

$$s_k f(k^*, h^*) - (\delta_k + g + n) k^* = 0, \tag{3.14}$$

and

$$s_h f(k^*, h^*) - (\delta_h + g + n) h^* = 0. \tag{3.15}$$

As in the basic Solow model, we focus on steady-state equilibria with $k^* > 0$ and $h^* > 0$ (if $f(0,0) = 0$, then there exists a trivial steady state with $k = h = 0$, which we ignore as we did in the previous chapter).

We can first prove that this steady-state equilibrium is in fact unique. To see this heuristically, consider Figure 3.1, which is drawn in the (k, h) space. The two curves represent the two equations (3.14) and (3.15). Both lines are upward sloping. For example, in (3.14) a higher level of h^* implies greater $f(k^*, h^*)$ from Assumption 1′, thus the level of k^* and that will satisfy the equation is higher. The same reasoning applies to (3.15). However, the proof of the next proposition shows that (3.15) is always shallower in the (k, h) space, so the two curves can only intersect once.

Proposition 3.1. *Suppose Assumptions 1′ and 2′ are satisfied. Then in the augmented Solow model with human capital, there exists a unique steady-state equilibrium* (k^*, h^*).

Proof. First consider the slope of the curve (3.14), corresponding to the $\dot{k} = 0$ locus, in the (k, h) space. Using the implicit function theorem, we have

$$\left.\frac{dh}{dk}\right|_{\dot{k}=0} = \frac{(\delta_k + g + n) - s_k f_k(k^*, h^*)}{s_k f_h(k^*, h^*)}, \tag{3.16}$$

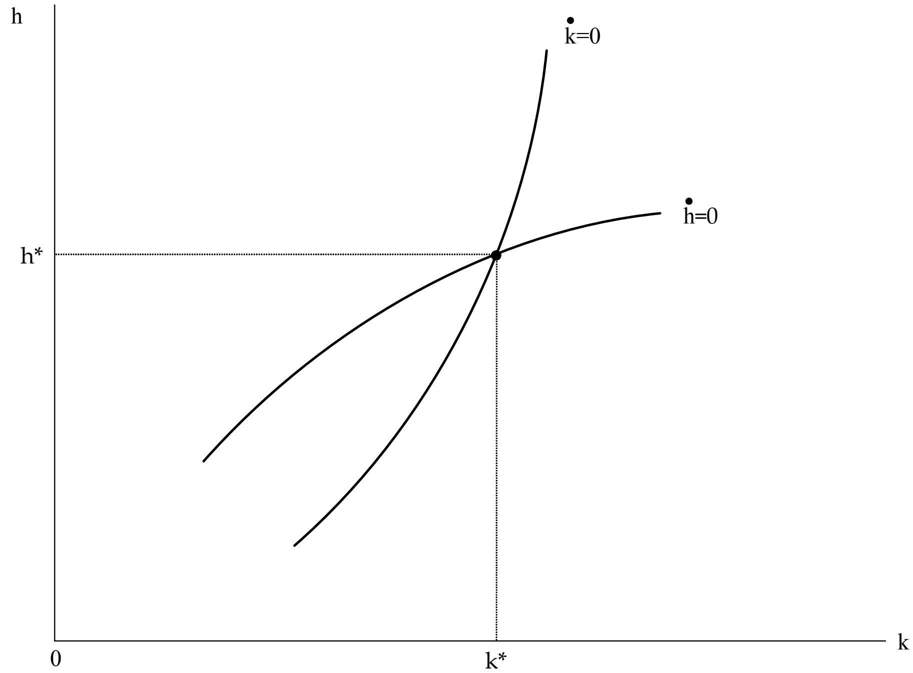

Figure 3.1. Steady-state equilibrium in the Solow model with human capital.

where $f_k \equiv \partial f / \partial k$. Rewriting (3.14), we have $s_k f(k^*, h^*) / k^* - (\delta_k + g + n) = 0$. Now recall that since f is strictly concave in k in view of Assumption 1′ and $f(0, h^*) \geq 0$, we have

$$\begin{aligned} f(k^*, h^*) &> f_k(k^*, h^*) k^* + f(0, h^*) \\ &> f_k(k^*, h^*) k^*. \end{aligned}$$

Therefore, $(\delta_k + g + n) - s_k f_k(k^*, h^*) > 0$, and (3.16) is strictly positive.

Similarly, defining $f_h \equiv \partial f / \partial h$ and applying the implicit function theorem to the $\dot{h} = 0$ locus, (3.15), we have

$$\left.\frac{dh}{dk}\right|_{\dot{h}=0} = \frac{s_h f_k(k^*, h^*)}{(\delta_h + g + n) - s_h f_h(k^*, h^*)}. \tag{3.17}$$

With the same argument as that used for (3.16), this expression is also strictly positive.

Next, we prove that (3.16) is steeper than (3.17) whenever (3.14) and (3.15) hold, so that can it most be one intersection. First, observe that

$$
\begin{aligned}
\left.\frac{dh}{dk}\right|_{\dot{h}=0} &< \left.\frac{dh}{dk}\right|_{\dot{k}=0} \\
&\Updownarrow \\
\frac{s_h f_k(k^*,h^*)}{(\delta_h+g+n)-s_h f_h(k^*,h^*)} &< \frac{(\delta_k+g+n)-s_k f_k(k^*,h^*)}{s_k f_h(k^*,h^*)} \\
&\Updownarrow \\
s_k s_h f_k f_h &< s_k s_h f_k f_h + (\delta_h+g+n)(\delta_k+g+n) \\
& \quad -(\delta_h+g+n)s_k f_k - (\delta_k+g+n)s_h f_h .
\end{aligned}
$$

Now using (3.14) and (3.15) and substituting for $(\delta_k+g+n) = s_k f(k^*,h^*)/k^*$ and $(\delta_h+g+n) = s_h f(k^*,h^*)/h^*$, this is equivalent to

$$
f(k^*,h^*) > f_k(k^*,h^*)\,k^* + f_h(k^*,h^*)\,h^*,
$$

which is satisfied by the fact that $f(k^*,h^*)$ is a strictly concave function.

Finally, to establish existence note that Assumption 2′ implies that $\lim_{h\to 0} f(k,h)/h = \infty$, $\lim_{k\to 0} f(k,h)/k = \infty$, $\lim_{h\to\infty} f(k,h)/h = 0$ and $\lim_{k\to\infty} f(k,h)/k = 0$, so that the curves look as in Figure 3.1, that is, (3.14) is below (3.15) as $k\to 0$ and $h\to\infty$, but it is above (3.15) as $k\to\infty$ and $h\to 0$. This implies that the two curves must intersect at least once. □

This proposition shows that a unique steady state exists when the Solow model is augmented with human capital. The comparative statics are similar to the basic Solow model (see Exercise 3.7). Most importantly, both greater s_k and greater s_h will translate into higher normalized output per capita, $\hat{y}^*$.

Now turning to cross-country behavior, consider two different countries that experience the same rate of labor-augmenting technological progress, g. This implies that the country with greater propensity to invest in physical and human capital will be relatively richer. This is the type of prediction can be investigated empirically to see whether the augmented Solow model gives us a useful way of looking at cross-country income differences.

Before doing this, however, we also need to check whether the unique steady state is globally stable. The next proposition shows that this is the case.

Proposition 3.2. *Suppose Assumptions 1′ and 2′ are satisfied. Then the unique steady-state equilibrium of the augmented Solow model with human capital,* (k^*,h^*), *is globally stable in the sense that starting with any* $k(0)>0$ *and* $h(0)$, *we have* $(k(t),h(t))\to(k^*,h^*)$.

A formal proof of this proposition is left to Exercise 3.6. Figure 3.2 gives a diagrammatic proof, by showing the law of motion of k and h depending on whether we are above or below the two curves representing the loci for $\dot{k} = 0$ and $\dot{h} = 0$, respectively, (3.14) and (3.15). When we are to the right of the (3.14) curve, there is too much physical capital relative to the amount of labor and human capital, and consequently, $\dot{k} < 0$. When we are to its left, we are in the converse situation and $\dot{k} > 0$. Similarly, when we are above the (3.15) curve, there is too little human capital relative to the amount of labor and physical capital, and thus $\dot{h} > 0$. When we are below it, $\dot{h} < 0$. Given these arrows, the global stability of the dynamics follows.

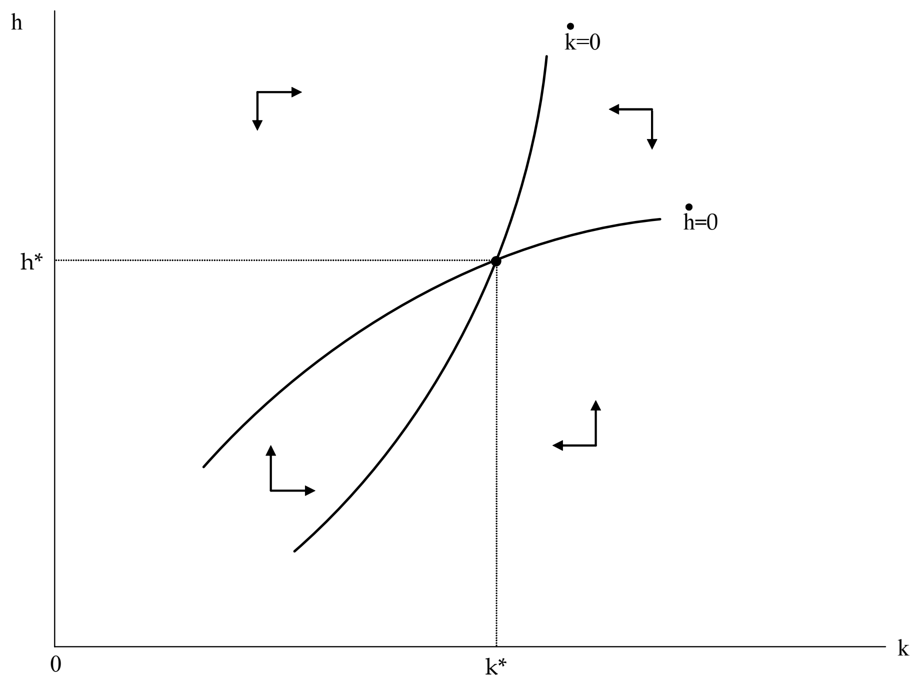

Figure 3.2. Dynamics of physical capital-labor and human capital-labor ratios in the Solow model with human capital.

We next characterize the equilibrium in greater detail when the production function (3.13) takes a Cobb-Douglas form.

Example 3.2. **(Augmented Solow model with Cobb-Douglas production functions)** Let us now work through a special case of the above model with Cobb-Douglas production function. In particular, suppose that the aggregate production function is

$$Y(t) = K(t)^{\alpha} H(t)^{\beta} (A(t) L(t))^{1-\alpha-\beta}, \tag{3.18}$$

where $0 < \alpha < 1$, $0 < \beta < 1$ and $\alpha + \beta < 1$. Output per effective unit of labor can then be written as

$$\hat{y}(t) = k^{\alpha}(t) h^{\beta}(t),$$

with the same definition of $\hat{y}(t)$, $k(t)$ and $h(t)$ as above. Using this functional form, (3.14) and (3.15) give the unique steady-state equilibrium as

$$k^* = \left(\left(\frac{s_k}{n+g+\delta_k}\right)^{1-\beta}\left(\frac{s_h}{n+g+\delta_h}\right)^{\beta}\right)^{\frac{1}{1-\alpha-\beta}} \tag{3.19}$$
$$h^* = \left(\left(\frac{s_k}{n+g+\delta_k}\right)^{\alpha}\left(\frac{s_h}{n+g+\delta_h}\right)^{1-\alpha}\right)^{\frac{1}{1-\alpha-\beta}},$$

which shows that higher saving rate in physical capital not only increases k^*, but also h^*. The same applies for a higher saving rate in human capital. This reflects the facts that the higher saving rate in physical capital, by increasing, k^*, raises overall output and thus the amount invested in schooling (since s_h is constant). Given (3.19), output per effective unit of labor in steady state is obtained as

$$\hat{y}^* = \left(\frac{s_k}{n+g+\delta_k}\right)^{\frac{\beta}{1-\alpha-\beta}}\left(\frac{s_h}{n+g+\delta_h}\right)^{\frac{\alpha}{1-\alpha-\beta}}. \tag{3.20}$$

This expression shows that the relative contributions of the saving rates for physical and human capital on (normalized) output per capita depends on the shares of physical and human capital—the larger is β, the more important is s_k and the larger is α, the more important is s_h.

In the next section, we will use the augmented Solow model to look at cross-country income differences.

3.4. Solow Model and Cross-Country Income Differences: Regression Analyses

3.4.1. A World of Augmented Solow Economies. An important paper by Mankiw, Romer and Weil (1992) used regression analysis to take the augmented Solow model, with human capital, to data. In line with our main emphasis here, let us focus on the cross-country part of Mankiw, Romer and Weil's (1992) analysis. To do this, we will use the Cobb-Douglas model in Example 3.2 and envisage a world consisting of $j = 1, ..., N$ countries.

Mankiw, Romer and Weil (1992), like many other authors, start with the assumption mentioned above, that "each country is an island"; in other words, they assume that countries do not interact (perhaps except for sharing some common technology growth, see below). This assumption enables us to analyze the behavior of each economy as a self-standing Solow model. Even though "each country is an island" is an unattractive assumption, it is a useful starting point both because of its simplicity and because this is where much of the literature started from (and in fact, it is still where much of the literature stands).

Following Example 3.2, we assume that country $j = 1, ..., N$ has the aggregate production function:

$$Y_j(t) = K_j(t)^{\alpha} H_j(t)^{\beta} (A_j(t) L_j(t))^{1-\alpha-\beta}.$$

This production function nests the basic Solow model without human capital when $\alpha = 0$. First, assume that countries differ in terms of their saving rates, $s_{k,j}$ and $s_{h,j}$, population growth rates, n_j, and technology growth rates $\dot{A}_j(t)/A_j(t) = g_j$. As usual, define $k_j \equiv K_j/A_jL_j$ and $h_j \equiv H_j/A_jL_j$.

Since our main interest here is cross-country income differences, rather than studying the dynamics of a particular country over time, let us focus on a world in which each country is in their steady state (thus ignoring convergence dynamics, which was the focus in the previous section). To the extent that countries are not too far from their steady state, there will be little loss of insight from this assumption, though naturally this approach will not be satisfactory when we think of countries experiencing very large growth spurts or growth collapses, as some of the examples discussed in Chapter 1.

Given the steady-state assumption, equivalents of equations (3.19) apply here and imply that the steady state physical and human capital to effective labor ratios of country j, $\left(k_j^*, h_j^*\right)$, are given by:

$$
\begin{aligned}
k_j^* &= \left(\left(\frac{s_{k,j}}{n_j + g_j + \delta_k}\right)^{1-\beta}\left(\frac{s_{h,j}}{n_j + g_j + \delta_h}\right)^{\beta}\right)^{\frac{1}{1-\alpha-\beta}} \\
h_j^* &= \left(\left(\frac{s_{k,j}}{n_j + g_j + \delta_k}\right)^{\alpha}\left(\frac{s_{h,j}}{n_j + g_j + \delta_h}\right)^{1-\alpha}\right)^{\frac{1}{1-\alpha-\beta}}.
\end{aligned}
$$

Consequently, using (3.20), the "steady-state"/balanced growth path income per capita of country j can be written as

$$
\begin{aligned}
y_j^*(t) &\equiv \frac{Y(t)}{L(t)} \\
&= A_j(t)\left(\frac{s_{k,j}}{n_j + g_j + \delta_k}\right)^{\frac{\alpha}{1-\alpha-\beta}}\left(\frac{s_{h,j}}{n_j + g_j + \delta_h}\right)^{\frac{\beta}{1-\alpha-\beta}}.
\end{aligned}
\tag{3.21}
$$

Here $y_j^*(t)$ stands for output per capita of country j along the balanced growth path. An immediate implication of this equation is that if g_j's are not equal across countries, income per capita will diverge, since the term in front, $A_j(t)$, will be growing at different rates for different countries. As we saw in Chapter 1, there is some evidence for this type of divergent behavior, but the world (per capita) income distribution can also be approximated by a relatively stable distribution. As mentioned there, this is an area of current research and debate whether we should model the world economy with an expanding or stable world income distribution. The former would be consistent with a specification in which the g_j's differ across countries, while the latter would require all countries to have the same rate of technological progress, g (recall the discussion in Chapter 1).

Since technological progress is taken as exogenous in the Solow model, it is, in many ways, more appropriate for the Solow model to assume a common rate of technical progress. Motivated by this, Mankiw, Romer and Weil (1992) make the following assumption:

Common technology advances assumption: $A_j(t) = \bar{A}_j \exp(gt)$.

That is, countries differ according to their technology *level*, in particular, according to their initial level of technology, $\bar{A}_j$, but they share the same common technology growth rate, g.

Now using this assumption together with (3.21) and taking logs, we obtain the following convenient log-linear equation for the balanced growth path of income for country $j = 1, ..., N$:

$$\ln y_j^*(t) = \ln \bar{A}_j + gt + \frac{\alpha}{1-\alpha-\beta} \ln\left(\frac{s_{k,j}}{n_j + g + \delta_k}\right) + \frac{\beta}{1-\alpha-\beta} \ln\left(\frac{s_{h,j}}{n_j + g + \delta_h}\right). \tag{3.22}$$

This is a simple and attractive equation. Most importantly, once we adopt values for the constants δ_k, δ_h and g (or estimate them from some other data sources), we can use cross-country data we can compute $s_{k,j}$, $s_{h,j}$, n_j, and thus construct measures of the two key right-hand side variables. Once we have these measures, equation (3.22) can be estimated by ordinary least squares (i.e., by regressing income per capita on these measures) to uncover the values of α and β.

Mankiw, Romer and Weil take $\delta_k = \delta_h = \delta$ and $\delta + g = 0.05$ as approximate depreciation rates for physical and human capital and growth rate for the world economy. These numbers are somewhat arbitrary, but their exact values are not important for the estimation. The literature typically approximates $s_{k,j}$ with average investment rates (investments/GDP). Investment rates, average population growth rates n_j, and log output per capita are from the Summers-Heston dataset discussed in Chapter 1. In addition, they use estimates of the fraction of the school-age population that is enrolled in secondary school as a measure of the investment rate in human capital, $s_{h,j}$. We return this variable below.

However, even with all of these assumptions, equation (3.22) can still not be estimated consistently. This is because the $\ln \bar{A}_j$ term is unobserved (at least to the econometrician) and thus will be captured by the error term. Most reasonable models of economic growth would suggest that technological differences, the $\ln \bar{A}_j$'s, should be correlated with investment rates in physical and human capital. Thus an estimation of (3.22) would lead to the most standard form of omitted variable bias and inconsistent estimates. Consistency would only follow under a stronger assumption than the common technology advances assumption introduced above. Therefore, implicitly, Mankiw, Romer and Weil make another *crucial* assumption:

Orthogonal technology assumption: $\bar{A}_j = \varepsilon_j A$, with ε_j orthogonal to all other variables.

Under the orthogonal technology assumption, $\ln \bar{A}_j$, which is part of the error term, is orthogonal to the key right-hand side variables and equation (3.22) can be estimated consistently.

3.4.2. Mankiw, Romer and Weil Estimation Results. Mankiw, Romer and Weil first estimate equation (3.22) without the human capital term (i.e., imposing $\alpha = 0$) for the cross-sectional sample of non-oil producing countries. In particular, their estimating equation is:

$$\ln y_j^* = \text{constant} + \frac{\alpha}{1-\alpha} \ln (s_{k,j}) - \frac{\alpha}{1-\alpha} \ln (n_j + g + \delta_k) + \varepsilon_j.$$

This equation is obtained from (3.22) by setting $\beta = 0$, dropping the time terms, since the equation refers to a single cross section and separating the terms $\ln (s_{k,j})$ and $\ln (n_j + g + \delta_k)$. Separating these two terms is useful to test the restriction that their coefficients should be equal in absolute value and of opposite signs. Finally, this equation also includes ε_j as the error term, capturing all omitted factors and influences on income per capita.

Their results on this estimation exercise are replicated in columns 1 of Table 3.1 using the original Mankiw, Romer and Weil data (standard errors in parentheses). Their estimates suggest a coefficient of around 1.4 for $\alpha / (1 - \alpha)$, which implies that α must be around 2/3. Since α is also the share of capital in national income, it should be around 1/3. Thus, the regression estimates without human capital appear to lead to overestimates of α. Columns 2 and 3 report the same results with updated data. The fit on the model is slightly less good than was the case with the Mankiw, Romer and Weil data, but the general pattern is similar. The implied values of α are also a little smaller than the original estimates, but still substantially higher than the 1/3 number one would expect on the basis of the underlying model.

Table 3.1
Estimates of the Basic Solow Model

	MRW	Updated data	
	1985	1985	2000
$\ln(s_k)$	1.42	1.01	1.22
	(.14)	(.11)	(.13)
$\ln(n+g+\delta)$	-1.97	-1.12	-1.31
	(.56)	(.55)	(.36)
Adj R^2	.59	.49	.49
Implied α	.59	.50	.55
No. of observations	98	98	107

The most natural reason for the high implied values of the parameter α in Table 3.1 is that ε_j is correlated with $\ln(s_{k,j})$, either because the orthogonal technology assumption is not a good approximation to reality or because there are also human capital differences correlated with $\ln(s_{k,j})$—so that there is an omitted variable bias.

Mankiw, Romer and Weil favor the second interpretation and estimate the augmented model, in particular the equation

$$\begin{aligned} \ln y_j^* &= \text{constant} + \frac{\alpha}{1-\alpha-\beta}\ln(s_{k,j}) - \frac{\alpha}{1-\alpha-\beta}\ln(n_j+g+\delta_k) \\ &\quad + \frac{\beta}{1-\alpha-\beta}\ln(s_{h,j}) - \frac{\beta}{1-\alpha-\beta}\ln(n_j+g+\delta_h) + \varepsilon_j. \end{aligned} \tag{3.23}$$

This requires a proxy for $\ln(s_{h,j})$. Mankiw, Romer and Weil use the fraction of the working age population that is in school. With this proxy and again under the orthogonal technology assumption, the original Mankiw, Romer and Weil estimates are given in column 1 of Table 3.2. Now the estimation is more successful. Not only is the Adjusted R^2 quite high (about 78%), the implied value for α is around 1/3. On the basis of this estimation result, Mankiw, Romer and Weil and others have interpreted the fit of the augmented Solow model to the data as a success: with common technology, human and physical capital investments appear to explain 78% of the cross-country income per capita differences and the implied parameter values are reasonable. Columns 2 and 3 of the table show the results with the updated data. The implied values of α are similar, though the Adjusted R^2 is somewhat lower.

Table 3.2
Estimates of the Augmented Solow Model

	MRW	Updated data	
	1985	1985	2000
$\ln(s_k)$	.69	.65	.96
	(.13)	(.11)	(.13)
$\ln(n+g+\delta)$	-1.73	-1.02	-1.06
	(.41)	(.45)	(.33)
$\ln(s_h)$	.66	.47	.70
	(.07)	(.07)	(.13)
Adj R^2	.78	.65	.60
Implied α	.30	.31	.36
Implied β	.28	.22	.26
No. of observations	98	98	107

To the extent that these regression results are reliable, they give a big boost to the augmented Solow model. In particular, the estimate of Adjusted R^2 suggests that over (or close to) three quarters of income per capita differences across countries can be explained by differences in their physical and human capital investment behavior. The immediate implication is that technology (TFP) differences have a somewhat limited role, confined to at most accounting for about a quarter of the cross-country income per capita differences. If this conclusion were appropriate, it would imply that, as far as the proximate causes of prosperity are concerned, we could confine our attention to physical and human capital, and assume that countries have access to more or less the same world technology. The implications for the modeling of economic growth are of course quite major.

In the next subsection, we will see why the conclusion that technology differences are minor and physical and human capital differences are the major proximate cause of income per capita differences should not be accepted without further investigation.

3.4.3. Challenges to the Regression Analyses of Growth Models. There are two major (and related) problems with this approach.

The first relates to the assumption that technology differences across countries are orthogonal to all other variables. While the constant technology advances assumption may be defended, the orthogonality assumption is too strong, almost untenable. We not only expect $\bar{A}_j$ to vary across countries, but also to be correlated with measures of s^h_j and s^k_j; countries that are more productive will also invest more in physical and human capital. This is for two reasons. The first is a version of the *omitted variable bias* problem; as we will discuss in detail later in the book, technology differences are also outcomes of investment decisions. Thus societies with high levels of $\bar{A}_j$ will be those that have invested more in technology for various reasons; it is then natural to expect the same reasons to induce greater investment in physical and human capital as well. Second, even ignoring the omitted variable bias problem, there is a *reverse causality* problem; complementarity between technology and physical or human capital imply that countries with high $\bar{A}_j$ will find it more beneficial to increase their stock of human and physical capital.

In terms of the regression equation (3.23), this implies that the key right-hand side variables are correlated with the error term, ε_j. Consequently, ordinary least squares regressions of equation (3.23) will lead to upwardly biased estimates of α and β. In addition, the estimate of the R^2, which is a measure of how much of the cross-country variability in income per capita can be explained by physical and human capital, will also be biased upwards.

The second problem relates to the magnitudes of the estimates of α and β in equation (3.23). The regression framework above is attractive in part because we can gauge whether the estimate of α was plausible. We should do the same for the estimate of β, the coefficient on the investment rate in human capital, s_j^h. We will now see that when we perform a similar analysis for β, we will find that it is too large relative to what we should expect on the basis of microeconometric evidence.

Recall first that Mankiw, Romer and Weil use the fraction of the working age population enrolled in school. This variable ranges from 0.4% to over 12% in the sample of countries used for this regression. Their estimates therefore imply that, holding all other variables constant, a country with approximately 12 for this variable should have income per capita about 9 times that of a country with $s_j^h = 0.4$. More explicitly, the predicted log difference in incomes between these two countries is

$$\frac{\alpha}{1-\alpha-\beta}\left(\ln 12 - \ln(0.4)\right) = 0.66 \times \left(\ln 12 - \ln(0.4)\right) \approx 2.24.$$

This implies that, holding all other factors constant, a country with schooling investment of over 12 should be about $\exp(2.24) - 1 \approx 8.5$ times richer than a country with a level of schooling investment of around 0.4.

In practice, the difference in average years of schooling between any two countries in the Mankiw-Romer-Weil sample is less than 12. In Chapter 10, we will see that there are good economic reasons to expect additional years of schooling to increase earnings proportionally, for example as in Mincer regressions of the form:

$$\ln w_i = \mathbf{X}_i'\boldsymbol{\gamma} + \phi S_i, \tag{3.24}$$

where w_i is wage earnings of individual i in some labor market, $\mathbf{X}_i$ is a set of demographic controls, and S_i is years of schooling. The estimate of the coefficient ϕ is the rate of returns to education, measuring the proportional increase in earnings resulting from one more year of schooling. The microeconometrics literature suggests that equation (3.24) provides a good approximation to the data and estimates ϕ to be between 0.06 and 0.10, implying that a worker with one more year of schooling earns about 6 to 10 percent more than a comparable worker with one less year of schooling. If labor markets are competitive, or at the very least, if wages are, on average, proportional to productivity, this also implies that one more year of schooling increases worker productivity by about 6 to 10 percent.

Can we deduce from this information how much richer a country with 12 more years of average schooling should be? The answer is yes, but with two caveats.

First, we need to assume that the micro-level relationship as captured by (3.24) applies identically to all countries. In other words, the implicit assumption in wage regressions in general and in equation (3.24) in particular is that the human capital (and the earnings capacity) of each individual is a function of his or her years of schooling. For example, ignoring other determinants of wage earnings, we can write the wage earnings of individual i is a function of his or her schooling as $w_i = \tilde{\phi}(S_i)$. The first key assumption is that this $\tilde{\phi}$ function is identical across countries and can be approximated by an exponential function of the form $\tilde{\phi}(S_i) \approx \exp(\phi S_i)$ so that we obtain equation (3.24). The reasons why this may be a reasonable assumption will be further discussed in Chapter 10.

Second, we need to assume that there are no *human capital externalities*—meaning that the human capital of a worker does not directly increase the productivity of other workers. There are reasons for why human capital externalities may exist and some economists believe that they are important. This issue will also be discussed in Chapter 10, where we will see that human capital externalities are unlikely to be very large. Thus it is reasonable to start without them. The key result which will enable us to go from the microeconometric wage regressions to cross-country differences is that, with constant returns to scale, perfectly competitive markets and no human capital externalities, differences in worker productivity directly translate into differences in income per capita. To see this, suppose that each firm f in country j has access to the production function

$$y_{fj} = K_f^{\alpha} (A_j H_f)^{1-\alpha},$$

where A_j is the productivity of all the firms in the country, K_f is the capital stock and H_f is effective units of human capital employed by firm f. Here the Cobb-Douglas production function is chosen for simplicity and does not affect the argument. Suppose also that firms in this country face a cost of capital equal to R_j. With perfectly competitive factor markets, profit maximization implies that the cost of capital must equal its marginal product,

$$R_j = \alpha \left(\frac{K_f}{A_j H_f} \right)^{-(1-\alpha)}. \tag{3.25}$$

This implies that all firms ought to function at the same physical to human capital ratio, and consequently, all workers, irrespective of their level of schooling, ought to work at the same physical to human capital ratio. Another direct implication of competitive labor markets is that in country j, wages per unit of human capital will be equal to

$$w_j = (1-\alpha)\,\alpha^{\alpha/(1-\alpha)} A_j R_j^{-\alpha/(1-\alpha)}.$$

Consequently, a worker with human capital h_i will receive a wage income of $w_j h_i$. Once again, this is a more general result; with aggregate constant returns to scale production technology, wage earnings are linear in the effective human capital of the worker, so that a worker with twice as much effective human capital as another should earn twice as much as this other worker (see Exercise 3.9). Next, substituting for capital from (3.25), we have total income in country j as

$$Y_j = (1-\alpha)\,\alpha^{\alpha/(1-\alpha)} R_j^{-\alpha/(1-\alpha)} A_j H_j,$$

where H_j is the total efficiency units of labor in country j. This equation implies that ceteris paribus (in particular, holding constant capital intensity corresponding to R_j and technology, A_j), a doubling of human capital will translate into a doubling of total income. Notice that in this exercise we are keeping not only A_j, but also R_j constant. While it may be reasonable to keep technology, A_j, constant, one may wonder whether R_j will change systematically in response to a change in H_j. While this is a possibility, any changes likely to be second-order. First, international capital flows may work towards equalizing the rates of returns across countries. Second, when capital-output ratio is constant, which Theorem 2.7 established as a requirement for a balanced growth path, then R_j will indeed be constant (irrespective of the exact form of the production function, see Exercise 3.10). Therefore, under constant returns and perfectly competitive factor markets, a doubling of human capital (efficiency units of labor) has the same effects on the earnings of an individual as the effect of a doubling of aggregate human capital has on total output.

This analysis implies that the estimated Mincerian rates of return to schooling can be used to calculate differences in the stock of human capital across countries. So in the absence of *human capital externalities*, a country with 12 more years of average schooling should have a stock of human capital somewhere between $\exp(0.10 \times 12) \simeq 3.3$ and $\exp(0.06 \times 12) \simeq 2.05$ times the stock of human capital of a county with fewer years of schooling. This implies that, holding other factors constant, this country should be about 2-3 times as rich as the country with zero years of average schooling, which is much less than the 8.5 fold difference implied by the Mankiw-Romer-Weil analysis.

The consequence of this discussion is that the estimate for β that is implied by the Mankiw-Romer-Weil regressions is too high relative to the estimates that would be implied by the microeconometric evidence and thus likely upwardly biased. The overestimation of the coefficient α is, in turn, most likely related to the possible correlation between the error term ε_j and the key right-hand side regressors in equation (3.23).

To recap, the comparison between the parameter estimates from the regression of (3.23) and the microeconometric Mincerian rates of return estimates to schooling imply that cross-country regression analysis is not necessarily giving us an accurate picture of the productivity differences and thus the proximate causes of income differences.

3.5. Calibrating Productivity Differences

What other approach can we use to gauge the importance of physical and human capital and technology differences? An alternative approach is to "calibrate" the (total factor) productivity differences across countries rather than estimating them using a regression framework. These total factor productivity differences are then interpreted as a measure of the contribution of "technology" to cross-country income differences.

The calibration approach was proposed and used by Klenow and Rodriguez (1997) and by Hall and Jones (1999). Here I follow Hall and Jones. The advantage of the calibration approach is that the omitted variable bias underlying the estimates of Mankiw, Romer and Weil will be less important (since micro-level evidence will be used to anchor the contribution of human capital to economic growth). The disadvantage is that certain assumptions on functional forms have to be taken much more seriously and we explicitly have to assume no human capital externalities.

3.5.1. Basics. Suppose that each country has access to the Cobb-Douglas aggregate production function:

$$Y_j = K_j^{\alpha} \left(A_j H_j\right)^{1-\alpha}, \tag{3.26}$$

where H_j is the stock of human capital of country j, capturing the amount of efficiency units of labor available to this country. K_j is its stock of physical capital and A_j is labor-augmenting technology. Since our focus is on cross-country comparisons, time arguments are omitted.

Suppose that each worker in country j has S_j years of schooling. Then using the Mincer equation (3.24) from the previous section, ignoring the other covariates and taking exponents, H_j can be estimated as

$$H_j = \exp\left(\phi S_j\right) L_j,$$

where L_j is employment in country j and ϕ is the rate on returns to schooling estimated from equation (3.24). This approach may not lead to very good estimates of the stock of human capital for a country, however. First, it does not take into account differences in other "human capital" factors, such as experience (which will be discussed in greater detail in Chapter 10). Second, countries may differ not only in the years of schooling of their labor forces, but in the quality of schooling and the amount of post-schooling human capital. Third, the rate of return to schooling may vary systematically across countries. As we will see in greater detail below, the rate of return to schooling may be lower in countries with a greater abundance of human capital. It is possible to deal with each of these problems to some extent by constructing better estimates of the stocks of human capital.

Following Hall and Jones, we make only a partial correction for the last factor. Let us assume that the rate of return to schooling does not vary by country, but is potentially different for different years of schooling. For example, one year of primary schooling may be more valuable than one year of graduate school (for example, because learning how to read might increase productivity more than a solid understanding of growth theory!). In particular, let the rate of return to acquiring the Sth year of schooling be $\phi(S)$. The above equation would be the special case where $\phi(S) = \phi$ for all S. With this assumption and with estimates of the returns to schooling for different years (e.g., primary schooling, second the schooling etc.), a somewhat better estimate of the stock of human capital can be constructed as

$$H_j = \sum_S \exp\{\phi(S)\, S\}\, L_j(S)$$

where $L_j(S)$ now refers to the total employment of workers with S years of schooling in country j.

A series for K_j can be constructed from Summers-Heston dataset using investment data and the perpetual invented method. In particular, recall that, with exponential depreciation, the stock of physical capital evolves according to

$$K_j(t+1) = (1-\delta)\, K_j(t) + I_j(t),$$

where $I_j(t)$ is the level of investment in country j at time j. Let us assume, following Hall and Jones that $\delta = 0.06$. With a complete series for $I_j(t)$, this equation can be used to calculate the stock of physical capital at any point in time. However, the Summers-Heston dataset does not contain investment information before the 1960s. This equation can still be used by assuming that each country's investment was growing at the same rate before the sample in order to compute the initial capital stock. Using this assumption, Hall and Jones calculate the physical capital stock for each country in the year 1985. We do the same here for various years. Finally, with the same arguments as before, we choose a value of $1/3$ for α.

Given series for H_j and K_j and a value for α, we can construct "predicted" incomes at a point in time using the following equation

$$\hat{Y}_j = K_j^{1/3}\, (A_{US} H_j)^{2/3}$$

for each country j, where A_{US} is the labor-augmenting technology level of the United States, computed so that this equation fits the United States perfectly, i.e., $Y_{US} =$

$K_{US}^{1/3}\left(A_{US}H_{US}\right)^{2/3}$. Throughout, time indices are dropped. In the Hall and Jones exercise, all values refer to 1985.

Once a series for $\hat{Y}_j$ has been constructed, it can be compared to the actual output series. The gap between the two series represents the contribution of technology. Alternatively, we could explicitly back out country-specific technology terms (relative to the United States) as

$$\frac{A_j}{A_{US}} = \left(\frac{Y_j}{Y_{US}}\right)^{3/2}\left(\frac{K_{US}}{K_j}\right)^{1/2}\left(\frac{H_{US}}{H_j}\right).$$

Figures 3.3-3.4 show the results of these exercises for 1980, 1990 and 2000.

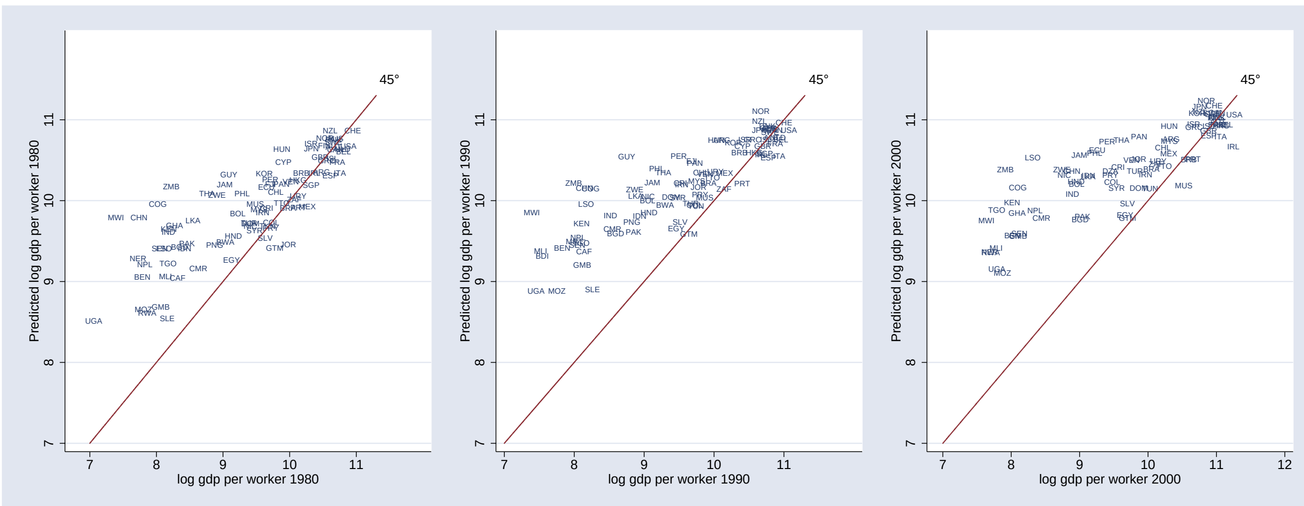

FIGURE 3.3. Calibrated technology levels relative to the US technology (from the Solow growth model with human capital) versus log GDP per worker, 1980, 1990 and 2000.

The following features are noteworthy:

(1) Differences in physical and human capital still matter a lot; the predicted and actual incomes are highly correlated. Thus the regression analysis was not entirely misleading in emphasizing the importance of physical and human capital.
(2) However, differently from the regression analysis, this exercise also shows that there are significant *technology (productivity) differences*. There are often large gaps between predicted and actual incomes, showing the importance of technology differences across countries. This can be most easily seen in the first three figures, where practically all observations are above the 45°, which implies that the neoclassical model is over predicting the income level of countries that are poorer than the United States.
(3) The same pattern is visible in the next three figures, which plot, the estimates of the technology differences, A_j/A_{US}, against log GDP per capita in the corresponding

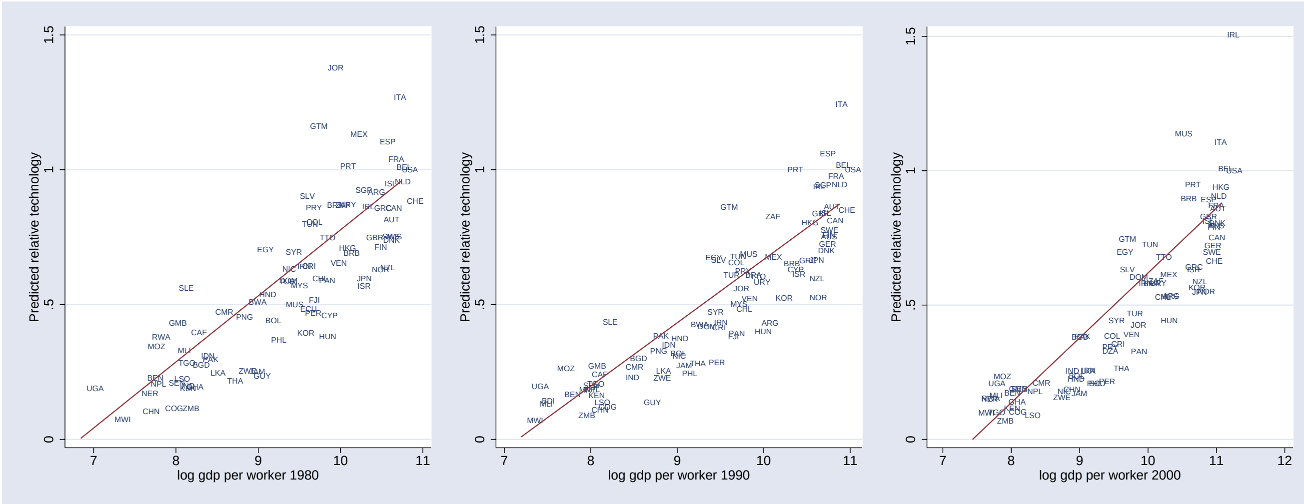

Figure 3.4. Calibrated technology levels relative to the US technology (from the Solow growth model with human capital) versus log GDP per worker, 1980, 1990 and 2000.

year. These differences are often substantial. More important, these differences are also strongly correlated with income per capita; richer countries appear to have "better technologies".

(4) Also interesting is the pattern that the empirical fit of the neoclassical growth model seems to deteriorate over time. In the first three figures, the observations are further above the 45° in the later years, and in the last three figures, the relative technology differences become larger. Why the fit of the simple neoclassical growth model is better in 1980 than in 2000 is an interesting and largely unanswered question.

3.5.2. Challenges. In the same way as the regression analysis was based on a number of stringent assumptions (in particular, the assumption that technology differences across countries were orthogonal to other factors), the calibration approach also relies on certain important assumptions. The above exposition highlighted several of those. In addition to the standard assumptions of competitive factor markets, we had to assume no human capital externalities, a Cobb-Douglas production function, and also make a range of approximations to measure cross-country differences in the stocks of physical and human capital.

Here let us focus on the functional form assumptions. Could we get away without the Cobb-Douglas production function? The answer is yes, but not perfectly. The reason for this is the same as the reason why the over-time TFP accounting approach may work without making functional form assumptions on the aggregate production function, but may also sometimes lead to misleading answers.

As a byproduct of investigating this question, we will see that the calibration approach is in fact a close cousin of the growth-accounting exercise (and for this reason, it is sometimes referred to as "levels accounting").

Recall equation (3.4), where we constructed TFP estimates from a general constant returns to scale production function (under competitive labor markets) by using average factor shares. Now instead imagine that the production function that applies to all countries in the world is given by

$$F(K_j, H_j, A_j),$$

and countries differ according to their physical and human capital as well as technology—but not according to F. Suppose also that we have data on K_j and H_j as well as capital and labor share for each country. Then a natural adaptation of equation (3.4) can be used across countries rather than over time. In particular, let us a rank countries in descending order according to their physical capital to human capital ratios, K_j/H_j (use Exercise 3.1 to think about why this is the right way to rank countries rather than doing so randomly). Then we can write

$$\hat{x}_{j,j+1} = g_{j,j+1} - \bar{\alpha}_{K,j,j+1} g_{K,j,j+1} - \bar{\alpha}_{Lj,j+1} g_{H,j,j+1}, \tag{3.27}$$

where $g_{j,j+1}$ is the proportional difference in output between countries j and $j+1$, $g_{K,j,j+1}$ is the proportional difference in capital stock between these countries and $g_{H,j,j+1}$ is the proportional difference in human capital stocks. In addition, $\bar{\alpha}_{K,j,j+1}$ and $\bar{\alpha}_{Lj,j+1}$ are the average capital and labor shares between the two countries. These can only be computed if we can observe capital and labor shares in national income by country. The estimate $\hat{x}_{j,j+1}$ is then the proportional TFP difference between the two countries.

Using this method, and taking one of the countries, for example the United States, as the base country, we can calculate relative technology differences across countries. While theoretically attractive, this levels-accounting exercise faces two challenges. One is data-related and the other one theoretical.

First, data on capital and labor shares across countries are not widely available. This makes the use of equation (3.27) far from straightforward. Consequently, almost all calibration or levels-accounting exercises that estimate technology (productivity) differences use the Cobb-Douglas approach of the previous subsection (i.e., a constant value of α_K equal to 1/3).

Second, even if data on capital and labor shares were available, the differences in factor proportions, e.g., differences in K_j/H_j, across countries are large. An equation like (3.27) is a good approximation when we consider small (infinitesimal) changes. As illustrated in Exercise 3.1, when differences in factor proportions are significant between the two observations, the use of this type of equation can lead to significant biases.

To sum up, the approach of calibrating productivity differences across countries is a useful alternative to the regression analysis, but has to rely on a range of stringent assumptions on the form of the production function and can also lead to biased estimates of technology differences when factors are mismeasured.

3.6. Estimating Productivity Differences

In the previous section, productivity/technology differences are obtained as "residuals" from a calibration exercise, so we have to trust the functional form assumptions used in this strategy. But if we are willing to trust the functional forms, we can also estimate these differences econometrically rather than rely on calibration. The great advantage of econometrics relative to calibration is that not only do we obtain estimates of the objects of interest, but we also have standard errors, which show us how much we can trust these estimates. In this section, we will briefly discuss two different approaches to estimating productivity differences.

3.6.1. A Naïve Approach. The first possibility is to take a production function of the form (3.26) as given and try to estimate this using cross country data. In particular, taking logs in this equation, we obtain:

$$\log Y_j = \alpha \log K_j + (1-\alpha)\log H_j + \alpha \log A_j. \tag{3.28}$$

Given series for Y_j, K_j and H_j, this equation can be estimated with ordinary least squares with the restriction that the coefficients on $\log K_j$ and $\log H_j$ sum to one, and the residuals can be interpreted as estimates of technology differences. Unfortunately, this approach is not particularly attractive, since the potential correlation between $\log A_j$ and $\log K_j$ or $\log H_j$ implies that the estimates of α need not be unbiased even though we impose the constant returns to scale assumption. Moreover, if we do not impose the assumption that these coefficients sum to one and test this restriction, it will be rejected. Thus, this regression approach runs into the same difficulties as the Mankiw, Romer and Weil approach discussed previously.

What this discussion highlights is that, even if we are willing to presume that we know the functional form of the aggregate production function, it is difficult to directly estimate productivity differences. So how can we improve over this naïve approach? The answer involves making more use of economic theory. Estimating an equation of the form (3.28) does not make use of the fact that we are looking at the equilibrium of an economic system. A more sophisticated approach would use more of the restrictions imposed by equilibrium behavior (and to bring additional relevant data). We next illustrate this using a specific attempt based on international trade. The reader who is not familiar with trade theory may want to skip this subsection.

3.6.2. Learning from International Trade*. We will discuss models of growth in trade in Chapter 19. Even without a detailed discussion of international trade theory, we can use data from international trade flows and some simple principles of international trade theory to obtain another way of estimating productivity differences across countries.

Let us follow an important paper by Trefler (1993), which uses an augmented version of the standard Heckscher-Ohlin approach to international trade. The Heckscher-Ohlin approach assumes that countries differ according to their factor proportions (e.g., some countries have much more physical capital relative to their labor supply than others). In a closed economy, this will lead to differences in relative factor costs and differences in the relative prices of products using these factors in different intensities. International trade results as a way of taking advantage of these relative price differences. The most extreme form of the theory assumes no cost of shipping goods and no policy impediments to trade, so that international trade can happen costlessly between countries.

Trefler starts from the standard Heckscher-Ohlin model of international trade, but allows for factor-specific productivity differences, so that capital in country j has productivity A_j^k, thus a stock of capital K_j in this country is equivalent to an effective supply of capital $A_j^k K_j$. Similarly for labor (human capital), country j has productivity A_j^h. In addition, Trefler assumes that all countries have the same homothetic preferences and there are sufficient factor intensity differences across goods to ensure international trade between countries to arbitrage relative price and relative factor costs differences (or in the jargon of international trade, countries will be in the *cone of diversification*). This latter assumption is important: when all countries have the same productivities both in physical and human capital, it leads to the celebrated *factor price equalization* result; all factor prices would be equal in all countries, because the world economy is sufficiently integrated. When there are productivity differences across countries, this assumption instead leads to *conditional factor price equalization*, meaning that factor prices are equalized once we take their different "effective" productivities into consideration.

Under these assumptions, a standard equation in international trade links the *net factor exports* of each country to the abundance of that factor in the country relative to the world as a whole. The term "net factor exports" needs some explanation. It does not refer to actual trade in factors (such as migration of people or capital flows). Instead trading goods is a way of trading the factors that are embodied in that particular good. For example, a country that exports cars made with capital and imports corn made with labor is implicitly exporting capital and importing labor. More specifically, the net export of capital by country j, X_j^K is calculated by looking at the total exports of country j and computing how much capital is necessary to produce these and then subtracting the amount of capital necessary to produce its total imports. For our purposes here, we do not need to get into issues of how this is calculated (suffice it to say that as with all things empirical, the devil is in the detail and these calculations are far from straightforward and require a range of assumptions). Then, the absence of trading frictions across countries and identical homothetic preferences imply that

$$X_j^K = A_j^k K_j - c_j^s \sum_{i=1}^{N} A_i^k K_i \quad (3.29)$$
$$X_j^H = A_j^h H_j - c_j^s \sum_{i=1}^{N} A_i^h H_i$$

where c_j^s is the share of country j in world consumption (the value of this country's consumption divided by world consumption) and N is the total number of countries in the world. These equations simply restate the conclusion in the previous paragraph that a country will be a net exporter of capital if its effective supply of capital, $A_j^k K_j$, exceeds a fraction, here c_j^s, of the world's effective supply of capital, $\sum_{i=1}^{N} A_i^k K_i$.

Consumption shares are easy to calculate. Then given estimates for X_j^K and X_j^H, the above system of $2 \times N$ equations can be solved for the same number of unknowns, the A_i^k and A_i^h's for N countries. If we stopped here, we would have obtained estimates for factor-specific productivity differences across countries from an entirely different source of variation than those exploited before. In addition, we would not have a single productivity parameter, but a separate labor-augmenting (or human-capital-augmenting) and a capital-augmenting productivity for each country, which is not an uninteresting achievement.

However, if we indeed stopped here, we would not know whether these numbers provide a good approximation to cross-country factor productivity differences. This is in some sense the same problem as we had in judging whether the calibrated productivity (technology) differences in the previous section were reliable. Fortunately, international trade theory gives us one more set of equations to check whether these numbers are reliable. As noted above, under the assumption that the world economy is sufficiently integrated, we have conditional factor price equalization. This implies that for any two countries j and j', we must have:

$$\frac{R_j}{A_j^k} = \frac{R_{j'}}{A_{j'}^k}, \quad (3.30)$$

$$\frac{w_j}{A_j^h} = \frac{w_{j'}}{A_{j'}^h}, \quad (3.31)$$

where R_j is the rental rate of capital in country j and w_j is the observed wage rate (which includes the compensation to human capital) in country j. Equation (3.31), for example, states that if workers in a particular country have, on average, half the efficiency units as those in the United States, their earnings should be roughly half of American workers.

With data on factor prices, we can therefore construct an alternative series for A_j^k and A_j^h's. It turns out that the series for A_j^k and A_j^h's implied by (3.29), (3.30) and (3.31) are

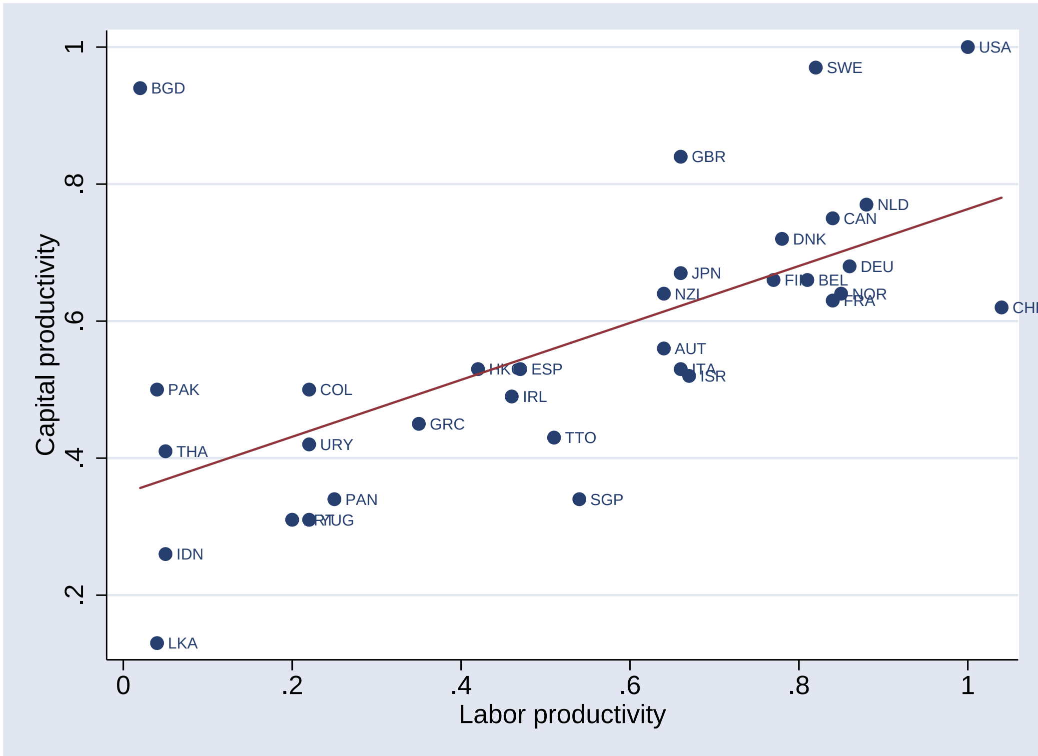

Figure 3.5. Comparison of labor-productivity and capital-productivity differences across countries.

very similar, so there appears to be some validity to this approach. Given this validation, we can presume that there is some information in the numbers that Trefler obtains.

Figure 3.5 shows Trefler's original estimates. The numbers in this figure imply that there are very large differences in labor productivity, and some substantial, but much smaller differences in capital productivity. For example, labor in Pakistan is 1/25th as productive as labor in the United States. In contrast, capital productivity differences are much more limited than labor productivity differences; capital in Pakistan is only half as productive as capital in the United States. This finding is not only intriguing in itself, but we will see that it is quite consistent with a class of models of technical change we will study in Chapter 15.

It is also informative to compare the productivity difference estimates from Trefler's approach to those from the previous section. Figures 3.6 and 3.7 undertake this comparison. The first plots the labor-productivity difference estimates from the Trefler approach against the calibrated overall productivity differences from the Cobb-Douglas specification in the previous section. The similarity between the two series is remarkable. This gives us some confidence that both approaches are capturing some features of reality and that in fact there are significant productivity (technology) differences across countries. Interestingly, however,

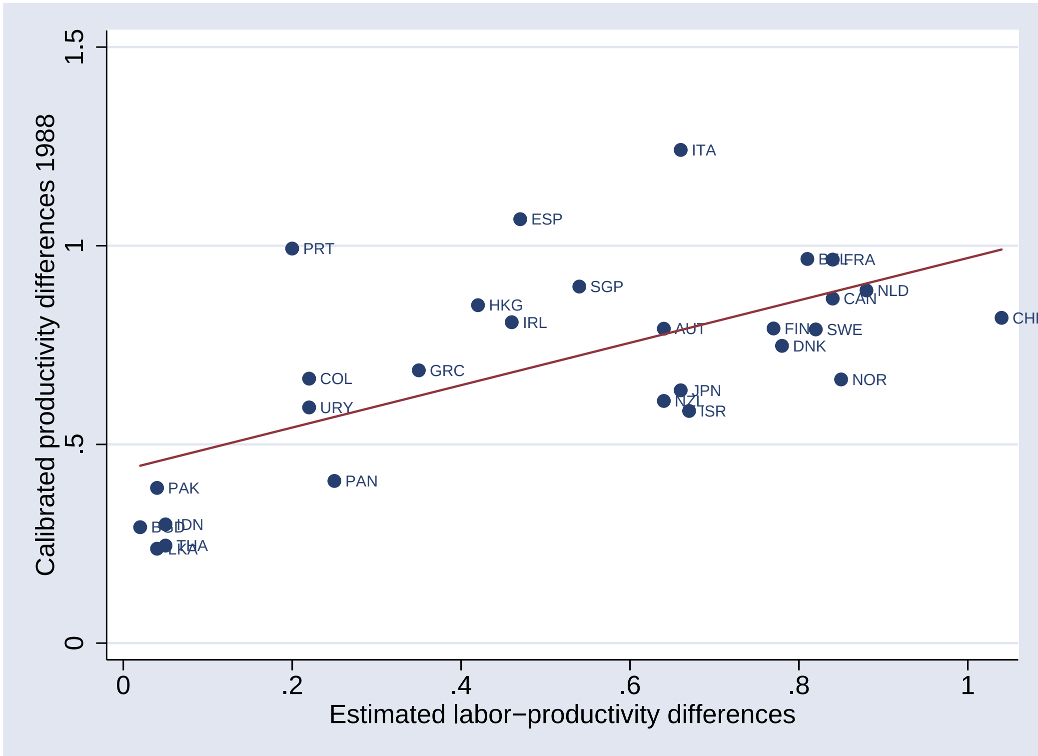

Figure 3.6. Comparison of the labor productivity estimates from the Trefler approach with the calibrated productivity differences from the Hall-Jones approach.

Figure 3.7 shows that the relationship between the calibrated productivity differences and the capital-productivity differences is considerably weaker than for labor productivity.

It is also important to emphasize that Trefler's approach relies on very stringent assumptions. To recap, the three major assumptions are:

(1) No international trading costs;
(2) Identical homothetic preferences;
(3) Sufficiently integrated world economy, leading to conditional factor price equalization.

All three of these assumptions are rejected in the data in one form or another. There are clearly international trading costs, including freight costs, tariff costs and other trading restrictions. There is very well-documented home bias in consumption violating the identical homothetic preferences assumption. Finally, most trade economists believe that conditional factor price equalization is not a good description of factor price differences across countries. In view of all of these, the results from the Trefler exercise have to be interpreted with

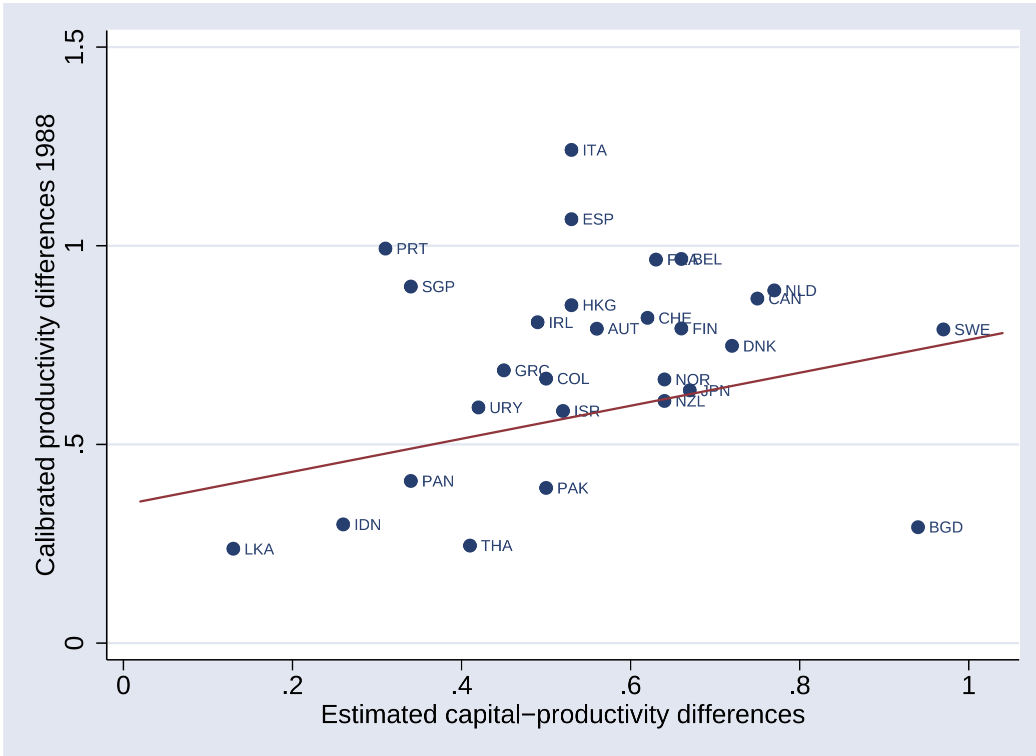

Figure 3.7. Comparison of the capital productivity estimates from the Trefler approach with the calibrated productivity differences from the Hall-Jones approach.

caution. Nevertheless, this approach is important in showing how different data and additional theory can be used to estimate cross-country technology differences and in providing a cross-validation for the calibration and estimation results discussed previously.

3.7. Taking Stock

What have we learned? The major point of this chapter has not been the development of new theory. Although we have extended the basic model of the previous chapter in a number of directions, if our interest were purely theoretical, we could have skipped the material in this chapter without much loss. Our major objective in this chapter has been to see whether we could use the Solow model to have a more informed interpretation of cross-country differences and also use data in order to understand the strengths and shortcomings of the Solow growth model.

At the end of this brief journey, the message is somewhat mixed. On the positive side, despite its simplicity, the Solow model has enough substance that we can take it to data in various different forms, including TFP accounting, regression analysis and calibration.

Moreover, each of these different methods gives us some idea about the sources of economic growth over time and of income differences across countries.

On the negative side, however, no single approach is entirely convincing. Each relies on a range of stringent auxiliary assumptions. Consequently, no firm conclusions can be drawn. The simplest applications of the Solow accounting framework suggest that technology is the main source of economic growth over time. However, this conclusion is disputed by those who point out that sufficient adjustments to the quality of physical and human capital substantially reduce or perhaps even totally eliminate residual TFP growth. The same debate is seen in the context of cross-country income differences; while some believe that accounting for differences in physical and human capital across countries leaves little need for technology differences, others show that, with reasonable models, most of the cross-country differences are due to technology.

While complete agreement is not possible, it is safe to say that the consensus in the literature today favors the interpretation that cross-country differences in income per capita cannot be understood solely on the basis of differences in physical and human capital; in other words there are technology differences across countries and these technology differences need to be understood.

Hence one important potential lesson from this data detour is that technological progress is not only important in generating economic growth in the basic Solow model, but also likely to be a major factor in cross-country differences in prosperity. A detailed study of technological progress and technology adoption decisions of households and firms is therefore necessary. This motivates the detailed analysis of technological progress and technology adoption later in the book. It is also useful to emphasize once again that differences in TFP are not necessarily due to technology in the narrow sense. If two countries have access to the same technology but make use of these techniques in different ways with different degrees of efficiency or if they are subject to different degrees of market or organizational failures, these differences will show up as TFP differences. One indication that TFP differences arising from market or organizational failures are important comes from episodes of severe crises. When countries have large drops in their income, due to civil wars, political instability, financial crises or other reasons, this is almost always associated with a corresponding decline in TFP (and little change in the capital stock and a much smaller change in labor inputs). Naturally, these drops in TFP are not caused by "technological regress" but result from the breakdown of the market or increases in some other sources of inefficiency. Therefore, when we talk of technology differences in the sense of this chapter, they should be construed rather broadly and we should pay special attention to cross-country differences in the efficiency of production. By implication, to understand TFP differences across countries, we must study not only differences in the techniques that they use but the way they organize markets and firms and how to incentivize different agents in the economy. This again shapes our agenda for the rest of the book, especially paving the way for our investigation of endogenous technological change in Part 4 and of differences in technology and productive efficiency across countries in Parts 6 and 7.

There is one more sense in which what we have learned in this chapter is limited. What the Solow model makes us focus on, physical capital, human capital and technology, are proximate causes of economic growth in cross-country differences. It is important to know which of these proximate causes are important and how they affect economic performance both to have a better understanding of the mechanics of economic growth and also to know which class of models to focus on. But at some level (and exaggerating somewhat) to say that a country is poor because it has insufficient physical capital, human capital and inefficient technology is like saying that a person is poor because he does not have money. There are, in turn, other reasons making some countries more abundant in physical capital, human capital and technology, in the same way as there are factors that make a person have more money than another. We have referred to these as the *fundamental causes* of differences in prosperity, contrasting with the proximate causes. A satisfactory understanding of economic growth and differences in prosperity across countries requires both an analysis of proximate causes and of fundamental causes of economic growth. The former is essential for us to understand the mechanics of economic growth and to develop the appropriate formal models incorporating these insights. The latter is important so that we can understand why some societies make choices that lead them to low physical capital, low human capital and inefficient technology and thus to relative poverty. This is the issue we turn to in the next chapter.

3.8. References and Literature

The growth accounting framework is introduced and applied in Solow (1957). Jorgensen, Gollop and Fraumeni (1987) give a comprehensive development of this framework, emphasizing how competitive markets are necessary and essentially sufficient for this approach to work. They also highlight the measurement difficulties and emphasize how underestimates of the quality improvements in physical and human capital will lead to overestimates of the contribution of technology to economic growth. Jorgensen (2005) contains a more recent survey.

Regression analysis based on the Solow model has a long history. More recent contributions include Baumol (1986), Barro (1991) and Barro and Sala-i-Martin (1992). Barro (1991) has done more than anybody else to popularize growth regressions, which have become a very commonly-used technique over the past two decades. See Durlauf (1996), Durlauf, Johnson and Temple (2005) and Quah (1993) for various critiques of growth regressions, especially focusing on issues of convergence. Wooldridge (2002) contains an excellent discussion of issues of omitted variable bias and how different approaches can be used (see, for example, Chapters 4, 5, 8, 9 and 10). The difficulties involved in estimating models with fixed effects and lagged dependent variables are discussed in Chapter 11.

The augmented Solow model with human capital is a generalization of the model presented in Mankiw, Romer and Weil (1992). As noted in the text, treating human capital as a separate factor of production may not be appropriate. Different ways of introducing human capital in the basic growth model are discussed in Chapter 10 below.

Mankiw, Romer and Weil (1992) also provide the first regression estimates of the Solow and the augmented Solow models. A detailed critique of the Mankiw, Romer and Weil is provided in Klenow and Rodriguez (1997). Hall and Jones (1999) and Klenow and Rodriguez (1997) provide the first calibrated estimates of productivity (technology) differences across countries. Caselli (2005) gives an excellent overview of this literature, with a detailed discussion of how one might correct for differences in the quality of physical and human capital across countries. He reaches the conclusion that such corrections will not change the basic conclusions of Klenow and Rodriguez and Hall and Jones, that cross-country technology differences are important.

The last subsection draws on Trefler (1993). Trefler does not emphasize the productivity estimates implied by this approach, focusing more on this method as a way of testing the Heckscher-Ohlin model. Nevertheless, these productivity estimates are an important input for growth economists. Trefler's approach has been criticized for various reasons, which are secondary for our focus here. The interested reader might also want to look at Gabaix (2000) and Davis and Weinstein (2001).

3.9. Exercises

Exercise 3.1. Suppose that output is given by the neoclassical production function $Y(t) = F[K(t), L(t), A(t)]$ satisfying Assumptions 1 and 2, and that we observe output, capital and labor at two dates t and $t+T$. Suppose that we estimate TFP growth between these two dates using the equation

$$\hat{x}(t, t+T) = g(t, t+T) - \alpha_K(t)\, g_K(t, t+T) - \alpha_L(t)\, g_L(t, t+T),$$

where $g(t, t+T)$ denotes output growth between dates t and $t+T$, etc., while $\alpha_K(t)$ and $\alpha_L(t)$ denote the factor shares at the beginning date. Let $x(t, t+T)$ be the true TFP growth between these two dates. Show that there exists functions F such that $\hat{x}(t, t+T)/x(t, t+T)$ can be arbitrarily large or small. Next show the same result when the TFP estimate is constructed using the end date factor shares, i.e., as

$$\hat{x}(t, t+T) = g(t, t+T) - \alpha_K(t+T)\, g_K(t, t+T) - \alpha_L(t+T)\, g_L(t, t+T).$$

Explain the importance of differences in factor proportions (capital-labor ratio) between the beginning and end dates in these results.

Exercise 3.2. Consider the economy with labor market imperfections as in the second part of Exercise 2.13 from the previous chapter, where workers were paid a fraction $\beta > 0$ of output. Show that in this economy the fundamental growth accounting equation leads to biased estimates of TFP.

Exercise 3.3. For the Cobb-Douglas production function from Example 3.1 $Y(t) = A(t) K(t)^a L(t)^{1-\alpha}$, derive an exact analog of (3.8) and show how the rate of convergence, i.e., the coefficient in front of $(\log y(t) - \log y^*(t))$, changes as a function of $\log y(t)$.

Exercise 3.4. Consider once again the production function in Example 3.1. Suppose that two countries, 1 and 2, have exactly the same technology and the same parameters α, n, δ and g, thus the same $y^*(t)$. Suppose that we start with $y_1(0) = 2y_2(0)$ at time $t = 0$. Using the parameter values in Example 3.1 calculate how long it would take for the income gap between the two countries to decline to 10%.

Exercise 3.5. Consider a collection of Solow economies, each with different levels of δ, s and n. Show that an equivalent of the conditional convergence regression equation (3.10) can be derived from an analog of (3.8) in this case.

Exercise 3.6. Prove Proposition 3.2.

Exercise 3.7. In the augmented Solow model (cfr Proposition 3.2) determine the impact of increase in s_k, s_h and n on h^* and k^*.

Exercise 3.8. Suppose the world is given by the augmented Solow growth model with the production function (3.13). Derive the equivalent of the fundamental growth accounting equation in this case and explain how one might use available data in order to estimate TFP growth using this equation.

Exercise 3.9. Consider the basic Solow model with no population growth and no technological progress, and a production function of the form $F(K, H)$, where H denotes the efficiency units of labor (human capital), given by $H = \sum_{i \in \mathcal{N}} h_i$, where $\mathcal{N}$ is the set of all individuals in the population and h_i is the human capital of individual i. Assume that H is fixed. Suppose there are no human capital externalities and factor markets are competitive.

(1) Calculate the steady-state equilibrium of this economy.
(2) Prove that if 10% higher h at the individual level is associated with a% higher earnings, then a 10% increase in the country's stock of human capital H will lead to a% increase in steady-state output. Compare this to the immediate impact of an unanticipated 10% increase in H (i.e., with the stock of capital unchanged).

Exercise 3.10. Consider a constant returns to scale production function for country j, $Y_j = F(K_j, A_j H_j)$, where K_j is physical capital, H_j denotes the efficiency units of labor and

A_j is labor-augmenting technology. Prove that if $K_j/Y_j = K_{j'}/Y_{j'}$ in two different countries j and j', than the rental rates of capital in the two countries, R_j and $R_{j'}$ will also be equal.

Exercise 3.11. Imagine you have a cross-section of countries, $i = 1, ..., N$, and for each country, at a single point in time, you observe labor L_i, capital K_i, total output Y_i, and the share of capital in national income, σ_i^K. Assume that all countries have access to a production technology of the following form

$$F(L, K, A)$$

where A is technology. Assume that F exhibits constant returns to scale in L and K, and all markets are competitive.

(1) Explain how you would estimate relative differences in technology/productivity across countries due to the term A without making any further assumptions. Write down the equations that are involved in estimating the contribution of A to cross-country income differences explicitly.
(2) Suppose that the exercise in part 1 leads to large differences in productivity due to the A term. How would you interpret this? Does it imply that countries have access to different production possibility sets?
(3) Now suppose that the true production function is $F(H, K, A)$ where H denotes efficiency units of labor. What other types of data would you need in order to estimate the contribution of technology/productivity across countries to output differences.
(4) Show that if H is calculated as in Section 3.5, but there are significant quality-of-schooling differences and no differences in A, this strategy will lead to significant differences in the estimates of A.

CHAPTER 4

Fundamental Determinants of Differences in Economic Performance

4.1. Proximate Versus Fundamental Causes

> "...the factors we have listed (innovation, economies of scale, education, capital accumulation etc.) are not causes of growth; *they are growth*." (North and Thomas, 1973, p. 2, italics in original).

The previous chapter illustrate how the Solow growth model can be used to understand cross-country income differences and the process of economic growth. In the context of the Solow growth model, the process of economic growth is driven by technological progress. Cross-country income differences, on the other hand, are due to a combination of technology differences, differences in physical capital per worker and in human capital per worker. While this approach provides us with a good starting point and delineates potential sources of economic growth and cross-country income differences, these sources are only *proximate causes* of economic growth and economic success. Let us focus on cross-country income differences, for example. As soon as we attempt to explain these differences with technology, physical capital and human capital differences, an obvious next question presents itself: if technology, physical capital and human capital are so important in understanding differences in the wealth of nations and if they can account for five-fold, ten-fold, twenty-fold or even thirty-fold differences in income per capita across countries, then why is it that societies do not improve their technologies, invest more in physical capital, and accumulate more human capital?

It appears therefore that any explanation that simply relies on technology, physical capital and human capital differences across countries is, at some level, incomplete. There must be some other reasons underneath those, reasons which we will refer to as *fundamental causes* of economic growth. It is these reasons that are preventing many countries from investing enough in technology, physical capital and human capital.

An investigation of fundamental causes of economic growth is important for at least two reasons. First, any theory that focuses on the intervening variables (proximate causes) alone, without understanding what the underlying driving forces are, would be incomplete. Thus growth theory will remain, in some essential sense, incomplete until it comes to grips with these fundamental causes. Second, if part of our study of economic growth is motivated by improving the growth performance of certain nations and the living standards of their citizens, understanding fundamental causes is central, since attempting to increase growth just focusing on proximate causes would be tantamount to dealing with symptoms of diseases without understanding what the diseases themselves are. While such attacks on symptoms can sometimes be useful, they are no substitute for a fuller understanding of the causes of the disease, which may allow a more satisfactory treatment. In the same way, we may hope that an understanding of the fundamental causes of economic growth could one day all for more satisfactory solutions to the major questions of social science concerning why some countries are poor and some are rich and how we can ensure that more nations grow faster.

What could these fundamental causes be? Can we make progress in understanding them? And, perhaps most relevant for this book, is growth theory useful in such an endeavor?

In this chapter, we will try to answer these questions. Let us start with the last two questions. The argument in this book is that a good understanding of the mechanics of economic growth, thus the detailed models of the growth process, are essential for a successful investigation of the fundamental causes of economic growth. This is for at least two reasons; first, we can only pose useful questions about the fundamental causes of economic growth by understanding what the major proximate causes are and how they impact economic outcomes. Second, only models that provide a good approximation to reality and are successful in qualitatively and quantitatively matching the major features of the growth process can inform us about whether the potential fundamental causes that are proposed could indeed play a significant role in generating the huge income per capita differences across countries. We will see that our analysis of the mechanics of economic growth will often be useful in discarding or refining certain proposed fundamental causes. As to the question of whether we can make progress, the vast economic growth literature is evidence that progress is being made and more progress is certainly achievable. In some sense, it is part of the objective of this book to convince you that the answer to this question is yes.

Returning to the first question, there are innumerable fundamental causes of economic growth that various economists, historians and social scientists have proposed over the ages. Clearly, listing them and cataloging them will be neither informative nor useful. Instead, we will classify the major candidate fundamental causes of economic growth into four categories of hypotheses. While such a classification undoubtedly fails to do justice to some of the nuances of the previous literature, it is satisfactory for our purposes of bringing out the main factors affecting cross-country income differences and economic growth. These are:

(1) The luck hypothesis.
(2) The geography hypothesis.
(3) The culture hypothesis.

(4) The institutions hypothesis.

By *luck*, we refer to the set of fundamental causes that explain divergent paths of economic performance among otherwise-identical countries, either because some small uncertainty or heterogeneity between them have led to different choices with far-ranging consequences, or because of different selection among multiple equilibria. By multiple equilibria, we refer to different equilibrium configurations that may be possible for the same underlying economic environment. When our models exhibit multiple equilibria, we are often unable to make specific predictions as to which of these equilibria will be selected by different countries and it is possible for two otherwise-identical countries to end up in different equilibria with quite different implications for economic growth and living standards. Luck and multiple equilibria can manifest themselves through any of the proximate causes we have discussed so far (and through some additional mechanisms that will be discussed later in the book). For example, multiple equilibria can exist in technology adoption, in models that focus on human capital or physical capital investments. Therefore, explanations based on luck or multiple equilibria are theoretically well grounded in the types of models we will study in this book. Whether they are empirically plausible is another matter.

By *geography*, we refer to all factors that are imposed on individuals as part of the physical, geographic and ecological environment in which they live. Geography can affect economic growth through a variety of proximate causes. Geographic factors that can influence the growth process include soil quality, which can affect agricultural productivity; natural resources, which directly contribute to the wealth of a nation and may facilitate industrialization by providing certain key resources, such as coal and iron ore during critical times; climate, which may affect productivity and attitudes directly; topography, which can affect the costs of transportation and communication; and disease environment, which can affect individual health, productivity and incentives to accumulate physical and human capital. For example, in terms of the aggregate production function of the Solow model, poor soil quality, lack of natural resources or an inhospitable climate may correspond to a low level of A, that is, to a type of "inefficient technology". Many philosophers and social scientists have suggested that climate also affects preferences in a fundamental way, so perhaps those in certain climates have a preference for earlier rather than later consumption, thus reducing their saving rates both in physical and human capital. Finally, differences in the disease burden across areas may affect the productivity of individuals and their willingness to accumulate human capital. Thus geography-based explanations can easily be incorporated into both the simple Solow model we have already studied and the more satisfactory models we will see later in the book.

By *culture*, we refer to beliefs, values and preferences that influence individual economic behavior. Differences in religious beliefs across societies are among the clearest examples of cultural differences that may affect economic behavior. Differences in preferences, for example, regarding how important wealth is relative to other status-generating activities and how patient individuals should be, might be as important as or even more important than luck, geography and institutions in affecting economic performance. Broadly speaking, culture can affect economic outcomes through two major channels. First, it can affect the willingness of individuals to tradeoff different activities or consumption today versus consumption tomorrow. Via this channel, culture will influence societies' occupational choices, market structure, saving rates and their willingness to accumulate physical and human capital. Second, culture may also affect the degree of cooperation among individuals, and cooperation and trust are often important foundations for productive activities in societies.

By *institutions*, we refer to rules, regulations, laws and policies that affect economic incentives and thus the incentives to invest in technology, physical capital and human capital. It is a truism of economic analysis that individuals will only take actions that are rewarded. Institutions, which shape these rewards, must therefore be important in affecting all three of the proximate causes of economic growth. What distinguishes institutions from geography, luck and culture is that they are *social choices*. While laws and regulations are not directly chosen by individuals and some institutional arrangements may be historically persistent, in the end the laws, policies and regulations under which a society lives are the choices of the members of that society. If the members of the society collectively decide to change them, they can change them. This implies that if institutions are a major fundamental cause of economic growth and cross-country differences in economic performance, they can be potentially reformed so as to achieve better outcomes. Such reforms may not be easy, they may encounter a lot of opposition, and often we may not exactly know which reforms will work. But they are still within the realm of the possible, and further research might help us understand how such reforms will affect economic incentives and how they can be implemented.

There is a clear parallel between institutions and culture. Both affect individual behavior and both are important determinants of incentives. Nevertheless, a crucial difference between the theories put into these two categories justifies their separation. Institutions are directly under the control of the members of the society, in the sense that by changing the distribution of resources, constitutions, laws and policies, individuals can collectively influence the institutions under which they live. In contrast, culture is a set of beliefs that have evolved over time and outside the direct control of individuals.[1] Even though institutions might be hard to change in practice, culture is much harder to influence, and any advice to a society that it should change its culture is almost vacuous.

[1] A major and important exception to this is the effect of education on the beliefs and values of individuals.

It is also important to emphasize that institutions themselves, even if they are a fundamental cause of economic growth and income differences across countries, are *endogenous*. They are equilibrium choices made either by the society at large or by some powerful groups in society. One can then argue that at some level luck, geography or culture should be more important, because they can be "more exogenous" in the sense that they are *not* equilibrium choices in the same way as institutions are and institutions will vary across societies largely because of geographic, cultural or random factors. While at some philosophical level this is correct, it is not a particularly useful observation. It neither obviates the need to understand the *direct* effects of luck, geography, culture and institutions (these direct effects have been the focus of much of the debate in this area) nor does it imply that understanding the specific role of institutions and economic development is secondary in any sense. After all, if we can understand what the effects of institutions are and which specific types of institutions matter, institutional reform can lead to major changes in economic behavior (even if part of the original variation in institutions was due to geography, luck or culture).

In the rest of this chapter, I will explain what the reasoning for these different hypotheses are and provide a brief overview of the empirical evidence pertaining to various fundamental causes of economic growth. The theoretical underpinnings and implications of the institutions view will be further developed in Part 8 of the book. At this point, the reader should be warned that the author of this book is not an objective outside observer in this debate, but a strong proponent of the institutions hypothesis. Therefore, not surprisingly, this chapter will conclude that the institutional differences are at the root of the important proximate causes that we have already listed. Nevertheless, the same evidence can be interpreted in different ways and the reader should feel free to draw his or her own conclusions.

Before delving into a discussion of the fundamental causes, one other topic deserves a brief discussion. This is where we start in the next section.

4.2. Economies of Scale, Population, Technology and World Growth

As we have emphasized in Chapter 1, cross-country income differences result from the differential growth experiences of countries over the past two centuries. This makes it important for us to understand the process of economic growth. Equally remarkable is the fact that world economic growth is, by and large, a phenomenon of the past 200 years or so. Thus another major question concerns why economic growth started so recently and why there was little economic growth before. The growth literature has provided a variety of interesting answers to this question. Many of them focus on the role of economies of scale and population. The argument goes as follows: in the presence of economies of scale (or increasing returns to scale), population needs to have reached a certain critical level so that technological progress can gather speed. Alternatively, some natural (steady) progress of technology that may have been going on in the background needs to reach a critical threshold for the process of growth to begin. These stories are quite plausible. World population has indeed increased tremendously over the past one million years and the world's inhabitants today have access to a pool of knowledge and technology unimaginable to our ancestors. Could these long-run developments of the world economy also account for cross-country differences? Is the increase in world population a good explanation for the take off of the world economy?

Let us focus on population to give a preliminary answer to these questions. The simplest way of thinking of the relationship between population and technological change is the Simon-Kremer model (after the demographer Julian Simon and the economist Michael Kremer). This model is implicitly one of the entire world economy, since there are no cross-country differences and proponents of this model do not try to explain differences across countries by their populations. Imagine that there is a small probability that each individual will discover a new idea that will contribute to the knowledge pool of the society. Crucially, these random discoveries are independent across individuals, so that a larger pool of individuals implies discovery of more new ideas, increasing aggregate productivity. Let output be determined simply by technology (this can be generalized so that technology and capital determine output as in the Solow model, but this does not affect the point we would like to make here):

$$Y(t) = A(t) L(t)^{\alpha} Z^{1-\alpha},$$

where $\alpha \in (0,1)$, $Y(t)$ is world output, $A(t)$ is the world stock of technology, $L(t)$ is world population, and Z is some other fixed factor of production, for example, land, which we normalized to $Z = 1$ without loss of any generality. Imagine we are in a continuous time world and suppose that

$$\dot{A}(t) = \lambda L(t), \tag{4.1}$$

where λ represents the rate at which random individuals make discoveries improving the knowledge pool of the society, and the initial level of world knowledge $A(0) > 0$ is taken as given. Population, in turn, is a function of output, for example because of the Malthusian channels discussed in Chapter 21 below. For example, we could assume that

$$L(t) = \phi Y(t). \tag{4.2}$$

Combining these three equations, we obtain (see Exercise 4.1):

$$\dot{A}(t) = \lambda \phi^{\frac{1}{1-\alpha}} A(t). \tag{4.3}$$

The solution to this differential equation involves

$$A(t) = \exp\left(\lambda \phi^{1/(1-\alpha)} t\right) A(0). \tag{4.4}$$

This shows how a model of economies of scale (increasing returns) in population can generate a steady increase in technology. It is also straightforward to verify that

$$Y(t) = \phi^{\frac{\alpha}{1-\alpha}} A(t), \tag{4.5}$$

so that aggregate income also grows at the constant level $\lambda\phi^{1/(1-\alpha)}$. Such a model would generate steady growth but no acceleration. Simon and Kremer, instead, assume that there are stronger externalities to population than in (4.1). They impose the following equation governing the accumulation of ideas:

$$\frac{\dot{A}(t)}{A(t)} = \lambda L(t). \tag{4.6}$$

This implies that the law of motion of technology is given by (see Exercise 4.2):

$$A(t) = \frac{1}{A(0)^{-1} - \lambda\phi^{1/(1-\alpha)}t}. \tag{4.7}$$

In contrast to (4.4), this equation implies an accelerating output level. Starting from a low-level of $A(0)$ (or $L(0)$), this model would generate a long period of low output, followed by an acceleration or a *take off*, reminiscent to the modern economic growth experience discussed in Chapter 1. Therefore, a model with significant economies of scale is capable of generating the pattern of take off we see in the data.

While such a story, which has been proposed by many economists, may have some appeal for accounting for world growth, it is important to emphasize that it has little to say about cross-country income differences or why modern economic growth started in some countries (Western Europe) and not others (Asia, South America, Africa). In fact, if we take Western Europe and Asia as the economic units, European population has consistently been less than that of Asia over the past 2000 years, thus it is unlikely that simple economies of scale to population are responsible for the economic takeoff in Western Europe while Asia stagnated. We will return to an explanation for why economic growth might have taken off in Western Europe in Chapter 24.

We conclude from this discussion that models based on economies of scale of one sort or another do not provide us with fundamental causes of cross-country income differences. At best, they are theories of world growth (the world taken as a whole). Moreover, once we recognize that the modern economic growth process was uneven, meaning that it took place in some parts of the world and not others, the appeal of such theories diminishes further. If economies of scale were responsible for modern economic growth, it should also be able to explain when and where this process of economic growth started. Existing models based on economies of scale do not. In this sense, they are unlikely to provide the fundamental causes of modern economic growth. Does this mean that these types of economies of scale and increasing returns to population are unimportant? Certainly not. They may well be part of the proximate causes of the growth process (for example, the part lying in the black box of technology). But this discussion suggests that these models need to be augmented by fundamental causes in order to explain why, when and where the takeoff occurred. This further motivates our investigation of the fundamental causes.

4.3. The Four Fundamental Causes

4.3.1. Luck and Multiple Equilibria. In Chapter 21, we will see a number of models in which multiple equilibria or multiple steady states can arise because of coordination failures in the product market or because of imperfections in credit markets. These models suggest that an economy, with given parameter values, can exhibit very different types of behavior, some with higher levels of income or perhaps sustained growth, while others correspond to poverty and stagnation. To give a flavor of these models, consider the following simple game of investment:

everybody else → individual ↓	high investment	low investment
high investment	y^H, y^H	$y^L - \varepsilon, y^L$
low investment	$y^L, y^L - \varepsilon$	y^L, y^L

The top row indicates whether all individuals in the society choose high or low investment (focusing on a symmetric equilibrium). The first column corresponds to high investment by all agents, while the second corresponds to low investment. The top row, on the other hand, corresponds to high investment by the individual in question, and the bottom row is for low investment. In each cell, the first number refers to the income of the individual in question, while the second number is the payoff to each of the other agents in the economy. Suppose that $y^H > y^L$ and $\varepsilon > 0$. This payoff matrix implies that high investment is more profitable when others are also undertaking high investment, because of technological complementarities or other interactions.

It is then clear that there are two (pure-strategy) symmetric equilibria in this game. In one, the individual expects all other agents to choose high investment and he does so himself as well. In the other, the individual expects all others to choose low investment and it is the best response for him to choose low investment. Since the same calculus applies to each agent, this argument establishes the existence of the two symmetric equilibria. This simple game captures, in a very reduced-form way, the essence of the "Big Push" models we will study in Chapter 21.

Two points are worth noting. First, depending on the extent of complementarities and other economic interactions, y^H can be quite large relative to y^L, so there may be significant income differences in the allocations implied by the two different equilibria. Thus if we believe that such a game is a good approximation to reality and different countries can end up in different equilibria, it could help in explaining very large differences in income per capita. Second, the two equilibria in this game are also "Pareto-ranked"—all individuals are better-off in the equilibrium in which everybody chooses high investment.

In addition to models of multiple equilibria, we will also study models in which the realization of stochastic variables determine when a particular economy transitions from low-productivity to high-productivity technologies and starts the process of takeoff (see Section 17.6 in Chapter 17).

Both models of multiple equilibria and those in which stochastic variables determine the long-run growth properties of the economy are attractive as descriptions of certain aspects of the development process. They are also informative about the mechanics of economic development in an interesting class of models. But do they inform us about the fundamental causes of economic growth? Can we say that the United States is rich today while Nigeria is poor because the former has been lucky in its equilibrium selection while the latter has been unlucky? Can we pinpoint their divergent development paths to some small stochastic events 200, 300 or 400 years ago? The answer seems to be no.

U.S. economic growth is the cumulative result of a variety of processes, ranging from innovations and free entrepreneurial activity to significant investments in human capital and rapid capital accumulation. It is difficult to reduce these to a simple lucky break or to the selection of the right equilibrium, while Nigeria ended up in a worse equilibrium. Even 400 years ago, the historical conditions were very different in the United States and in Nigeria, and as will discuss further below, this led to different opportunities, different institutional paths and different incentives. It is the combination of the historical experiences of countries and different economic incentives that underlies their different processes of economic growth.

Equally important, models based on luck or multiple equilibria can explain why there might be a 20-year or perhaps a 50-year divergence between two otherwise-identical economies. But how are we to explain a 500-year divergence? It certainly does not seem plausible to imagine that Nigeria, today, can suddenly switch equilibria and quickly achieve the level of income per capita in the United States.[2] Most models of multiple equilibria are unsatisfactory in another fundamental sense. As in the simple example discussed above, most models of multiple equilibria involve the presence of Pareto-ranked equilibria. This implies that one equilibrium gives higher utility or welfare to *all* agents than another. While such Pareto-ranked equilibria are a feature of our parsimonious models, which do not specify many relevant dimensions of heterogeneity that are important in practice, it is not clear whether they are useful in thinking about why some countries are rich and some are poor. If indeed

[2]Naturally, one can argue that reforms or major changes in the growth trajectory are always outcomes of a switch from one equilibrium to another. But such an explanation would not have much empirical content, unless it is based on a well-formulated model of *equilibrium selection* and can make predictions about when we might expect such switches.

it were possible for Nigerians to change their behavior and for all individuals in the nation to become better-off (say by switching from low to high investment in terms of the game above), it is very difficult to believe that for 200 years they have not been able to coordinate on such a better action. Most readers will be aware that Nigerian history is shaped by religious and ethnic conflict, by the civil war that ravaged the nation, and is still adversely affected by the extreme corruption of politicians, bureaucrats and soldiers that have enriched themselves at the expense of the population at large. That an easy Pareto improving change exists against this historical and social background seems improbable to say the least.

To be fair, not all models of "multiple equilibria" allow easy transitions from a Pareto inferior equilibrium to a superior equilibrium. In the literature, a useful distinction is between models of multiple equilibria, where different equilibria can be reached if individuals change their beliefs and behaviors simultaneously, versus models of *multiple steady states with history dependence*, where once a particular path of equilibrium is embarked upon, it becomes much harder (perhaps impossible) to transition to the other steady state equilibrium (see Chapter 21). Such models are much more attractive for understanding persistent differences in economic performance across countries. Nevertheless, unless some other significant source of conflict of interest or distortions are incorporated, it seems unlikely that the difference between the United States and Nigeria can be explained by using models where the two countries have identical parameters, but have made different choices and stuck with them. The mechanics of how a particular steady-state equilibrium can be maintained would be the most important element of such a theory, and other fundamental causes of economic growth, including institutions, policies or perhaps culture, must play a role in explaining this type of persistence. Put differently, in today's world of free information, technology and capital flows, if Nigeria had the same parameters, the same opportunities and the same "institutions" as the United States, there should exist some arrangement such that these new technologies can be imported and everybody could be made better-off.

Another challenge to models of multiple steady states concerns the ubiquity of growth miracles such as South Korea and Singapore, which we discussed in Chapter 1. If cross-country income differences are due to multiple steady states, from which escape is impossible, then how can we explain countries that embark upon a very rapid growth process? The example of China may be even more telling here. While China stagnated under communism until Mao's death, the changes in economic institutions and policies that took place thereafter have led to very rapid economic growth. If China was in a low-growth steady state before Mao's death, then we need to explain how it escaped from the steady state after 1978, and why it did not do so before? Inevitably this takes us to the role of other fundamental causes, such as institutions, policies and culture.

A different, and perhaps more promising, argument about the importance of luck can be made by emphasizing the role of leaders. Perhaps it was Mao who held back China, and his death and the identity, beliefs and policies of his successor were at the root of its subsequent growth. Perhaps the identity of the leader of a country can thus be viewed as a stochastic event, shaping economic performance. This point of view probably has a lot of merit. Recent empirical work by Jones and Olken (2005) shows that leaders seem to matter for the economic performance of nations. Thus luck could play a major role in cross-country income and growth differences by determining whether growth-enhancing or growth-retarding leaders are selected. Nevertheless, such an explanation is closer to the institutional approaches than the pure luck category. First of all, leaders will often influence the economic performance of their societies by the policies they set and the institutions they develop. Thus, the selection and behavior of leaders and the policies that they pursue should be a part of the institutional explanations. Second, Jones and Olken's research points to an important interaction between the effect of leaders and a society's institutions. Leaders seem to matter for economic growth only in countries where institutions are non-democratic or weak (in the sense of not placing constraints on politicians or elites). In democracies and in societies where other institutions appear to place checks on the behavior of politicians and leaders, the identity of leaders seems to play almost no role in economic performance.

Given these considerations, we conclude that models emphasizing luck and multiple equilibria are useful for our study of the mechanics of economic development, but they are unlikely to provide us with the fundamental causes of why world economic growth started 200 years ago and why some countries are rich while others are poor toady.

4.3.2. Geography. While the approaches in the last subsection emphasize the importance of luck and multiple equilibria among otherwise-identical societies, an alternative is to emphasize the deep heterogeneity across societies. The geography hypothesis is, first and foremost, about the fact that not all areas of the world are created equal. "Nature", that is, the physical, ecological and geographical environment of nations, plays a major role in their economic experiences. As pointed out above, geographic factors can play this role by determining both the preferences and the opportunity set of individual economic agents in different societies. There are at least three main versions of the geography hypothesis, each emphasizing a different mechanism for how geography affects prosperity.

The first and earliest version of the geography hypothesis goes back to Montesquieu ([1748], 1989). Montesquieu, who was a brilliant French philosopher and an avid supporter of Republican forms of government, was also convinced that climate was among the main determinants of the fate of nations. He believed that climate, in particular heat, shaped human attitudes and effort, and via this channel, affected both economic and social outcomes. He wrote in his classic book *The Spirit of the Laws*:

> "The heat of the climate can be so excessive that the body there will be absolutely without strength. So, prostration will pass even to the spirit; no curiosity, no noble enterprise, no generous sentiment; inclinations will all be passive there; laziness there will be happiness,"
>
> "People are ... more vigorous in cold climates. The inhabitants of warm countries are, like old men, timorous; the people in cold countries are, like young men, brave..."

Today some of the pronouncements in these passages appear somewhat naïve and perhaps bordering on "political incorrectness". They still have many proponents, however. Even though Montesquieu's eloquence makes him stand out among those who formulated this perspective, he was neither the first nor the last to emphasize such geographic fundamental causes of economic growth. Among economists a more revered figure is one of the founders of our discipline, Alfred Marshall. Almost a century and a half after Montesquieu, Marshall wrote:

> "...vigor depends partly on race qualities: but these, so far as they can be explained at all, seem to be chiefly due to climate." (1890, p. 195).

While the first version of the geography hypothesis appears naïve and raw to many of us, its second version, which emphasizes the impact of geography on the technology available to a society, especially in agriculture, is more palatable and has many more supporters. This view is developed by an early Nobel Prize winner in economics, Gunnar Myrdal, who wrote

> "...serious study of the problems of underdevelopment ... should take into account the climate and its impacts on soil, vegetation, animals, humans and physical assets—in short, on living conditions in economic development." (1968, volume 3, p. 2121).

More recently, Jared Diamond, in his widely popular *Guns, Germs and Steel,* espouses this view and argues that geographical differences between the Americas and Europe (or more appropriately, Eurasia) have determined the timing and nature of settled agriculture and via this channel, shaped whether societies have been able to develop complex organizations and advanced civilian and military technologies (1997, e.g., p. 358). The economist Jeffrey Sachs has been a recent and forceful proponent of the importance of geography in agricultural productivity, stating that

> "By the start of the era of modern economic growth, if not much earlier, temperate-zone technologies were more productive than tropical-zone technologies ..." (2001, p. 2).

There are a number of reasons for questioning this second, and more widely-held view, of geographic determinism as well. Most of the technological differences emphasized by these authors refer to agriculture. But as we have seen in Chapter 1 and will encounter again below, the origins of differential economic growth across countries goes back to the age of industrialization. Modern economic growth came with industry, and it is the countries that have failed to industrialize that are poor today. Low agricultural productivity, if anything, should create a comparative advantage in industry, and thus encourage those countries with the "unfavorable geography" to start investing in industry before others. One might argue that reaching a certain level of agricultural productivity is a prerequisite for industrialization. While this is plausible (at least possible), we will see below that many of the societies that have failed to industrialize had already achieved a certain level of agricultural productivity, and in fact were often ahead of those who later industrialized very rapidly. Thus a simple link between unfavorable agricultural conditions and the failure to take off seems to be absent.[3]

The third variant of the geography hypothesis, which has become particularly popular over the past decade, links poverty in many areas of the world to their "disease burden," emphasizing that: "The burden of infectious disease is ... higher in the tropics than in the temperate zones" (Sachs, 2000, p. 32). Bloom and Sachs (1998) and Gallup and Sachs (2001, p. 91) claim that the prevalence of malaria alone reduces the annual growth rate of sub-Saharan African economies by as much as 2.6 percent a year. Such a magnitude implies that had malaria been eradicated in 1950, income per capita in sub-Saharan Africa would be double of what it is today. If we add to this the effect of other diseases, we would obtain even larger effects (perhaps implausibly large effects). The World Health Organization also subscribes to this view and in its recent report writes:

> "...in today's world, poor health has particularly pernicious effects on economic development in sub-Saharan Africa, South Asia, and pockets of high disease and intense poverty elsewhere..." (p. 24) and
>
> "...extending the coverage of crucial health services... to the world's poor could save millions of lives each year, reduce poverty, spur economic development and promote global security." (p. i).

This third version of the geography hypothesis may be much more plausible than the first two, especially since it is well documented in the microeconomics literature that unhealthy individuals are less productive and perhaps less able to learn and thus accumulate human capital. We will discuss both the general geography hypothesis and this specific version of it in greater detail below. But even at this point, an important caveat needs to be mentioned.

[3] Ex post, one can in fact tell the opposite story: perhaps poor nations of today had agriculturally *superior* land, and this created a comparative advantage against industry and they failed to benefit from the increasing returns to scale in manufacturing.

The fact that the burden of disease is heavier in poor nations today is as much a consequence as a cause of poverty. European nations in the 18th and even 19th centuries were plagued by many diseases. The process of economic development enabled them to eradicate these diseases and create healthier environments for living. The fact that many poor countries have unhealthy environments is, at least in part, a consequence of their failure to develop economically.

4.3.3. Institutions. An alternative fundamental cause of differences in economic growth and income per capita is institutions. One problem with the institutions hypothesis is that it is somewhat difficult to define what "institutions" are. In daily usage, the word institutions refers to may different things, and the academic literature is sometimes not clear about its definition.

The economic historian Douglass North was awarded the Nobel Prize in economics largely because of his work emphasizing the importance of institutions in the historical development process. North (1990, p. 3) offers the following definition:

> "Institutions are the rules of the game in a society or, more formally, are the humanly devised constraints that shape human interaction."

He goes on to emphasize the key implications of institutions:

> "In consequence [institutions] structure incentives in human exchange, whether political, social, or economic."

This definition encapsulates the three important elements that make up institutions. First, they are "humanly devised"; that is, in contrast to geography, which is outside human control, institutions refer to man-made factors. Institutions are about the effect of the societies' own choices on their own economic fates. Second, institutions are about placing constraints on individuals. These do not need to be unassailable constraints. Any law can be broken, any regulation can be ignored. Nevertheless, policies, regulations and laws that punish certain types of behavior while rewarding others will naturally have an effect on behavior. And this brings the third important element in the definition. The constraints placed on individuals by institutions will shape human interaction and affect incentives. In some deep sense, institutions, much more than the other candidate fundamental causes, are about the importance of incentives.

The reader may have already noted that the above definition makes institutions a rather broad concept. In fact, this is precisely the sense in which we will use the concept of institutions throughout this book; institutions will refer to a *broad cluster* of arrangements that influence various economic interactions among individuals. These include economic, political and social relations among households, individuals and firms. The importance of political institutions, which determine the process of collective decision-making in society, cannot be overstated and will be the topic of analysis in Part 8 of this book. But this is not where we will begin.

A more natural starting point for the study of the fundamental causes of income differences across countries is with *economic institutions*, which comprise such things as the structure of property rights, the presence and (well or ill) functioning of markets, and the contractual opportunities available to individuals and firms. Economic institutions are important because they influence the structure of economic incentives in society. Without property rights, individuals will not have the incentive to invest in physical or human capital or adopt more efficient technologies. Economic institutions are also important because they ensure the allocation of resources to their most efficient uses, and they determine who obtains profits, revenues and residual rights of control. When markets are missing or ignored (as was the case in many former socialist societies, for example), gains from trade go unexploited and resources are misallocated. We therefore expect societies with economic institutions that facilitate and encourage factor accumulation, innovation and the efficient allocation of resources to prosper relative to societies that do not have such institutions.

The hypothesis that differences in economic institutions are a fundamental cause of different patterns of economic growth is intimately linked to the models we will develop in this book. In all of our models, especially in those that endogenize physical capital, human capital and technology accumulation, individuals will respond to (profit) incentives. Economic institutions shape these incentives. Therefore, we will see that the way that humans themselves decide to organize their societies determines whether or not incentives to improve productivity and increase output will be forthcoming. Some ways of organizing societies encourage people to innovate, to take risks, to save for the future, to find better ways of doing things, to learn and educate themselves, to solve problems of collective action and to provide public goods. Others do not. Our theoretical models will then pinpoint exactly what specific policy and institutional variables are important in retarding or encouraging economic growth.

We will see in Part 8 of the book that theoretical analysis will be useful in helping us determine what are "good economic institutions" that encourage physical and human capital accumulation and the development and adoption of better technologies (though "good economic institutions" may change from environment to environment and from time to time). It should already be intuitive to the reader that economic institutions that tax productivity-enhancing activities will not encourage economic growth. Economic institutions that ban innovation will not lead to technological improvements. Therefore, enforcement of some basic *property rights* will be an indispensable element of good economic institutions. But other aspects of economic institutions matter as well. We will see, for example, that human capital is important both for increasing productivity and for technology adoption. However, for a broad cross-section of society to be able to accumulate human capital we need some degree of equality of opportunity. Economic institutions that only protect a rich elite or the already-privileged will not achieve such equality of opportunity and will often create other distortions, potentially retarding economic growth. We will also see in Chapter 14 that the process of Schumpeterian creative destruction, where new firms improve over and destroy incumbents, is an essential element of economic growth. Schumpeterian creative destruction requires a level playing field, so that incumbents are unable to block technological progress. Economic growth based on creative destruction therefore also requires economic institutions that guarantee some degree of equality of opportunity in the society.

Another question may have already occurred to the reader: why should any society have economic and political institutions that retard economic growth? Would it not be better for all parties to maximize the size of the national pie (level of GDP, economic growth etc.)? There are two possible answers to this question. The first takes us back to multiple equilibria. It may be that the members of the society cannot coordinate on the "right," i.e., growth-enhancing, institutions. This answer is not satisfactory for the same reasons as other broad explanations based on multiple equilibria are unsatisfactory; if there exists an equilibrium institutional improvement that will make *all* members of a society richer and better-off, it seems unlikely that the society will be unable to coordinate on this improvement for extended periods of time.

The second answer, instead, recognizes that there are inherent *conflicts of interest* within the society. There are no reforms, no changes, no advances that would make everybody better-off; as in the Schumpeterian creative destruction stories, every reform, every change and every advance creates winners and losers. Our theoretical investigations in Part 8 will show that institutional explanations are intimately linked with the conflicts of interests in society. Put simply, the distribution of resources cannot be separated from the aggregate economic performance of the economy—or perhaps in a more familiar form, efficiency and distribution cannot be separated. This implies that institutions that fail to maximize the growth potential of an economy may nevertheless create benefits for some segments of the society, who will then form a constituency in favor of these institutions. Thus to understand the sources of institutional variations we have to study the winners and losers of different institutional reforms and why winners are unable to buy off or compensate losers, and why they are not powerful enough to overwhelm the losers, even when the institutional change in question may increase the size of the national pie. Such a study will not only help us understand why some societies choose or end up with institutions that do not encourage economic growth, but will also enable us to make predictions about institutional change. After all, the fact that institutions can and do change is a major difference between the institutions hypothesis and the geography and culture hypotheses. Questions of equilibrium institutions and endogenous institutional change are central for the institutions hypothesis, but we have to postpone their discussion to Part 8. For now, however, we can note that the endogeneity of institutions has another important implication; endogeneity of institutions makes empirical work on assessing the role of institutions more challenging, because it implies that the standard "simultaneity" biases in econometrics will be present when we look at the effect of institutions on economic outcomes.[4]

In this chapter, we will focus on the empirical evidence in favor and against the various different hypotheses. We will argue that this evidence, by and large, suggests that institutional differences that societies choose and end up with are a primary determinant of their economic fortunes. The further discussion below and a summary of recent empirical work will try to bolster this case. Nevertheless, it is important to emphasize that this does not mean that only institutions matter and luck, geography and culture are not important. The four fundamental causes are potentially complementary. The evidence we will provide suggests that institutions are the most important one among these four causes, but does not deny the potential role of other factors, such as cultural influences.

4.3.4. Culture. The final fundamental explanation for economic growth emphasizes the idea that different societies (or perhaps different races or ethnic groups) have different cultures, because of different shared experiences or different religions. Culture is viewed, by some social scientists, as a key determinant of the values, preferences and beliefs of individuals and societies and, the argument goes, these differences play a key role in shaping economic performance.

At some level, culture can be thought of as influencing equilibrium outcomes for a given set of institutions. Recall that in the presence of multiple equilibria, there is a central question of equilibrium selection. For example, in the simple game discussed above, will society coordinate on the high-investment or the low-investment equilibrium? Perhaps culture may be related to this process of equilibrium selection. "Good" cultures can be thought of as ways of coordinating on better (Pareto superior) equilibria. Naturally, the arguments discussed above, that an entire society could being stuck in an equilibrium in which *all* individuals are worse-off than in an alternative equilibrium is implausible, would militate against the importance of this particular role of culture. Alternatively, different cultures generate different sets of beliefs about how people behave and this can alter the set of equilibria for a given specification of institutions (for example, some beliefs will allow punishment strategies to be used whereas others will not).

[4]Note also that geography is indeed "exogenous" in the sense that, with some notable exceptions such as global warming, it is not much influenced by economic decisions. But this does not make it "econometrically exogenous". Geographic characteristics may still be (and in fact likely are) correlated with other factors that influence economic growth.

The most famous link between culture and economic development is that proposed by Weber (1930), who argued that the origins of industrialization in Western Europe could be traced to a cultural factor—the Protestant reformation and particularly the rise of Calvinism. Interestingly, Weber provided a clear summary of his views as a comment on Montesquieu's arguments:

> "Montesquieu says of the English that they 'had progressed the farthest of all peoples of the world in three important things: in piety, in commerce, and in freedom'. Is it not possible that their commercial superiority and their adaptation to free political institutions are connected in some way with that record of piety which Montesquieu ascribes to them?"

Weber argued that English piety, in particular, Protestantism, was an important driver of capitalists development. Protestantism led to a set of beliefs that emphasized hard work, thrift, saving. It also interpreted economic success as consistent with, even as signalling, being chosen by God. Weber contrasted these characteristics of Protestantism with those of other religions, such as Catholicism and other religions, which he argued did not promote capitalism. More recently, similar ideas have been applied to emphasize different implications of other religions. Many historians and scholars have argued that not only the rise of capitalism, but also the process of economic growth and industrialization are intimately linked to cultural and religious beliefs. Similar ideas are also proposed as explanations for why Latin American countries, with their Iberian heritage, are poor and unsuccessful, while their North American neighbors are more prosperous thanks to their Anglo-Saxon culture.

A related argument, originating in anthropology, argues that societies may become "dysfunctional" because their cultural values and their system of beliefs do not encourage cooperation. An original and insightful version of this argument is developed in Banfield's (1958) analysis of poverty in Southern Italy. His ideas were later picked up and developed by Putnam (1993), who suggested the notion of "social capital," as a stand-in for cultural attitudes that lead to cooperation and other "good outcomes". Many versions of these ideas are presented in one form or another in the economics literature as well.

Two challenges confront theories of economic growth based on culture. The first is the difficulty of measuring culture. While both Putnam himself and some economists have made some progress in measuring certain cultural characteristics with self-reported beliefs and attitudes in social surveys, simply stating that the North of Italy is rich because it has good social capital while the South is poor because it has poor social capital runs the risk of circularity. The second difficulty confronting cultural explanations is for accounting for growth miracles, such as those of South Korea and Singapore. As mentioned above, if some Asian cultural values are responsible for the successful growth experiences of these countries, it becomes difficult to explain why these Asian values did not lead to growth before. Why do these values not spur economic growth in North Korea? If Asian values are important for Chinese growth today, why did they not lead to a better economic performance under Mao's dictatorship? Both of these challenges are, in principle, surmountable. One may be able to develop models of culture, with better mapping to data, and also with an associated theory of when and how culture may change rapidly under certain circumstances, to allow stagnation to be followed by a growth miracle. While possible in principle, such theories have not been developed yet. Moreover, the evidence presented in the next section suggests that cultural effects are not the major force behind the large differences in economic growth experienced by many countries over the past few centuries.

4.4. The Effect of Institutions on Economic Growth

We now argue that there is convincing empirical support for the hypothesis that differences in economic institutions, rather than luck, geography or culture, *cause* differences in incomes per-capita. Let us start by looking at the simplest correlation between a measure of economic institutions and income per capita.

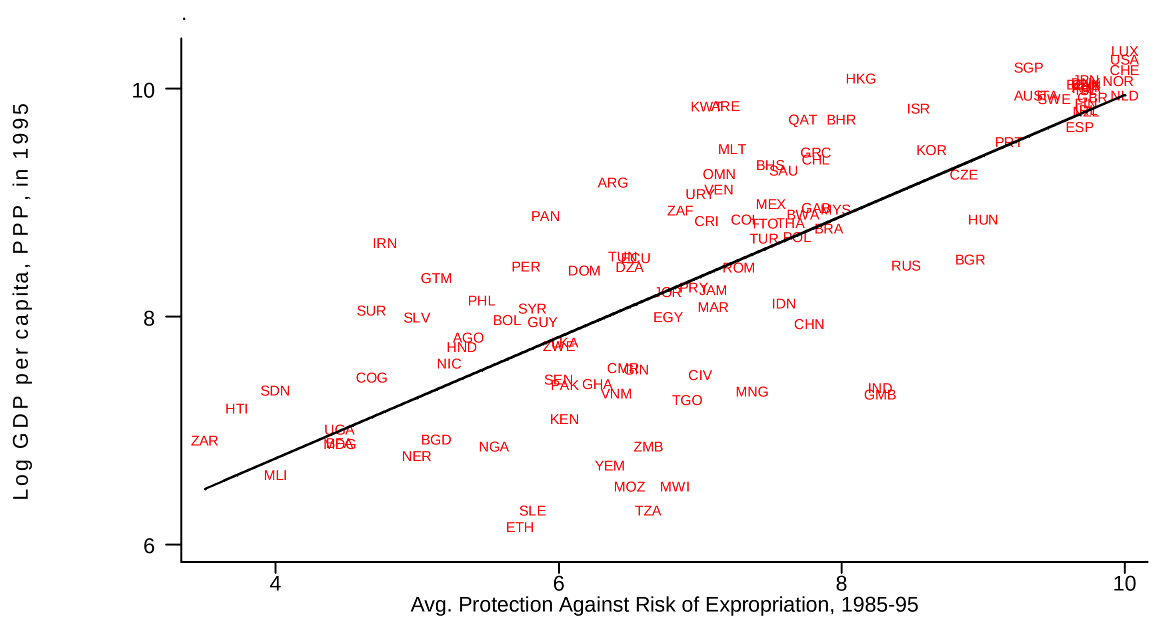

FIGURE 4.1. Relationship between economic institutions, as measured by average expropriation risk 1985-1995, and GDP per capita.

Figure 4.1 shows the cross-country correlation between the log of GDP per-capita in 1995 and a broad measure of property rights, "protection against expropriation risk", averaged over the period 1985 to 1995. The data on this measure of economic institutions come from

Political Risk Services, a private company which assesses the risk that foreign investments will be expropriated in different countries. These data are not perfect. They reflect the subjective assessment of some analysts about how secure property rights are. Nevertheless, they are useful for our purposes. First, they emphasize the security of property rights, which is an essential aspect of economic institutions, especially in regards to their effect on economic incentives. Second, these measures are purchased by businessmen contemplating investment in these countries, thus they reflect the "market assessment" of security of property rights.

Figure 4.1 shows that countries with more secure property rights—thus better economic institutions—have higher average incomes. One should not interpret the correlation in this figure as depicting a causal relationship—that is, as establishing that secure property rights cause prosperity. First, the correlation might reflect reverse causation; it may be that only countries that are sufficiently wealthy can afford to enforce property rights. Second and more importantly, there might be a problem of omitted variable bias. It could be something else, for example, geography or culture, that explains both why countries are poor and why they have insecure property rights. Thus if omitted factors determine institutions and incomes, we would spuriously infer the existence of a causal relationship between economic institutions and incomes when in fact no such relationship exists. This is the standard identification problem in economics resulting from simultaneity or omitted variable biases. Finally, security of property rights—or other proxy measures of economic institutions—are themselves equilibrium outcomes, presumably resulting from the underlying political institutions and political conflict. While this last point is important, a satisfactory discussion requires us to develop models of political economy of institutions, which will have to wait until Part 8 of the book.

To further illustrate these potential identification problems, suppose that climate or geography matter for economic performance. In fact, a simple scatterplot shows a positive association between latitude (the absolute value of distance from the equator) and income per capita consistent with the views of Montesquieu and other proponents of the geography hypothesis. Interestingly, Montesquieu not only claimed that warm climate makes people lazy and thus unproductive, but also unfit to be governed by democracy. He argued that despotism would be the political system in warm climates. Therefore, a potential explanation for the patterns we see in Figure 4.1 is that there is an omitted factor, geography, which explains both economic institutions and economic performance. Ignoring this potential third factor would lead to mistaken conclusions.

Even if Montesquieu's story appears both unrealistic and condescending to our modern sensibilities, the general point should be taken seriously: the correlations depicted in Figure 4.1, and for that matter that shown in Figure 4.2, do not necessarily reflect causal relationships . As we pointed out in the context of the effect of religion or social capital on economic

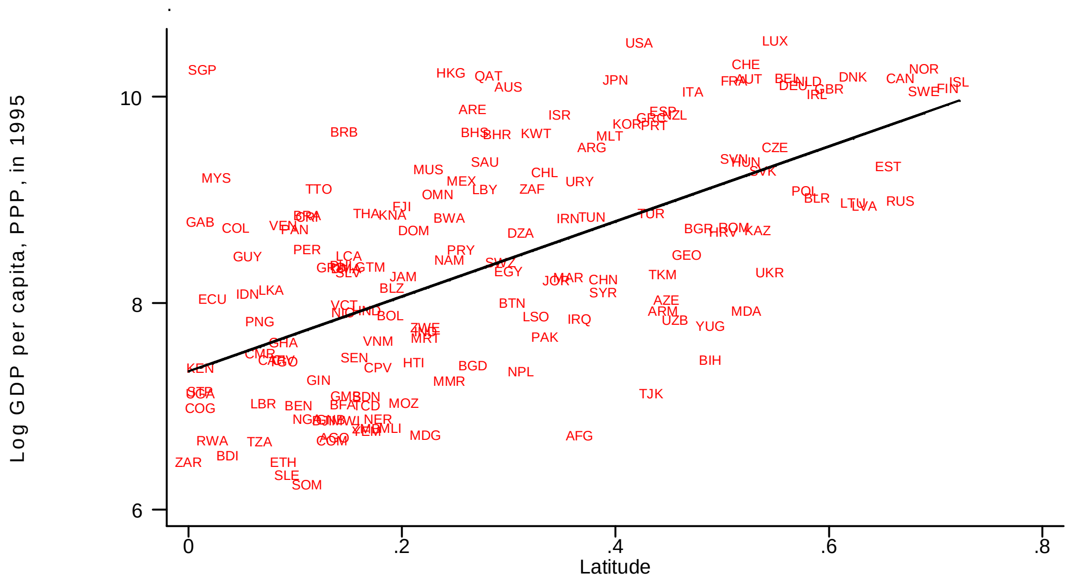

FIGURE 4.2. Relationship between latitude (distance of capital from the equator) and income per capita in 1995.

performance, these types of scatterplots, correlations, or their multidimensional version in ordinary least squares regressions, *cannot* establish causality.

How can we overcome the challenge of establishing a causal relationship between (economic) institutions and economic outcomes? The answer to this question is to specify econometric approaches based on convincing identifying restrictions. This can be done by using estimating structural econometric models or by using more reduced-form approaches, based on instrumental-variables strategies. At the moment we do not know enough about the evolution of economic institutions and their impact on economic outcomes to be able to specify and estimate fully-structural econometric models. Thus as a first step, we can look at more reduced-form evidence that might still be informative about the causal relationship between institutions and economic growth. One way of doing so is to learn from history, in particular from the "natural experiments", which are unusual historical events where, while other fundamental causes of economic growth are held constant, institutions change because of potentially-exogenous reasons. We now discuss lessons from two such natural experiments.

4.4.1. The Korean Experiment. Until the end of World War II, Korea was under Japanese occupation. Korean independence came shortly after the war. The major fear of the United States during this time period was the takeover of the entire Korean peninsula either by the Soviet Union or by communist forces under the control of the former guerrilla fighter, Kim Il Sung. U.S. authorities therefore supported the influential nationalist leader Syngman Rhee, who was in favor of separation rather than a united communist Korea. Elections in the South were held in May 1948, amidst a widespread boycott by Koreans opposed to separation. The newly elected representatives proceeded to draft a new constitution and established the Republic of Korea to the south of the 38th parallel. The North became the Democratic People's Republic of Korea, under the control of Kim Il Sung.

These two independent countries organized themselves in radically different ways and adopted completely different sets of (economic and political) institutions. The North followed the model of Soviet socialism and the Chinese Revolution in abolishing private property in land and capital. Economic decisions were not mediated by the market, but by the communist state. The South instead maintained a system of private property and capitalist economic institutions.

Before this "natural experiment" in institutional change, North and South Korea shared the same history and cultural roots. In fact, Korea exhibited an unparalleled degree of ethnic, linguistic, cultural, geographic and economic homogeneity. There are few geographic distinctions between the North and South, and both share the same disease environment. Moreover, before the separation the North and the South were at the same level of development. If anything, there was slightly more industrialization in the North. Maddison (2001) estimates that at the time of separation, North and South Korea had approximately the same income per capita.

We can therefore think of the splitting on the Koreas 50 years ago as a natural experiment that we can use to identify the causal influence of institutions on prosperity. Korea was split into two, with the two halves organized in radically different ways, and with geography, culture and many other potential determinants of economic prosperity held fixed. Thus any differences in economic performance can plausibly be attributed to differences in institutions.

Figure 4.3 uses data from Maddison (2001) and shows that the two Koreas have experienced dramatically diverging paths of economic development since separation:

By the late 1960's South Korea was transformed into one of the Asian "miracle" economies, experiencing one of the most rapid surges of economic prosperity in history while North Korea stagnated. By 2000 the level of income in South Korea was $16,100 while in North Korea it was only $1,000. There is only one plausible explanation for the radically different economic experiences of the two Koreas after 1950: their very different institutions led to divergent economic outcomes. In this context, it is noteworthy that the two Koreas not only shared the same geography, but also the same culture, so that neither geographic nor cultural differences could have much to do with the divergent paths of the two Koreas. Of course one can say that South Korea was lucky while the North was unlucky (even though this was not due to any kind of multiple equilibria, but a result of the imposition of different

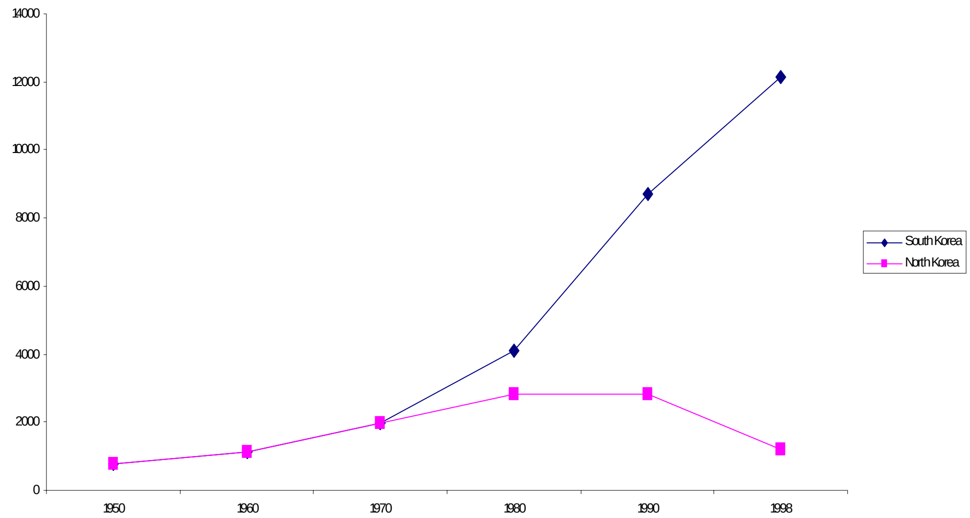

Figure 4.3. Evolution of income per capita North and South Korea after the separation.

institutions). Nevertheless, the perspective of "luck" is unlikely to be particularly useful in this context, since what is remarkable is the persistence of the dysfunctional North Korean institutions. Despite convincing evidence that the North Korean system has been generating poverty and famine, the leaders of the Communist Party in North Korea have opted to use all the means available to them to maintain their regime.

However convincing on its own terms, the evidence from this natural experiment is not sufficient for the purposes of establishing the importance of economic institutions as the primary factor shaping cross-country differences in economic prosperity. First, this is only one case, and in the better-controlled experiments in the natural sciences, a relatively large sample is essential. Second, here we have an example of an extreme case, the difference between a market-oriented economy and an extreme communist one. Few social scientists today would deny that a lengthy period of totalitarian centrally-planned rule has significant economic costs. And yet, many might argue that differences in economic institutions among capitalist economies or among democracies are not the major factor leading to differences in their economic trajectories. To establish the major role of economic institutions in the prosperity and poverty of nations we need to look at a larger scale "natural experiment" in institutional divergence.

4.4.2. The Colonial Experiment: The Reversal of Fortune. The colonization of much of the world by Europeans provides such a large scale natural experiment. Beginning in the early fifteenth century and especially after 1492, Europeans conquered many other nations. The colonization experience transformed the institutions in many diverse lands conquered or controlled by Europeans. Most importantly, Europeans imposed very different sets of institutions in different parts of their global empire, as exemplified most sharply by the contrast of the institutional structure that developed in the Northeastern United States, based on small-holder private property and democracy, versus the institutions in the Caribbean plantation economies, based on repression and slavery. As a result, while geography was held constant, Europeans initiated very significant changes in the economic institutions of different societies.

The impact of European colonialism on economic institutions is perhaps most dramatically conveyed by a single fact—historical evidence shows that there has been a remarkable Reversal of Fortune in economic prosperity within former European colonies. Societies like the Mughals in India, and the Aztecs and the Incas in the Americas were among the richest civilizations in 1500, yet the nation-states that now coincide with the boundaries of these empires are among the poorer nations of today. In contrast, countries occupying the territories of the less-developed civilizations of North America, New Zealand and Australia are now much richer than those in the lands of the Mughals, Aztecs and Incas.

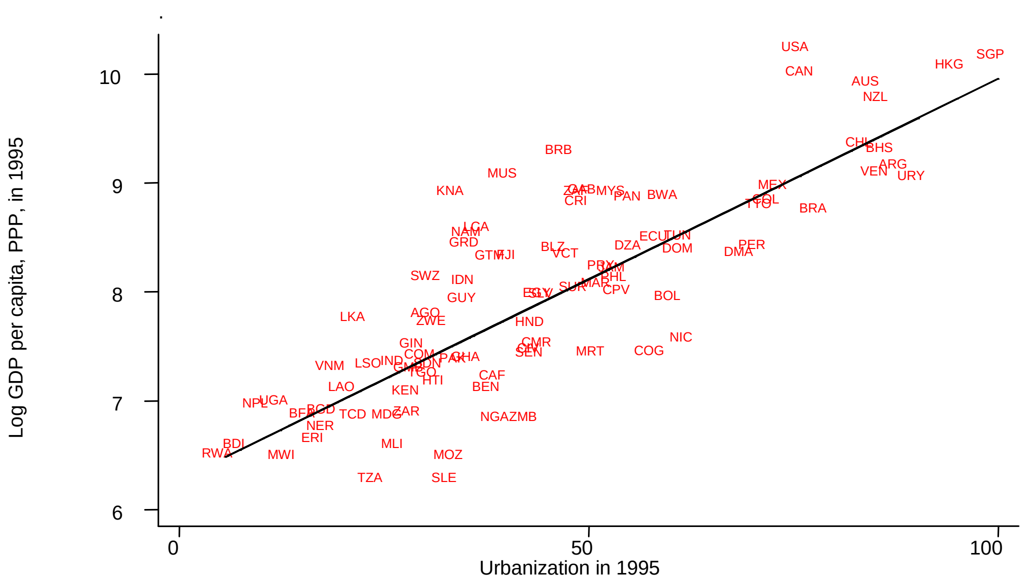

Figure 4.4. Urbanization and Income today.

The Reversal of Fortune is not confined to such comparisons. To document the reversal more broadly, we need a proxy for prosperity 500 years ago. Fortunately, urbanization rates and population density can serve the role of such proxies. Only societies with a certain level of productivity in agriculture and a relatively developed system of transport and commerce can sustain large urban centers and a dense population. Figure 4.4 shows the relationship between income per capita and urbanization (fraction of the population living in urban centers with greater than 5,000 inhabitants) today, and demonstrates that even today, long after industrialization, there is a significant relationship between urbanization and prosperity.

Naturally, high rates of urbanization do not mean that the majority of the population lived in prosperity. In fact, before the twentieth century urban areas were often centers of poverty and ill health. Nevertheless, urbanization is a good proxy for average prosperity and closely corresponds to the GDP per capita measures we are using to look at prosperity today. Another variable that is useful for measuring pre-industrial prosperity is the density of the population, which is closely related to urbanization.

Figures 4.5 and 4.6 show the relationship between income per capita today and urbanization rates and (log) population density in 1500 for the sample of European colonies. Let us focus on 1500 since it is before European colonization had an effect on any of these societies. A strong negative relationship, indicating a reversal in the rankings in terms of economic prosperity between 1500 and today, is clear in both figures. In fact, the figures show that in 1500 the temperate areas were generally less prosperous than the tropical areas, but this pattern too was reversed by the twentieth century.

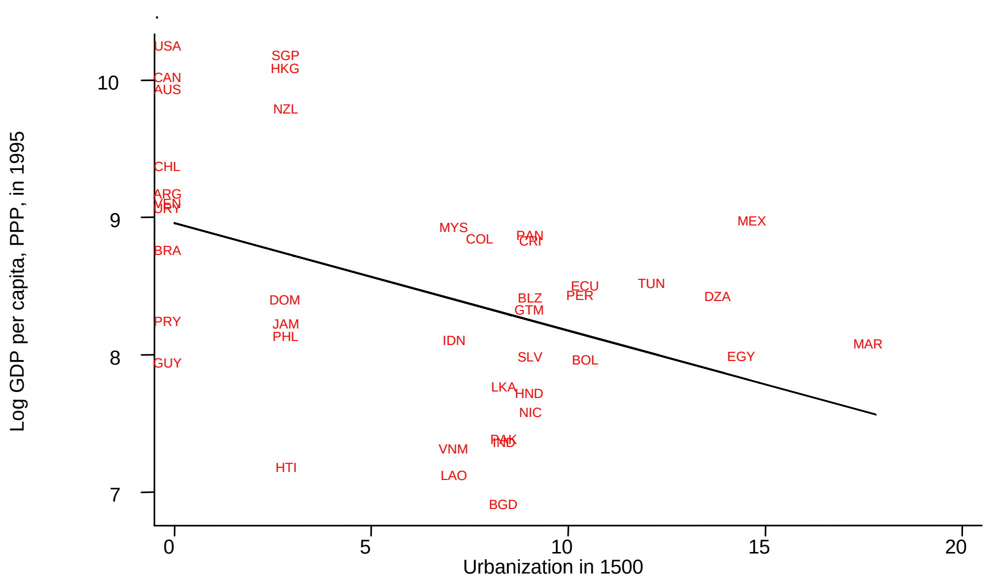

Figure 4.5. Reversal of Fortune: urbanization in 1500 versus income per capita in 1995 among the former European colonies.

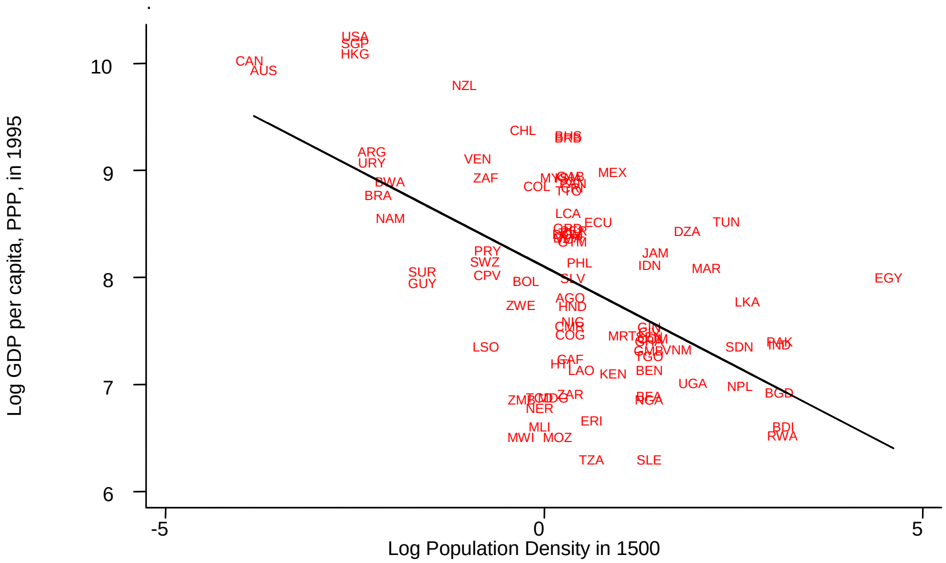

Figure 4.6. Reversal of Fortune: population density in 1500 versus income per capita in 1995 them on the former European colonies.

There is something extraordinary and unusual about this reversal. A wealth of evidence shows that after the initial spread of agriculture there was remarkable persistence in urbanization and population density for all countries, including those that were subsequently colonized by Europeans. Extending the data on urbanization to earlier periods shows that both among former European colonies and non-colonies, urbanization rates and prosperity persisted for 500 years or longer. Even though there are prominent examples of the decline and fall of empires, such as Ancient Egypt, Athens, Rome, Carthage and Venice, the overall pattern was one of persistence. It is also worth noting that reversal was not the general pattern in the world after 1500. When we look at Europe as a whole or at the entire world excluding the former European colonies, there is no evidence of a similar reversal between 1500 and 1995.

There is therefore no reason to think that what is going on in Figures 4.5 and 4.6 is some sort of natural reversion to the mean. Instead, the Reversal of Fortune among the former European colonies reflects something unusual, something related to the intervention that these countries experienced. The major intervention, of course, was related to the change in institutions. As discussed above, not only did the Europeans impose a different order in almost all countries they conquered, there were also tremendous differences between the types of institutions they imposed on in the different colonies.[5] These institutional differences among the former colonies are likely at the root of the reversal in economic fortunes.

To bolster this case, let us look at the timing and the nature of the reversal a little more closely. When did the reversal occur? One possibility is that it arose shortly after the conquest of societies by Europeans but Figure 4.7 shows that the previously-poor colonies surpassed the former highly-urbanized colonies starting in the late eighteenth and early nineteenth centuries. Moreover, a wealth of qualitative and quantitative evidence suggests that this went hand in hand with industrialization (see, for example, Acemoglu, Johnson and Robinson, 2002). Figure 4.7 shows average urbanization in colonies with relatively low and high urbanization in 1500. The initially high-urbanization countries have higher levels of urbanization and prosperity until around 1800. At that time the initially low-urbanization countries start to grow much more rapidly and a prolonged period of divergence begins.

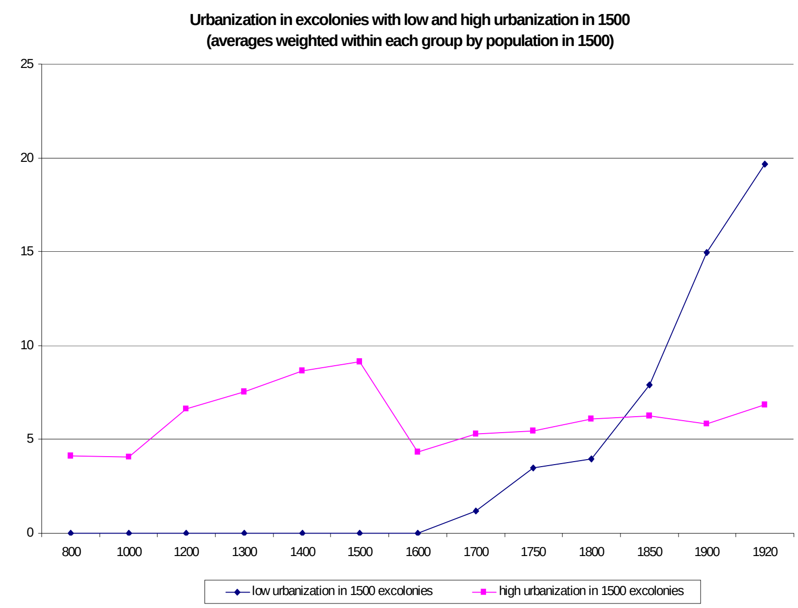

Figure 4.7. The Timing of the Reversal of Fortune: Evolution of average urbanization between initially-high and initially-low urbanization former colonies.

These patterns are clearly inconsistent with simple geography based views of relative prosperity. In 1500 it was the countries in the tropics which were relatively prosperous,

[5]In some instances, including those in Central America and India, the colonial institutions were built on the pre-colonial institutions. In these cases, the issue becomes one of whether Europeans maintained and further developed existing hierarchical institutions, such as those in the Aztec, the Inca or the Mughal Empires, or whether they introduced or imposed political and economic institutions encouraging broad-based participation and investment.

today it is the reverse. This makes it implausible to base a theory of relative prosperity on the intrinsic poverty of the tropics, climate, disease environments or other fixed characteristics.

Nevertheless, following Diamond (1997), one could propose what Acemoglu, Johnson and Robinson (2002a) call a "sophisticated geography hypothesis," which claims that geography matters but in a time varying way. For example, Europeans created "latitude specific" technology, such as heavy metal ploughs, that only worked in temperate latitudes and not with tropical soils. Thus when Europe conquered most of the world after 1492, they introduced specific technologies that functioned in some places (the United States, Argentina, Australia) but not others (Peru, Mexico, West Africa). However, the timing of the reversal in the nineteenth century is inconsistent with the most natural types of sophisticated geography hypotheses. Europeans may have had latitude specific technologies, but the timing implies that these technologies must have been industrial, not agricultural, and it is difficult to see why industrial technologies do not function in the tropics (and in fact, they have functioned quite successfully in tropical Singapore and Hong Kong).

Similar considerations weigh against the culture hypothesis. Although culture is slow-changing the colonial experiment was sufficiently radical to have caused major changes in the cultures of many countries that fell under European rule. In addition, the destruction of many indigenous populations and immigration from Europe are likely to have created new cultures or at least modified existing cultures in major ways. Nevertheless, the culture hypothesis does not provide a natural explanation for the reversal, and has nothing to say on the timing of the reversal. Moreover, we will discuss below how econometric models that control for the effect of institutions on income do not find any evidence of an effect of religion or culture on prosperity.

The importance of luck is also limited. The different institutions imposed by the Europeans were not random. They were instead very much related to the conditions they encountered in the colonies. In other words, the types of institutions that were imposed and developed in the former colonies were endogenous outcomes, outcomes of equilibria that we need to study.

4.4.3. The Reversal and the Institutions Hypothesis. Is the Reversal of Fortune consistent with a dominant role for economic institutions in comparative development? The answer is yes. In fact, once we recognize the variation in economic institutions created by colonization, we see that the Reversal of Fortune is exactly what the institutions hypothesis predicts.

The evidence in Acemoglu, Johnson and Robinson (2002a) shows a close connection between initial population density, urbanization, and the creation of good economic institutions.

In particular, the evidence points out that, others things equal, the higher the initial population density or the greater initial urbanization, the worse were subsequent institutions, including both institutions right after independence and also institutions today. Figures 4.8 and 4.9 illustrate these relationships using the same measure of current economic institutions used in Figure 4.1, protection against expropriation risk today. They document that the relatively densely settled and highly urbanized colonies ended up with worse institutions, while sparsely-settled and non-urbanized areas received an influx of European migrants and developed institutions protecting the property rights of a broad cross-section of society. European colonialism therefore led to an "institutional reversal," in the sense that the previously-richer and more-densely settled places ended up with worse institutions. The institutional reversal does not mean that institutions were better in the previously more densely-settled areas. It only implies a tendency for the relatively poorer and less densely-settled areas to end up with better institutions than previously-rich and more densely-settled areas.

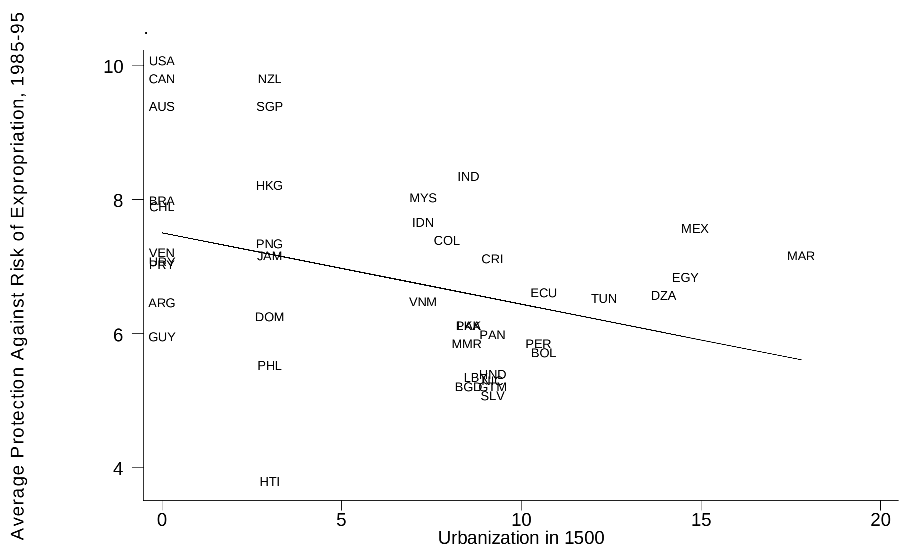

Figure 4.8. The Institutional Reversal: urbanization in 1500 and economic institutions today among the former European colonies.

As discussed in footnote 5 above, it is possible that the Europeans did not actively introduce institutions discouraging economic progress in many of these places, but inherited them from previous civilizations there. The structure of the Mughal, Aztec and Inca empires were already very hierarchical with power concentrated in the hands of narrowly based ruling elites and structured to extract resources from the majority of the population for the benefit of a minority. Often Europeans simply took over these existing institutions. What is important in any case is that in densely-settled and relatively-developed places it was in the interests of

Europeans to have institutions facilitating the extraction of resources, without any respect for the property rights of the majority of the populace. In contrast, in the sparsely-settled areas it was in their interests to develop institutions protecting property rights. These incentives led to an institutional reversal.

The institutional reversal, combined with the institutions hypothesis, predicts the Reversal of Fortune: relatively rich places ended up with relatively worse economic institutions, and if these institutions are important, we should see them become relatively poor over time. This is what the Reversal of Fortune shows.

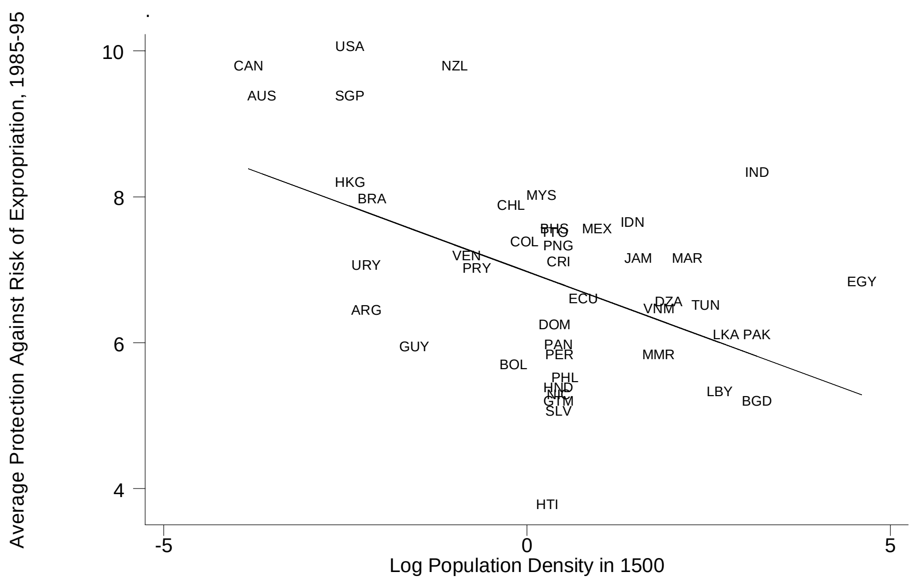

Figure 4.9. The Institutional Reversal: population density in 1500 and economic institutions today among the former European colonies.

Moreover, the institutions hypothesis is consistent with the timing of the reversal. Recall that the institutions hypothesis links incentives to invest in physical and human capital and in technology to economic institutions, and argues that economic prosperity results from these investments. Therefore, we expect economic institutions to play a more important role in shaping economic outcomes when there are major new investment opportunities—thus creating greater need for entry by new entrepreneurs and for the process of creative destruction. The opportunity to industrialize was the major investment opportunity of the 19th century. As documented in Chapter 1, countries that are rich today, both among the former European colonies and other countries, are those that industrialized successfully during this critical period.

The explanation for the reversal that emerges from the discussion so far is one in which the economic institutions in various colonies were shaped by Europeans to serve their own (economic) interests. Moreover, because conditions and endowments differed between colonies, Europeans consciously created different economic institutions, which persisted and continue to shape economic performance. Why did Europeans introduce better institutions in previously-poor and unsettled areas than in previously-rich and densely-settled areas? Without going into details, a number of obvious ideas that have emerged from the research in this area can be mentioned.

Europeans were more likely to introduce or maintain economic institutions facilitating the extraction of resources in areas where they would benefit from the extraction of resources. This typically meant areas controlled by a small group of Europeans, as well as areas offering resources to be extracted. These resources included gold and silver, valuable agricultural commodities such as sugar, but most importantly, what is perhaps the most valuable commodity overall, human labor. In places with a large indigenous population, Europeans could exploit the population. This was achieved in various forms, using taxes, tributes or employment as forced labor in mines or plantations. This type of colonization was incompatible with institutions providing economic or civil rights to the majority of the population. Consequently, a more developed civilization and a denser population structure made it more profitable for the Europeans to introduce worse economic institutions.

In contrast, in places with little to extract, and in sparsely-settled places where the Europeans themselves became the majority of the population, it was in their interests to introduce economic institutions protecting their own property rights (and also to attract further settlers).

4.4.4. Settlements, Mortality and Development. The initial conditions of the colonies we have emphasized so far, indigenous population density and urbanization, are not the only factors affecting Europeans' colonization strategy. In addition, the disease environments differed markedly among the colonies, with obvious consequences on the attractiveness of European settlement. Since, as we noted above, when Europeans settled, they established institutions that they themselves had to live under, whether Europeans could settle or not had a major effect on the subsequent path of institutional development. In other words, we expect the disease environment 200 or more years ago, especially the prevalence of malaria and yellow fever which crucially affected potential European mortality, to have shaped the path of institutional development in the former European colonies and via this channel, current institutions and current economic outcomes. If in addition, the disease environment of the colonial times affects economic outcomes today only through its effect on institutions, then we can use this historical disease environment as an exogenous source of variation in current institutions. From an econometric point of view, this will correspond to a valid instrument to estimate the casual effect of economic institutions on prosperity. Although mortality rates of potential European settlers could be correlated with indigenous mortality, which may determine income today, in practice local populations had developed much greater immunity to malaria and yellow fever. Acemoglu, Johnson and Robinson (2001) present a variety of evidence suggesting that the major effect of European settler mortality is through institutions.

In particular, Acemoglu, Johnson and Robinson (2001) argue that:

(1) There were different types of colonization policies which created different sets of institutions. At one extreme, European powers set up "extractive states", exemplified by the Belgian colonization of the Congo. These institutions did not introduce much protection for private property, nor did they provide checks and balances against government expropriation. At the other extreme, many Europeans migrated and settled in a number of colonies. The settlers in many areas tried to replicate European institutions, with strong emphasis on private property and checks against government power. Primary examples of this include Australia, New Zealand, Canada, and the United States.

(2) The colonization strategy was influenced by the feasibility of settlements. In places where the disease environment was not favorable to European settlement, extractive policies were more likely.

(3) The colonial state and institutions persisted to some degree and make it more likely that former European colonies that suffered extractive colonization have worse institutions today.

Summarizing schematically, the argument is:

(potential) settler mortality ⇒ settlements ⇒ early institutions ⇒ current institutions ⇒ current performance

Based on these three premises, Acemoglu, Johnson and Robinson (2001) use the mortality rates expected by the first European settlers in the colonies as an instrument for current institutions in the sample of former European colonies. Their instrumental-variables estimates show a large and robust effect of institutions on economic growth and income per capita. Figures 4.10 and 4.11 provide an overview of the evidence. Figure 4.10 shows the cross-sectional relationship between income per capita and the measure of economic institutions we encountered in Figure 4.1, protection against expropriation risk. It shows a very strong relationship between the historical mortality risk faced by Europeans and the current extent to which property rights are enforced. A bivariate regression has an R^2 of 0.26. It also shows that there were very large differences in European mortality. Countries such as Australia, New Zealand and the United States were very healthy, and existing evidence suggests that life expectancy in Australia and New Zealand was in fact greater than in Britain. In contrast, all Europeans faced extremely high mortality rates in Africa, India and South-East Asia. Differential mortality was largely due to tropical diseases such as malaria and yellow fever and at the time it was not understood how these diseases arose nor how they could be prevented or cured.

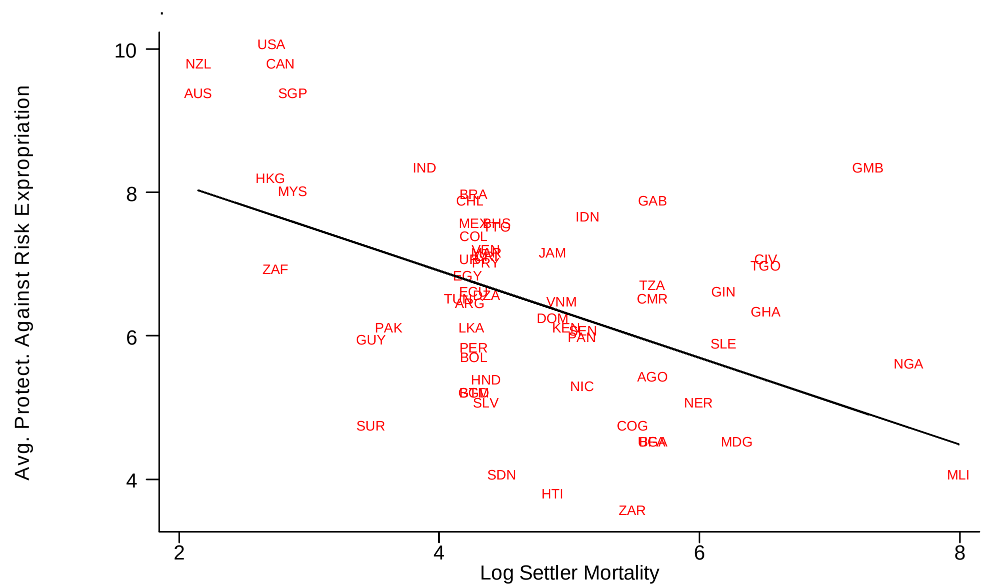

Figure 4.10. The relationship between mortality of potential European settlers and current economic institutions.

Figures 4.10 and 4.11 already show that, if we accept the exclusion restriction that the mortality rates of potential European settlers should have no effect on current economic outcomes other than through institutions, there is a large impact of economic institutions on economic performance. This is documented in detail in Acemoglu, Johnson and Robinson (2001), who present a range of robustness checks confirming this result. Their estimates suggest that most of the gap between rich and poor countries today is due to differences in economic institutions. For example, the evidence suggests that over 75% of the income gap between relatively rich and relatively poor countries can be explained by differences in their economic institutions (as proxied by security of property rights). Equally important, the evidence indicates that once the effect of institutions is estimated via this methodology, there appears to be no effect of geographical variables; neither latitude, nor whether or not a country is land-locked nor the current disease environment appear to have much effect on current economic outcomes. This evidence again suggests that institutional differences across

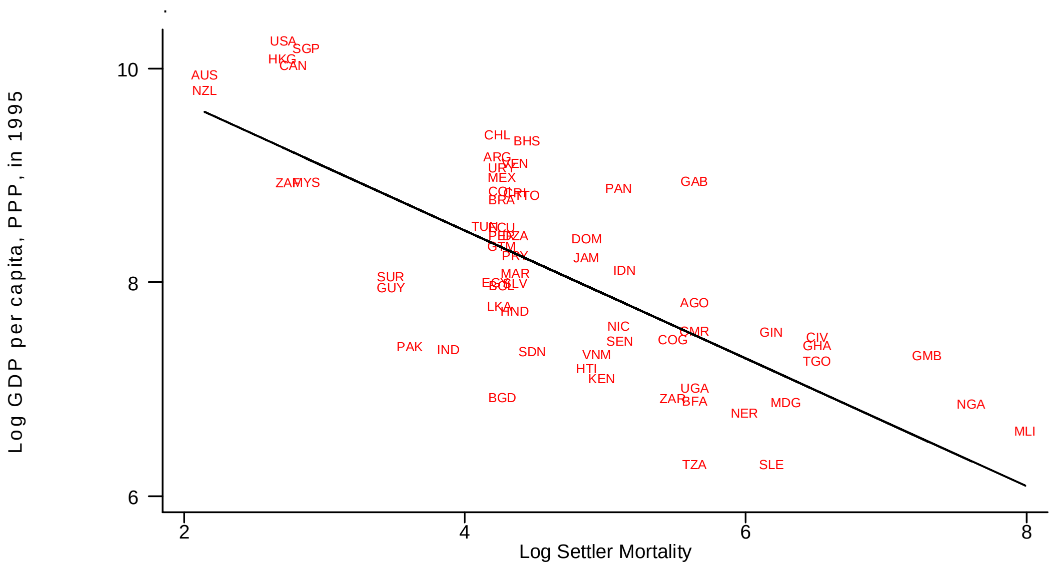

Figure 4.11. The relationship between mortality of potential European settlers and GDP per capita in 1995.

countries are a major determinant of their economic fortunes, while geographic differences are much less important.

These results also provide an interpretation for why Figure 4.2 showed a significant correlation between latitude and income per-capita. This is because of the correlation between latitude and the determinants of European colonization strategies. Europeans did not have immunity to tropical diseases during the colonial period and thus settler colonies tended, other things equal, to be created in temperate latitudes. Thus the historical creation of economic institutions was correlated with latitude. Without considering the role of economic institutions, one would find a spurious relationship between latitude and income per capita. However, once economic institutions are properly controlled for, these relationships go away and there appears to be no causal effect of geography on prosperity today (though geography may have been important historically in shaping economic institutions).

4.4.5. Culture, Colonial Identity and Economic Development. One might think that culture may have played an important role in the colonial experience, since Europeans not only brought new institutions, but also their own "cultures". European culture might have affected the economic development of former European colonies through three different channels. First, as already mentioned above, cultures may be systematically related to the national identity of the colonizing power. For example, the British may have implanted a "good" Anglo-Saxon culture into colonies such as Australia and the United States, while the Spanish may have condemned Latin America by endowing it with an Iberian culture. Second, Europeans may have had a culture, work ethic or set of beliefs that were conducive to prosperity. Finally, Europeans also brought different religions with different implications for prosperity. Many of these hypotheses have been suggested as explanations for why Latin America, with its Roman Catholic religion and Iberian culture, is poor relative to the Anglo-Saxon Protestant North America.

However, the econometric evidence in Acemoglu, Johnson and Robinson (2001) is not consistent with any of these views either. Similar to the evidence related to geographical variables, the econometric strategy discussed above suggests that, once the effect of economic institutions is taken into account, neither the identity of the colonial power, nor the contemporary fraction of Europeans in the population, nor the proportions of the populations of various religions appear to have a direct effect on economic growth and income per capita.

These econometric results are supported by historical examples. Although no Spanish colony has been as successful economically as British colonies such as the United States, many former British colonies, such as those in Africa, India and Bangladesh, are poor today. It is also clear that the British in no way simply re-created British institutions in their colonies. For example, by 1619 the North American colony of Virginia had a representative assembly with universal male suffrage, something that did not arrive in Britain itself until 1919. Another telling example is that of the Puritan colony in Providence Island in the Caribbean. While the Puritan values are often credited with the arrival of democracy and equality of opportunity in Northeastern United States, the Puritan colony in Providence Island quickly became just like any other Caribbean slave colony despite its Puritanical inheritance.

Similarly, even though the 17th century Dutch had perhaps the best domestic economic institutions in the world, their colonies in South-East Asia ended up with institutions designed for the extraction of resources, providing little economic or civil rights to the indigenous population. These colonies consequently experienced slow growth relative to other countries.

To emphasize that the culture or the religion of the colonizer was not at the root of the divergent economic performances of the colonies, Figure 4.12 shows the reversal among the British colonies (with population density in 1500 on the horizontal axis). Just as in Figure 4.6, there is a strong negative relationship between population density in 1500 and income per capita today.

With respect to the role of Europeans, Singapore and Hong Kong are now two of the richest countries in the world, despite having negligible numbers of Europeans. Moreover, Argentina and Uruguay have as high proportions of people of European descent as the United States and Canada, but are much less rich. To further document this, Figure 4.13 shows a pattern similar to the Reversal of Fortune, but now among countries where the fraction of

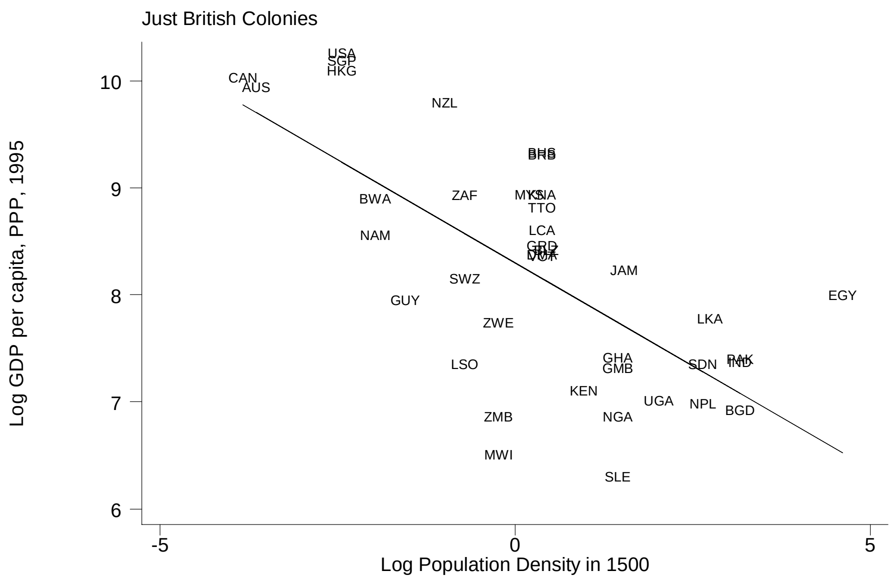

Figure 4.12. The Reversal of Fortune among British Colonies: population density in 1500 versus GDP per capita in 1995 among former British colonies.

those of European descent in 1975 is less than 5 percent of the population—thus a sample of countries in which European values or culture cannot have much direct effect today.

Overall, the evidence is not consistent with a major role of geography, religion or culture transmitted by the identity of the colonizer or the presence of Europeans. Instead, differences in economic institutions appear to be the robust causal factor underlying the differences in income per capita across countries. Institutions therefore appear to be the most important fundamental cause of income differences and long-run growth.

4.5. What Types of Institutions?

As already noted above the notion of institutions used in this chapter and in much of the literature is rather broad. It encompasses different types of social arrangements, laws, regulations, enforcement of property rights and so on. One may, perhaps rightly, complain that we are learning relatively little by emphasizing the importance of such a broad cluster of institutions. It is therefore important to try to understand what types of institutions are more important. This will not only be useful in our empirical analysis of fundamental causes, but can provide us a better sense of what types of models to develop in order to link fundamental causes to growth mechanics and to ultimate economic outcomes.

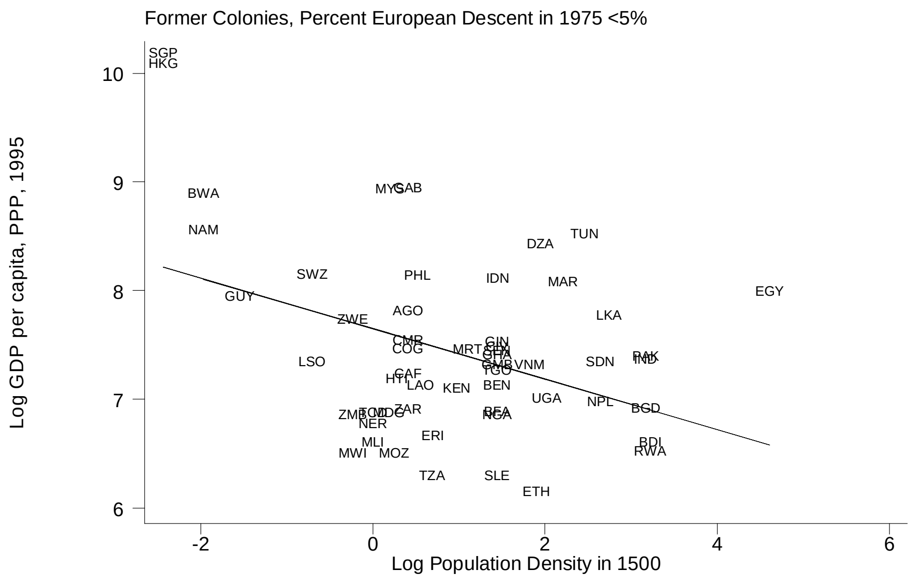

Figure 4.13. The Reversal of Fortune among former European colonies with two current European inhabitants.

There is relatively little work on "unbundling" the broad cluster of institutions in order to understand what specific types of institutions might be more important for economic outcomes. Much of this type of work remains to be done in the future. Here we can mention some existing work attempting to distinguish the impact of "contracting institutions" from the influence of "property rights institutions". One of the important roles of institutions is to facilitate contracting between lenders and borrowers or between different firms, to facilitate the functioning of markets and the allocation of resources. Such contracting is only possible if laws, courts and regulations uphold contracts in the appropriate way. Let us refer to institutional arrangements of this sort that support private contracts as *contracting institutions*. The other cluster of institutions emphasized above relates to those constraining government and elite expropriation. Let us refer to these as *property rights institutions*. Although in many situations contracting institutions and property right institutions will be intimately linked, they are nonetheless conceptually different. While contracting institutions regulate "horizontal" relationships in society between regular citizens, property rights institutions are about the protection of citizens against the power of elites, politicians and privileged groups. These two sets of institutions are potentially distinct and can thus have distinct effects.

Acemoglu and Johnson (2005) investigate the relative roles of these two sets of institutions. Their strategy is again to make use of the natural experiment of colonial history. What helps this particular unbundling exercise is that in the sample of former European colonies, the legal system imposed by colonial powers appears to have a strong effect on contracting institutions, but little impact on the available measures of property rights institutions. At the same time, both mortality rates for potential European settlers and population density in 1500, which we have seen above as important determinants of European colonization strategy, have a large effect on current property rights institutions, and no impact on contracting institutions. Using these different sources of variation in the sample of former European colonies, it is possible to estimate the separate effects of contracting institutions and property rights institutions.

Consistent with the pattern shown in Figure 4.12, which suggests that the identity of the colonizer is not a major determinant of future economic success of the colony, the empirical evidence estimating the different sources of variation in colonial history finds that property rights institutions are much more important for current economic outcomes than contracting institutions. Countries with greater constraints on politicians and elites and more protection against expropriation by these powerful groups appear to have substantially higher long-run growth rates and higher levels of current income. They also have significantly greater investment levels and generate more credit for the private sector. In contrast, the role of contracting institutions is more limited. Once the effects of property rights institutions are controlled for, contracting institutions seem to have no impact on income per capita, the investment to GDP ratio, and the private credit to GDP ratio. Contracting institutions appear to have some effect on stock market development, however.

These results suggest that contracting institutions affect the form of financial intermediation, but have less impact on economic growth and investment. It seems that economies can function in the face of weak contracting institutions without disastrous consequences, but not in the presence of a significant risk of expropriation from the government or other powerful groups. A possible interpretation is that private contracts or other reputation-based mechanisms can, at least in part, alleviate the problems originating from weak contracting institutions. For example, when it is more difficult for lenders to collect on their loans, interest rates increase, banks that can monitor effectively play a more important role, or reputation-based credit relationships may emerge. In contrast, property rights institutions relate to the relationship between the state and citizens. When there are no checks on the state, on politicians, and on elites, private citizens do not have the security of property rights necessary for investment.

Nevertheless, interpreting the evidence in Acemoglu and Johnson (2005) one should also bear in mind that the sources of variation in income per capita and investment rates identifying the different effects of contracting and property rights institutions relate to very large differences discussed in Chapter 1. It is possible that contracting institutions have relatively small affects, so that they are hard to detect when we look at countries with thirty-fold differences in income per capita. Therefore, this evidence should be interpreted as suggesting that contracting institutions are less important in generating the large differences in economic development than the property rights institutions, not necessarily as suggesting that contracting institutions do not matter for economic outcomes.

4.6. Disease and Development

The evidence presented above already militates against a major role of geographic factors in economic development. One version of the geography hypothesis deserves further analysis, however. A variety of evidence suggests that unhealthy individuals are less productive and often less successful in acquiring human capital. Could the differences in the disease environments across countries have an important effect on economic development? Could they be a major factor in explaining the very large income differences across countries? A recent paper by David Weil (2006), for example, argues that the framework used in the previous chapter, with physical capital, human capital and technology, should be augmented by including health capital. In other words, we may want to think over production function of the form $F(K, H, Z, A)$, where H denotes efficiency units of labor (human capital as conventionally measured), while Z is "health capital". Weil suggests a methodology for measuring the contribution of health capital to productivity from micro estimates and argues that differences in health capital emerge as an important factor in accounting for cross-country differences in income levels.

The idea that part of the low productivity of less-developed nations is due to the unhealthy state of their workforces has obvious appeal. The micro evidence and the work by David Weil shows that it has some empirical validity as well. But does it imply that geographic factors are an important fundamental cause of economic growth? Not necessarily. As already mentioned above, the burden of disease is endogenous. Today's unhealthy countries are unhealthy precisely because they are poor and are unable to invest in health care, clean water and other health-improving technologies. After all, much of Europe was very unhealthy and suffering from low life expectancy only 200 years ago. This changed *with* economic growth. In this sense, even if "health capital" is a useful concept and does contribute to accounting for cross-country income differences, it may itself be a proximate cause, affected by other factors, such as institutions or culture.

A recent paper by Acemoglu and Johnson (2006) directly investigates the impact of changes in disease burdens on economic development. They exploit the large improvements in life expectancy, particularly among the relatively poor nations, that took place starting in the 1940s. These health improvements were the direct consequence of significant international health interventions, more effective public health measures, and the introduction of

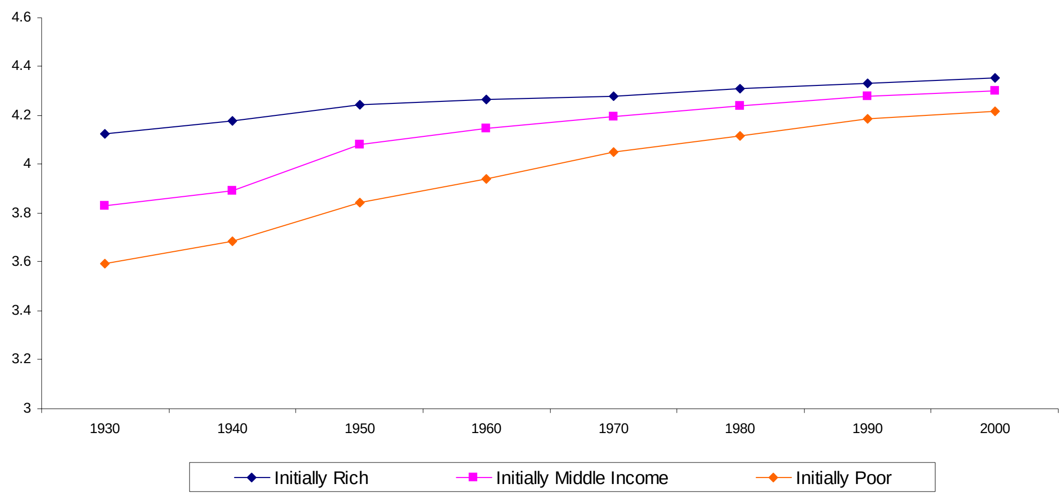

Figure 4.14. Evolution of life expectancy at birth among initially-poor, initially-middle-income and initially-rich countries, 1940-2000.

new chemicals and drugs. More important for the purposes of understanding the effect of disease on economic growth, these health improvements were by and large exogenous from the viewpoint of individual nations. Moreover, their impact on specific nations also varied, depending on whether the country in question was affected by the specific diseases for which the cures and the drugs became internationally available. The impact of these health improvements was major, in fact so major that it may deserve to be called the *international epidemiological transition*, since it led to an unprecedented improvement in life expectancy in a large number of countries. Figure 4.14 shows this unprecedented convergence in life expectancy by plotting life expectancy in countries that were initially (circa 1940) poor, middle income, and rich. It illustrates that while in the 1930s life expectancy was low in many poor and middle-income countries, this transition brought their levels of life expectancy close to those prevailing in richer parts of the world. As a consequence of these developments, health conditions in many parts of the less-developed world today, though still in dire need of improvement, are significantly better than the corresponding health conditions were in the West at the same stage of development.

The international epidemiological transition allows a promising empirical strategy to isolate potentially-exogenous changes in health conditions. The effects of the international epidemiological transition on a country's life expectancy were related to the extent to which its population was initially (circa 1940) affected by various specific diseases, for example, tuberculosis, malaria, and pneumonia, and to the timing of the various health interventions.

This reasoning suggests that potentially-exogenous variation in the health conditions of the country can be measured by calculating a measure of predicted mortality, driven by the interaction of baseline cross-country disease prevalence with global intervention dates for specific diseases. Acemoglu and Johnson (2006) show that such measures of predicted mortality have a large and robust effect on changes in life expectancy starting in 1940, but have *no* effect on changes in life expectancy *prior* to this date (i.e., before the key interventions). This suggests that the large increases in life expectancy experience by many countries after 1940 were in fact related to the global health interventions.

Perhaps not surprisingly, Acemoglu and Johnson (2006) find that predicted mortality and the changes in life expectancy that it causes have a fairly large effect on population; a 1% increase in life expectancy is related to an approximately 1.3-1.8% increase in population. However, there is no evidence of a positive effect on GDP per capita. This is depicted in Figure 4.15, shows no convergence in income per capita between initially-poor, initially-middle-income and initially-rich countries. Similarly, there appears to be no evidence of an increase in human capital investments associated with improvements in life expectancy.

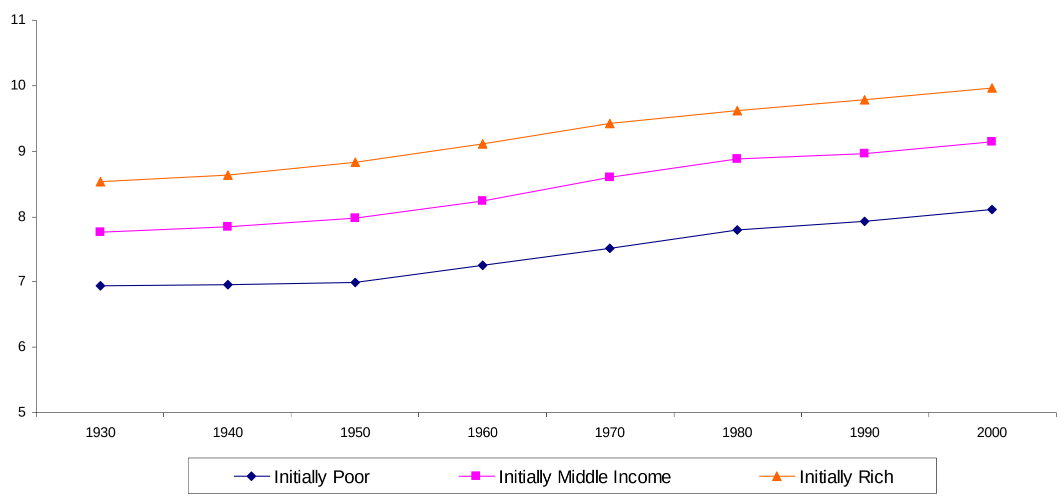

Figure 4.15. Evolution of GDP per capita among initially-poor, initially-middle-income and initially-rich countries, 1940-2000.

Why did the very significant increases in life expectancy and health not cause improvements in GDP per capita? The most natural answer to this question comes from neoclassical growth theory (presented in the previous two chapters and in Chapter 8 below). The first-order effect of increased life expectancy is to increase population, which initially reduces capital-to-labor and land-to-labor ratios, depressing income per capita. This initial decline is later compensated by higher output as more people enter the labor force. However, there is no reason to expect either a complete offset of the initial decline in income per capita or a large significant increase, especially when many of the effect countries are heavily vested in agriculture, so that land-to-labor ratios may change permanently. Consequently, small beneficial effects of health on productivity may not be sufficient to offset or reverse the negative effects of population pressures on income per capita.

4.7. Political Economy of Institutions: First Thoughts

The evidence presented in this chapter suggests that institutions are the most important fundamental cause of economic growth. We must therefore think about why institutions and policies differ across countries in order to understand why some countries are poor and some are rich. We will also argue below that understanding institutional changes holds clues about why the process of world economic growth started 200 years ago or so.

However, an explanation of differences in income across countries and over time in terms of institutional differences is also incomplete. If, as this chapter has documented, some institutions are conducive to rapid economic growth and others to stagnation, why would any society collectively choose institutions that condemn them to stagnation? The answer to this question relates to the nature of collective choices in societies. Institutions and policies, like other collective choices, are not taken for the good of the society at large, but are a result of a political equilibrium. In order to understand such political equilibria, we need to understand the conflicting interests of different individuals and groups in societies, and how they will be mediated by different political institutions. Thus, a proper understanding of how institutions affect economic outcomes and why institutions differ across countries (and why they sometimes change and pave the way for growth miracles) requires models of *political economy*, which explicitly studies how the conflicting interests of different individuals are aggregated into collective choices. Models of political economy also specify why certain individuals and groups may be opposed to economic growth or prefer institutions that eschew growth opportunities.

The discussion in this chapter therefore justifies why a study of political economy has to be part of any investigation of economic growth. Much of the study of economic growth has to be about the structure of models, so that we understand the mechanics of economic growth and the proximate causes of income differences. But part of this broad study must also confront the fundamental causes of economic growth, which relate to policies, institutions and other factors that lead to different investment, accumulation and innovation decisions.

4.8. Taking Stock

This chapter has emphasized the differences between the proximate causes of economic growth, related to physical capital accumulation, human capital and technology, and the fundamental causes, which influence the incentives to invest in these factors of production. We have argued that many of the questions motivating our study of economic growth must lead us to an investigation of the fundamental causes. But an understanding of fundamental causes is most useful when we can link them to the parameters of fully-worked-out model of economic growth to see how they affect the mechanics of growth and what types of predictions they generate.

When we turn to the institutions hypothesis, which we have argued in this chapter that the available evidence favors, the role of theory becomes even more important. As already pointed out above, the institutions view makes sense only when there are groups in society that favor institutions that do not necessarily enhance the growth potential of the economy. They will do so because they will not directly or indirectly benefit from the process of economic growth. Thus it is important to develop a good understanding of the distributional implications of economic growth (for example, how it affects relative prices and relative incomes, and how it may destroy the rents of incumbents). This theoretical understanding of the implications of the growth process than needs to be combined with political economy models of collective decision-making, to investigate under what circumstances groups opposed to economic growth can be powerful enough to maintain non-growth-enhancing institutions in place.

In this chapter, our objective has been more limited (since many of the more interesting growth models will be developed later in the book) and we have focused on the broad outlines of a number of alternative fundamental causes of economic growth and had a first look at the long-run empirical evidence relevance to these hypotheses. We argued that approaches emphasizing institutional differences (and differences in policies, laws and regulations) across societies are most promising for understanding both the current growth experiences of countries and the historical process of economic growth. We have also emphasized the importance of studying the political economy of institutions, as a way of understanding why institutions differ across societies and lead to divergent economic paths.

4.9. References and Literature

The early part of this chapter builds on Acemoglu, Johnson and Robinson (2006), who discuss the distinction between proximate and fundamental causes and the various different approaches to the fundamental causes of economic growth. North and Thomas (1973) appear to be the first to implicitly criticize growth theory for focusing on proximate causes alone and ignoring fundamental cause of economic growth. Diamond (1997) also draws a distinction between proximate and fundamental explanations.

The importance of population in generating economies of scale was first articulated by Julian Simon (1990). The model presented in Section 4.2 draws on Simon's work and work by Michael Kremer (1993). Kremer (1993) argues for the importance of economies of scale and increasing returns to population based on the acceleration in the growth rate of world population. Another important argument relating population to technological change is proposed by Esther Boserup (1965) and is based on the idea that increases in population creates scarcity, inducing societies to increase their productivity. Other models that build economies of scale to population and discuss the transition of the world economy from little or no growth to one of rapid economic growth include Hanson and Prescott (2001), Galor and Weil (2001), Galor and Moav (2002) and Jones (2004). Some of these papers also try to reconcile the role of population in generating technological progress with the later demographic transition. Galor (2006) provides an excellent summary of this literature and an extensive discussion. McEvedy and Jones (1978) provide a concise history of world population and relatively reliable information going back to 10,000 B.C. Their data indicate that, as claimed in the text, total population in Asia has been consistently greater than in Western Europe over this time period.

The geography hypothesis has many proponents. In addition to Montesquieu, Machiavelli was an early proponent of the importance of climate and geographic characteristics. Marshall (1890), Kamarck (1976), and Myrdal (1986) are among the economists who have most clearly articulated various different versions of the geography hypothesis. It has more recently been popularized by Sachs (2000, 2001), Bloom and Sachs (1998) and Gallup and Sachs (2001). Diamond (1997) offers a more sophisticated version of the geography hypothesis, where the availability of different types of crops and animals, as well as the axes of communication of continents, influence the timing of settled agriculture and thus the possibility of developing complex societies. Diamond's thesis is therefore based on geographic differences, but also relies on such institutional factors as intervening variables.

Scholars emphasizing the importance of various types of institutions in economic development include John Locke, Adam Smith, John Stuart Mill, Arthur Lewis, Douglass North and Robert Thomas. The recent economics literature includes many models highlighting the importance of property rights, for example, Skaperdas (1992), Tornell and Velasco (1992), Acemoglu (1995), Grossman and Kim (1995, 1996), Hirsleifer (2001) and Dixit (2004). Other models emphasize the importance of policies within a given institutional framework. Well-known examples of this approach include Perotti (1993), Saint-Paul and Verdier (1993), Alesina and Rodrik (1994), Persson and Tabellini (1994), Ades and Verdier (1996), Krusell and Rios-Rull (1999), and Bourguignon and Verdier (2000). There is a much smaller literature on endogenous institutions and the effect of these institutions on economic outcomes. Surveys of this work can be found in Acemoglu (2007) and Acemoglu and Robinson (2006). The literature on the effect of economic institutions on economic growth is summarized and discussed in greater detail in Acemoglu, Johnson and Robinson (2006), which also provides an overview of the empirical literature on the topic. We will return to many of these issues and Part 8 of the book.

The importance of religion for economic development is most forcefully argued in Max Weber's work, for example (1930, 1958). Many other scholars since then have picked up on this idea and have argued about the importance of religion. Prominent examples include Huntington (2001) and Landes (2001). Landes, for example, tries to explain the rise of the West based on cultural and religious variables. This evidence is criticized in Acemoglu, Johnson and Robinson (2005). Barro and McCleary (2003) provide evidence of a positive correlation between the prevalence of religious beliefs and economic growth. One has to be careful in interpreting this evidence as showing a causal effect of religion on economic growth, since religious beliefs are endogenous both to economic outcomes and to other fundamental causes of income differences.

The emphasis on the importance of cultural factors or "social capital" goes back to Banfield (1958), and is popularized by Putnam (1993). The essence of these interpretations appears to be related to the role of culture or social capital in ensuring the selection of better equilibrium. Similar ideas are also advanced in Greif (2006). Many scholars, including Véliz (1994), North, Summerhill and Weingast (2000), and Wiarda (2001), emphasize the importance of cultural factors in explaining the economic backwardness of Latin American countries. Knack and Keefer (1997) and Durlauf and Fafchamps (2003) document positive correlations between measures of social capital and various economic outcomes. None of this work establishes a causal effect of social capital because of the potential endogeneity of social capital and culture. A number of recent papers attempt to overcome these difficulties. Notable contributions here include Guiso, Sapienza and Zingales (2004) and Tabellini (2006).

The discussion of the Puritan colony in the Providence Island is based on Newton (1914) and Kupperman (1993).

The literature on the effect of economic institutions and policies on economic growth is vast. Most growth regressions include some controls for institutions or policies and find them to be significant (see, for example, those reported in Barro and Sala-i-Martin, 2004). One of the first papers looking at the cross-country correlation between property rights measures and economic growth is Knack and Keefer (1995). This literature does not establish causal effect either, since simultaneity and endogeneity concerns are not dealt with. Mauro (1998) and Hall and Jones (1999) present the first instrumental-variable estimates on the effect of institutions (or corruption) on long-run economic development.

The evidence reported here, which exploits differences in colonial experience to create an instrumental-variables strategy, is based on Acemoglu, Johnson and Robinson (2001, 2002). The urbanization and population density data used here are from Acemoglu, Johnson and Robinson (2002), which compiled these based on work by Bairoch (1988), Bairoch, Batou and Chèvre (1988), Chandler (1987), Eggimann (1999), McEvedy and Jones (1978). Further details and econometric results are presented in Acemoglu, Johnson and Robinson (2002). The data on mortality rates of potential settlers is from Acemoglu, Johnson and Robinson (2001), who compiled the data based on work by Curtin (1989, 1998) and Gutierrez (1986). That paper also provides a large number of robustness checks, documenting the influence of economic institutions on economic growth and showing that other factors, including religion and geography, have little effect on long-run economic development once the effect of institutions is controlled for.

The details of the Korean experiment and historical references are provided in Acemoglu (2003) and Acemoglu, Johnson and Robinson (2006).

The discussion of distinguishing the effects of different types of institutions draws on Acemoglu and Johnson (2005).

The discussion of the effect of disease on development is based on Weil (2006) and especially on Acemoglu and Johnson (2006), which used the econometric strategy described in the text. Figures 4.14 and 4.15 are from Acemoglu and Johnson (2006). In these figures, initially-poor countries are those that are poorer than Spain in 1940, and include China, Bangladesh, India, Pakistan, Myanmar, Thailand, El Salvador, Honduras, Indonesia, Brazil, Sri Lanka, Malaysia, Nicaragua, Korea, Ecuador, and the Philippines, while initially-rich countries are those that are richer than Argentina in 1940 and include Belgium, Netherlands, Sweden, Denmark, Canada, Germany, Australia, New Zealand, Switzerland, the United Kingdom and the United States. Young (2004) investigates the effect of the HIV epidemic in South Africa and reaches a conclusion similar to that reported here, though his analysis relies on a calibration of the neoclassical growth model rather than econometric estimation.

4.10. Exercises

Exercise 4.1. Derive equations (4.3) and (4.4).

Exercise 4.2. Derive equation (4.7). Explain how the behavior implied for technology by this equation differs from (4.4). Why is this? Do you find the assumptions leading to (4.4) or to (4.7) more plausible?

Exercise 4.3.

(1) Show that the models leading to both (4.4) or to (4.7) imply a constant income per capita throughout.

(2) Modify equation (4.2) to

$$L(t) = \phi Y(t)^{\beta},$$

for some $\beta \in (0,1)$. Justify this equation and derive the law of motion technology and income per capita under the two scenarios considered in the text. Are the implications of this model more reasonable than those considered in the text?

Exercise 4.4. In his paper "Tropical Underdevelopment", Jeff Sachs notes that differences in income per capita between tropical and temperate zones have widened over the past 150 years. He interprets this pattern as evidence indicating that the "geographical burden" of the tropical areas has been getting worse over the process of recent development. Discuss this thesis. If you wish, offer alternative explanations. How would you go about testing different approaches? (If possible, suggest original ways, rather than approaches that were already tried).

Part 2

Towards Neoclassical Growth

This part of the book is a preparation for what is going to come next. In some sense, it can be viewed as the "preliminaries" for the rest of the book. Our ultimate purpose is to enrich the basic Solow model by introducing well-defined consumer preferences and consumer optimization, and in the process, clarify the relationship between growth theory and general equilibrium theory. This will enable us to open the blackbox of savings and capital accumulation, turning these decisions into forward-looking investment decisions. It will also enable us to make welfare statements about whether the rate of growth of an economy is too slow, too fast or just right from a welfare-maximizing (Pareto optimality) viewpoint. This will then open the way for us to study technology as another forward-looking investment by firms, researchers and individuals. However, much of this will have to wait for Parts 3 and 4 of the book, where we will study these models in detail. In the next three chapters, we will instead do the work necessary to appreciate what is to come then. The next chapter will set up the problem and make the relationship between models of economic growth and general equilibrium theory more explicit. It will also highlight some of the assumptions implicit in the growth models. The two subsequent chapters develop the mathematical tools for dynamic optimization in discrete and continuous time. To avoid making these chapters purely about mathematics, we will use a variety of economic models of some relevance to growth theory as examples and also include the analysis of the equilibrium and optimal growth.

CHAPTER 5

Foundations of Neoclassical Growth

The Solow growth model is predicated on a constant saving rate. Instead, it would be much more satisfactory to specify the *preference orderings* of individuals, as in standard general equilibrium theory, and derive their decisions from these preferences. This will enable us both to have a better understanding of the factors that affect savings decisions and also to discuss the "optimality" of equilibria—in other words, to pose and answer questions related to whether the (competitive) equilibria of growth models can be "improved upon". The notion of improvement here will be based on the standard concept of Pareto optimality, which asks whether some households can be made better-off without others being made worse-off. Naturally, we can only talk of individuals or households being "better-off" if we have some information about well-defined preference orderings.

5.1. Preliminaries

To prepare for this analysis, let us consider an economy consisting of a unit measure of infinitely-lived households. By a unit measure of households we mean an uncountable number of households, for example, the set of households $\mathcal{H}$ could be represented by the unit interval $[0,1]$. This is an abstraction adopted for simplicity, to emphasize that each household is infinitesimal and will have no effect on aggregates. Nothing we do in this book hinges on this assumption. If the reader instead finds it more convenient to think of the set of households, $\mathcal{H}$, as a countable set of the form $\mathcal{H} = \{1, 2, ..., M\}$ with $M = \infty$, this can be done without any loss of generality. The advantage of having a unit measure of households is that averages and aggregates are the same, enabling us to economize on notation. It would be even simpler to have $\mathcal{H}$ as a finite set in the form $\{1, 2, ..., M\}$ with M large but finite. For many models, this would also be acceptable, but as we will see below, models with overlapping generations require the set of households to be infinite.

We can either assume that households are truly "infinitely lived" or that they consist of overlapping generations with full (or partial) altruism linking generations within the household. Throughout, we equate households with individuals, and thus ignore all possible sources of conflict or different preferences within the household. In other words, we assume that households have well-defined preference orderings.

As in basic general equilibrium theory, we make enough assumptions on preference orderings (in particular, reflexivity, completeness and transitivity) so that these preference orderings can be represented by utility functions. In particular, suppose that each household i has an *instantaneous utility function* given by

$$u_i\left(c_i\left(t\right)\right),$$

where $u_i : \mathbb{R}_+ \rightarrow \mathbb{R}$ is increasing and concave and $c_i\left(t\right)$ is the consumption of household i. Here and throughout, we take the domain of the utility function to be $\mathbb{R}_+$ rather than $\mathbb{R}$, so that negative levels of consumption are not allowed. Even though some well-known economic models allow negative consumption, this is not easy to interpret in general equilibrium or in growth theory, thus this restriction is sensible.

The instantaneous utility function captures the utility that an individual derives from consumption at time t. It is therefore *not* the same as a utility function specifying a complete preference ordering over all commodities—here consumption levels in all dates. For this reason, the instantaneous utility function is sometimes also referred to as the "felicity function".

There are two major assumptions in writing an instantaneous utility function. First, it imposes that the household does not derive any utility from the consumption of other households, so consumption externalities are ruled out. Second, in writing the instantaneous utility function, we have already imposed that overall utility is *time separable*, that is, instantaneous utility at time t is independent of the consumption levels at past or future dates. This second feature is important in enabling us to develop tractable models of dynamic optimization.

Finally, let us introduce a third assumption and suppose that households discount the future "exponentially"—or "proportionally". In discrete time, and ignoring uncertainty, this implies that household preferences at time $t = 0$ can be represented as

$$\sum_{t=0}^{\infty} \beta_i^t u_i\left(c_i\left(t\right)\right), \tag{5.1}$$

where $\beta_i \in (0,1)$ is the discount factor of household i. This functional form implies that the weight given to tomorrow's utility is a fraction β_i of today's utility, and the weight given to the utility the day after tomorrow is a fraction β_i^2 of today's utility, and so on. Exponential discounting and time separability are convenient for us because they naturally ensure "time-consistent" behavior.

We call a solution $\{x\left(t\right)\}_{t=0}^{T}$ (possibly with $T = \infty$) to a dynamic optimization problem *time-consistent* if the following is true: whenever $\{x\left(t\right)\}_{t=0}^{T}$ is an optimal solution starting at time $t = 0$, $\{x\left(t\right)\}_{t=t'}^{T}$ is an optimal solution to the continuation dynamic optimization problem starting from time $t = t' \in [0, T]$. If a problem is not time-consistent, we refer to it as *time-inconsistent*. Time-consistent problems are much more straightforward to work with and satisfy all of the standard axioms of rational decision-making. Although time-inconsistent preferences may be useful in the modeling of certain behaviors we observe in practice, such as problems of addiction or self-control, time-consistent preferences are ideal for the focus in this book, since they are tractable, relatively flexible and provide a good approximation to reality in the context of aggregative models. It is also worth noting that many classes of preferences that do not feature exponential and time separable discounting nonetheless lead to time-consistent behavior. Exercise 5.1 discusses issues of time consistency further and shows how certain other types of utility formulations lead to time-inconsistent behavior, while Exercise 5.2 introduces some common non-time-separable preferences that lead to time-consistent behavior.

There is a natural analog to (5.1) in continuous time, again incorporating exponential discounting, which is introduced and discussed below (see Section 5.10 and Chapter 7).

The expression in (5.1) ignores uncertainty in the sense that it assumes the sequence of consumption levels for individual i, $\{c_i(t)\}_{t=0}^{\infty}$ is known with certainty. If instead this sequence were uncertain, we would need to look at expected utilitymaximization. Most growth models do not necessitate an analysis of growth under uncertainty, but a stochastic version of the neoclassical growth model is the workhorse of much of the rest of modern macroeconomics and will be presented in Chapter 17. For now, it suffices to say that in the presence of uncertainty, we interpret $u_i(\cdot)$ as a Bernoulli utility function, so that the preferences of household i at time $t = 0$ can be represented by the following von Neumann-Morgenstern expected utility function:

$$\mathbb{E}_0^i \sum_{t=0}^{\infty} \beta_i^t u_i\left(c_i(t)\right),$$

where $\mathbb{E}_0^i$ is the expectation operator with respect to the information set available to household i at time $t = 0$.

The formulation so far indexes individual utility function, $u_i(\cdot)$, and the discount factor, β_i, by "i" to emphasize that these preference parameters are potentially different across households. Households could also differ according to their income processes. For example, each household could have effective labor endowments of $\{e_i(t)\}_{t=0}^{\infty}$, thus a sequence of labor income of $\{e_i(t)\,w(t)\}_{t=0}^{\infty}$, where $w(t)$ is the equilibrium wage rate per unit of effective labor.

Unfortunately, at this level of generality, this problem is not tractable. Even though we can establish some existence of equilibrium results, it would be impossible to go beyond that. Proving the existence of equilibrium in this class of models is of some interest, but our focus is on developing workable models of economic growth that generate insights about the process of growth over time and cross-country income differences. We will therefore follow the standard approach in macroeconomics and assume the existence of a *representative household*.

5.2. The Representative Household

When we say that an economy *admits a representative household*, this means that the preference (demand) side of the economy can be represented *as if* there were a single household making the aggregate consumption and saving decisions (and also the labor supply decisions when these are endogenized) subject to a single budget constraint. The major convenience of the representative household assumption is that instead of thinking of the preference side of the economy resulting from equilibrium interactions of many heterogeneous households, we will be able to model it as a solution to a single maximization problem. Note that, for now, the description concerning a representative household is purely positive—it asks the question of whether the aggregate behavior can be represented as if it were generated by a single household. We can also explore the stronger notion of whether and when an economy admits a "normative" representative household. If this is the case, not only aggregate behavior can be represented as if it were generated by a single household, but we can also use the utility function of the normative representative household for welfare comparisons. We return to a further discussion of these issues below.

Let us start with the simplest case that will lead to the existence of a representative household. Suppose that each household is identical, i.e., it has the same discount factor β, the same sequence of effective labor endowments $\{e(t)\}_{t=0}^{\infty}$ and the same instantaneous utility function

$$u(c_i(t))$$

where $u : \mathbb{R}_+ \to \mathbb{R}$ is increasing and concave and $c_i(t)$ is the consumption of household i. Therefore, there is really a representative household in this case. Consequently, again ignoring uncertainty, the preference side of the economy can be represented as the solution to the following maximization problem starting at time $t = 0$:

$$\max \sum_{t=0}^{\infty} \beta^t u(c(t)), \tag{5.2}$$

where $\beta \in (0, 1)$ is the common discount factor of all the households, and $c(t)$ is the consumption level of the representative household.

The economy described so far admits a representative household rather trivially; all households are identical. In this case, the representative household's preferences, (5.2), can be used not only for positive analysis (for example, to determine what the level of savings will be), but also for normative analysis, such as evaluating the optimality of different types of equilibria.

Often, we may not want to assume that the economy is indeed inhabited by a set of identical households, but instead assume that the behavior of the households can be modeled *as if* it were generated by the optimization decision of a representative household. Naturally, this would be more realistic than assuming that all households are identical. Nevertheless, this is not without any costs. First, in this case, the representative household will have positive meaning, but not always a normative meaning (see below). Second, it is not in fact true that most models with heterogeneity lead to a behavior that can be represented as if it were generated by a representative household.

In fact most models do not admit a representative household. To illustrate this, let us consider a simple exchange economy with a finite number of commodities and state an important theorem from general equilibrium theory. In preparation for this theorem, recall that in an exchange economy, we can think of the object of interest as the excess demand functions (or correspondences) for different commodities. Let these be denoted by $\mathbf{x}(p)$ when the vector of prices is p. An economy will admit a representative household if these excess demands, $\mathbf{x}(p)$, can be modeled as if they result from the maximization problem of a single consumer.

Theorem 5.1. ***(Debreu-Mantel-Sonnenschein Theorem)*** *Let $\varepsilon > 0$ be a scalar and $N < \infty$ be a positive integer. Consider a set of prices $\mathbf{P}_\varepsilon = \left\{p \in \mathbb{R}^N_+ : p_j/p_{j'} \geq \varepsilon \text{ for all } j \text{ and } j'\right\}$ and any continuous function $\mathbf{x} : \mathbf{P}_\varepsilon \to \mathbb{R}^N_+$ that satisfies Walras' Law and is homogeneous of degree 0. Then there exists an exchange economy with N commodities and $H < \infty$ households, where the aggregate demand is given by $\mathbf{x}(p)$ over the set $\mathbf{P}_\varepsilon$.*

Proof. See Debreu (1974) or Mas-Colell, Winston and Green (1995), Proposition 17.E.3. □

This theorem states the following result: the fact that excess demands come from the optimizing behavior of households puts no restrictions on the form of these demands. In particular, $\mathbf{x}(p)$ does not necessarily possess a negative-semi-definite Jacobian or satisfy the weak axiom of revealed preference (which are requirements of demands generated by individual households). This implies that, without imposing further structure, it is impossible to derive the aggregate excess demand, $\mathbf{x}(p)$, from the maximization behavior of a single household. This theorem therefore raises a severe warning against the use of the representative household assumption.

Nevertheless, this result is partly an outcome of very strong income effects. Special but approximately realistic preference functions, as well as restrictions on the distribution of income across individuals, enable us to rule out arbitrary aggregate excess demand functions. To show that the representative household assumption is not as hopeless as Theorem 5.1 suggests, we will now show a special and relevant case in which aggregation of individual preferences is possible and enables the modeling of the economy as if the demand side was generated by a representative household.

To prepare for this theorem, consider an economy with a finite number N of commodities and recall that an indirect utility function for household i, $v_i\left(p, y^i\right)$, specifies the household's (ordinal) utility as a function of the price vector $p = (p_1, ..., p_N)$ and the household's income y^i. Naturally, any indirect utility function $v_i\left(p, y^i\right)$ has to be homogeneous of degree 0 in p and y.

Theorem 5.2. ***(Gorman's Aggregation Theorem)*** *Consider an economy with a finite number $N < \infty$ of commodities and a set $\mathcal{H}$ of households. Suppose that the preferences of household $i \in \mathcal{H}$ can be represented by an indirect utility function of the form*

$$v^i\left(p, y^i\right) = a^i\left(p\right) + b\left(p\right) y^i, \tag{5.3}$$

then these preferences can be aggregated and represented by those of a representative household, with indirect utility

$$v\left(p, y\right) = \int_{i \in \mathcal{H}} a^i\left(p\right) di + b\left(p\right) y,$$

where $y \equiv \int_{i \in \mathcal{H}} y^i di$ is aggregate income.

Proof. See Exercise 5.3. □

This theorem implies that when preferences admit this special quasi-linear form, we can represent aggregate behavior as if it resulted from the maximization of a single household. This class of preferences are referred to as Gorman preferences after Terrence Gorman, who was among the first economists studying issues of aggregation and proposed the special class of preferences used in Theorem 5.2. The quasi-linear structure of these preferences limits the extent of income effects and enables the aggregation of individual behavior. Notice that instead of the summation, this theorem used the integral over the set $\mathcal{H}$ to allow for the possibility that the set of households may be a continuum. The integral should be thought of as the "Lebesgue integral," so that when $\mathcal{H}$ is a finite or countable set, $\int_{i \in \mathcal{H}} y^i di$ is indeed equivalent to the summation $\sum_{i \in \mathcal{H}} y^i$. Note also that this theorem is stated for an economy with a finite number of commodities. This is only for simplicity, and the same result can be generalized to an economy with an infinite or even a continuum of commodities. However, for most of this chapter, we restrict attention to economies with either a finite or a countable number of commodities to simplify notation and avoid technical details.

Note also that for preferences to be represented by an indirect utility function of the Gorman form does not necessarily mean that this utility function will give exactly the indirect utility in (5.3). Since in basic consumer theory a monotonic transformation of the utility function has no effect on behavior (but affects the indirect utility function), all we require is that there exists a monotonic transformation of the indirect utility function that takes the form given in (5.3).

Another attractive feature of Gorman preferences for our purposes is that they contain some commonly-used preferences in macroeconomics. To illustrate this, let us start with the following example:

EXAMPLE 5.1. **(Constant Elasticity of Substitution Preferences)** A very common class of preferences used in industrial organization and macroeconomics are the constant elasticity of substitution (CES) preferences, also referred to as Dixit-Stiglitz preferences after the two economists who first used these preferences. Suppose that each household denoted by $i \in \mathcal{H}$ has total income y^i and preferences defined over $j = 1, ..., N$ goods given by

$$U^i\left(x_1^i, ..., x_N^i\right) = \left[\sum_{j=1}^{N}\left(x_j^i - \xi_j^i\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}, \tag{5.4}$$

where $\sigma \in (0,\infty)$ and $\xi_j^i \in [-\bar{\xi}, \bar{\xi}]$ is a household specific term, which parameterizes whether the particular good is a necessity for the household. For example, $\xi_j^i > 0$ may mean that household i needs to consume a certain amount of good j to survive. The utility function (5.4) is referred to as constant elasticity of substitution (CES), since if we define the level of consumption of each good as $\hat{x}_j^i = x_j^i - \xi_j^i$, the elasticity of substitution between any two $\hat{x}_j^i$ and $\hat{x}_{j'}^i$ would be equal to σ.

Each consumer faces a vector of prices $p = (p_1, ..., p_N)$, and we assume that for all i,

$$\sum_{j=1}^{N} p_j \bar{\xi} < y^i,$$

so that the household can afford a bundle such that $\hat{x}_j^i \geq 0$ for all j. In Exercise 5.6, you will be asked to derive the optimal consumption levels for each household and show that their indirect utility function is given by

$$v^i\left(p, y^i\right) = \frac{\left[-\sum_{j=1}^{N} p_j \xi_j^i + y^i\right]}{\left[\sum_{j=1}^{N} p_j^{1-\sigma}\right]^{\frac{1}{1-\sigma}}}, \tag{5.5}$$

which satisfies the Gorman form (and is also homogeneous of degree 0 in p and y). Therefore, this economy admits a representative household with indirect utility:

$$v(p, y) = \frac{\left[-\sum_{j=1}^{N} p_j \xi_j + y\right]}{\left[\sum_{j=1}^{N} p_j^{1-\sigma}\right]^{\frac{1}{1-\sigma}}}$$

where y is aggregate income given by $y \equiv \int_{i\in\mathcal{H}} y^i di$ and $\xi_j \equiv \int_{i\in\mathcal{H}} \xi_j^i di$. It is also straightforward to verify that the utility function leading to this indirect utility function is

$$U(x_1, ..., x_N) = \left[\sum_{j=1}^{N}\left(x_j - \xi_j\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}. \tag{5.6}$$

We will see below that preferences closely related to the CES preferences will play a special role not only in aggregation but also in ensuring *balanced growth* in neoclassical growth models.

It is also possible to prove the converse to Theorem 5.2. Since this is not central to our focus, we state this result in the text rather than stating and proving it formally. The essence of this converse is that unless we put some restrictions on the distribution of income across households, Gorman preferences are not only sufficient for the economy to admit a representative household, but they are also *necessary.* In other words, if the indirect utility functions of some households do not take the Gorman form, there will exist some distribution of income such that aggregate behavior cannot be represented as if it resulted from the maximization problem of a single representative household.

In addition to the aggregation result in Theorem 5.2, Gorman preferences also imply the existence of a normative representative household. Recall that an allocation is *Pareto optimal* if no household can be made strictly better-off without some other household being made worse-off (see Definition 5.2 below). We then have:

Theorem 5.3. ***(Existence of a Normative Representative Household)*** *Consider an economy with a finite number $N < \infty$ of commodities and a set $\mathcal{H}$ of households. Suppose that the preferences of each household $i \in \mathcal{H}$ take the Gorman form, $v^i\left(p, y^i\right) = a^i(p) + b(p)\, y^i$.*

(1) *Then any allocation that maximizes the utility of the representative household, $v(p, y) = \sum_{i \in \mathcal{H}} a^i(p) + b(p)\, y$, with $y \equiv \sum_{i \in \mathcal{H}} y^i$, is Pareto optimal.*
(2) *Moreover, if $a^i(p) = a^i$ for all p and all $i \in \mathcal{H}$, then any Pareto optimal allocation maximizes the utility of the representative household.*

Proof. We will prove this result for an exchange economy. Suppose that the economy has a total endowment vector of $\omega = (\omega_1, ..., \omega_N)$. Then we can represent a Pareto optimal allocation as:

$$\max_{\{p_j\}_{j=1}^N, \{y^i\}_{i \in \mathcal{H}}} \sum_{i \in \mathcal{H}} \alpha^i v^i\left(p, y^i\right) = \sum_{i \in \mathcal{H}} \alpha^i \left(a^i(p) + b(p)\, y^i\right)$$

subject to

$$\begin{aligned} -\left(\sum_{i \in \mathcal{H}} \frac{\partial a^i(p)}{\partial p_j} + \frac{\partial b(p)}{\partial p_j} y\right) &= b(p)\, \omega_j \text{ for } j = 1, ..., N \\ \sum_{j=1}^N p_j \omega_j &= y, \\ p_j &\geq 0 \text{ for all } j, \end{aligned}$$

where $\left\{\alpha^i\right\}_{i \in \mathcal{H}}$ are nonnegative Pareto weights with $\sum_{i \in \mathcal{H}} \alpha^i = 1$. The first set of constraints use Roy's identity to express the total demand for good j and set it equal to the supply of good j, which is the endowment ω_j. The second equation makes sure that total income in the economy is equal to the value of the endowments. The third set of constraints requires that all prices are nonnegative.

Now compare the above maximization problem to the following problem:

$$\max \sum_{i \in \mathcal{H}} a^i (p) + b (p) y$$

subject to the same set of constraints. The only difference between the two problems is that in the latter each household has been assigned the same weight.

Let (p^*, y^*) be a solution to the second problem. By definition it is also a solution to the first problem with $\alpha^i = \alpha$, and therefore it is Pareto optimal, which establishes the first part of the theorem.

To establish the second part, suppose that $a^i (p) = a^i$ for all p and all $i \in \mathcal{H}$. To obtain a contradiction, let $\mathbf{y} \in \mathbb{R}^{|\mathcal{H}|}$ and suppose that $(p^{**}_\alpha, \mathbf{y}^{**}_\alpha)$ is a solution to the first problem for some weights $\{\alpha^i\}_{i \in \mathcal{H}}$ and suppose that it is not a solution to the second problem. Let

$$\alpha^M = \max_{i \in \mathcal{H}} \alpha^i$$

and

$$\mathcal{H}^M = \{ i \in \mathcal{H} \ | \alpha^i = \alpha^M \}$$

be the set of households given the maximum Pareto weight. Let (p^*, y^*) be a solution to the second problem such that

$$y^i = 0 \text{ for all } i \notin \mathcal{H}. \tag{5.7}$$

Note that such a solution exists since the objective function and the constraint set in the second problem depend only on the vector $(y^1, .., y^{|\mathcal{H}|})$ through $y = \sum_{i \in \mathcal{H}} y^i$.

Since, by definition, $(p^{**}_\alpha, \mathbf{y}^{**}_\alpha)$ is in the constraint set of the second problem and is not a solution, we have

$$\begin{aligned} \sum_{i \in \mathcal{H}} a^i + b(p^*) y^* &> \sum_{i \in \mathcal{H}} a^i + b(p^{**}_\alpha) y^{**}_\alpha \\ b(p^*) y^* &> b(p^{**}_\alpha) y^{**}_\alpha. \end{aligned}$$

The hypothesis that it is a solution to the first problem also implies that

$$\begin{aligned} \sum_{i \in \mathcal{H}} \alpha^i a^i + \sum_{i \in \mathcal{H}} \alpha^i b(p^{**}_\alpha) (y^{**}_\alpha)^i &\geq \sum_{i \in \mathcal{H}} \alpha^i a^i + \sum_{i \in \mathcal{H}} \alpha^i b(p^*) (y^*)^i \\ \sum_{i \in \mathcal{H}} \alpha^i b(p^{**}_\alpha) (y^{**}_\alpha)^i &\geq \sum_{i \in \mathcal{H}} \alpha^i b(p^*) (y^*)^i . \end{aligned} \tag{5.8}$$

Then, it can be seen that the solution (p^{**}, y^{**}) to the Pareto optimal allocation problem satisfies $y^i = 0$ for any $i \notin \mathcal{H}^M$. In view of this and the choice of (p^*, y^*) in (5.7), equation

(5.8) implies

$$\begin{aligned}\alpha^M b\left(p_\alpha^{**}\right)\sum_{i\in\mathcal{H}}\left(y_\alpha^{**}\right)^i &\geq \alpha^M b\left(p^*\right)\sum_{i\in\mathcal{H}}\left(y^*\right)^i\\ b\left(p_\alpha^{**}\right)\left(y_\alpha^{**}\right) &\geq b\left(p^*\right)\left(y^*\right),\end{aligned}$$

which contradicts equation (5.8), and establishes that, under the stated assumptions, any Pareto optimal allocation maximizes the utility of the representative household. □

5.3. Infinite Planning Horizon

Another important microfoundation for the standard preferences used in growth theory and macroeconomics concerns the planning horizon of individuals. While, as we will see in Chapter 9, some growth models are formulated with finitely-lived individuals, most growth and macro models assume that individuals have an infinite-planning horizon as in equation (5.2) or equation (5.22) below. A natural question to ask is whether this is a good approximation to reality. After all, most individuals we know are not infinitely-lived.

There are two reasonable microfoundations for this assumption. The first comes from the "Poisson death model" or the *perpetual youth model*, which will be discussed in greater detail in Chapter 9. The general justification for this approach is that, while individuals are finitely-lived, they are not aware of when they will die. Even somebody who is 95 years old will recognize that he cannot consume all his assets, since there is a fair chance that he will live for another 5 or 10 years. At the simplest level, we can consider a discrete-time model and assume that each individual faces a constant probability of death equal to ν. This is a strong simplifying assumption, since the likelihood of survival to the next age in reality is not a constant, but a function of the age of the individual (a feature best captured by actuarial life tables, which are of great importance to the insurance industry). Nevertheless, this is a good starting point, since it is relatively tractable and also implies that individuals have an expected lifespan of $1/\nu < \infty$ periods, which can be used to get a sense of what the value of ν should be.

Suppose also that each individual has a standard instantaneous utility function $u : \mathbb{R}_+ \rightarrow \mathbb{R}$, and a "true" or "pure" discount factor $\hat{\beta}$, meaning that this is the discount factor that he would apply between consumption today and tomorrow if he were sure to live between the two dates. Moreover, let us normalize $u(0) = 0$ to be the utility of death. Now consider an individual who plans to have a consumption sequence $\{c(t)\}_{t=0}^{\infty}$ (conditional on living). Clearly, after the individual dies, the future consumption plans do not matter. Standard arguments imply that this individual would have an *expected* utility at time $t = 0$ given by

$$\begin{aligned} U(0) &= u(c(0)) + \hat{\beta}(1-\nu)\, u(c(0)) + \hat{\beta}\nu u(0) \\ &\quad + \hat{\beta}^2 (1-\nu)^2 u(c(1)) + \hat{\beta}^2 (1-\nu)\, \nu u(0) + ... \\ &= \sum_{t=0}^{\infty} \left(\hat{\beta}(1-\nu)\right)^t u(c(t)) \\ &= \sum_{t=0}^{\infty} \beta^t u(c(t)), \end{aligned} \tag{5.9}$$

where the second line collects terms and uses $u(0) = 0$, while the third line defines $\beta \equiv \hat{\beta}(1-\nu)$ as the "effective discount factor" of the individual. With this formulation, the model with finite-lives and random death, would be isomorphic to the model of infinitely-lived individuals, but naturally the reasonable values of β may differ. Note also the emphasized adjective "expected" utility here. While until now agents faced no uncertainty, the possibility of death implies that there is a non-trivial (in fact quite important!) uncertainty in individuals' lives. As a result, instead of the standard ordinal utility theory, we have to use the expected utility theory as developed by von Neumann and Morgenstern. In particular, equation (5.9) is already the expected utility of the individual, since probabilities have been substituted in and there is no need to include an explicit expectations operator. Throughout, except in the stochastic growth analysis in Chapter 17, we do not introduce expectations operators and directly specified the expected utility.

In Exercise 5.7, you are asked to derive a similar effective discount factor for an individual facing a constant death rate in continuous time.

A second justification for the infinite planning horizon comes from intergenerational altruism or from the "bequest" motive. At the simplest level, imagine an individual who lives for one period and has a single offspring (who will also live for a single period and will beget a single offspring etc.). We may imagine that this individual not only derives utility from his consumption but also from the bequest he leaves to his offspring. For example, we may imagine that the utility of an individual living at time t is given by

$$u(c(t)) + U^b(b(t)),$$

where $c(t)$ is his consumption and $b(t)$ denotes the bequest left to his offspring. For concreteness, let us suppose that the individual has total income $y(t)$, so that his budget constraint is

$$c(t) + b(t) \le y(t).$$

The function $U^b(\cdot)$ contains information about how much the individual values bequests left to his offspring. In general, there may be various reasons why individuals leave bequests (including accidental bequests like the individual facing random death probability just discussed). Nevertheless, a natural benchmark might be one in which the individual is "purely altruistic" so that he cares about the utility of his offspring (with some discount factor).[1] Let the discount factor apply between generations be β. Also assume that the offspring will have an income of w without the bequest. Then the utility of the individual can be written as

$$u\left(c\left(t\right)\right)+\beta V\left(b\left(t\right)+w\right),$$

where $V\left(\cdot\right)$ can now be interpreted as the continuation value, equal to the utility that the offspring will obtain from receiving a bequest of $b\left(t\right)$ (plus his own income of w). Naturally, the value of the individual at time t can in turn be written as

$$V\left(y\left(t\right)\right)=\max_{c(t)+b(t)\leq y(t)}\left\{u\left(c\left(t\right)\right)+\beta V\left(b\left(t\right)+w\left(t+1\right)\right)\right\},$$

which defines the current value of the individual starting with income $y\left(t\right)$ and takes into account what the continuation value will be. We will see in the next chapter that this is the canonical form of a dynamic programming representation of an infinite-horizon maximization problem. In particular, under some mild technical assumptions, this dynamic programming representation is equivalent to maximizing

$$\sum_{s=0}^{\infty}\beta^{s}u\left(c_{t+s}\right)$$

at time t. Intuitively, while each individual lives for one period, he cares about the utility of his offspring, and realizes that in turn his offspring cares about the utility of his own offspring, etc. This makes each individual internalize the utility of all future members of the "dynasty". Consequently, fully altruistic behavior within a dynasty (so-called "dynastic" preferences) will also lead to an economy in which decision makers act as if they have an infinite planning horizon.

5.4. The Representative Firm

The previous section discussed how the general equilibrium economy admits a representative household only under special circumstances. The other assumption commonly used in growth models, and already introduced in Chapter 2, is the "representative firm" assumption. In particular, recall from Chapter 2 that the entire production side of the economy was represented by an aggregate production possibilities set, which can be thought of as the production facility set or the "production function" of a representative firm. One may think that this representation also requires quite stringent assumptions on the production structure of the economy. This is not the case, however. While not all economies would

[1] The alternative to "purely altruistic" preferences are those in which a parent receives utility from specific types of bequests or from a subcomponent of the utility of his or her offspring. Models with such "impure altruism" are sometimes quite convenient and will be discussed in Chapters 9 and 21.

admit a representative household, the standard assumptions we adopt in general equilibrium theory or a dynamic general equilibrium analysis (in particular no production externalities and competitive markets) are sufficient to ensure that the formulation with a representative firm is without loss of any generality.

This result is stated in the next theorem.

THEOREM 5.4. ***(The Representative Firm Theorem)*** *Consider a competitive production economy with $N \in \mathbb{N} \cup \{+\infty\}$ commodities and a countable set $\mathcal{F}$ of firms, each with a convex production possibilities set $Y^f \subset \mathbb{R}^N$. Let $p \in \mathbb{R}_+^N$ be the price vector in this economy and denote the set of profit maximizing net supplies of firm $f \in \mathcal{F}$ by $\hat{Y}^f(p) \subset Y^f$ (so that for any $\hat{y}^f \in \hat{Y}^f(p)$, we have $p \cdot \hat{y}^f \geq p \cdot y^f$ for all $y^f \in Y^f$). Then there exists a* representative firm *with production possibilities set $Y \subset \mathbb{R}^N$ and set of profit maximizing net supplies $\hat{Y}(p)$ such that for any $p \in \mathbb{R}_+^N$, $\hat{y} \in \hat{Y}(p)$ if and only if $\hat{y}(p) = \sum_{f \in \mathcal{F}} \hat{y}^f$ for some $\hat{y}^f \in \hat{Y}^f(p)$ for each $f \in \mathcal{F}$.*

PROOF. Let Y be defined as follows:

$$Y = \left\{ \sum_{f \in \mathcal{F}} y^f \, : \, y^f \in Y^f \text{ for each } f \in \mathcal{F} \right\}.$$

To prove the "if" part of the theorem, fix $p \in \mathbb{R}_+^N$ and construct $\hat{y} = \sum_{f \in \mathcal{F}} \hat{y}^f$ for some $\hat{y}^f \in \hat{Y}^f(p)$ for each $f \in \mathcal{F}$. Suppose, to obtain a contradiction, that $\hat{y} \notin \hat{Y}(p)$, so that there exists y' such that $p \cdot y' > p \cdot \hat{y}$. By definition of the set Y, this implies that there exists $\{y^f\}_{f \in \mathcal{F}}$ with $y^f \in Y^f$ such that

$$\begin{aligned} p \cdot \left(\sum_{f \in \mathcal{F}} y^f \right) &> p \cdot \left(\sum_{f \in \mathcal{F}} \hat{y}^f \right) \\ \sum_{f \in \mathcal{F}} p \cdot y^f &> \sum_{f \in \mathcal{F}} p \cdot \hat{y}^f, \end{aligned}$$

so that there exists at least one $f' \in \mathcal{F}$ such that

$$p \cdot y^{f'} > p \cdot \hat{y}^{f'},$$

which contradicts the hypothesis that $\hat{y}^f \in \hat{Y}^f(p)$ for each $f \in \mathcal{F}$ and completes this part of the proof.

To prove the "only if" part of the theorem, let $\hat{y} \in \hat{Y}(p)$ be a profit maximizing choice for the representative firm. Then, since $\hat{Y}(p) \subset Y$, we have that

$$\hat{y} = \sum_{f \in \mathcal{F}} y^f$$

for some $y^f \in Y^f$ for each $f \in \mathcal{F}$. Let $\hat{y}^f \in \hat{Y}^f(p)$. Then,

$$\sum_{f \in \mathcal{F}} p \cdot y^f \leq \sum_{f \in \mathcal{F}} p \cdot \hat{y}^f,$$

which implies that

$$p \cdot \hat{y} \leq p \cdot \sum_{f \in \mathcal{F}} \hat{y}^f. \tag{5.10}$$

Since, by hypothesis, $\sum_{f \in \mathcal{F}} \hat{y}^f \in Y$ and $\hat{y} \in \hat{Y}(p)$, we also have

$$p \cdot \hat{y} \geq p \cdot \sum_{f \in \mathcal{F}} \hat{y}^f.$$

Therefore, inequality (5.10) must hold with equality, so that

$$p \cdot y^f = p \cdot \hat{y}^f,$$

for each $f \in \mathcal{F}$, and thus $y^f \in \hat{Y}^f(p)$. This completes the proof of the theorem. □

This theorem implies that, given the assumptions that there are "no externalities" and that all factors are priced competitively, our focus on the aggregate production possibilities set of the economy or on the representative firm is without loss of any generality. Why is there such a difference between the representative household and representative firm assumptions? The answer is related to income effects. The reason why the representative household assumption is restrictive is that changes in prices create income effects, which affect different households differently. A representative household exists only when these income effects can be ignored, which is what the Gorman preferences guarantee. Since there are no income effects in producer theory, the representative firm assumption is without loss of any generality.

Naturally, the fact that we can represent the production side of an economy by a representative firm does not mean that heterogeneity among firms is uninteresting or unimportant. On the contrary, many of the models of endogenous technology we will see below will feature productivity differences across firms as a crucial part of equilibrium process, and individual firms' attempts to increase their productivity relative to others will often be an engine of economic growth. Theorem 5.4 simply says that when we take the production possibilities sets of the firms in the economy as given, these can be equivalently represented by a single representative firm or an aggregate production possibilities set.

5.5. Problem Formulation

Let us now consider a discrete time infinite-horizon economy and suppose that the economy admits a representative household. In particular, once again ignoring uncertainty, the representative household has the $t = 0$ objective function

$$\sum_{t=0}^{\infty} \beta^t u(c(t)), \tag{5.11}$$

with a discount factor of $\beta \in (0,1)$.

In continuous time, this utility function of the representative household becomes

$$\int_0^\infty \exp(-\rho t)\, u(c(t))\, dt \tag{5.12}$$

where $\rho > 0$ is now the discount rate of the individuals.

Where does the exponential form of the discounting in (5.12) come from? At some level, we called discounting in the discrete time case also "exponential", so the link should be apparent.

To see it more precisely, imagine we are trying to calculate the value of \$1 in T periods, and divide the interval $[0,T]$ into $T/\Delta t$ equally-sized subintervals. Let the interest rate in each subinterval be equal to $\Delta t \cdot r$. It is important that the quantity r is multiplied by Δt, otherwise as we vary Δt, we would be changing the interest rate. Using the standard compound interest rate formula, the value of \$1 in T periods at this interest rate is given by

$$v(T \mid \Delta t) \equiv (1 + \Delta t \cdot r)^{T/\Delta t}.$$

Now we want to take the continuous time limit by letting $\Delta t \to 0$, i.e., we wish to calculate

$$v(T) \equiv \lim_{\Delta t \to 0} v(T \mid \Delta t) \equiv \lim_{\Delta t \to 0} (1 + \Delta t \cdot r)^{T/\Delta t}.$$

Since the limit operator is continuous, we can write

$$\begin{aligned} v(T) &\equiv \exp\left[\lim_{\Delta t \to 0} \ln (1 + \Delta t \cdot r)^{T/\Delta t}\right] \\ &= \exp\left[\lim_{\Delta t \to 0} \frac{T}{\Delta t} \ln (1 + \Delta t \cdot r)\right]. \end{aligned}$$

However, the term in square brackets has a limit of the form $0/0$. Let us next write this as

$$\lim_{\Delta t \to 0} \frac{\ln (1 + \Delta t \cdot r)}{\Delta t / T} = \lim_{\Delta t \to 0} \frac{r/(1 + \Delta t \cdot r)}{1/T} = rT,$$

where the first equality follows from L'Hospital's rule. Therefore,

$$v(T) = \exp(rT).$$

Conversely, \$1 in T periods from now, is worth $\exp(-rT)$ today. The same reasoning applies to discounting utility, so the utility of consuming $c(t)$ in period t evaluated at time $t = 0$ is $\exp(-\rho t)\, u(c(t))$, where ρ denotes the (subjective) discount rate.

5.6. Welfare Theorems

We are ultimately interested in equilibrium growth. But in general competitive economies such as those analyzed so far, we know that there should be a close connection between Pareto optima and competitive equilibria. So far we did not exploit these connections, since without explicitly specifying preferences we could not compare locations. We now introduce these theorems and develop the relevant connections between the theory of economic growth and dynamic general equilibrium models.

Let us start with models that have a finite number of consumers, so that in terms of the notation above, the set $\mathcal{H}$ is finite. However, we allow an infinite number of commodities, since in dynamic growth models, we are ultimately interested in economies that have an infinite number of time periods, thus an infinite number of commodities. The results stated in this section have analogs for economies with a continuum of commodities (corresponding to dynamic economies in continuous time), but for the sake of brevity and to reduce technical details, we focus on economies with a countable number of commodities.

Therefore, let the commodities be indexed by $j \in \mathbb{N}$ and $x^i \equiv \left\{x_j^i\right\}_{j=0}^{\infty}$ be the consumption bundle of household i, and $\omega^i \equiv \left\{\omega_j^i\right\}_{j=0}^{\infty}$ be its endowment bundle. In addition, let us assume that feasible x^i's must belong to some consumption set $X^i \subset \mathbb{R}_+^{\infty}$. The most relevant interpretation for us is that at each date $j = 0, 1, ...$, each individual consumes a finite dimensional vector of products, so that $x_j^i \in X_j^i \subset \mathbb{R}_+^K$ for some integer K. I introduced the consumption set in order to allow for situations in which an individual may not have negative consumption of certain commodities. The consumption set is a subset of $\mathbb{R}_+^{\infty}$since consumption bundles are represented by infinite sequences and we have imposed the restriction that consumption levels cannot be negative (this can be extended by allowing some components of the vector, corresponding to different types of labor supply, to be negative; this is straightforward and I do not do this to conserve notation).

Let $\mathbf{X} \equiv \prod_{i \in \mathcal{H}} X^i$ be the Cartesian product of these consumption sets, which can be thought of as the aggregate consumption set of the economy. We also use the notation $\mathbf{x} \equiv \left\{x^i\right\}_{i \in \mathcal{H}}$ and $\boldsymbol{\omega} \equiv \left\{\omega^i\right\}_{i \in \mathcal{H}}$ to describe the entire consumption allocation and endowments in the economy. Feasibility of a consumption allocation requires that $\mathbf{x} \in \mathbf{X}$.

Each household in $\mathcal{H}$ has a well defined preference ordering over consumption bundles. At the most general level, this preference ordering can be represented by a relationship $\succsim_i$ for household i, such that $x' \succsim_i x$ implies that household i weakly prefers $\mathbf{x}'$ to $\mathbf{x}$. When these preferences satisfy some relatively weak properties (completeness, reflexivity and transitivity), they can equivalently be represented by a real-valued utility function $u^i : X^i \to \mathbb{R}$, such that whenever $x' \succsim_i x$, we have $u^i(x') \geq u^i(x)$. The domain of this function is $X^i \subset \mathbb{R}_+^{\infty}$. Let $\mathbf{u} \equiv \left\{u^i\right\}_{i \in \mathcal{H}}$ be the set of utility functions.

Let us next describe the production side. As already noted before, everything in this book can be done in terms of aggregate production sets. However, to keep in the spirit of general equilibrium theory, let us assume that there is a finite number of firms represented by the set $\mathcal{F}$ and that each firm $f \in \mathcal{F}$ is characterized by a production set Y^f, which specifies what levels of output firm f can produce from specified levels of inputs. In other words,

$y^f \equiv \left\{y_j^f\right\}_{j=0}^{\infty}$ is a feasible production plan for firm f if $y^f \in Y^f$. For example, if there were only two commodities, labor and a final good, Y^f would include pairs $(-l, y)$ such that with labor input l (hence a negative sign), the firm can produce at most as much as y. As is usual in general equilibrium theory, let us take each Y^f to be a *cone,* so that if $y \in Y^f$, then $\lambda y \in Y^f$ for any $\lambda \in \mathbb{R}_+$. This implies two important features: first, $0 \in Y^f$ for each $f \in \mathcal{F}$; second, each Y^f exhibits constant returns to scale. If there are diminishing returns to scale because of some scarce factors, such as entrepreneurial talent, this is added as an additional factor of production and Y^f is still interpreted as a cone.

Let $\mathbf{Y} \equiv \prod_{f\in\mathcal{F}} Y^f$ represent the aggregate production set in this economy and $\mathbf{y} \equiv \left\{y^f\right\}_{f\in\mathcal{F}}$ such that $y^f \in Y^f$ for all f, or equivalently, $\mathbf{y} \in \mathbf{Y}$.

The final object that needs to be described is the ownership structure of firms. In particular, if firms make profits, they should be distributed to some agents in the economy. We capture this by assuming that there exists a sequence of numbers (profit shares) represented by $\boldsymbol{\theta} \equiv \left\{\theta_f^i\right\}_{f\in\mathcal{F}, i\in\mathcal{H}}$ such that $\theta_f^i \geq 0$ for all f and i, and $\sum_{i\in\mathcal{H}} \theta_f^i = 1$ for all $f \in \mathcal{F}$. The number θ_f^i is the share of profits of firm f that will accrue to household i.

An economy $\mathcal{E}$ is described by preferences, endowments, production sets, consumption sets and allocation of shares, i.e., $\mathcal{E} \equiv (\mathcal{H}, \mathcal{F}, \mathbf{u}, \boldsymbol{\omega}, \mathbf{Y}, \mathbf{X}, \boldsymbol{\theta})$. An allocation in this economy is $(\mathbf{x}, \mathbf{y})$ such that $\mathbf{x}$ and $\mathbf{y}$ are feasible, that is, $\mathbf{x} \in \mathbf{X}$, $\mathbf{y} \in \mathbf{Y}$, and $\sum_{i\in\mathcal{H}} x_j^i \leq \sum_{i\in\mathcal{H}} \omega_j^i + \sum_{f\in\mathcal{F}} y_j^f$ for all $j \in \mathbb{N}$. The last requirement implies that the total consumption of each commodity has to be less than the sum of its total endowment and net production.

A price system is a sequence $p \equiv \{p_j\}_{j=0}^{\infty}$, such that $p_j \geq 0$ for all j. We can choose one of these prices as the numeraire and normalize it to 1. We also define $p \cdot x$ as the inner product of p and x, i.e., $p \cdot x \equiv \sum_{j=0}^{\infty} p_j x_j$.[2]

A competitive economy refers to an environment without any externalities and where all commodities are traded competitively. In a competitive equilibrium, all firms maximize profits, all consumers maximize their utility given their budget set and all markets clear. More formally:

Definition 5.1. *A competitive equilibrium for the economy* $\mathcal{E} \equiv (\mathcal{H}, \mathcal{F}, \mathbf{u}, \boldsymbol{\omega}, \mathbf{Y}, \mathbf{X}, \boldsymbol{\theta})$ *is given by an allocation* $\left(\mathbf{x}^* = \left\{x^{i*}\right\}_{i\in\mathcal{H}}, \mathbf{y}^* = \left\{y^{f*}\right\}_{f\in\mathcal{F}}\right)$ *and a price system* p^* *such that*

(1) *The allocation* $(\mathbf{x}^*, \mathbf{y}^*)$ *is feasible, i.e.,* $x^{i*} \in X^i$ *for all* $i \in \mathcal{H}$, $y^{f*} \in Y^f$ *for all* $f \in \mathcal{F}$ *and*

$$\sum_{i\in\mathcal{H}} x_j^{i*} \leq \sum_{i\in\mathcal{H}} \omega_j^i + \sum_{f\in\mathcal{F}} y_j^{f*} \text{ for all } j \in \mathbb{N}.$$

[2]You may note that such an inner product may not always exist in infinite dimensional spaces. But this technical detail does not concern us here, since whenever p corresponds to equilibrium prices, this inner product representation will exist.

(2) *For every firm* $f \in \mathcal{F}$, y^{f*} *maximizes profits, i.e.,*

$$p^* \cdot y^{f*} \leq p^* \cdot y \text{ for all } y \in Y^f.$$

(3) *For every consumer* $i \in \mathcal{H}$, x^{i*} *maximizes utility, i.e.,*

$$u^i\left(x^{i*}\right) \geq u^i(x) \text{ for all } x \text{ such that } x \in X^i \text{ and } p^* \cdot x \leq p^* \cdot \left(\omega^i + \sum_{f \in \mathcal{F}} \theta_f^i y^f\right).$$

A major focus of general equilibrium theory is to establish the existence of a competitive equilibrium under reasonable assumptions. When there is a finite number of commodities and standard convexity assumptions are made on preferences and production sets, this is straightforward (in particular, the proof of existence involves simple applications of Theorems A.13, A.14, and A.16 in Appendix Chapter A). When there is an infinite number of commodities, as in infinite-horizon growth models, proving the existence of a competitive equilibrium is somewhat more difficult and requires more sophisticated arguments. Nevertheless, for our focus here proving the existence of a competitive equilibrium under general conditions is not central (since the typical growth models will have sufficient structure to ensure the existence of a competitive equilibrium in a relatively straightforward manner). Instead, the efficiency properties of competitive equilibria, when they exist, and the decentralization of certain desirable (efficient) allocations as competitive equilibria are more important. For this reason, let us recall the standard definition of Pareto optimality.

Definition 5.2. *A feasible allocation* $(\mathbf{x}, \mathbf{y})$ *for economy* $\mathcal{E} \equiv (\mathcal{H}, \mathcal{F}, \mathbf{u}, \boldsymbol{\omega}, \mathbf{Y}, \mathbf{X}, \boldsymbol{\theta})$ *is* Pareto optimal *if there exists no other feasible allocation* $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ *such that* $\hat{x}^i \in X^i$, $\hat{y}^f \in Y^f$ *for all* $f \in \mathcal{F}$,

$$\sum_{i \in \mathcal{H}} \hat{x}_j^i \leq \sum_{i \in \mathcal{H}} \omega_j^i + \sum_{f \in \mathcal{F}} \hat{y}_j^f \text{ for all } j \in \mathbb{N},$$

and

$$u^i\left(\hat{x}^i\right) \geq u^i\left(x^i\right) \text{ for all } i \in \mathcal{H}$$

with at least one strict inequality.

Our next result is the celebrated *First Welfare Theorem* for competitive economies. Before presenting this result, we need the following definition.

Definition 5.3. *Household* $i \in \mathcal{H}$ *is* locally non-satiated *at* x^i *if* $u^i\left(x^i\right)$ *is strictly increasing in at least one of its arguments at* x^i *and* $u^i\left(x^i\right) < \infty$.

The latter requirement in this definition is already implied by the fact that $u^i : X^i \to \mathbb{R}$, but it is included for additional emphasis, since it is important for the proof and also because if in fact we had $u^i\left(x^i\right) = \infty$, we could not meaningfully talk about $u^i\left(x^i\right)$ being strictly increasing.

Theorem 5.5. ***(First Welfare Theorem I)*** *Suppose that* $(\mathbf{x}^*, \mathbf{y}^*, p^*)$ *is a competitive equilibrium of economy* $\mathcal{E} \equiv (\mathcal{H}, \mathcal{F}, \mathbf{u}, \boldsymbol{\omega}, \mathbf{Y}, \mathbf{X}, \boldsymbol{\theta})$ *with* $\mathcal{H}$ *finite. Assume that all households are locally non-satiated at* $\mathbf{x}^*$*. Then* $(\mathbf{x}^*, \mathbf{y}^*)$ *is Pareto optimal.*

Proof. To obtain a contradiction, suppose that there exists a feasible $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ such that $u^i(\hat{x}^i) \geq u^i(x^i)$ for all $i \in \mathcal{H}$ and $u^i(\hat{x}^i) > u^i(x^i)$ for all $i \in \mathcal{H}'$, where $\mathcal{H}'$ is a non-empty subset of $\mathcal{H}$.

Since $(\mathbf{x}^*, \mathbf{y}^*, p^*)$ is a competitive equilibrium, it must be the case that for all $i \in \mathcal{H}$,

$$
\begin{aligned}
p^* \cdot \hat{x}^i &\geq p^* \cdot x^{i*} \\
&= p^* \cdot \left(\omega^i + \sum_{f \in \mathcal{F}} \theta^i_f y^{f*}\right)
\end{aligned}
\tag{5.13}
$$

and for all $i \in \mathcal{H}'$,

$$
p^* \cdot \hat{x}^i > p^* \cdot \left(\omega^i + \sum_{f \in \mathcal{F}} \theta^i_f y^{f*}\right). \tag{5.14}
$$

The second inequality follows immediately in view of the fact that x^{i*} is the utility maximizing choice for household i, thus if $\hat{x}^i$ is strictly preferred, then it cannot be in the budget set. The first inequality follows with a similar reasoning. Suppose that it did not hold. Then by the hypothesis of local-satiation, u^i must be strictly increasing in at least one of its arguments, let us say the j'th component of x. Then construct $\hat{x}^i(\varepsilon)$ such that $\hat{x}^i_j(\varepsilon) = \hat{x}^i_j$ and $\hat{x}^i_{j'}(\varepsilon) = \hat{x}^i_{j'} + \varepsilon$. For $\varepsilon \downarrow 0$, $\hat{x}^i(\varepsilon)$ is in household i's budget set and yields strictly greater utility than the original consumption bundle x^i, contradicting the hypothesis that household i was maximizing utility.

Also note that local non-satiation implies that $u^i(x^i) < \infty$, and thus the right-hand sides of (5.13) and (5.14) are finite (otherwise, the income of household i would be infinite, and the household would either reach a point of satiation or infinite utility, contradicting the local non-satiation hypothesis).

Now summing over (5.13) and (5.14), we have

$$
\begin{aligned}
p^* \cdot \sum_{i \in \mathcal{H}} \hat{x}^i &> p^* \cdot \sum_{i \in \mathcal{H}} \left(\omega^i + \sum_{f \in \mathcal{F}} \theta^i_f y^{f*}\right), \\
&= p^* \cdot \left(\sum_{i \in \mathcal{H}} \omega^i + \sum_{f \in \mathcal{F}} y^{f*}\right),
\end{aligned}
\tag{5.15}
$$

where the second line uses the fact that the summations are finite, so that we can change the order of summation, and that by definition of shares $\sum_{i \in \mathcal{H}} \theta^i_f = 1$ for all f. Finally, since $\mathbf{y}^*$

is profit-maximizing at prices p^*, we have that

$$p^* \cdot \sum_{f\in\mathcal{F}} y^{f*} \geq p^* \cdot \sum_{f\in\mathcal{F}} y^f \text{ for any } \left\{y^f\right\}_{f\in\mathcal{F}} \text{ with } y^f \in Y^f \text{ for all } f \in \mathcal{F}. \tag{5.16}$$

However, by feasibility of $\hat{x}^i$ (Definition 5.1, part 1), we have

$$\sum_{i\in\mathcal{H}} \hat{x}^i_j \leq \sum_{i\in\mathcal{H}} \omega^i_j + \sum_{f\in\mathcal{F}} \hat{y}^f_j,$$

and therefore, by multiplying both sides by p^* and exploiting (5.16), we have

$$\begin{aligned} p^* \cdot \sum_{i\in\mathcal{H}} \hat{x}^i_j &\leq p^* \cdot \left(\sum_{i\in\mathcal{H}} \omega^i_j + \sum_{f\in\mathcal{F}} \hat{y}^f_j\right) \\ &\leq p^* \cdot \left(\sum_{i\in\mathcal{H}} \omega^i_j + \sum_{f\in\mathcal{F}} y^{f*}_j\right), \end{aligned}$$

which contradicts (5.15), establishing that any competitive equilibrium allocation $(\mathbf{x}^*, \mathbf{y}^*)$ is Pareto optimal. □

The proof of the First Welfare Theorem is both intuitive and simple. The proof is based on two intuitive ideas. First, if another allocation Pareto dominates the competitive equilibrium, then it must be non-affordable in the competitive equilibrium. Second, profit-maximization implies that any competitive equilibrium already contains the maximal set of affordable allocations. It is also simple since it only uses the summation of the values of commodities at a given price vector. In particular, it makes no convexity assumption. However, the proof also highlights the importance of the feature that the relevant sums exist and are finite. Otherwise, the last step would lead to the conclusion that "$\infty < \infty$" which may or may not be a contradiction. The fact that these sums exist, in turn, followed from two assumptions: finiteness of the number of individuals and non-satiation. However, as noted before, working with economies that have only a finite number of households is not always sufficient for our purposes. For this reason, the next theorem turns to the version of the First Welfare Theorem with an infinite number of households. For simplicity, here we take $\mathcal{H}$ to be a countably infinite set, e.g., $\mathcal{H} = \mathbb{N}$. The next theorem generalizes the First Welfare Theorem to this case. It makes use of an additional assumption to take care of infinite sums.

THEOREM 5.6. ***(First Welfare Theorem II)*** *Suppose that $(\mathbf{x}^*, \mathbf{y}^*, p^*)$ is a competitive equilibrium of the economy $\mathcal{E} \equiv (\mathcal{H}, \mathcal{F}, \mathbf{u}, \boldsymbol{\omega}, \mathbf{Y}, \mathbf{X}, \boldsymbol{\theta})$ with $\mathcal{H}$ countably infinite. Assume that all households are locally non-satiated at $\mathbf{x}^*$ and that $\sum_{j=0}^{\infty} p^*_j < \infty$. Then $(\mathbf{x}^*, \mathbf{y}^*, p^*)$ is Pareto optimal.*

PROOF. The proof is the same as that of Theorem 5.5, with a major difference. Local non-satiation does not guarantee that the summations are finite (5.15), since we have the sum over an infinite number of households. However, since endowments are finite, the assumption that $\sum_{j=0}^{\infty} p_j^* < \infty$ ensures that the sums in (5.15) are indeed finite and the rest of the proof goes through exactly as in the proof of Theorem 5.5. $\square$

Theorem 5.6 will be particularly useful when we discuss overlapping generation models.

We next briefly discuss the Second Welfare Theorem, which is the converse of the First Welfare Theorem. It answers the question of whether a Pareto optimal allocation can be decentralized as a competitive equilibrium. Interestingly, for the Second Welfare Theorem whether or not $\mathcal{H}$ is finite is not as important as for the First Welfare Theorem. Nevertheless, the Second Welfare Theorem requires a number of assumptions on preferences and technology, such as the convexity of consumption and production sets and preferences, and a number of additional requirements (which are trivially satisfied when the number of commodities is finite). This is because the Second Welfare Theorem implicitly contains an "existence of equilibrium argument," which runs into problems in the presence of non-convexities. Before stating this theorem, recall that the consumption set of each individual $i \in \mathcal{H}$ is $X^i \subset \mathbb{R}_+^{\infty}$, so a typical element of X^i is $x^i = \left(x_0^i, x_1^i, x_2^i, ...\right)$, where x_t^i can be interpreted as the vector of consumption of individual i at time t. Similarly, a typical element of the production set of firm $f \in \mathcal{F}$, Y^f, is $y^f = \left(y_1^f, y_2^f, ...\right)$. Let us define $x^i[T] = \left(x_0^i, x_1^i, x_2^i, ..., x_T^i, 0, 0, ...\right)$ and $y^f[T] = \left(y_0^f, y_1^f, y_2^f, ..., y_T^f, 0, 0, ...\right)$. In other words, these are truncated sequences which involves zero consumption or zero production after some date T. It can be verified that $\lim_{T\to\infty} x^i[T] = x^i$ and $\lim_{T\to\infty} y^f[T] = y^f$ in the product topology (see Section A.4 in Appendix Chapter A).

Theorem 5.7. ***(Second Welfare Theorem)*** *Consider a Pareto optimal allocation* $(\mathbf{x}^{**}, \mathbf{y}^{**})$ *in economy with endowment vector* ω*, production sets* $\left\{Y^f\right\}_{f\in\mathcal{F}}$*, consumption sets* $\left\{X^i\right\}_{i\in\mathcal{H}}$*, and utility functions* $\left\{u^i(\cdot)\right\}_{i\in\mathcal{H}}$*. Suppose that all production and consumption sets are convex, all production sets are cones, and all utility functions* $\left\{u^i(\cdot)\right\}_{i\in\mathcal{H}}$ *are continuous and quasi-concave and satisfy local non-satiation. Moreover, suppose also that* $0 \in X^i$ *for each* $i \in \mathcal{H}$*, that for each* $x, x' \in X^i$ *with* $u^i(x) > u^i(x')$ *for all* $i \in \mathcal{H}$*, there exists* $\bar{T}$ *such that* $u^i(x[T]) > u^i(x'[T])$ *for all* $T \geq \bar{T}$ *and for all* $i \in \mathcal{H}$ *and that for each* $y \in Y^f$*, there exists* $\tilde{T}$ *such that* $y[T] \in Y^f$ *for all* $T \geq \tilde{T}$ *and for all* $f \in \mathcal{F}$*. Then there exist a price vector* p^{**} *and an endowment and share allocations* $(\boldsymbol{\omega}^{**}, \boldsymbol{\theta}^{**})$ *such that in economy* $\mathcal{E} \equiv (\mathcal{H}, \mathcal{F}, \mathbf{u}, \boldsymbol{\omega}^{**}, \mathbf{Y}, \mathbf{X}, \boldsymbol{\theta}^{**})$*,*

(1) *the endowment allocation* $\boldsymbol{\omega}^{**}$ *satisfies* $\omega = \sum_{i\in\mathcal{H}} \omega^{i**}$*;*
(2) *for all* $f \in \mathcal{F}$*,*

$$p^{**} \cdot y^{f**} \leq p^{**} \cdot y \text{ for all } y \in Y^f;$$

(3) *for all* $i \in \mathcal{H}$,

$$\textit{if } x^i \in X^i \textit{ involves } u^i\left(x^i\right) > u^i\left(x^{i**}\right), \textit{ then } p^{**} \cdot x^i \geq p^{**} \cdot w^{i**},$$

where $w^{i**} \equiv \omega^{i**} + \sum_{f \in \mathcal{F}} \theta_f^{i**} y^{f**}$.

Moreover, if $p^{**} \cdot \mathbf{w}^{**} > 0$ *[i.e.,* $p^{**} \cdot w^{i**} > 0$ *for each* $i \in \mathcal{H}$*], then economy* $\mathcal{E}$ *has a competitive equilibrium* $(\mathbf{x}^{**}, \mathbf{y}^{**}, p^{**})$.

The proof of this theorem involves the application of the Geometric Hahn-Banach Theorem, Theorem A.25, from Appendix Chapter A. It is somewhat long and involved. For this reason, a sketch of this proof as provided in the next (starred) section. Here notice that if instead of an infinite-dimensional economy, we were dealing with an economy with a finite commodity space, say with K commodities, then the hypothesis in the theorem, that $0 \in X^i$ for each $i \in \mathcal{H}$ and that for each $x, x' \in X^i$ with $u^i(x) > u^i(x')$ for all $i \in \mathcal{H}$, there exists $\bar{T}$ such that $u^i(x[T]) > u^i(x'[T])$ for all $T \geq \bar{T}$ and for all $i \in \mathcal{H}$ (and also that for each $y \in Y^f$, there exists $\tilde{T}$ such that $y[T] \in Y^f$ for all $T \geq \tilde{T}$ and for all $f \in \mathcal{F}$) would be satisfied automatically, by taking $\bar{T} = \tilde{T} = K$. In fact, this condition is not imposed in the statement of the Second Welfare Theorem in economies with a finite number of commodities. Its role in dynamic economies, with which we are concerned with in this book, is that changes in allocations that are very far in the future should not have a "large" effect on preferences. This is naturally satisfied when we look at infinite-horizon economies with discounted utility and separable production structure. Intuitively, if a sequence of consumption levels x is strictly preferred to x', then setting the elements of x and x' to 0 in the very far (and thus heavily discounted) future should not change this conclusion (since discounting implies that x could not be strictly preferred to x' because of higher consumption under x in the arbitrarily far future). Similarly, if some production vector y is feasible, the separable production structure implies that $y[T]$, which involves zero production after some date T, must also be feasible. Exercise 5.13 demonstrates these claims more formally. One difficulty in applying this theorem is that u^i may not be defined when x has zero elements (so that the consumption set X^i does not contain 0). Exercise 5.14 shows that the theorem can be generalized to the case in which there exists a strictly positive vector $\varepsilon \in \mathbb{R}^K$ with each element sufficiently small and with $(\varepsilon, ..., \varepsilon) \in X^i$ for all $i \in \mathcal{H}$.

The conditions for the Second Welfare Theorem are more difficult to satisfy than those for the First Welfare Theorem because of the convexity requirements. In many ways, it is also the more important of the two theorems. While the First Welfare Theorem is celebrated as a formalization of Adam Smith's invisible hand, the Second Welfare Theorem establishes the stronger results that any Pareto optimal allocation can be *decentralized* as a competitive equilibrium. An immediate corollary of this is an existence result; since the Pareto optimal allocation can be decentralized as a competitive equilibrium, a competitive equilibrium must exist (at least for the endowments leading to Pareto optimal allocations).

The Second Welfare Theorem motivates many macroeconomists to look for the set of Pareto optimal allocations instead of explicitly characterizing competitive equilibria. This is especially useful in dynamic models where sometimes competitive equilibria can be quite difficult to characterize or even to specify, while social welfare maximizing allocations are more straightforward.

The real power of the Second Welfare Theorem in dynamic macro models comes when we combine it with models that admit a representative household. Recall that Theorem 5.3 shows an equivalence between Pareto optimal allocations and optimal allocations for the representative household. In certain models, including many—but not all—growth models studied in this book, the combination of a representative household and the Second Welfare Theorem enables us to characterize *the optimal growth allocation* that maximizes the utility of the representative household and assert that this will correspond to a competitive equilibrium.

5.7. Sketch of the Proof of the Second Welfare Theorem, Theorem 5.7*

In this section, I provide a proof of the Second Welfare Theorem. The most important part of the theorem is proved by using the Geometric Hahn-Banach Theorem, Theorem A.25, from Appendix Chapter A

Proof of Theorem 5.7: (Sketch)

First, I establish that there exists a price vector p^{**} and an endowment and share allocation $(\boldsymbol{\omega}^{**}, \boldsymbol{\theta}^{**})$ that satisfy conditions 1-3. This has two parts.

(Part 1) This part follows from the Geometric Hahn-Banach Theorem, Theorem A.25. Define the "more preferred" sets for each $i \in \mathcal{H}$:

$$P^i = \left\{x^i \in X^i : u^i\left(x^i\right) > u^i\left(x^{i**}\right)\right\}.$$

Clearly, each P^i is convex. Let $P = \sum_{i \in \mathcal{H}} P^i$ and $Y' = \sum_{f \in \mathcal{F}} Y^f + \{\omega\}$, where recall that $\omega = \sum_{i \in \mathcal{H}} \omega^{i**}$, so that Y' is the sum of the production sets shifted by the endowment vector. Both P and Y' are convex (since each P^i and each Y^f are convex). Consider the sequences of production plans for each firm to be subsets of ℓ_∞^K, i.e., vectors of the form $y^f = \left(y_0^f, y_1^f, ...\right)$, with each $y_j^f \in \mathbb{R}_+^K$. Moreover, since each production set is a cone, $Y' = \sum_{f \in \mathcal{F}} Y^f + \{\omega\}$ has an interior point (the argument is identical to that of Exercise A.32 in Appendix Chapter A). Moreover, let $x^{**} = \sum_{i \in \mathcal{H}} x^{i**}$. By feasibility and local non-satiation, $x^{**} = \sum_{f \in \mathcal{F}} y^{i**} + \omega$. Then $x^{**} \in Y'$ and also $x^{**} \in \overline{P}$ (where recall that $\overline{P}$ is the closure of P).

Next, observe that $P \cap Y' = \varnothing$. Otherwise, there would exist $\tilde{y} \in Y'$, which is also in P. This implies that if distributed appropriately across the households, $\tilde{y}$ would make all households equally well off and at least one of them would be strictly better off (i.e., by the definition of the set P, there would exist $\left\{\tilde{x}^i\right\}_{i\in\mathcal{H}}$ such that $\sum_{i\in\mathcal{H}}\tilde{x}^i = \tilde{y}$, $\tilde{x}^i \in X^i$, and $u^i\left(\tilde{x}^i\right) \geq u^i\left(x^{i**}\right)$ for all $i \in \mathcal{H}$ with at least one strict inequality). This would contradict the hypothesis that (x^{**}, y^{**}) is a Pareto optimum.

Since Y' has an interior point, P and Y' are convex, and $P \cap Y' = \varnothing$, Theorem A.25 implies that there exists a nonzero continuous linear functional ϕ such that

$$\phi\left(y\right) \leq \phi\left(x^{**}\right) \leq \phi\left(x\right) \text{ for all } y \in Y' \text{ and all } x \in P. \tag{5.17}$$

(Part 2) We next need to show that this linear functional can be interpreted as a price vector (i.e., that it does have an inner product representation). Consider the functional

$$\bar{\phi}\left(x\right) = \lim_{T\to\infty}\phi\left(x\left[T\right]\right),$$

where recall that for $x^i = \left(x^i_0, x^i_1, x^i_2, ...\right)$, $x^i\left[T\right] = \left(x^i_0, x^i_1, x^i_2, ..., x^i_T, 0, 0, ...\right)$. Moreover, let $\bar{x}^i_T$ be defined as $\bar{x}^i_J = \left(0, 0, ..., x^i_J, 0, ...,\right)$, i.e., as the sequence x^i with zeros everywhere except its Jth element. Finally, define

$$\bar{z}^{\phi}_t = \begin{cases} \bar{x}_t & \text{if } \phi\left(\bar{x}_t\right) \geq 0 \\ -\bar{x}_t & \text{if } \phi\left(\bar{x}_t\right) < 0 \end{cases}$$

and $\bar{z}^{\phi} = \left(\bar{z}^{\phi}_0, \bar{z}^{\phi}_1, ...\right)$. Then, by the linearity of ϕ, we have that

$$\begin{aligned} \phi\left(x\left[T\right]\right) = \sum_{t=0}^{T}\phi\left(\bar{x}_t\right) &\leq \sum_{t=0}^{T}\left|\phi\left(\bar{x}_t\right)\right| \\ &= \phi\left(\bar{z}^{\phi}\left[T\right]\right) \\ &\leq \left\|\phi\right\|\left\|\bar{z}^{\phi}\left[T\right]\right\| \\ &= \left\|\phi\right\|\left\|x\left[T\right]\right\| \\ &\leq \left\|\phi\right\|\left\|x\right\|, \end{aligned}$$

where the first line holds by definition, the second line uses the definition of $\bar{z}^{\phi}$ introduced above, the third line uses the fact that ϕ is a linear functional, the fourth line exploits the fact that the norm $\left\|\cdot\right\|$ does not depend on whether the elements are negative or positive, and the final line uses the fact that $\left\|x\right\| \geq \left\|x\left[T\right]\right\|$. This string of relationships implies that the sequence $\left\{\sum_{t=0}^{T}\left|\phi\left(\bar{x}_t\right)\right|\right\}_{T=1}^{\infty}$ dominates the sequence $\left\{\phi\left(x\left[T\right]\right)\right\}_{T=1}^{\infty}$ and is also bounded by $\left\|\phi\right\|\left\|x\right\|$. Therefore, $\left\{\phi\left(x\left[T\right]\right)\right\}_{T=1}^{\infty}$ converges to some well-defined linear functional $\bar{\phi}\left(x\right)$. The last inequality above also implies that $\bar{\phi}\left(x\right) \leq \left\|\phi\right\|\left\|x\right\|$, so $\bar{\phi}$ is a bounded and thus continuous linear functional (Theorem A.24 in Appendix Chapter A).

Next, define $\bar{\phi}_J : \mathbb{R}^K \to \mathbb{R}$ as $\bar{\phi}_J : x_J \mapsto \phi\left(\bar{x}_J\right)$. Clearly, $\bar{\phi}_J$ is a linear functional (since ϕ is a linear) and moreover, since the domain of $\bar{\phi}_J$ is a Euclidean space, it has an inner product representation, and in particular, there exists $p^{**}_J \in \mathbb{R}^K$ such that

$$\bar{\phi}_J\left(x_J\right) = p^{**}_J \cdot x_J \text{ for all } x_J \in \mathbb{R}^K.$$

This also implies that

$$\bar{\phi}(x) = \lim_{T\to\infty} \phi(x[T]) = \lim_{T\to\infty} \sum_{J=0}^{T} \bar{\phi}_J(x_J)$$
$$= \lim_{T\to\infty} \sum_{J=0}^{T} p_J^{**} \cdot x_J,$$

so that $\bar{\phi}$ is a continuous linear functional within inner productive presentation.

To complete this part of the proof, we only need to show that $\bar{\phi}(x) = \sum_{j=0}^{\infty} \bar{\phi}_j(x_j)$ can be used instead of ϕ as the continuous linear functional in (5.17). This follows immediately from the hypothesis that $0 \in X^i$ for each $i \in \mathcal{H}$, that for each $x, x' \in X^i$ with $u^i(x) > u^i(x')$ for all $i \in \mathcal{H}$, there exists $\bar{T}$ such that $u^i(x[T]) > u^i(x'[T])$ for all $T \geq \bar{T}^i$ and for all $i \in \mathcal{H}$, and that for each $y \in Y^f$, there exists $\tilde{T}$ such that $y[T] \in Y^f$ for all $T \geq \tilde{T}^f$ and for all $f \in \mathcal{F}$. In particular, take $T' = \max\left\{\bar{T}^i, \tilde{T}^f\right\}$ and fix $x \in P$. Since x has the property that $u^i\left(x^i\right) > u^i\left(x^{i**}\right)$ for all $i \in \mathcal{H}$, we also have that $u^i\left(x^i[T]\right) > u^i\left(x^{i**}[T]\right)$ for all $i \in \mathcal{H}$ and $T \geq T'$. Therefore,

$$\phi(x^{**}[T]) \leq \phi(x[T]) \text{ for all } x \in P.$$

Not taking limits, we obtain that

$$\bar{\phi}(x^{**}) \leq \bar{\phi}(x) \text{ for all } x \in P.$$

A similar argument establishes that $\bar{\phi}(x^{**}) \geq \bar{\phi}(y)$ for all $y \in Y'$, so that $\bar{\phi}(x)$ can be used as the continuous linear functional separating P and Y'. Since $\bar{\phi}_j(x_j)$ is a linear functional on $X_j \subset \mathbb{R}_+^K$, it has an inner product representation, $\bar{\phi}_j(x_j) = p_j^{**} \cdot x_j$ and therefore so does $\bar{\phi}(x) = \sum_{j=0}^{\infty} \bar{\phi}_j(x_j) = p^{**} \cdot x$.

Parts 1 and 2 have therefore established that there exists a price vector (functional) p^{**} such that conditions 2 and 3 hold. Condition 1 is satisfied by construction. Condition 2 is sufficient to establish that all firms maximize profits at the price vector p^{**}. To show that all consumers maximize utility at the price vector p^{**}, use the hypothesis that $p^{**} \cdot w^{i**} > 0$ for each $i \in \mathcal{H}$. We know from Condition 3 that if $x^i \in X^i$ involves $u^i\left(x^i\right) > u^i\left(x^{i**}\right)$, then $p^{**} \cdot x^i \geq p^{**} \cdot w^{i**}$. This implies that if there exists x^i that is strictly preferred to x^{i**} and satisfies $p^{**} \cdot x^i = p^{**} \cdot w^{i**}$ (which would amount to the consumer not maximizing utility), then there exists $x^i - \varepsilon$ for ε small enough, such that $u^i\left(x^i - \varepsilon\right) > u^i\left(x^{i**}\right)$, then $p^{**} \cdot \left(x^i - \varepsilon\right) < p^{**} \cdot w^{i**}$, thus violating Condition 3. Therefore, consumers also maximize utility at the price p^{**}, establishing that $(\mathbf{x}^{**}, \mathbf{y}^{**}, p^{**})$ is a competitive equilibrium. □

5.8. Sequential Trading

A final issue that is useful to discuss at this point relates to sequential trading. Standard general equilibrium models assume that all commodities are traded at a given point in time—and once and for all. That is, once trading takes place at the initial date, there is no more trade or production in the economy. This may be a good approximation to reality when different commodities correspond to different goods. However, when different commodities correspond to the same good in different time periods or in different states of nature, trading once and for all at a single point is much less reasonable. In models of economic growth, we typically assume that trading takes place at different points in time. For example, in the Solow growth model of Chapter 2, we envisaged firms hiring capital and labor at each t. Does the presence of sequential trading make any difference to the insights of general equilibrium analysis? If the answer to this question were yes, then the applicability of the lessons from general equilibrium theory to dynamic macroeconomic models would be limited. Fortunately, in the presence of complete markets, which we assume in most of our models, sequential trading gives the same result as trading at a single point in time.

More explicitly, the *Arrow-Debreu equilibrium* of a dynamic general equilibrium model involves all the households trading at a single market at time $t = 0$ and purchasing and selling irrevocable claims to commodities indexed by date and state of nature. This means that at time $t = 0$, households agree on all future trades (including trades of goods that are not yet produced). Sequential trading, on the other hand, corresponds to separate markets opening at each t, and households trading labor, capital and consumption goods in each such market at each period. Clearly, both for mathematical convenience and descriptive realism, we would like to think of macroeconomic models as involving sequential trading, with separate markets at each date.

The key result concerning the comparison of models with trading at a single point in time and those with sequential trading is due to Arrow (1964). Arrow showed that with complete markets (and time consistent preferences), trading at a single point in time and sequential trading are equivalent. The easiest way of seeing this is to consider the Arrow securities already discussed in Chapter 2. *(Basic) Arrow Securities* provide an economical means of transferring resources across different dates and different states of nature. Instead of completing all trades at a single point in time, say at time $t = 0$, households can trade Arrow securities and then use these securities to purchase goods at different dates or after different states of nature have been revealed. While Arrow securities are most useful when there is uncertainty as well as a temporal dimension, for our purposes it is sufficient to focus on the transfer of resources across different dates.

The reason why sequential trading with Arrow securities achieves the same result as trading at a single point in time is simple: by the definition of a competitive equilibrium, households correctly anticipate all the prices that they will be facing at different dates (and under different states of nature) and purchase sufficient Arrow securities to cover the expenses that they will incur once the time to trade comes. In other words, instead of buying claims at time $t = 0$ for $x^h_{i,t'}$ units of commodity $i = 1, ..., N$ at date t' at prices $\left(p_{1,t'}, ..., p_{N,t}\right)$, it is sufficient for household h to have an income of $\sum_{i=1}^{N} p_{i,t'} x_{i,t'}^{h}$ and know that it can purchase as many units of each commodity as it wishes at time t' at the price vector $\left(p_{1,t'}, ..., p_{N,t'}\right)$.

This result can be stated in a slightly more formal manner. Let us consider a dynamic exchange economy running across periods $t = 0, 1, ..., T$, possibly with $T = \infty$.[3] Nothing here depends on the assumption that we are focusing on an exchange economy, but suppressing production simplifies notation. Imagine that there are N goods at each date, denoted by $(x_{1,t}, ..., x_{N,t})$, and let the consumption of good i by household h at time t be denoted by $x_{i,t}^{h}$. Suppose that these goods are perishable, so that they are indeed consumed at time t. Denote the set of households by $\mathcal{H}$ and suppose that each household $h \in \mathcal{H}$ has a vector of endowment $\left(\omega_{1,t}^{h}, ..., \omega_{N,t}^{h}\right)$ at time t, and preferences given by the time separable function of the form

$$\sum_{t=0}^{T} \beta_h^t u^h \left(x_{1,t}^h, ..., x_{N,t}^h\right),$$

for some $\beta_h \in (0,1)$. These preferences imply that there are no externalities and preferences are time consistent. We also assume that all markets are open and competitive.

Let an Arrow-Debreu equilibrium be given by $(\mathbf{p}^*, \mathbf{x}^*)$, where $\mathbf{x}^*$ is the complete list of consumption vectors of each household $h \in \mathcal{H}$, that is,

$$\mathbf{x}^* = (x_{1,0}, ...x_{N,0}, ..., x_{1,T}, ...x_{N,T}),$$

with $x_{i,t} = \left\{x_{i,t}^h\right\}_{h\in\mathcal{H}}$ for each i and t, and $\mathbf{p}^*$ is the vector of complete prices $\mathbf{p}^* = \left(p_{1,0}^*, ..., p_{N,0}^*, ..., p_{1,T}, ..., p_{N,T}\right)$, with one of the prices, say $p_{1,0}^*$, chosen as the numeraire, i.e., $p_{1,0}^* = 1$. In the Arrow-Debreu equilibrium, each individual purchases and sells claims on each of the commodities, thus engages in trading only at $t = 0$ and chooses an allocation that satisfies the budget constraint

$$\sum_{t=0}^{T} \sum_{i=1}^{N} p_{i,t}^* x_{i,t}^h \leq \sum_{t=0}^{T} \sum_{i=1}^{N} p_{i,t}^* \omega_{i,t}^h \text{ for each } h \in \mathcal{H}.$$

Market clearing then requires

$$\sum_{h\in\mathcal{H}} \sum_{i=1}^{N} x_{i,t}^h \leq \sum_{h\in\mathcal{H}} \sum_{i=1}^{N} \omega_{i,t}^h \text{ for each } i = 1, ..., N \text{ and } t = 0, 1, ..., T.$$

In the equilibrium with sequential trading, markets for goods dated t open at time t. Instead, there are T bonds—*Arrow securities*—that are in zero net supply and can be traded among the households at time $t = 0$. The bond indexed by t pays one unit of one of the goods, say good $i = 1$ at time t. Let the prices of bonds be denoted by $(q_1, ..., q_T)$, again expressed in units of good $i = 1$ (at time $t = 0$). This implies that a household can purchase a unit of bond t at time 0 by paying q_t units of good 1 and then will receive one unit of good 1 at time t (or conversely can sell short one unit of such a bond) The purchase of bond t by

[3] When $T = \infty$, we assume that all the summations take a finite value.

household h is denoted by $b_t^h \in \mathbb{R}$, and since each bond is in zero net supply, market clearing requires that

$$\sum_{h\in\mathcal{H}} b_t^h = 0 \text{ for each } t = 0, 1, ..., T.$$

Notice that in this specification we have assumed the presence of only T bonds (Arrow securities). More generally, we could have allowed additional bonds, for example bonds traded at time $t > 0$ for delivery of good 1 at time $t' > t$. This restriction to only T bonds is without loss of any generality (see Exercise 5.10).

Sequential trading corresponds to each individual using their endowment plus (or minus) the proceeds from the corresponding bonds at each date t. Since there is a market for goods at each t, it turns out to be convenient (and possible) to choose a separate numeraire for each date t, and let us again suppose that this numeraire is good 0, so that $p_{1,t}^{**} = 1$ for all t. Therefore, the budget constraint of household $h \in \mathcal{H}$ at time t, given the equilibrium price vector for goods and bonds, $(\mathbf{p}^{**}, \mathbf{q}^{**})$, can be written as:

$$\sum_{i=1}^{N} p_{i,t}^{**} x_{i,t}^h \leq \sum_{i=1}^{N} p_{i,t}^{**} \omega_{i,t}^h + q_t^{**} b_t^h \text{ for } t = 0, 1, ..., T, \tag{5.18}$$

with the normalization that $q_0^{**} = 1$. Let an equilibrium of the sequential trading economy be denoted by $(\mathbf{p}^{**}, \mathbf{q}^{**}, \mathbf{x}^{**}, \mathbf{b}^{**})$, where once again $\mathbf{p}^{**}$ and $\mathbf{x}^{**}$ denote the entire lists of prices and quantities of consumption by each household, and $\mathbf{q}^{**}$ and $\mathbf{b}^{**}$ denote the vector of bond prices and bond purchases by each household. Given this specification, the following theorem can be established.

Theorem 5.8. ***(Sequential Trading)*** *For the above-described economy, if* $(\mathbf{p}^*, \mathbf{x}^*)$ *is an Arrow-Debreu equilibrium, then there exists a sequential trading equilibrium* $(\mathbf{p}^{**}, \mathbf{q}^{**}, \mathbf{x}^{**}, \mathbf{b}^{**})$*, such that* $\mathbf{x}^* = \mathbf{x}^{**}$*,* $p_{i,t}^{**} = p_{i,t}^*/p_{1,t}^*$ *for all* i *and* t *and* $q_t^{**} = p_{1,t}^*$ *for all* $t > 0$*. Conversely, if* $(\mathbf{p}^{**}, \mathbf{q}^{**}, \mathbf{x}^{**}, \mathbf{b}^{**})$ *is a sequential trading equilibrium, then there exists an Arrow-Debreu equilibrium* $(\mathbf{p}^*, \mathbf{x}^*)$ *with* $\mathbf{x}^* = \mathbf{x}^{**}$*,* $p_{i,t}^* = p_{i,t}^{**} p_{1,t}^*$ *for all* i *and* t*, and* $p_{1,t}^* = q_t^{**}$ *for all* $t > 0$*.*

Proof. See Exercise 5.9. □

This theorem implies that all the results concerning Arrow–Debreu equilibria apply to economies with sequential trading. In most of the models studied in this book (unless we are explicitly concerned with endogenous financial markets), we will focus on economies with sequential trading and assume that there exist Arrow securities to transfer resources across dates. These securities might be riskless bonds in zero net supply, or in models without uncertainty, this role will typically be played by the capital stock. We will also follow the approach leading to Theorem 5.8 and normalize the price of one good at each date to 1. This implies that in economies with a single consumption good, like the Solow or the neoclassical growth models, the price of the consumption good in each date will be normalized to 1 and the interest rates will directly give the intertemporal relative prices. This is the justification for focusing on interest rates as the key relative prices in macroeconomic (economic growth) models.

5.9. Optimal Growth in Discrete Time

Motivated by the discussion in the previous section let us start with an economy characterized by an aggregate production function, and a representative household. The optimal growth problem in discrete time with no uncertainty, no population growth and no technological progress can be written as follows:

$$\max_{\{c(t),k(t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u\left(c\left(t\right)\right) \tag{5.19}$$

subject to

$$k\left(t+1\right) = f(k\left(t\right)) + \left(1-\delta\right) k\left(t\right) - c\left(t\right), \tag{5.20}$$

$k\left(t\right) \geq 0$ and given $k\left(0\right) = k_0 > 0$. The objective function is familiar and represents the discounted sum of the utility of the representative household. The constraint (5.20) is also straightforward to understand; total output per capita produced with capital-labor ratio $k\left(t\right)$, $f\left(k\left(t\right)\right)$, together with a fraction $1-\delta$ of the capital that is undepreciated make up the total resources of the economy at date t. Out of this resources $c\left(t\right)$ is spent as consumption per capita $c\left(t\right)$ and the rest becomes next period's capital-labor ratio, $k\left(t+1\right)$.

The optimal growth problem imposes that the social planner chooses an entire sequence of consumption levels and capital stocks, only subject to the resource constraint, (5.20). There are no additional equilibrium constraints.

We have also specified that the initial level of capital stock is $k\left(0\right)$, but this gives a single initial condition. We will see later that, in contrast to the basic Solow model, the solution to this problem will correspond to two, not one, differential equations. We will therefore need another boundary condition, but this will not take the form of an initial condition. Instead, this additional boundary condition will come from the optimality of a dynamic plan in the form of a *transversality condition*.

This maximization problem can be solved in a number of different ways, for example, by setting up an infinite dimensional Lagrangian. But the most convenient and common way of approaching it is by using *dynamic programming.*

It is also useful to note that even if we wished to bypass the Second Welfare Theorem and directly solve for competitive equilibria, we would have to solve a problem similar to the maximization of (5.19) subject to (5.20). In particular, to characterize the equilibrium, we would need to start with the maximizing behavior of households. Since the economy admits a representative household, we only need to look at the maximization problem of this consumer. Assuming that the representative household has one unit of labor supplied inelastically, this problem can be written as:

$$\max_{\{c(t),k(t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u\left(c\left(t\right)\right)$$

subject to some given $a\left(0\right)$ and

$$a\left(t+1\right) = r\left(t\right)\left[a\left(t\right) - c\left(t\right) + w\left(t\right)\right], \tag{5.21}$$

where $a\left(t\right)$ denotes the assets of the representative household at time t, $r\left(t\right)$ is the rate of return on assets and $w\left(t\right)$ is the equilibrium wage rate (and thus the wage earnings of the representative household). The constraint, (5.21) is the flow budget constraint, meaning that it links tomorrow's assets to today's assets. Here we need an additional condition so that this flow budget constraint eventually converges (i.e., so that $a\left(t\right)$ should not go to negative infinity). This can be ensured by imposing a lifetime budget constraint. Since a flow budget constraint in the form of (5.21) is both more intuitive and often more convenient to work with, we will not work with the lifetime budget constraint, but augment the flow budget constraint with another condition to rule out the level of wealth going to negative infinity. This condition will be introduced below.

5.10. Optimal Growth in Continuous Time

The formulation of the optimal growth problem in continuous time is very similar. In particular, we have

$$\max_{[c(t),k(t)]_{t=0}^{\infty}} \int_0^{\infty} \exp\left(-\rho t\right) u\left(c\left(t\right)\right) dt \tag{5.22}$$

subject to

$$\dot{k}\left(t\right) = f\left(k\left(t\right)\right) - c\left(t\right) - \delta k\left(t\right), \tag{5.23}$$

$k\left(t\right) \geq 0$ and given $k\left(0\right) = k_0 > 0$. The objective function (5.22) is the direct continuous-time analog of (5.19), and (5.23) gives the resource constraint of the economy, similar to (5.20) in discrete time.

Once again, this problem lacks one boundary condition which will come from the transversality condition.

The most convenient way of characterizing the solution to this problem is via *optimal control theory.* Dynamic programming and optimal control theory will be discussed briefly in the next two chapters.

5.11. Taking Stock

This chapter introduced the preliminaries necessary for an in-depth study of equilibrium and optimal growth theory. At some level it can be thought of as an "odds and ends" chapter, introducing the reader to the notions of representative household, dynamic optimization, welfare theorems and optimal growth. However, what we have seen is more than odds and ends, since a good understanding of the general equilibrium foundations on economic growth and the welfare theorems should enable the reader to better understand and appreciate the material that will be introduced in Part 3 below.

The most important take-away messages from this chapter are as follows. First, the set of models we study in this book are examples of more general dynamic general equilibrium models. It is therefore important to understand which features of the growth models are general (in the sense that they do not depend on the specific simplifying assumptions we make) and which results depend on the further simplifying assumptions we adopt. In this respect, the First and the Second Welfare Theorems are essential. They show that provided that all product and factor markets are competitive and there are no externalities in production or consumption (and under some relatively mild technical assumptions), dynamic competitive equilibrium will be Pareto optimal and that any Pareto optimal allocation can be decentralized as a dynamic competitive equilibrium. These results will be relevant for the first part of the book, where our focus will be on competitive economies. They will not be as relevant (at least if used in their current form), when we turn to models of technological change, where product markets will be monopolistic or when we study certain classes of models of economic development, where various market imperfections will play an important role.

Second, the most general class of dynamic general equilibrium models will not be tractable enough for us to derive sharp results about the process of economic growth. For this reason, we we will often adopt a range of simplifying assumptions. The most important of those is the representative household assumption, which enables us to model the demand side of the economy as if it were generated by the optimizing behavior of a single household. We saw how this assumption is generally not satisfied, but also how a certain class of preferences, the Gorman preferences, enable us to model economies as if they admit a representative household. We also discussed how typical general equilibrium economies can be modeled as if they admit a representative firm.

In addition, in this chapter we introduced the first formulation of the optimal growth problems in discrete and in continuous time, which will be useful as examples in the next two chapters where we discuss the tools necessary for the study of dynamic optimization problems.

5.12. References and Literature

This chapter covered a lot of ground and in most cases, many details were omitted for brevity. Most readers will be familiar with much of the material in this chapter. Mas-Colell, Winston and Green (1995) have an excellent discussion of issues of aggregation and what types of models admit representative households. They also have a version of the Debreu-Mantel-Sonnenschein theorem, with a sketch proof. The representative firm theorem, Theorem 5.4, presented here is rather straightforward, but I am not aware of any other discussion of this theorem in the literature. It is important to distinguish the subject matter of this theorem from the Cambridge controversy in early growth theory, which revolved around the issue of whether different types of capital goods could be aggregated into a single capital index (see, for example, Wan, 1969). The representative firm theorem says nothing about this issue.

The best reference for existence of competitive equilibrium and the welfare theorems with a finite number of consumers and a finite number of commodities is still Debreu's (1959) *Theory of Value*. This short book introduces all of the mathematical tools necessary for general equilibrium theory and gives a very clean exposition. Equally lucid and more modern are the treatments of the same topics in Mas-Colell, Winston and Green (1995) and Bewley (2006). The reader may also wish to consult Mas-Colell, Winston and Green (1995, Chapter 16) for a full proof of the Second Welfare Theorem with a finite number of commodities (which was only sketched in Theorem 5.7 above). Both of these books also have an excellent discussion of the necessary restrictions on preferences so that they can be represented by utility functions. Mas-Colell, Winston and Green (1995) also has an excellent discussion of expected utility theory of von Neumann and Morgenstern, which we have touched upon. Mas-Colell, Winston and Green (1995, Chapter 19) also gives a very clear discussion of the role of Arrow securities and the relationship between trading at the single point in time and sequential trading. The classic reference on Arrow securities is Arrow (1964).

Neither of these two references discuss infinite-dimensional economies. The seminal reference for infinite dimensional welfare theorems is Debreu (1954). Stokey, Lucas and Prescott (1989, Chapter 15) presents existence and welfare theorems for economies with a finite number of consumers and countably infinite number of commodities. The mathematical prerequisites for their treatment are greater than what has been assumed here, but their treatment is both thorough and straightforward to follow once the reader makes the investment in the necessary mathematical techniques. The most accessible reference for the Hahn-Banach Theorem, which is necessary for a proof of Theorem 5.7 in infinite-dimensional spaces are Kolmogorov and Fomin (1970), Kreyszig (1978) and Luenberger (1969). The latter is also an excellent source for all the mathematical techniques used in Stokey, Lucas and Prescott (1989) and also contains much material useful for appreciating continuous time optimization. Finally, a version of Theorem 5.6 is presented in Bewley (2006), which contains an excellent discussion of overlapping generations models.

5.13. Exercises

Exercise 5.1. Recall that a solution $\{x(t)\}_{t=0}^{T}$ to a dynamic optimization problem is *time-consistent* if the following is true: whenever $\{x(t)\}_{t=0}^{T}$ is an optimal solution starting at time $t=0$, $\{x(t)\}_{t=t'}^{T}$ is an optimal solution to the continuation dynamic optimization problem starting from time $t=t' \in [0,T]$.

(1) Consider the following optimization problem

$$\max_{\{x(t)\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t u(x(t))$$
$$\text{subject to}$$
$$x(t) \in [0,\bar{x}]$$
$$G(x(0),...,x(T)) \leq 0.$$

Although you do not need to, you may assume that G is continuous and convex, and u is continuous and concave.

Prove that any solution $\{x^*(t)\}_{t=0}^{T}$ to this problem is time consistent.

(2) Now Consider the optimization problem

$$\max_{\{x(t)\}_{t=0}^{T}} u(x(0)) + \delta \sum_{t=1}^{T} \beta^t u(x(t))$$
$$\text{subject to}$$
$$x(t) \in [0,\bar{x}]$$
$$G(x(0),...,x(T)) \leq 0.$$

Suppose that the objective function at time $t = 1$ becomes $u(x(1)) + \delta \sum_{t=2}^{T} \beta^{t-1} u(x(t))$.

Interpret this objective function (sometimes referred to as "hyperbolic discounting").

(3) Let $\{x^*(t)\}_{t=0}^{T}$ be a solution to this maximization problem. Assume that the individual chooses $x^*(0)$ at $t=0$, and then is allowed to reoptimize at $t=1$, i.e., solve

the problem

$$\max_{\{x(t)\}_{t=1}^{T}} u(x(1)) + \delta \sum_{t=2}^{T} \beta^{t-1} u(x(t))$$

subject to

$$x(t) \in [0, \bar{x}]$$

$$G(x^*(0), ..., x(T)) \leq 0.$$

Prove that the solution from $t = 1$ onwards, $\{x^{**}(t)\}_{t=1}^{T}$ is not necessarily the same as $\{x^*(t)\}_{t=1}^{T}$.

(4) Explain which standard axioms of preferences in basic general equilibrium theory are violated by those in parts 2 and 3 of this exercise.

Exercise 5.2. This exercise asks you to work through an example that illustrates the difference between the coefficient of relative risk aversion and the intertemporal elasticity of substitution. Consider a household with the following non-time-separable preferences over consumption levels at two dates:

$$V(c_1, c_2) = \mathbb{E}\left[\left(\frac{c_1^{1-\theta} - 1}{1-\theta}\right)^{\frac{\alpha-1}{\alpha}} + \beta\left(\frac{c_2^{1-\theta} - 1}{1-\theta}\right)^{\frac{\alpha-1}{\alpha}}\right]^{\frac{\alpha}{\alpha-1}},$$

where $\mathbb{E}$ is the expectations operator. The budget constraint of the household is

$$c_1 + \frac{1}{1+r} c_2 \leq W,$$

where r is the interest rate and W is its total wealth, which may be stochastic.

(1) Let us first suppose that W is nonstochastic and equal to $W_0 > 0$. Characterize the utility maximizing choice of c_1 and c_2.
(2) Compute the intertemporal elasticity of substitution.
(3) Now suppose that W is distributed over the support $[\underline{W}, \overline{W}]$ with some distribution function $G(W)$, where $0 < \underline{W} < \overline{W} < \infty$. Characterize the utility maximizing choice of c_1 and compute the coefficient of relative risk aversion. Provide conditions under which the coefficient of relative risk aversion is the same as the intertemporal elasticity of substitution. Explain why the two differ and interpret the conditions under which they are the same.

Exercise 5.3. Prove Theorem 5.2.

Exercise 5.4. Prove Theorem 5.3 when there is also production.

Exercise 5.5. * Generalize Theorem 5.3 to an economy with a continuum of commodities.

Exercise 5.6. (1) Derive the utility maximizing demands for consumers in Example 5.1 and show that the resulting indirect utility function for each consumer is given by (5.5).

(2) Show that maximization of (5.6) leads to the indirect utility function corresponding to the representative household.

(3) Now suppose that $U^i\left(x_1^i, ..., x_N^i\right) = \sum_{j=1}^{N}\left(x_j^i - \xi_j^i\right)^{\frac{\sigma-1}{\sigma}}$. Repeat the same computations and verify that the resulting indirect utility function is homogeneous of degree 0 in p and y, but does not satisfy the Gorman form. Show, however, that a monotonic transformation of the indirect utility function satisfies the Gorman form. Is this sufficient to ensure that the economy admits a representative household?

Exercise 5.7. Construct a continuous-time version of the model with finite lives and random death. In particular suppose that an individual faces a constant (Poisson) flow rate of death equal to $\nu > 0$ and has a true discount factor equal to ρ. Show that this individual will behave as if he is infinitely lived with an effective discount factor of $\rho + \nu$.

Exercise 5.8. (1) Will dynastic preferences as those discussed in Section 5.2 lead to infinite-horizon maximization if the instantaneous utility function of future generations are different (i.e., $u_t(\cdot)$ potentially different for each generation t)?

(2) How would the results be different if an individual cares about the continuation utility of his offspring with discount factor β, but also cares about the continuation utility of the offspring of his offspring with a smaller discount factor δ?

Exercise 5.9. Prove Theorem 5.8.

Exercise 5.10. Consider the sequential trading model discussed above and suppose now that individuals can trade bonds at time t that deliver one unit of good 0 at time t'. Denote the price of such bonds by $q_{t,t'}$.

(1) Rewrite the budget constraint of household h at time t, (5.18), including these bonds.
(2) Prove an equivalent of Theorem 5.8 in this environment with the extended set of bonds.

Exercise 5.11. Consider a two-period economy consisting of two types of households. N_A households have the utility function

$$u\left(c_1^i\right) + \beta_A u\left(c_2^i\right),$$

where c_1^i and c_2^i denotes the consumption of household i into two periods. The remaining N_B households have the utility function

$$u\left(c_1^i\right) + \beta_B u\left(c_2^i\right),$$

with $\beta_B < \beta_A$. Each group, respectively, has income y_A and y_B at date 1, and can save this to the second date at some exogenously given gross interest rate R. Show that for general $u(\cdot)$, this economy does not admit a representative household.

Exercise 5.12. Consider an economy consisting of N households each with utility function at time $t = 0$ given by

$$\sum_{t=0}^{\infty} \beta^t u\left(c^i(t)\right),$$

with $\beta \in (0, 1)$, where $c^i(t)$ denotes the consumption of household i at time t. The economy starts with an endowment of Y units of the final good and has access to no production technology. This endowment can be saved without depreciating or gaining interest rate between periods.

(1) What are the Arrow-Debreu commodities in this economy?
(2) Characterize the set of Pareto optimal allocations of this economy.
(3) Does Theorem 5.7 apply to this economy?
(4) Now consider an allocation of Y units to the households, $\left\{y^i\right\}_{i=1}^N$, such that $\sum_{i=1}^N y^i = Y$. Given this allocation, find the unique competitive equilibrium price vector and the corresponding consumption allocations.
(5) Are all competitive equilibria Pareto optimal?
(6) Now derive a redistribution scheme for decentralizing the entire set of Pareto optimal allocations?

Exercise 5.13. (1) Suppose that utility of individual i given by $\sum_{t=0}^{\infty} \beta^t u^i\left(x^i(t)\right)$, where $x^i(t) \in X \subset \mathbb{R}_+^K$, u^i is continuous, X is compact, and $\beta < 1$. Show that the hypothesis that for any $x, x' \in X^i$ with $u^i(x) > u^i(x')$, there exists $\bar{T}$ such that $u^i(x[T]) > u^i(x'[T])$ for all $T \geq \bar{T}$ in Theorem 5.7 is satisfied.
(2) Suppose that the production structure is given by a neoclassical production function, where the production vector at time t is only a function of inputs at time t and capital stock chosen at time $t-1$, and that higher capital so contributes to greater production and there is free disposal. Then show that the second hypothesis in Theorem 5.7 that for each $y \in Y^f$, there exists $\tilde{T}$ such that $y[T] \in Y^f$ for all $T \geq \tilde{T}$ is satisfied.

Exercise 5.14. *

(1) Show that Theorem 5.7 does not cover the one good neoclassical growth model with instantaneous preferences given by $u(c) = \left(c^{1-\theta} - 1\right) / (1-\theta)$ with $\theta \geq 1$.
(2) Prove Theorem 5.7 under the assumption that there exists a strictly positive vector $\varepsilon \in \mathbb{R}_+^K$ with each element sufficiently small and $(\varepsilon, ..., \varepsilon) \in X^i$ for all $i \in \mathcal{H}$.
(3) Show that this modified version of Theorem 5.7 covers the economy in case 1 of this exercise.

CHAPTER 6

Dynamic Programming and Optimal Growth

This chapter will provide a brief introduction to infinite horizon optimization in discrete time, focusing particularly on *stationary dynamic programming* problems under certainty. The main purpose of the chapter is to introduce the reader to dynamic programming techniques, which will be used in the rest of the book. Since dynamic programming has become an important tool in many areas of economics and especially in macroeconomics, a good understanding of these techniques is a prerequisite not only for economic growth, but also for the study of many diverse topics in economics.

The material in this chapter is presented in three parts. The first part provides a number of results necessary for applications of dynamic programming techniques in infinite-dimensional optimization problems. However, since understanding how these results are derived is important for a more thorough appreciation of the theory of dynamic programming and its applications, the second part, in particular, Sections 6.3 and 6.4, will provide additional tools necessary for a deeper understanding of dynamic programming and for the proofs of the main theorems. The material in these two sections is not necessary for the rest of the course and it is clearly marked, so that those who only wish to acquire a working knowledge of dynamic programming techniques can skip them. The third part then provides a more detailed discussion on how dynamic programming techniques can be used in applications and also presents a number of results on optimal growth using these tools.

Throughout this chapter, the focus is on discounted maximization problems under certainty, similar to the maximization problems introduced in the previous chapter. Dynamic optimization problems under uncertainty are discussed in Chapter 16.

6.1. Brief Review of Dynamic Programming

Using abstract but simple notation, the canonical dynamic optimization program in discrete time can be written as

$$\begin{aligned} \textbf{Problem A1} &: \\ V^*(x(0)) &= \sup_{\{x(t+1)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(x(t), x(t+1)) \\ & \quad \text{subject to} \\ x(t+1) &\in G(x(t)), \qquad \text{for all } t \geq 0 \\ & \quad x(0) \text{ given.} \end{aligned}$$

where $\beta \in (0,1)$, and $x(t)$ is a vector of variables, or more formally, $x(t) \in X \subset \mathbb{R}^K$ for some $K \geq 1$. $G(x)$ is a set-valued mapping, or a correspondence, also written as

$$G : X \rightrightarrows X$$

(see Appendix Chapter A), thus the first constraint basically specifies what values of $x(t+1)$ are allowed given the value $x(t)$. For this reason, we can think of $x(t)$ as the *state variable* (state vector) of the problem, while $x(t+1)$ plays the role of the *control variable* (control vector) at time t. Therefore, the constraint $x(t+1) \in G(x(t))$ determines which control variables can be chosen given the state variable. The real-valued function $U : X \times X \to \mathbb{R}$ is the instantaneous payoff function of this problem, and we have imposed that overall payoff (objective function) is a discounted sum of instantaneous payoffs.

In the problem formulation, we used "sup," i.e., the supremum, rather than max, since there is no guarantee that the maximal value is attained by any feasible plan. However, in all cases studied in this book the maximal value will be attained, so the reader may wish to substitute "max" for "sup". When the maximal value is attained by some sequence $\{x^*(t+1)\}_{t=0}^{\infty} \in X^{\infty}$, we refer to this as a solution or as an *optimal plan* (where X^{∞} is the infinite product of the set X, so that an element of X^{∞} is a sequence with each member in X).

Notice that this problem is *stationary* in the sense that the instantaneous payoff function U is not time-dependent; it only depends on $x(t)$ and $x(t+1)$. A more general formulation would be to have $U(x(t), x(t+1), t)$, but for most economic problems this added level of generality is not necessary. Yet another more general formulation would be to relax the discounted objective function, and write the objective function as

$$\sup_{\{x(t)\}_{t=0}^{\infty}} U(x(0), x(1), ...).$$

Again the added generality in this case is not particularly useful for most of the problems we are interested in, and the discounted objective function ensures *time-consistency* as discussed in the previous chapter.

Of particular importance for us in this chapter is the function $V^*(x(0))$, which can be thought of as the *value function*, meaning the value of pursuing the optimal strategy starting with initial state $x(0)$.

Problem A1 is somewhat abstract. However, it has the advantage of being tractable and general enough to nest many interesting economic applications. The next example shows how our canonical optimal growth problem can be put into this language.

Example 6.1. Recall the optimal growth problem from the previous chapter:

$$\max_{\{c(t),k(t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c(t))$$

subject to

$$k(t+1) = f(k(t)) - c(t) + (1-\delta)k(t),$$

$k(t) \geq 0$ and given $k(0)$. This problem maps into the general formulation here with a simple one-dimensional state and control variables. In particular, let $x(t) = k(t)$ and $x(t+1) = k(t+1)$. Then use the constraint to write:

$$c(t) = f(k(t)) - k(t+1) + (1-\delta)k(t),$$

and substitute this into the objective function to obtain:

$$\max_{\{k(t+1)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(f(k(t)) - k(t+1) + (1-\delta)k(t))$$

subject to $k(t) \geq 0$. Now it can be verified that this problem is a special case of Problem A1 with $U(k(t), k(t+1)) = u(f(k(t)) - k(t+1) + (1-\delta)k(t))$ and with the constraint correspondence $G(k(t))$ given by $k(t+1) \in [0, f(k(t)) + (1-\delta)k(t)]$.

Problem A1, also referred to as the *sequence problem*, is one of choosing an infinite sequence $\{x(t)\}_{t=0}^{\infty}$ from some (vector) space of infinite sequences (for example, $\{x(t)\}_{t=0}^{\infty} \in X^{\infty} \subset \mathcal{L}^{\infty}$, where $\mathcal{L}^{\infty}$ is the vector space of infinite sequences that are bounded with the $\|\cdot\|_{\infty}$ norm, which we will denote throughout by the simpler notation $\|\cdot\|$). Sequence problems sometimes have nice features, but their solutions are often difficult to characterize both analytically and numerically.

The basic idea of dynamic programming is to turn the sequence problem into a *functional equation*. That is, it is to transform the problem into one of finding a function rather than a sequence. The relevant functional equation can be written as follows:

Problem A2 :

$$V(x) = \sup_{y \in G(x)} \{U(x,y) + \beta V(y)\}, \text{ for all } x \in X, \tag{6.1}$$

where $V : X \to \mathbb{R}$ is a real-valued function. Intuitively, instead of explicitly choosing the sequence $\{x(t)\}_{t=0}^{\infty}$, in (6.1), we choose a *policy*, which determines what the control vector $x(t+1)$ should be for a given value of the state vector $x(t)$. Since instantaneous payoff function $U(\cdot,\cdot)$ does not depend on time, there is no reason for this policy to be time-dependent either, and we denote the control vector by y and the state vector by x. Then the problem can be written as making the right choice of y for any value of x. Mathematically, this corresponds to maximizing $V(x)$ for any $x \in X$. The only subtlety in (6.1) is the presence of the $V(\cdot)$ on the right-hand side, which will be explained below. This is also the reason why (6.1) is also called the *recursive formulation*—the function $V(x)$ appears both on the left and the right-hand sides of equation (6.1) and is thus defined recursively.

The functional equation in Problem A2 is also called the *Bellman equation*, after Richard Bellman, who was the first to introduce the dynamic programming formulation, though this formulation was anticipated by the economist Lloyd Shapley in his study of stochastic games.

At first sight, the recursive formulation might appear not as a great advance over the sequence formulation. After all, functions might be trickier to work with than sequences. Nevertheless, it turns out that the functional equation of dynamic programming is easy to work with in many instances. In applied mathematics and engineering, it is favored because it is computationally convenient. In economics, perhaps the major advantage of the recursive formulation is that it often gives better economic insights, similar to the logic of comparing today to tomorrow. In particular, $U(x,y)$ is the" return for today" and $\beta V(y)$ is the continuation return from tomorrow onwards, equivalent to the "return for tomorrow". Consequently, in many applications we can use our intuitions from two-period maximization or economic problems. Finally, in some special but important cases, the solution to Problem A2 is simpler to characterize analytically than the corresponding solution of the sequence problem, Problem A1.

In fact, the form of Problem A2 suggests itself naturally from the formulation Problem A1. Suppose Problem A1 has a maximum starting at $x(0)$ attained by the optimal sequence $\{x^*(t)\}_{t=0}^{\infty}$ with $x^*(0) = x(0)$. Then under some relatively weak technical conditions, we have that

$$\begin{aligned} V^*(x(0)) &= \sum_{t=0}^{\infty} \beta^t U(x^*(t), x^*(t+1)) \\ &= U(x(0), x^*(1)) + \beta \sum_{s=0}^{\infty} \beta^s U(x^*(s+1), x^*(s+2)) \\ &= U(x(0), x^*(1)) + \beta V^*(x^*(1)). \end{aligned}$$

This equation encapsulates the basic idea of dynamic programming: *the Principle of Optimality*, and it is stated more formally in Theorem 6.2.

Essentially, an optimal plan can be broken into two parts, what is optimal to do today, and the optimal continuation path. Dynamic programming exploits this principle and provides us with a set of powerful tools to analyze optimization in discrete-time infinite-horizon problems.

As noted above, the particular advantage of this formulation is that the solution can be represented by a time invariant *policy function* (or policy mapping),

$$\pi : X \to X,$$

determining which value of $x(t+1)$ to choose for a given value of the state variable $x(t)$. In general, however, there will be two complications: first, a control reaching the optimal value may not exist, which was the reason why we originally used the notation sup; second, we may not have a policy function, but a policy correspondence, $\Pi : X \rightrightarrows X$, because there may be more than one maximizer for a given state variable. Let us ignore these complications for now and present a heuristic exposition. These issues will be dealt with below.

Once the value function V is determined, the policy function is given straightforwardly. In particular, by definition it must be the case that if optimal policy is given by a policy function $\pi(x)$, then

$$V(x) = U(x, \pi(x)) + \beta V(\pi(x)), \text{ for all } x \in X,$$

which is one way of determining the policy function. This equation simply follows from the fact that $\pi(x)$ is the optimal policy, so when $y = \pi(x)$, the right-hand side of (6.1) reaches the maximal value $V(x)$.

The usefulness of the recursive formulation in Problem A2 comes from the fact that there are some powerful tools which not only establish existence of the solution, but also some of its properties. These tools are not only employed in establishing the existence of a solution to Problem A2, but they are also useful in a range of problems in economic growth, macroeconomics and other areas of economic dynamics.

The next section states a number of results about the relationship between the solution to the sequence problem, Problem A1, and the recursive formulation, Problem A2. These results will first be stated informally, without going into the technical details. Section 6.3 will then present these results in greater formality and provide their proofs.

6.2. Dynamic Programming Theorems

Let us start with a number of assumptions on Problem A1. Since these assumptions are only relevant for this section, we number them separately from the main assumptions used throughout the book. Consider first a sequence $\{x^*(t)\}_{t=0}^{\infty}$ which attains the supremum in Problem A1. Our main purpose is to ensure that this sequence will satisfy the recursive equation of dynamic programming, written here as

$$V(x^*(t)) = U(x^*(t), x^*(t+1)) + \beta V(x^*(t+1)), \text{ for all } t = 0, 1, 2, ..., \tag{6.2}$$

and that any solution to (6.2) will also be a solution to Problem A1, in the sense that it will attain its supremum. In other words, we are interested in in establishing equivalence results between the solutions to Problem A1 and Problem A2.

To prepare for these results, let us define the set of feasible sequences or *plans* starting with an initial value $x(t)$ as:

$$\mathbf{\Phi}(x(t)) = \{\{x(s)\}_{s=t}^{\infty} : x(s+1) \in G(x(s)), \text{ for } s = t, t+1, ...\}.$$

Intuitively, $\Phi(x(t))$ is the set of feasible choices of vectors starting from $x(t)$. Let us denote a typical element of the set $\Phi(x(0))$ by $\mathbf{x} = (x(0), x(1), ...) \in\Phi(x(0))$. Our first assumption is:

Assumption 6.1. $G(x)$ is nonempty for all $x \in X$; and for all $x(0) \in X$ and $\mathbf{x} \in\Phi(x(0))$, $\lim_{n\to\infty} \sum_{t=0}^{n} \beta^t U(x(t), x(t+1))$ exists and is finite.

This assumption is stronger than what is necessary to establish the results that will follow. In particular, for much of the theory of dynamic programming, it is sufficient that the limit in Assumption 6.1 exists. However, in economic applications, we are not interested in optimization problems where households or firms achieve infinite value. This is for two obvious reasons. First, when some agents can achieve infinite value, the mathematical problems are typically not well defined. Second, the essence of economics, tradeoffs in the face of scarcity, would be absent in these cases. In cases, where households can achieve infinite value, economic analysis is still possible, by using methods sometimes called "overtaking criteria," whereby different sequences that give infinite utility are compared by looking at whether one of them gives higher utility than the other one at each date after some finite threshold. None of the models we study in this book require us to consider these more general optimality concepts.

Assumption 6.2. X is a compact subset of $\mathbb{R}^K$, G is nonempty, compact-valued and continuous. Moreover, $U : \mathbf{X}_G \to \mathbb{R}$ is continuous, where $\mathbf{X}_G = \{(x, y) \in X \times X : y \in G(x)\}$.

This assumption is also natural. We need to impose that $G(x)$ is compact-valued, since optimization problems with choices from non–compact sets are not well behaved (see Appendix Chapter A). In addition, the assumption that U is continuous leads to little loss of generality for most economic applications. In all the models we will encounter in this book, U will be continuous. The most restrictive assumption here is that the state variable lies in a compact set, i.e., that X is compact. The most important results in this chapter can be generalized to the case in which X is not compact, though this requires additional notation and somewhat more difficult analysis. The case in which X is not compact is important in the analysis of economic growth, since most interesting models of growth will involve the state variable, e.g., the capital stock, growing steadily. In many cases, with a convenient normalization, the mathematical problem can be turned in to one in which the state variable lies in a compact set. One class of important problems that cannot be treated without allowing for non-compact X are those with endogenous growth. However, since the methods developed in the next chapter do not require this type of compactness assumption and since I will often use continuous-time methods to study endogenous growth models, I simplify the discussion here by assuming that X is compact.

Note also that since X is compact, $G(x)$ is continuous and compact-valued, $\mathbf{X}_G$ is also compact. Since a continuous function from a compact domain is also bounded, Assumption 6.2 also implies that U is bounded, which will be important for some of the results below.

Assumptions 6.1 and 6.2 together ensure that in both Problems A1 and A2, the supremum (the maximal value) is attained for some feasible plan $\mathbf{x}$. We state all the relevant theorems incorporating this fact.

To obtain sharper results, we will also impose:

Assumption 6.3. U is strictly concave, in the sense that for any $\alpha \in (0,1)$ and any (x,y), $(x',y') \in \mathbf{X}_G$, we have

$$U\left(\alpha x + (1-\alpha)x', \alpha y + (1-\alpha)y'\right) \geq \alpha U(x,y) + (1-\alpha)U(x',y'),$$

and if $x \neq x'$,

$$U\left(\alpha x + (1-\alpha)x', \alpha y + (1-\alpha)y'\right) > \alpha U(x,y) + (1-\alpha)U(x',y').$$

Moreover, G is convex in the sense that for any $\alpha \in [0,1]$, and $x, x' \in X$, whenever $y \in G(x)$ and $y' \in G(x')$, then we have

$$\alpha y + (1-\alpha)y' \in G\left(\alpha x + (1-\alpha)x'\right).$$

This assumption imposes conditions similar to those used in many economic applications: the constraint set is assumed to be convex and the objective function is concave or strictly concave.

Our next assumption puts some more structure on the objective function, in particular it ensures that the objective function is increasing in the state variables (its first K arguments), and that greater levels of the state variables are also attractive from the viewpoint of relaxing the constraints; i.e., a greater x means more choice.

Assumption 6.4. For each $y \in X$, $U(\cdot, y)$ is strictly increasing in each of its first K arguments, and G is monotone in the sense that $x \leq x'$ implies $G(x) \subset G(x')$.

The final assumption we will impose is that of differentiability and is also common in most economic models. This assumption will enable us to work with first-order necessary conditions.

Assumption 6.5. U is continuously differentiable on the interior of its domain $\mathbf{X}_G$.

Given these assumptions, the following sequence of results can be established. The proofs for these results are provided in Section 6.4.

Theorem 6.1. ***(Equivalence of Values)*** *Suppose Assumptions 6.1 and 6.2 hold. Then for any* $x \in X$, $V^*(x)$ *defined in Problem A1 is also a solution to Problem A2. Moreover, any* $V(x)$ *defined in Problem A2 that satisfies* $\lim_{t\to\infty} \beta^t V(x(t)) = 0$ *for all* $(x, x(1), x(2), ...) \in \Phi(x)$ *is also a solution to Problem A1, so that* $V^*(x) = V(x)$ *for all* $x \in X$.

Therefore, both the sequence problem and the recursive formulation achieve the same value. While important, this theorem is not of direct relevance in most economic applications, since we do not care about the value but we care about the optimal plans (actions). This is dealt with in the next theorem.

Theorem 6.2. ***(Principle of Optimality)*** *Suppose Assumption 6.1 holds. Let* $\mathbf{x}^* \in \Phi(x(0))$ *be a feasible plan that attains* $V^*(x(0))$ *in Problem A1. Then we have that*

$$V^*(x^*(t)) = U(x^*(t), x^*(t+1)) + \beta V^*(x^*(t+1)) \tag{6.3}$$

for $t = 0, 1, ...$ *with* $x^*(0) = x(0)$.

Moreover, if any $\mathbf{x}^* \in \Phi(x(0))$ *satisfies (6.3), then it attains the optimal value in Problem A1.*

This theorem is the major conceptual result in the theory of dynamic programming. It states that the returns from an optimal plan (sequence) $\mathbf{x}^* \in \Phi(x(0))$ can be broken into two parts; the current return, $U(x^*(t), x^*(t+1))$, and the continuation return $\beta V^*(x^*(t+1))$, where the continuation return is identically given by the discounted value of a problem starting from the state vector from tomorrow onwards, $x^*(t+1)$. In view of the fact that V^* in Problem A1 and V in Problem A2 are identical from Theorem 6.1, (6.3) also implies

$$V(x^*(t)) = U(x^*(t), x^*(t+1)) + \beta V(x^*(t+1)).$$

Notice also that the second part of Theorem 6.2 is equally important. It states that if any feasible plan $\mathbf{x}^*$ starting with $x(0)$, that is, $\mathbf{x}^* \in \Phi(x(0))$, satisfies (6.3), then $\mathbf{x}^*$ attains $V^*(x(0))$.

Therefore, this theorem states that we can go from the solution of the recursive problem to the solution of the original problem and vice versa. Consequently, under Assumptions 6.1 and 6.2, there is no risk of excluding solutions in writing the problem recursively.

The next results summarize certain important features of the value function V in Problem A2. These results will be useful in characterizing qualitative features of optimal plans in dynamic optimization problems without explicitly finding the solutions.

Theorem 6.3. ***(Existence of Solutions)*** *Suppose that Assumptions 6.1 and 6.2 hold. Then there exists a unique continuous and bounded function $V : X \to \mathbb{R}$ that satisfies (6.1). Moreover, an optimal plan $\mathbf{x}^* \in \Phi(x(0))$ exists for any $x(0) \in X$.*

This theorem establishes two major results. The first is the uniqueness of the value function (and hence of the Bellman equation) in dynamic programming problems. Combined with Theorem 6.1, this result naturally implies that an optimal solution that achieves the supremum V^* in Problem A1 and also that like V, V^* is continuous and bounded. The optimal solution to Problem A1 or A2 may not be unique, however, even though the value function is unique. This may be the case when two alternative feasible sequences achieve the same maximal value. As in static optimization problems, non-uniqueness of solutions is a consequence of lack of strict concavity of the objective function. When the conditions are strengthened by including Assumption 6.3, uniqueness of the optimum will plan is guaranteed. To obtain this result, we first prove:

Theorem 6.4. ***(Concavity of the Value Function)*** *Suppose that Assumptions 6.1, 6.2 and 6.3 hold. Then the unique $V : X \to \mathbb{R}$ that satisfies (6.1) is strictly concave.*

Combining the previous two theorems we have:

Corollary 6.1. *Suppose that Assumptions 6.1, 6.2 and 6.3 hold. Then there exists a unique optimal plan $\mathbf{x}^* \in \Phi(x(0))$ for any $x(0) \in X$. Moreover, the optimal plan can be expressed as $x^*(t+1) = \pi(x^*(t))$, where $\pi : X \to X$ is a continuous policy function.*

The important result in this corollary is that the "policy function" π is indeed a function, not a correspondence. This is a consequence of the fact that x^* is uniquely determined. This result also implies that the policy mapping π is continuous in the state vector. Moreover, if there exists a vector of parameters $\mathbf{z}$ continuously affecting either the constraint correspondence Φ or the instantaneous payoff function U, then the same argument establishes that π is also continuous in this vector of parameters. This feature will enable qualitative analysis of dynamic macroeconomic models under a variety of circumstances.

Our next result shows that under Assumption 6.4, we can also establish that the value function V is strictly increasing.

Theorem 6.5. ***(Monotonicity of the Value Function)*** *Suppose that Assumptions 6.1, 6.2 and 6.4 hold and let $V : X \to \mathbb{R}$ be the unique solution to (6.1). Then V is strictly increasing in all of its arguments.*

Finally, our purpose in developing the recursive formulation is to use it to characterize the solution to dynamic optimization problems. As with static optimization problems, this is often made easier by using differential calculus. The difficulty in using differential calculus with (6.1) is that the right-hand side of this expression includes the value function V, which is endogenously determined. We can only use differential calculus when we know from more primitive arguments that this value function is indeed differentiable. The next theorem ensures that this is the case and also provides an expression for the derivative of the value function, which corresponds to a version of the familiar Envelope Theorem. Recall that $\text{Int}X$ denotes the interior of the set X, $D_x f$ denotes the gradient of the function f with respect to the vector x, and Df denotes the gradient of the function f with respect to all of its arguments (see Appendix Chapter A).

Theorem 6.6. ***(Differentiability of the Value Function)*** *Suppose that Assumptions 6.1, 6.2, 6.3 and 6.5 hold. Let π be the policy function defined above and assume that $x' \in \text{Int}X$ and $\pi(x') \in \text{Int}G(x')$, then $V(x)$ is continuously differentiable at x', with derivative given by*

$$DV\left(x'\right) = D_x U\left(x', \pi\left(x'\right)\right). \tag{6.4}$$

These results will enable us to use dynamic programming techniques in a wide variety of dynamic optimization problems. Before doing so, we discuss how these results are proved. The next section introduces a number of mathematical tools from basic functional analysis necessary for proving some of these theorems and Section 6.4 provides the proofs of all the results stated in this section.

6.3. The Contraction Mapping Theorem and Applications*

In this section, I present a number of mathematical results that are necessary for making progress with the dynamic programming formulation. In this sense, the current section is a "digression" from the main story line. Therefore, this section and the next can be skipped without interfering with the study of the rest of the book. Nevertheless, the material in this and the next section are useful for a good understanding of foundations of dynamic programming and should enable the reader to achieve a better understanding of these methods. The reader may also wish to consult Appendix Chapter A before reading this section.

Recall from Appendix Chapter A that (S, d) is a metric space, if S is a space and d is a metric defined over this space with the usual properties. The metric is referred to as "d" since it loosely corresponds to the "distance" between two elements of S. A metric space is more general than a finite dimensional Euclidean space such as a subset of $\mathbb{R}^K$. But as with the Euclidean space, we are most interested in defining "functions" from the metric space into itself. We will refer to these functions as *operators* or *mappings* to distinguish them from real-valued functions. Such operators are often denoted by the letter T and standard notation often involves writing Tz for the image of a point $z \in S$ under T (rather than the more intuitive and familiar $T(z)$), and using the notation $T(Z)$ when the operator T is applied to a subset Z of S. We will use this standard notation here.

Definition 6.1. *Let (S, d) be a metric space and $T : S \to S$ be an operator mapping S into itself. T is a* contraction mapping *(with* modulus *β) if for some $\beta \in (0, 1)$,*

$$d(Tz_1, Tz_2) \leq \beta d(z_1, z_2), \text{ for all } z_1, z_2 \in S.$$

In other words, a contraction mapping brings elements of the space S "closer" to each other.

Example 6.2. Let us take a simple interval of the real line as our space, $S = [a, b]$, with usual metric of this space $d(z_1, z_2) = |z_1 - z_2|$. Then $T : S \to S$ is a contraction if for some $\beta \in (0, 1)$,

$$\frac{|Tz_1 - Tz_2|}{|z_1 - z_2|} \leq \beta < 1, \quad \text{all } z_1, z_2 \in S \text{ with } z_1 \neq z_2.$$

Definition 6.2. *A* fixed point *of T is any element of S satisfying $Tz = z$.*

Recall also that a metric space (S, d) is complete if every Cauchy sequence (whose elements are getting closer) in S converges to an element in S (see Appendix Chapter A). Despite its simplicity, the following theorem is one of the most powerful results in functional analysis.

Theorem 6.7. ***(Contraction Mapping Theorem)*** *Let (S, d) be a complete metric space and suppose that $T : S \to S$ is a contraction. Then T has a unique fixed point, $\hat{z}$, i.e., there exists a unique $\hat{z} \in S$ such that*

$$T\hat{z} = \hat{z}.$$

Proof. *(Existence)* Note $T^n z = T(T^{n-1}z)$ for any $n = 1, 2, \ldots$. Choose $z_0 \in S$, and construct a sequence $\{z_n\}_{n=0}^{\infty}$ with each element in S, such that $z_{n+1} = Tz_n$ so that

$$z_n = T^n z_0.$$

Since T is a contraction, we have that

$$d(z_2, z_1) = d(Tz_1, Tz_0) \leq \beta d(z_1, z_0).$$

Repeating this argument

$$d(z_{n+1}, z_n) \leq \beta^n d(z_1, z_0), \quad n = 1, 2, ... \tag{6.5}$$

Hence, for any $m > n$,

$$\begin{aligned} d(z_m, z_n) &\leq d(z_m, z_{m-1}) + ... + d(z_{n+2}, z_{n+1}) + d(z_{n+1}, z_n) \\ &\leq \left(\beta^{m-1} + ... + \beta^{n+1} + \beta^n\right) d(z_1, z_0) \\ &= \beta^n \left(\beta^{m-n-1} + ... + \beta + 1\right) d(z_1, z_0) \\ &\leq \frac{\beta^n}{1-\beta} d(z_1, z_0), \end{aligned} \tag{6.6}$$

where the first inequality uses the triangle inequality (which is true for any metric d, see Appendix Chapter A). The second inequality uses (6.5). The last inequality uses the fact that $1/(1-\beta) = 1 + \beta + \beta^2 + ... > \beta^{m-n-1} + ... + \beta + 1$.

The string of inequalities in (6.6) imply that as $n \to \infty$, $m \to \infty$, z_m and z_n will be approaching each other, so that $\{z_n\}_{n=0}^{\infty}$ is a Cauchy sequence. Since S is complete, every Cauchy sequence in S has an limit point in S, therefore:

$$z_n \to \hat{z} \in S.$$

The next step is to show that $\hat{z}$ is a fixed point. Note that for any $z_0 \in S$ and any $n \in \mathbb{N}$, we have

$$\begin{aligned} d(T\hat{z}, \hat{z}) &\leq d(T\hat{z}, T^n z_0) + d(T^n z_0, \hat{z}) \\ &\leq \beta d(\hat{z}, T^{n-1} z_0) + d(T^n z_0, \hat{z}), \end{aligned}$$

where the first relationship again uses the triangle inequality, and the second inequality utilizes the fact that T is a contraction. Since $z_n \to \hat{z}$, both of the terms on the right tend to zero as $n \to \infty$, which implies that $d(T\hat{z}, \hat{z}) = 0$, and therefore $T\hat{z} = \hat{z}$, establishing that $\hat{z}$ is a fixed point.

(Uniqueness) Suppose, to obtain a contradiction, that there exist $\hat{z}, z \in S$, such that $Tz = z$ and $T\hat{z} = \hat{z}$ with $\hat{z} \neq z$. This implies

$$0 < d(\hat{z}, z) = d(T\hat{z}, Tz) \leq \beta d(\hat{z}, z),$$

which delivers a contradiction in view of the fact that $\beta < 1$ and establishes uniqueness. □

The Contraction Mapping Theorem can be used to prove many well-known results. The next example and Exercise 6.4 show how it can be used to prove existence of unique solutions to differential equations. Exercise 6.5 shows how it can be used to prove the Implicit Function Theorem (Theorem A.23 in Appendix Chapter A).

Example 6.3. Consider the following one-dimensional differential equation

$$\dot{x}(t) = f(x(t)), \tag{6.7}$$

with a boundary condition $x(0) = c \in \mathbb{R}$. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous in the sense that it is continuous and also for some $M < \infty$, it satisfies the following boundedness condition, $|f(x'') - f(x')| \leq M |x'' - x'|$ for all $x', x'' \in \mathbb{R}$. The Contraction Mapping Theorem, Theorem 6.7, can be used to prove the existence of a continuous function $x^*(t)$ that is the unique solution to this differential equation on any compact interval, in particular on $[0, s]$ for some $s \in \mathbb{R}_+$. To do this, consider the space of continuous functions on $[0, s]$, $\mathbf{C}[0, s]$, and define the following operator, T such that for any $g \in \mathbf{C}[0, s]$,

$$Tg(z) = c + \int_0^z f(g(x))\, dx.$$

Notice that T is a mapping from the space of continuous functions on $[0, s]$ into itself, i.e., $T : \mathbf{C}[0, s] \to \mathbf{C}[0, s]$. Moreover, it can be verified T is a contraction for some s. This follows because for any $z \in [0, s]$, we have

$$\left| \int_0^z f(g(x))\, dx - \int_0^z f(\tilde{g}(x))\, dx \right| \leq \int_0^z M |g(x) - \tilde{g}(x)|\, dx \tag{6.8}$$

by the Lipschitz continuity of $f(\cdot)$. This implies that

$$\|Tg(z) - T\tilde{g}(z)\| \leq M \times s \times \|g - \tilde{g}\|,$$

where recall that $\|\cdot\|$ denotes the sup norm, now defined over the space of functions. Choosing $s < 1/M$ establishes that for s sufficiently small, T is indeed a contraction. Then applying Theorem 6.7, we can conclude that there exists a unique fixed point of T over $\mathbf{C}[0, s]$. This fixed point is the unique solution to the differential equation and it is also continuous. Exercise 6.4 will ask you to verify some of these steps and also suggest how the result can be extended so that it applies to $\mathbf{C}[0, s]$ for any $s \in \mathbb{R}_+$.

The main use of the Contraction Mapping Theorem for us is that it can be applied to any metric space, so in particular to the space of functions. Applying it to equation (6.1) will establish the existence of a unique value function V in Problem A2, greatly facilitating the analysis of such dynamic models. Naturally, for this we have to prove that the recursion in (6.1) defines a contraction mapping. We will see below that this is often straightforward.

Before doing this, let us consider another useful result. Recall that if (S, d) is a complete metric space and S' is a closed subset of S, then (S', d) is also a complete metric space.

Theorem 6.8. ***(Applications of Contraction Mappings)*** *Let (S, d) be a complete metric space, $T : S \to S$ be a contraction mapping with $T\hat{z} = \hat{z}$.*

1. *If S' is a closed subset of S, and $T(S') \subset S'$, then $\hat{z} \in S'$.*
2. *Moreover, if $T(S') \subset S'' \subset S'$, then $\hat{z} \in S''$.*

PROOF. Take $z_0 \in S'$, and construct the sequence $\{T^n z_0\}_{n=0}^{\infty}$. Each element of this sequence is in S' by the fact that $T(S') \subset S'$. Theorem 6.7 implies that $T^n z_0 \to \hat{z}$. Since S' is closed, $\hat{z} \in S'$, proving part 1 in the theorem.

We know that $\hat{z} \in S'$. Then the fact that $T(S') \subset S'' \subset S'$ implies that $\hat{z} = T\hat{z} \in T(S') \subset S''$, establishing part 2. □

The second part of this theorem is very important to prove results such as strict concavity or that a function is strictly increasing. This is because the set of strictly concave functions or the set of the strictly increasing functions are not closed (and complete). Therefore, we cannot apply the Contraction Mapping Theorem to these spaces of functions. The second part of this theorem enables us to circumvent this problem.

The previous two theorems show that the contraction mapping property is both simple and powerful. We will see how powerful it is as we apply to obtain several important results below. Nevertheless, beyond some simple cases, such as Example 6.2, it is difficult to check whether an operator is indeed a contraction. This may seem particularly difficult in the case of spaces whose elements correspond to functions, which are those that are relevant in the context of dynamic programming. The next theorem provides us with sufficient conditions for an operator to be a contraction that are typically straightforward to check. For this theorem, let us use the following notation: for a real valued function $f(\cdot)$ and some constant $c \in \mathbb{R}$ we define $(f + c)(x) \equiv f(x) + c$. Then:

THEOREM 6.9. ***(Blackwell's Sufficient Conditions For a Contraction)*** *Let $X \subseteq \mathbb{R}^K$, and $\mathbf{B}(X)$ be the space of bounded functions $f : X \to \mathbb{R}$ defined on X. Suppose that $T : \mathbf{B}(X) \to \mathbf{B}(X)$ is an operator satisfying the following two conditions:*

(1) ***(monotonicity)*** *For any $f, g \in \mathbf{B}(X)$ and $f(x) \leq g(x)$ for all $x \in X$ implies $(Tf)(x) \leq (Tg)(x)$ for all $x \in X$.*
(2) ***(discounting)*** *There exists $\beta \in (0, 1)$ such that*

$$[T(f + c)](x) \leq (Tf)(x) + \beta c, \quad \text{for all } f \in B(X),\ c \geq 0 \text{ and } x \in X.$$

Then, T is a contraction with modulus β.

PROOF. Let $\|\cdot\|$ denote the sup norm, so that $\|f - g\| = \max_{x \in X} |f(x) - g(x)|$. Then, by definition for any $f, g \in \mathbf{B}(X)$,

$$\begin{aligned} f(x) &\leq g(x) + \|f - g\| && \text{for any } x \in X, \\ (Tf)(x) &\leq T[g + \|f - g\|](x) && \text{for any } x \in X, \\ (Tf)(x) &\leq (Tg)(x) + \beta \|f - g\| && \text{for any } x \in X, \end{aligned}$$

where the second line applies the operator T on both sides and uses monotonicity, and the third line uses discounting (together with the fact that $\|f - g\|$ is simply a number). By the converse argument,

$$\begin{aligned} g(x) &\leq f(x) + \|g-f\| && \text{for any } x \in X, \\ (Tg)(x) &\leq T[f + \|g-f\|](x) && \text{for any } x \in X, \\ (Tg)(x) &\leq (Tf)(x) + \beta\|g-f\| && \text{for any } x \in X. \end{aligned}$$

Combining the last two inequalities implies

$$\|Tf - Tg\| \leq \beta \|f-g\|,$$

proving that T is a contraction. □

We will see that Blackwell's sufficient conditions are straightforward to check in many economic applications, including the models of optimal or equilibrium growth.

6.4. Proofs of the Main Dynamic Programming Theorems*

We now prove Theorems 6.1-6.6. We start with a straightforward lemma, which will be useful in these proofs. For a feasible infinite sequence $\mathbf{x} = (x(0), x(1), ...) \in \Phi(x(0))$ starting at $x(0)$, let

$$\bar{\mathbf{U}}(\mathbf{x}) \equiv \sum_{t=0}^{\infty} \beta^t U(x(t), x(t+1))$$

be the value of choosing this potentially non-optimal infinite feasible sequence. In view of Assumption 6.1, $\bar{\mathbf{U}}(\mathbf{x})$ exists and is finite. The next lemma shows that $\bar{\mathbf{U}}(\mathbf{x})$ can be separated into two parts, the current return and the continuation return.

Lemma 6.1. *Suppose that Assumption 6.1 holds. Then for any $x(0) \in X$ and any $\mathbf{x} \in \Phi(x(0))$, we have that*

$$\bar{\mathbf{U}}(\mathbf{x}) = U(x(0), x(1)) + \beta \bar{\mathbf{U}}(\mathbf{x}')$$

where $\mathbf{x}' = (x(1), x(2), ...)$.

Proof. Since under Assumption 6.1 $\bar{\mathbf{U}}(\mathbf{x})$ exists and is finite, we have

$$\begin{aligned} \bar{\mathbf{U}}(\mathbf{x}) &= \sum_{t=0}^{\infty} \beta^t U(x(t), x(t+1)) \\ &= U(x(0), x(1)) + \beta \sum_{s=0}^{\infty} \beta^s U(x(s+1), x(s+2)) \\ &= U(x(0), x(1)) + \beta \bar{\mathbf{U}}(\mathbf{x}') \end{aligned}$$

as defined in the lemma. □

We start with the proof of Theorem 6.1. Before providing this proof, it is useful to be more explicit about what it means for V and V^* to be solutions to Problems A1 and A2. Let us start with Problem A1. Using the notation introduced in this section, we can write that for any $x(0) \in X$,

$$V^*(x(0)) = \sup_{\mathbf{x} \in \mathbf{\Phi}(x(0))} \bar{\mathbf{U}}(\mathbf{x}).$$

In view of Assumption 6.1, which ensures that all values are bounded, this immediately implies

$$V^*(x(0)) \geq \bar{\mathbf{U}}(\mathbf{x}) \text{ for all } \mathbf{x} \in \mathbf{\Phi}(x(0)), \tag{6.9}$$

since no other feasible sequence of choices can give higher value than the supremum, $V^*(x(0))$. However, if some function $\tilde{V}$ satisfies condition (6.9), so will $\alpha\tilde{V}$ for $\alpha > 1$. Therefore, this condition is not sufficient. In addition, we also require that

$$\text{for any } \varepsilon > 0, \text{ there exists } \mathbf{x}' \in \mathbf{\Phi}(x(0)) \text{ s.t. } V^*(x(0)) \leq \bar{\mathbf{U}}(\mathbf{x}') + \varepsilon, \tag{6.10}$$

The conditions for $V(\cdot)$ to be a solution to Problem A2 are similar. For any $x(0) \in X$,

$$V(x(0)) \geq U(x(0), y) + \beta V(y), \quad \text{all } y \in G(x(0)), \tag{6.11}$$

and

$$\text{for any } \varepsilon > 0, \text{ there exists } y' \in G(x(0)) \text{ s.t. } V(x(0)) \leq U(x(0), y') + \beta V(y) + \varepsilon. \tag{6.12}$$

We now have:

Proof of Theorem 6.1. If $\beta = 0$, Problems A1 and A2 are identical, thus the result follows immediately. Suppose that $\beta > 0$ and take an arbitrary $x(0) \in X$ and some $x(1) \in G(x(0))$. The objective function in Problem A1 is continuous in the product topology in view of Assumptions 6.1 and 6.2 (see Theorem A.11 in Appendix Chapter A). Moreover, the constraint set $\Phi(x(0))$ is a closed subset of X^∞ (infinite product of X). From Assumption 6.1, X is compact. By Tychonoff's Theorem, Theorem A.12, X^∞ is compact in the product topology. A closed subset of a compact set is compact (Fact A.2 in Appendix Chapter A), which implies that $\Phi(x(0))$ is compact. Thus we can apply Weierstrass' Theorem, Theorem A.9, to Problem A1, to conclude that there exists $\mathbf{x} \in \Phi(x(0))$ attaining $V^*(x(0))$. Moreover, the constraint set is a continuous correspondence (again in the product topology), so Berge's Maximum Theorem, Theorem A.13, implies that $V^*(x(0))$ is continuous. Since $x(0) \in X$ and X is compact, this implies that $V^*(x(0))$ is bounded (Corollary A.1 in Appendix Chapter A). A similar reasoning implies that there exists $\mathbf{x}' \in \Phi(x(1))$ attaining $V^*(x(1))$. Next, since $(x(0), \mathbf{x}') \in \Phi(x(0))$ and $V^*(x(0))$ is the supremum in Problem A1 starting with $x(0)$, Lemma 6.1 implies

$$\begin{aligned} V^*(x(0)) &\geq U(x(0), x(1)) + \beta \bar{\mathbf{U}}(\mathbf{x}'), \\ &= U(x(0), x(1)) + \beta V^*(x(1)), \end{aligned}$$

thus verifying (6.11).

Next, take an arbitrary $\varepsilon > 0$. By (6.10), there exists $\mathbf{x}'_\varepsilon = (x(0), x'_\varepsilon(1), x'_\varepsilon(2), ...) \in \Phi(x(0))$ such that

$$\bar{\mathbf{U}}\left(\mathbf{x}'_\varepsilon\right) \geq V^*(x(0)) - \varepsilon.$$

Now since $\mathbf{x}''_\varepsilon = (x'_\varepsilon(1), x'_\varepsilon(2), ...) \in \Phi(x'_\varepsilon(1))$ and $V^*(x'_\varepsilon(1))$ is the supremum in Problem A1 starting with $x'_\varepsilon(1)$, Lemma 6.1 implies

$$\begin{aligned} U\left(x(0), x'_\varepsilon(1)\right) + \beta\bar{U}\left(\mathbf{x}''_\varepsilon\right) &\geq V^*(x(0)) - \varepsilon \\ U\left(x(0), x'_\varepsilon(1)\right) + \beta V^*\left(x'_\varepsilon(1)\right) &\geq V^*(x(0)) - \varepsilon, \end{aligned}$$

The last inequality verifies (6.12) since $x'_\varepsilon(1) \in G(x(0))$ for any $\varepsilon > 0$. This proves that any solution to Problem A1 satisfies (6.11) and (6.12), and is thus a solution to Problem A2.

To establish the reverse, note that (6.11) implies that for any $x(1) \in G(x(0))$,

$$V(x(0)) \geq U(x(0), x(1)) + \beta V(x(1)).$$

Now substituting recursively for $V(x(1))$, $V(x(2))$, etc., and defining $\mathbf{x} = (x(0), x(1), ...)$, we have

$$V(x(0)) \geq \sum_{t=0}^{n} U(x(t), x(t+1)) + \beta^{n+1} V(x(n+1)).$$

Since $n \to \infty$, $\sum_{t=0}^{n} U(x(t), x(t+1)) \to \bar{\mathbf{U}}(\mathbf{x})$ and $\beta^{n+1} V(x(n+1)) \to 0$ (by hypothesis), we have that

$$V(x(0)) \geq \bar{\mathbf{U}}(\mathbf{x}),$$

for any $\mathbf{x} \in \Phi(x(0))$, thus verifying (6.9).

Next, let $\varepsilon > 0$ be a positive scalar. From (6.12), we have that for any $\varepsilon' = \varepsilon(1-\beta) > 0$, there exists $x_\varepsilon(1) \in G(x(0))$ such that

$$V(x(0)) \leq U(x(0), x_\varepsilon(1)) + \beta V(x_\varepsilon(1)) + \varepsilon'.$$

Let $x_\varepsilon(t) \in G(x(t-1))$, with $x_\varepsilon(0) = x(0)$, and define $\mathbf{x}_\varepsilon \equiv (x(0), x_\varepsilon(1), x_\varepsilon(2), ...)$. Again substituting recursively for $V(x_\varepsilon(1))$, $V(x_\varepsilon(2))$,..., we obtain

$$\begin{aligned} V(x(0)) &\leq \sum_{t=0}^{n} U(x_\varepsilon(t), x_\varepsilon(t+1)) + \beta^{n+1} V(x(n+1)) + \varepsilon' + \varepsilon'\beta + ... + \varepsilon'\beta^n \\ &\leq \bar{\mathbf{U}}(\mathbf{x}_\varepsilon) + \varepsilon, \end{aligned}$$

where the last step follows using the definition of ε (in particular that $\varepsilon = \varepsilon' \sum_{t=0}^{\infty} \beta^t$) and because as $n \to \infty$, $\sum_{t=0}^{n} U(x_\varepsilon(t), x_\varepsilon(t+1)) \to \bar{\mathbf{U}}(\mathbf{x}_\varepsilon)$. This establishes that $V(0)$ satisfies (6.10), and completes the proof. □

In economic problems, we are often interested not in the maximal value of the program but in the optimal plans that achieve this maximal value. Recall that the question of whether the optimal path resulting from Problems A1 and A2 are equivalent was addressed by Theorem 6.2. We now provide a proof of this theorem.

PROOF OF THEOREM 6.2. By hypothesis $\mathbf{x}^* \equiv (x(0), x^*(1), x^*(2), ...)$ is a solution to Problem A1, i.e., it attains the supremum, $V^*(x(0))$ starting from $x(0)$. Let $\mathbf{x}^*_t \equiv (x^*(t), x^*(t+1), ...)$.

We first show that for any $t \geq 0$, $\mathbf{x}^*_t$ attains the supremum starting from $x^*(t)$, so that

$$\bar{\mathbf{U}}(\mathbf{x}^*_t) = V^*(x(t)). \tag{6.13}$$

The proof is by induction. The base step of induction, for $t = 0$, is straightforward, since, by definition, $\mathbf{x}^*_0 = \mathbf{x}^*$ attains $V^*(x(0))$.

Next suppose that the statement is true for t, so that (6.13) is true for t, and we will establish it for $t+1$. Equation (6.13) implies that

$$\begin{aligned} V^*(x^*(t)) &= \bar{\mathbf{U}}(\mathbf{x}^*_t) \\ &= U(x^*(t), x^*(t+1)) + \beta \bar{\mathbf{U}}(\mathbf{x}^*_{t+1}). \end{aligned} \tag{6.14}$$

Let $\mathbf{x}_{t+1} = (x^*(t+1), x(t+2), ...) \in \Phi(x^*(t+1))$ be any feasible plan starting with $x^*(t+1)$. By definition, $\mathbf{x}_t = (x^*(t), \mathbf{x}_{t+1}) \in \Phi(x^*(t))$. Since $V^*(x^*(t))$ is the supremum starting with $x^*(t)$, we have

$$\begin{aligned} V^*(x^*(t)) &\geq \bar{\mathbf{U}}(\mathbf{x}_t) \\ &= U(x^*(t), x^*(t+1)) + \beta \bar{\mathbf{U}}(\mathbf{x}_{t+1}). \end{aligned}$$

Combining this inequality with (6.14), we obtain

$$V^*(x^*(t+1)) = \bar{\mathbf{U}}(\mathbf{x}^*_{t+1}) \geq \bar{\mathbf{U}}(\mathbf{x}_{t+1})$$

for all $\mathbf{x}_{t+1} \in \Phi(x^*(t+1))$. This establishes that $\mathbf{x}^*_{t+1}$ attains the supremum starting from $x^*(t+1)$ and completes the induction step, proving that equation (6.13) holds for all $t \geq 0$.

Equation (6.13) then implies that

$$\begin{aligned} V^*(x^*(t)) &= \bar{\mathbf{U}}(\mathbf{x}^*_t) \\ &= U(x^*(t), x^*(t+1)) + \beta \bar{\mathbf{U}}(\mathbf{x}^*_{t+1}) \\ &= U(x^*(t), x^*(t+1)) + \beta V^*(x^*(t+1)), \end{aligned}$$

establishing (6.3) and thus completing the proof of the first part of the theorem.

Now suppose that (6.3) holds for $\mathbf{x}^* \in \Phi(x(0))$. Then substituting repeatedly for $\mathbf{x}^*$, we obtain

$$V^*(x(0)) = \sum_{t=0}^{n} \beta^t U(x^*(t), x^*(t+1)) + \beta^{n+1} V^*(x(n+1)).$$

In view of the fact that $V^*(\cdot)$ is bounded, we have that

$$\begin{aligned} \bar{\mathbf{U}}(\mathbf{x}^*) &= \lim_{n\to\infty} \sum_{t=0}^{n} \beta^t U(x^*(t), x^*(t+1)) \\ &= V^*(x(0)), \end{aligned}$$

thus $\mathbf{x}^*$ attains the optimal value in Problem A1, completing the proof of the second part of the theorem. □

We have therefore established that under Assumptions 6.1 and 6.2, we can freely interchange Problems A1 and A2. Our next task is to prove that a policy achieving the optimal path exists for both problems. We will provide two alternative proofs for this to show how this conclusion can be reached either by looking at Problem A1 or at Problem A2, and then exploiting their equivalence. The first proof is more abstract and works directly on the sequence problem, Problem A1.

Proof of Theorem 6.3. *(Version 1)* Consider Problem A1. The argument at the beginning of the proof of Theorem 6.1 again enables us to apply Weierstrass's Theorem, Theorem A.9 to conclude that an optimal path $\mathbf{x} \in \Phi(0)$ exists. □

Proof of Theorem 6.3. *(Version 2)* Let $\mathbf{C}(X)$ be the set of continuous functions defined on X, endowed with the sup norm, $\|f\| = \sup_{x\in X} |f(x)|$. In view of Assumption 6.2, the relevant set X is compact and therefore all functions in $\mathbf{C}(X)$ are bounded since they are continuous and X is compact (Corollary A.1 in Chapter A). For $V \in \mathbf{C}(X)$, define the operator T as

$$TV(x) = \max_{y\in G(x)} \{U(x,y) + \beta V(y)\}. \tag{6.15}$$

A fixed point of this operator, $V = TV$, will be a solution to Problem A2. We first prove that such a fixed point (solution) exists. The maximization problem on the right-hand side of (6.15) is one of maximizing a continuous function over a compact set, and by Weierstrass's Theorem, Theorem A.9, it has a solution. Consequently, T is well-defined. Moreover, because $G(x)$ is a nonempty and continuous correspondence by Assumption 6.1 and $U(x,y)$ and $V(y)$ are continuous by hypothesis, Berge's Maximum Theorem, Theorem A.13, implies that $\max_{y\in G(x)} \{U(x,y) + \beta V(y)\}$ is continuous in x, thus $TV(x) \in \mathbf{C}(X)$ and T maps $\mathbf{C}(X)$ into itself.

Next, it is also straightforward to verify that T satisfies Blackwell's sufficient conditions for a contraction in Theorem 6.9 (see Exercise 6.6). Therefore, applying Theorem 6.7, a unique fixed point $V \in \mathbf{C}(X)$ to (6.15) exists and this is also the unique solution to Problem A2.

Now consider the maximization in Problem A2. Since U and V are continuous and $G(x)$ is compact-valued, we can apply Weierstrass's Theorem once more to conclude that $y \in G(x)$ achieving the maximum exists. This defines the set of maximizers $\Pi(x)$ for Problem A2. Let $\mathbf{x}^* = (x(0), x^*(1), ...)$ with $x^*(t+1) \in \Pi(x^*(t))$ for all $t \geq 0$. Then from Theorems 6.1 and 6.2, $\mathbf{x}^*$ is also and optimal plan for Problem A1. □

These two proofs illustrate how different approaches can be used to reach the same conclusion, once the equivalences in Theorems 6.1 and 6.2 have been established.

An additional result that follows from the second version of the theorem (which can also be derived from version 1, but would require more work), concerns the properties of the correspondence of maximizing values

$$\Pi : X \rightrightarrows X.$$

An immediate application of the Theorem of the Maximum, Theorem A.13 in Appendix Chapter A, implies that Π is a upper hemi-continuous and compact-valued correspondence. This observation will be used in the proof of Corollary 6.1. Before turning to this corollary, we provide a proof of Theorem 6.4, which shows how Theorem 6.8 can be useful in establishing a range of results in dynamic optimization problems.

Proof of Theorem 6.4. Recall that $\mathbf{C}(X)$ is the set of continuous (and bounded) functions over the compact set X. Let $\mathbf{C}'(X) \subset \mathbf{C}(X)$ be the set of bounded, continuous, (weakly) concave functions on X, and let $\mathbf{C}''(X) \subset \mathbf{C}'(X)$ be the set of strictly concave functions. Clearly, $\mathbf{C}'(X)$ is a closed subset of the complete metric space $\mathbf{C}(X)$, but $\mathbf{C}''(X)$ is not a closed subset. Let T be as defined in (6.15). Since it is a contraction, it has a unique fixed point in $\mathbf{C}(X)$. By Theorem 6.8, proving that $T[\mathbf{C}'(X)] \subset \mathbf{C}''(X) \subset \mathbf{C}'(X)$ would be sufficient to establish that this unique fixed point is in $\mathbf{C}''(X)$ and hence the value function is strictly concave. Let $V \in \mathbf{C}'(X)$ and for $x' \neq x''$ and $\alpha \in (0,1)$, let

$$x_\alpha \equiv \alpha x' + (1-\alpha)x''.$$

Let $y' \in G(x')$ and $y'' \in G(x'')$ be solutions to Problem A2 with state vectors x' and x''. This implies that

$$\begin{aligned} TV(x') &= U\left(x', y'\right) + \beta V\left(y'\right) \text{ and} \\ TV(x'') &= U\left(x'', y''\right) + \beta V\left(y''\right). \end{aligned} \tag{6.16}$$

In view of Assumption 6.3 (that G is convex valued) $y_\alpha \equiv \alpha y' + (1-\alpha)\, y'' \in G(x_\alpha)$, so that

$$\begin{aligned} TV(x_\alpha) &\geq U(x_\alpha, y_\alpha) + \beta V(y_\alpha), \\ &> \alpha\left[U\left(x', y'\right) + \beta V\left(y'\right)\right] \\ &\quad +(1-\alpha)[U\left(x'', y''\right) + \beta V\left(y''\right)] \\ &= \alpha TV(x') + (1-\alpha)TV(x''), \end{aligned}$$

where the first line follows by the fact that $y_\alpha \in G(x_\alpha)$ is not necessarily the maximizer. The second line uses Assumption 6.3 (strict concavity of U), and the third line is simply the definition introduced in (6.16). This argument implies that for any $V \in \mathbf{C}'(X)$, TV is strictly concave, thus $T[\mathbf{C}'(X)] \subset \mathbf{C}''(X)$. Then Theorem 6.8 implies that the unique fixed point V^* is in $\mathbf{C}''(X)$, and hence it is strictly concave. □

Proof of Corollary 6.1. Assumption 6.3 implies that $U(x, y)$ is concave in y, and under this assumption, Theorem 6.4 established that $V(y)$ is strictly concave in y. The sum of a concave function and a strictly concave function is strictly concave, thus the right-hand side of Problem A2 is strictly concave in y. Therefore, combined with the fact that $G(x)$ is convex for each $x \in X$ (again Assumption 6.3), there exists a unique maximizer $y \in G(x)$ for each $x \in X$. This implies that the policy correspondence $\Pi(x)$ is single-valued, thus a function, and can thus be expressed as $\pi(x)$. Since $\Pi(x)$ is upper hemi-continuous as observed above, so is $\pi(x)$. Since an upper hemi-continuous function is continuous, the corollary follows. □

Proof of Theorem 6.5. The proof again follows from Theorem 6.8. Let $\mathbf{C}'(X) \subset \mathbf{C}(X)$ be the set of bounded, continuous, nondecreasing functions on X, and let $\mathbf{C}''(X) \subset \mathbf{C}'(X)$ be the set of strictly increasing functions. Since $\mathbf{C}'(X)$ is a closed subset of the complete metric space $\mathbf{C}(X)$, Theorem 6.8 implies that if $T[\mathbf{C}'(X)] \subset \mathbf{C}''(X)$, then the fixed point to (6.15), i.e., V, is in $\mathbf{C}''(X)$, and therefore, it is a strictly increasing function. To see that this is the case, consider any $V \in \mathbf{C}'(X)$, i.e., any nondecreasing function. In view of Assumption 6.4, $\max_{y \in G(x)} \{U(x, y) + \beta V(y)\}$ is strictly increasing. This establishes that $TV \in \mathbf{C}''(X)$ and completes the proof. □

Proof of Theorem 6.6. From Corollary 6.1, $\Pi(x)$ is single-valued, thus a function that can be represented by $\pi(x)$. By hypothesis, $\pi(x(0)) \in \text{Int}G(x(0))$ and from Assumption 6.2 G is continuous. Therefore, there exists a neighborhood $\mathcal{N}(x(0))$ of $x(0)$ such that $\pi(x(0)) \in \text{Int}G(x)$, for all $x \in N(x(0))$. Define $W(\cdot)$ on $\mathcal{N}(x(0))$ by

$$W(x) = U(x, \pi(x(0))) + \beta V(\pi(x(0))).$$

In view of Assumptions 6.3 and 6.5, the fact that $V[\pi(x(0))]$ is a number (independent of x), and the fact that U is concave and differentiable, $W(\cdot)$ is concave and differentiable. Moreover, since $\pi(x(0)) \in G(x)$ for all $x \in \mathcal{N}(x(0))$, it follows that

$$W(x) \leq \max_{y \in G(x)} \{U(x, y) + \beta V(y)\} = V(x), \qquad \text{for all } x \in \mathcal{N}(x(0)) \tag{6.17}$$

with equality at $x(0)$.

Since $V(\cdot)$ is concave, $-V(\cdot)$ is convex, and by a standard result in convex analysis, it possesses subgradients. Moreover, any subgradient p of $-V$ at $x(0)$ must satisfy

$$p \cdot (x - x(0)) \geq V(x) - V(x(0)) \geq W(x) - W(x(0)), \qquad \text{for all } x \in \mathcal{N}(x(0)),$$

where the first inequality uses the definition of a subgradient and the second uses the fact that $W(x) \leq V(x)$, with equality at $x(0)$ as established in (6.17). This implies that every subgradient p of $-V$ is also a subgradient of $-W$. Since W is differentiable at $x(0)$, its subgradient p must be unique, and another standard result in convex analysis implies that any convex function with a unique subgradient at an interior point $x(0)$ is differentiable at $x(0)$. This establishes that $-V(\cdot)$, thus $V(\cdot)$, is differentiable as desired.

The expression for the gradient (6.4) is derived in detail in the next section. □

6.5. Fundamentals of Dynamic Programming

In this section, we return to the fundamentals of dynamic programming and show how they can be applied in a range of problems. The main result in this section is Theorem 6.10, which shows how dynamic first-order conditions, the Euler equations, together with the transversality condition are sufficient to characterize solutions to dynamic optimization problems. This theorem is arguably more useful in practice than the main dynamic programming theorems presented above.

6.5.1. Basic Equations. Consider the functional equation corresponding to Problem A2:

$$V(x) = \max_{y \in G(x)} \{U(x,y) + \beta V(y)\}, \text{ for all } x \in X. \tag{6.18}$$

Let us assume throughout that Assumptions 6.1-6.5 hold. Then from Theorem 6.4, the maximization problem in (6.18) is strictly concave, and from Theorem 6.6, the maximand is also differentiable. Therefore for any interior solution $y \in \text{Int}G(x)$, the first-order conditions are necessary and sufficient for an optimum. In particular, optimal solutions can be characterized by the following convenient *Euler equations*:

$$D_y U(x, y^*) + \beta D_x V(y^*) = 0, \tag{6.19}$$

where we use $*$'s to denote optimal values and once again D denotes gradients (recall that, in the general case, x is a vector not a scalar, thus $D_x U$ is a vector of partial derivatives and we denote the vector of partial derivatives of the value function V evaluated at y by $D_x V(y)$).

The set of first-order conditions in equation (6.19) would be sufficient to solve for the optimal policy, y^*, if we knew the form of the $V(\cdot)$ function. Since this function is determined recursively as part of the optimization problem, there is a little more work to do before we obtain the set of equations that can be solved for the optimal policy.

Fortunately, we can use the equivalent of the Envelope Theorem for dynamic programming and differentiate (6.18) with respect to the state vector, x, to obtain:

$$D_x V(x) = D_x U(x, y^*). \tag{6.20}$$

The reason why this is the equivalent of the Envelope Theorem is that the term $D_y U(x, y^*) + \beta D_x V(y^*)$ times the induced change in y in response to the change in x is absent from the expression. This is because the term $D_y U(x, y^*) + \beta D_x V(y^*) = 0$ from (6.19).

Now using the notation $y^* = \pi(x)$ to denote the optimal policy function (which is single-valued in view of Assumption 6.3) and the fact that $D_x V(y) = D_x V(\pi(x))$, we can combine these two equations to write

$$D_y U(x, \pi(x)) + \beta D_x U(\pi(x), \pi(\pi(x))) = 0, \tag{6.21}$$

where $D_x U$ represents the gradient vector of U with respect to its first K arguments, and $D_y U$ represents its gradient with respect to the second set of K arguments. Notice that (6.21) is a functional equation in the unknown function $\pi(\cdot)$ and characterizes the optimal policy function.

These equations become even simpler and more transparent in the case where both x and y are scalars. In this case, (6.19) becomes:

$$\frac{\partial U(x, y^*)}{\partial y} + \beta V'(y^*) = 0, \tag{6.22}$$

where V' the notes the derivative of the V function with respect to it single scalar argument.

This equation is very intuitive; it requires the sum of the marginal gain today from increasing y and the discounted marginal gain from increasing y on the value of all future returns to be equal to zero. For instance, as in Example 6.1, we can think of U as decreasing in y and increasing in x; equation (6.22) would then require the current cost of increasing y to be compensated by higher values tomorrow. In the context of growth, this corresponds to current cost of reducing consumption to be compensated by higher consumption tomorrow. As with (6.19), the value of higher consumption in (6.22) is expressed in terms of the derivative of the value function, $V'(y^*)$, which is one of the unknowns. To make more progress, we use the one-dimensional version of (6.20) to find an expression for this derivative:

$$V'(x) = \frac{\partial U(x, y^*)}{\partial x}. \tag{6.23}$$

Now in this one-dimensional case, combining (6.23) together with (6.22), we have the following very simple condition:

$$\frac{\partial U(x, \pi(x))}{\partial y} + \beta \frac{\partial U(\pi(x), \pi(\pi(x)))}{\partial x} = 0$$

where ∂x denotes the derivative with respect to the first argument and ∂y with respect to the second argument.

Alternatively, we could write the one-dimensional Euler equation with the time arguments as

$$\frac{\partial U(x(t), x^*(t+1))}{\partial x(t+1)} + \beta \frac{\partial U(x^*(t+1), x^*(t+2))}{\partial x(t+1)} = 0. \tag{6.24}$$

However, this Euler equation is not sufficient for optimality. In addition we need the *transversality condition.* The transversality condition is essential in infinite-dimensional problems, because it makes sure that there are no beneficial simultaneous changes in an infinite number of choice variables. In contrast, in finite-dimensional problems, there is no need for such a condition, since the first-order conditions are sufficient to rule out possible gains when we change many or all of the control variables at the same time. The role that the transversality condition plays in infinite-dimensional optimization problems will become more apparent after we see Theorem 6.10 and after the discussion in the next subsection.

In the general case, the transversality condition takes the form:

$$\lim_{t\to\infty} \beta^t D_{x(t)} U(x^*(t), x^*(t+1)) \cdot x^*(t) = 0, \tag{6.25}$$

where "$\cdot$" denotes the inner product operator. In the one-dimensional case, we have the simpler transversality condition:

$$\lim_{t\to\infty} \beta^t \frac{\partial U(x^*(t), x^*(t+1))}{\partial x(t)} \cdot x^*(t) = 0. \tag{6.26}$$

In words, this condition requires that the product of the marginal return from the state variable x times the value of this state variable does not increase asymptotically at a rate faster than $1/\beta$.

The next theorem shows that the transversality condition together with the transformed Euler equations in (6.21) are sufficient to characterize an optimal solution to Problem A1 and therefore to Problem A2. Exercise 6.11 in fact shows that a stronger version of this result applies even when the problem is nonstationary, and I will discuss an application of this more general version of the theorem below.

Theorem 6.10. ***(Euler Equations and the Transversality Condition)*** *Let $X \subset \mathbb{R}^K_+$, and suppose that Assumptions 6.1-6.5 hold. Then a sequence $\{x^*(t+1)\}_{t=0}^{\infty}$, with $x^*(t+1) \in IntG(x^*(t))$, $t = 0, 1, \ldots$, is optimal for Problem A1 given $x(0)$, if it satisfies (6.21) and (6.25).*

Proof. Consider an arbitrary $x(0)$ and $\mathbf{x}^* \equiv (x(0), x^*(1), ...) \in \Phi(x(0))$ be a feasible (nonnegative) sequence satisfying (6.21) and (6.25). We first show that $\mathbf{x}^*$ yields higher value than any other $\mathbf{x} \equiv (x(0), x(1), ...) \in \Phi(x(0))$. For any $\mathbf{x} \in \Phi(x(0))$, define

$$\Delta_{\mathbf{x}} \equiv \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t [U(x^*(t), x^*(t+1)) - U(x(t), x(t+1))]$$

as the difference of the objective function between the feasible sequences $\mathbf{x}^*$ and $\mathbf{x}$.

From Assumptions 6.2 and 6.5, U is continuous, concave, and differentiable. By definition of a concave function, we have

$$\begin{aligned} \Delta_{\mathbf{x}} \geq & \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t [D_x U(x^*(t), x^*(t+1)) \cdot (x^*(t) - x(t)) \\ & + D_x U(x^*(t), x^*(t+1)) \cdot (x^*(t+1) - x(t+1))] \end{aligned}$$

for any $\mathbf{x} \in \Phi(x(0))$. Using the fact that $x^*(0) = x(0)$ and rearranging terms, we obtain

$$\begin{aligned}
&\Delta_{\mathbf{x}} \geq \\
&\lim_{T\to\infty}\sum_{t=0}^{T}\beta^t \left[D_y U(x^*(t), x^*(t+1)) + \beta D_x U(x^*(t+1), x^*(t+2))\right]\cdot(x^*(t+1) - x(t+1)) \\
&-\lim_{T\to\infty}\beta^T D_x U(x^*(T), x^*(T+1))\cdot x^*(T+1) \\
&+\lim_{T\to\infty}\beta^T D_x U(x^*(T), x^*(T+1))\cdot x(T+1)).
\end{aligned}$$

Since $\mathbf{x}^*$ satisfies (6.21), the terms in first line are all equal to zero. Moreover, since it satisfies (6.25), the second line is also equal to zero. Finally, from Assumption 6.4, U is increasing in x, i.e., $D_x U \geq 0$ and $x \geq 0$, so the last term is nonnegative, establishing that $\Delta_{\mathbf{x}} \geq 0$ for any $\mathbf{x} \in \Phi(x(0))$. Consequently, $\mathbf{x}^*$ yields higher value than any feasible $\mathbf{x} \in \Phi(x(0))$ and is therefore optimal. □

We now illustrate how the tools that have been developed so far can be used in the context of the problem of optimal growth, which will be further discussed in Section 6.6.

Example 6.4. Consider the following optimal growth, with log preferences, Cobb-Douglas technology and full depreciation of capital stock

$$\begin{aligned}
&\max_{\{c(t),k(t+1)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty}\beta^t \ln c(t) \\
&\text{subject to} \\
&k(t+1) = k(t)^{\alpha} - c(t) \\
&k(0) = k_0 > 0,
\end{aligned}$$

where, as usual, $\beta \in (0,1)$, k denotes the capital-labor ratio (capital stock), and the resource constraint follows from the production function $K^{\alpha}L^{1-\alpha}$, written in per capita terms.

This is one of the canonical examples which admits an explicit-form characterization. To derive this, let us follow Example 6.1 and set up the maximization problem in its recursive form as

$$V(x) = \max_{y\geq 0}\left\{\ln(x^{\alpha} - y) + \beta V(y)\right\},$$

with x corresponding to today's capital stock and y to tomorrow's capital stock. Our main objective is to find the policy function $y = \pi(x)$, which determines tomorrow's capital stock as a function of today's capital stock. Once this is done, we can easily determine the level of consumption as a function of today's capital stock from the resource constraint.

It can be verified that this problem satisfies Assumptions 6.1-6.5. The only non-obvious feature here is whether x and y indeed belong to a compact set. The argument used in Section 6.6 for Proposition 6.1 can be used to verify that this is the case, and we will not repeat the argument here. Consequently, Theorems 6.1-6.6 apply. In particular, since $V(\cdot)$

is differentiable, the Euler equation for the one-dimensional case, (6.22), implies

$$\frac{1}{x^{\alpha}-y}=\beta V'(y).$$

The envelope condition, (6.23), gives:

$$V'(x)=\frac{\alpha x^{\alpha-1}}{x^{\alpha}-y}.$$

Thus using the notation $y=\pi(x)$ and combining these two equations, we have

$$\frac{1}{x^{\alpha}-\pi(x)}=\beta\frac{\alpha\pi(x)^{\alpha-1}}{\pi(x)^{\alpha}-\pi(\pi(x))}\text{ for all }x,$$

which is a functional equation in a single function, $\pi(x)$. There are no straightforward ways of solving functional equations, but in most cases guess-and-verify type methods are most fruitful. For example in this case, let us conjecture that

$$\pi(x)=ax^{\alpha}. \tag{6.27}$$

Substituting for this in the previous expression, we obtain

$$\begin{aligned}\frac{1}{x^{\alpha}-ax^{\alpha}} &= \beta\frac{\alpha a^{\alpha-1}x^{\alpha(\alpha-1)}}{a^{\alpha}x^{\alpha^2}-a^{1+\alpha}x^{\alpha^2}},\\ &= \frac{\beta}{a}\frac{\alpha}{x^{\alpha}-ax^{\alpha}},\end{aligned}$$

which implies that, with the policy function (6.28), $a=\beta\alpha$ satisfies this equation. Recall from Corollary 6.1 that, under the assumptions here, there is a unique policy function. Since we have established that the function

$$\pi(x)=\beta\alpha x^{\alpha}$$

satisfies the necessary and sufficient conditions (Theorem 6.10), it must be the unique policy function. This implies that the law of motion of the capital stock is

$$k(t+1)=\beta\alpha k(t)^{\alpha} \tag{6.28}$$

and the optimal consumption level is

$$c(t)=(1-\beta a)\,k(t)^{\alpha}.$$

Exercise 6.7 continues with some of the details of this example, and also shows how the optimal growth equilibrium involves a sequence of capital-labor ratios converging to a unique steady state.

Finally, we now have a brief look at the intertemporal utility maximization problem of a consumer facing a certain income sequence.

Example 6.5. Consider the problem of an infinitely-lived consumer with instantaneous utility function defined over consumption $u(c)$, where $u:\mathbb{R}_{+}\to\mathbb{R}$ is strictly increasing, continuously differentiable and strictly concave. The individual discounts the future exponentially with the constant discount factor $\beta\in(0,1)$. He also faces a certain (nonnegative) labor income stream of $\{w(t)\}_{t=0}^{\infty}$, and moreover starts life with a given amount of assets $a(0)$. He receives a constant net rate of interest $r > 0$ on his asset holdings (so that the gross rate of return is $1 + r$). To start with, let us suppose that wages are constant, that is, $w(t) = w$.

Then, the utility maximization problem of the individual can be written as

$$\max_{\{c(t),a(t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c(t))$$

subject to the flow budget constraint

$$a(t+1) = (1+r)(a(t) + w - c(t)),$$

with $a(0) > 0$ given. In addition, we impose the assumption that the individual cannot have a negative asset holdings, so $a(t) \geq 0$ for all t. This example is one of the most common applications of dynamic optimization in economics. Unfortunately, the reader will notice that the feasible set for the state variable $a(t)$ is not necessarily compact, so the theorems developed so far cannot be directly applied to this problem. One way of proceeding is to strengthen these theorems, so that they cover situations in which the feasible set (X in terms of the notation above) is potentially unbounded. While such a strengthening is possible, it requires additional arguments. An alternative approach is to make use of the economic structure of the model. In particular, the following method works in general (but not always, see Exercise 6.13). Let us choose some $\bar{a}$ and limit $a(t)$ to lie in the set $[0, \bar{a}]$. Solve the problem and then verify that indeed $a(t)$ is in the interior of this set. In this example, we can choose $\bar{a} \equiv a(0) + w/r$, which is the sum of the net present discounted value of the labor earnings of the individual and his initial wealth, and will be assumed to be finite below. This strategy of finding an upper bound for the state variable and thus ensuring that it will lie in a compact set is used often in applications.

A couple of additional comments are useful at this point. First the budget constraint could have been written alternatively as $a(t+1) = (1+r)a(t) + w - c(t)$. The difference between these two alternative budget constraints involes the timing of interest payments. The first one presumes that the individual starts the period with assets $a(t)$, then receives his labor income, $w(t)$, and then consumes $c(t)$. Whatever is left is saved for the next date and earns the gross interest rate $(1+r)$. In this formulation, $a(t)$ refers to asset holdings at the beginning of time t. The alternative formulation instead interprets $a(t)$ as asset holdings at the end of time t. The choice between these two formulations has no bearing on the results. Observe also that the flow budget constraint does not capture all of the constraints that individual must satisfy. In particular, an individual can satisfy the flow budget constraint, but run his assets position to $-\infty$. In general to prevent this, we need to impose an additional restriction, for example, that the asset position of the individual does not become "too negative" at infinity. However, here we do not need this additional restriction, since we have already imposed that $a(t) \geq 0$ for all t.

Let us also focus on the case where the wealth of the individual is finite, so $a(0) < \infty$ and $w/r < \infty$. With these assumptions, let us now write the recursive formulation of the individual's maximization problem. The state variable is $a(t)$, and consumption can be expressed as

$$c(t) = a(t) + w - (1+r)^{-1} a(t+1).$$

With standard arguments and denoting the current value of the state variable by a and its future value by a', the recursive form of this dynamic optimization problem can be written as

$$V(a) = \max_{a' \in [0,\bar{a}]} \left\{ u\left(a + w - (1+r)^{-1} a'\right) + \beta V\left(a'\right) \right\}.$$

Clearly $u(\cdot)$ is strictly increasing in a, continuously differentiable in a and a' and is strictly concave in a. Moreover, since $u(\cdot)$ is continuously differentiable in $a \in (0, \bar{a})$ and the individual's wealth is finite, $V(a(0))$ is also finite. Thus all of the results from our analysis above, in particular Theorems 6.1-6.6, apply and imply that $V(a)$ is differentiable and a continuous solution $a' = \pi(a)$ exists. Moreover, we can use the Euler equation (6.19), or its more specific form (6.22) for one-dimensional problems to characterize the optimal consumption plan. In particular,

$$\begin{aligned} u'\left(a + w - (1+r)^{-1} a'\right) &= \\ u'(c) &= \beta(1+r) V'\left(a'\right). \end{aligned} \tag{6.29}$$

This important equation is often referred to as the "consumption Euler" equation. It states that the marginal utility of current consumption must be equal to the marginal increase in the continuation value multiplied by the product of the discount factor, β, and the gross rate of return to savings, $(1+r)$. It captures the essential economic intuition of dynamic programming approach, which reduces the complex infinite-dimensional optimization problem to one of comparing today to "tomorrow". Naturally, the only difficulty here is that tomorrow itself will involve a complicated maximization problem and hence tomorrow's value function and its derivative are endogenous. But here the envelope condition, (6.23), again comes to our rescue and gives us

$$V'\left(a'\right) = u'\left(c'\right),$$

where c' refers to next period's consumption. Using this relationship, the consumption Euler equation becomes

$$u'(c) = \beta(1+r) u'\left(c'\right). \tag{6.30}$$

This form of the consumption Euler equation is more familiar and requires the marginal utility of consumption today to be equal to the marginal utility of consumption tomorrow multiplied by the product of the discount factor and the gross rate of return. Since we have assumed that β and $(1+r)$ are constant, the relationship between today's and tomorrow's consumption never changes. In particular, since $u(\cdot)$ is assumed to be continuously differentiable and strictly concave, $u'(\cdot)$ always exists and is strictly decreasing. Therefore, the intertemporal consumption maximization problem implies the following simple rule:

$$\begin{array}{lll} & \text{if } r = \beta^{-1}-1 & c = c' \text{ and consumption is constant over time} \\ (6.31) & \text{if } r > \beta^{-1}-1 & c < c' \text{ and consumption increases over time} \\ & \text{if } r < \beta^{-1}-1 & c > c' \text{ and consumption decreases over time.} \end{array}$$

The remarkable feature is that these statements have been made without any reference to the initial level of asset holdings $a(0)$ and the wage rate w. It turns out that these only determine the initial level of consumption. The "slope" of the optimal consumption path is independent of the wealth of the individual. Exercise 6.12 asks you to determine the level of initial consumption using the transversality condition and the intertemporal budget constraint, while Exercise 6.13 asks you to verify that whenever $r \leq \beta - 1$, $a(t) \in (0, \bar{a})$ for all t (so that the artificial bounds on asset holdings that I imposed have no bearing on the results).

The problem so far is somewhat restrictive, because of the assumption that wages are constant over time. What happens if instead there is an arbitrary sequence of wages $\{w(t)\}_{t=0}^{\infty}$? Let us assume that these are known in advance, so that there is no uncertainty. Under this assumption, all of the results derived in this example, in particular, the characterization in (6.31), still apply, but some additional care is necessary, since in this case the budget constraint of the individual, thus the constraint correspondence G in terms of the general formulation above, is no longer "autonomous" (independent of time). In this case, two approaches are possible. The first is to introduce an additional state variable. For example, one can introduce the net present discounted value of future labor earnings, $h(t) = \sum_{s=0}^{\infty} (1+r)^{-s} w(s)$, as an additional state variable. In this case, the budget constraint of the individual can be written as

$$a(t+1) + h(t+1) \leq (1+r)(a(t) + h(t) - c(t)),$$

and a similar analysis can be applied with the value function defined over two state variables, $V(a,h)$. This approach is economically meaningful, since the net present discounted value of future earnings is a relevant state variable. But it does not always solve our problems. First, $h(t)$ is now a state variable that has its own non-autonomous evolution (rather than being directly controlled by the individual), so our previous analysis needs to be modified. Second, in many problems, it is difficult to find an economically meaningful additional state variable. Fortunately, however, one can directly apply Theorem 6.10, even when Theorems 6.1-6.6 do not hold. Exercise 6.11 contains the generalization of Theorem 6.10 that enables us to do that, and Exercise 6.12 applies this result to the consumption problem with a time-varying sequence of labor income $\{w(t)\}_{t=0}^{\infty}$. It also shows that the exact shape of this labor income sequence has no effect on the slope or level of the consumption profile.

6.5.2. Dynamic Programming Versus the Sequence Problem. To get more insights into dynamic programming, let us return to the sequence problem. Also, let us suppose that x is one dimensional and that there is a finite horizon T. Then the problem becomes

$$\max_{\{x(t+1)\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t U(x(t), x(t+1))$$

subject to $x(t+1) \geq 0$ with $x(0)$ as given. Moreover, let $U(x(T), x(T+1))$ be the last period's utility, with $x(T+1)$ as the state variable left after the last period (this utility could be thought of as the "salvage value" for example).

In this case, we have a finite-dimensional optimization problem and we can simply look at first-order conditions. Moreover, let us again assume that the optimal solution lies in the interior of the constraint set, i.e., $x^*(t) > 0$, so that we do not have to worry about boundary conditions and complementary-slackness type conditions. Given these, the first-order conditions of this finite-dimensional problem are exactly the same as the above Euler equation. In particular, we have

$$\text{for any } 0 \leq t \leq T-1, \ \frac{\partial U(x^*(t), x^*(t+1))}{\partial x(t+1)} + \beta \frac{\partial U(x^*(t+1), x^*(t+2))}{\partial x(t+1)} = 0,$$

which are identical to the Euler equations for the infinite-horizon case. In addition, for $x(T+1)$, we have the following boundary condition

$$x^*(T+1) \geq 0, \text{ and } \beta^T \ \frac{\partial U(x^*(T), x^*(T+1))}{\partial x(T+1)} x^*(T+1) = 0. \tag{6.32}$$

Intuitively, this boundary condition requires that $x^*(T+1)$ should be positive only if an interior value of it maximizes the salvage value at the end. To provide more intuition for this expression, let us return to the formulation of the optimal growth problem in Example 6.1.

Example 6.6. Recall that in terms of the optimal growth problem, we have

$$U(x(t), x(t+1)) = u(f(x(t)) + (1-\delta)x(t) - x(t+1)),$$

with $x(t) = k(t)$ and $x(t+1) = k(t+1)$. Suppose we have a finite-horizon optimal growth problem like the one discussed above where the world comes to an end at date T. Then at the last date T, we have

$$\frac{\partial U(x^*(T), x^*(T+1))}{\partial x(T+1)} = -u'(c^*(T+1)) < 0.$$

From (6.32) and the fact that U is increasing in its first argument (Assumption 6.4), an optimal path must have $k^*(T+1) = x^*(T+1) = 0$. Intuitively, there should be no capital left at the end of the world. If any resources were left after the end of the world, utility could be improved by consuming them either at the last date or at some earlier date.

Now, heuristically we can derive the transversality condition as an extension of condition (6.32) to $T \to \infty$. Take this limit, which implies

$$\lim_{T\to\infty} \beta^T \frac{\partial U(x^*(T), x^*(T+1))}{\partial x(T+1)} x^*(T+1) = 0.$$

Moreover, as $T \to \infty$, we have the Euler equation

$$\frac{\partial U(x^*(T), x^*(T+1))}{\partial x(T+1)} + \beta \frac{\partial U(x^*(T+1), x^*(T+2))}{\partial x(T+1)} = 0.$$

Substituting this relationship into the previous equation, we obtain

$$-\lim_{T\to\infty} \beta^{T+1} \frac{\partial U(x^*(T+1), x^*(T+2))}{\partial x(T+1)} x^*(T+1) = 0.$$

Canceling the negative sign, and without loss of any generality, changing the timing:

$$\lim_{T\to\infty} \beta^T \frac{\partial U(x^*(T), x^*(T+1))}{\partial x(T)} x^*(T) = 0,$$

which is exactly the transversality condition in (6.26). This derivation also highlights that alternatively we could have had the transversality condition as

$$\lim_{T\to\infty} \beta^T \frac{\partial U(x^*(T), x^*(T+1))}{\partial x(T+1)} x^*(T+1) = 0,$$

which emphasizes that there is no unique transversality condition, but we generally need a boundary condition at infinity to rule out variations that change an infinite number of control variables at the same time. A number of different boundary conditions at infinity can play this role. We will return to this issue when we look at optimal control in continuous time.

6.6. Optimal Growth in Discrete Time

We are now in a position to apply the methods developed so far to characterize the solution to the standard discrete time optimal growth problem introduced in the previous chapter. Example 6.4 already showed how this can be done in the special case with logarithmic utility, Cobb-Douglas production function and full depreciation. In this section, we will see that the results apply more generally to the canonical optimal growth model introduced in Chapter 5.

Recall the optimal growth problem for a one-sector economy admitting a representative household with instantaneous utility function u and discount factor $\beta \in (0,1)$. This can be written as

$$\max_{\{c(t),k(t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c(t)) \tag{6.33}$$

subject to

$$k(t+1) = f(k(t)) + (1-\delta)k(t) - c(t) \text{ and } k(t) \geq 0, \tag{6.34}$$

with the initial capital stock given by $k(0)$.

We continue to make the standard assumptions on the production function as in Assumptions 1 and 2. In addition, we assume that:

Assumption 3′. $u : [\underline{c}, \infty) \rightarrow \mathbb{R}$ is continuously differentiable and strictly concave for $\underline{c} \in [0, \infty)$.

This is considerably stronger than what we need. In fact, concavity or even continuity is enough for most of the results. But this assumption helps us avoid inessential technical details. The lower bound on consumption is imposed to have a compact set of consumption possibilities. We refer to this as Assumption 3′ to distinguish it from the very closely related Assumption 3 that will be introduced and used in Chapter 8 and thereafter.

The first step is to write the optimal growth problem as a (stationary) dynamic programming problem. This can be done along the lines of the above formulations. In particular, let the choice variable be next date's capital stock, denoted by s. Then the resource constraint (6.34) implies that current consumption is given by $c = f(k) + (1-\delta)k - s$, and thus we can write the open growth problem in the following recursive form:

$$V(k) = \max_{s \in G(k)} \{u(f(k) + (1-\delta)k - s) + \beta V(s)\} \tag{6.35}$$

where $G(k)$ is the constraint correspondence, given by the interval $[0, f(k) + (1-\delta)k - \underline{c}]$, which imposes that consumption cannot fall below $\underline{c}$ and that the capital stock cannot be negative.

It can be verified that under Assumptions 1, 2 and 3′, the optimal growth problem satisfies Assumptions 6.1-6.5 of the dynamic programming problems. The only non-obvious feature is that the level of consumption and capital stock belong to a compact set. To verify that this is the case, note that the economy can never settle into a level of capital-labor ratio greater than $\bar{k}$, defined by

$$\delta \bar{k} = f\left(\bar{k}\right),$$

since this is the capital-labor ratio that would sustain itself when consumption is set equal to 0. If the economy starts with $k(0) < \bar{k}$, it can never exceed $\bar{k}$. If it starts with $k(0) > \bar{k}$, it can never exceed $k(0)$. Therefore, without loss of any generality, we can restrict consumption and capital stock to lie in the compact set $\left[0, \vec{k}\right]$, where

$$\vec{k} \equiv f\left(\max\left\{k(0), \bar{k}\right\}\right) + (1-\delta)\max\left\{k(0), \bar{k}\right\}.$$

Consequently, Theorems 6.1-6.6 immediately apply to this problem and we can use these results to derive the following proposition to characterize the optimal growth path of the one-sector infinite-horizon economy.

Proposition 6.1. *Given Assumptions 1, 2 and 3′, the optimal growth model as specified in (6.33) and (6.34) has a solution characterized by the value function $V(k)$ and consumption* *function* $c(k)$*. The capital stock of the next period is given by* $s(k) = f(k) + (1-\delta)k - c(k)$*. Moreover,* $V(k)$ *is strictly increasing and concave and* $s(k)$ *is nondecreasing in* k*.*

PROOF. Optimality of the solution to the value function (6.35) for the problem (6.33) and (6.34) follows from Theorems 6.1 and 6.2. That $V(k)$ exists follows from Theorem 6.3, and the fact that it is increasing and strictly concave, with the policy correspondence being a policy function follows from Theorem 6.4 and Corollary 6.1.

Thus we only have to show that $s(k)$ is nondecreasing. This can be proved by contradiction. Suppose, to arrive at a contradiction, that $s(k)$ is decreasing, i.e., there exists k and $k' > k$ such that $s(k) > s(k')$. Since $k' > k$, $s(k)$ is feasible when the capital stock is k'. Moreover, since, by hypothesis, $s(k) > s(k')$, $s(k')$ is feasible at capital stock k.

By optimality and feasibility, we must have:

$$\begin{aligned} V(k) &= u(f(k) + (1-\delta)k - s(k)) + \beta V(s(k)) \\ &\geq u\left(f(k) + (1-\delta)k - s\left(k'\right)\right) + \beta V\left(s\left(k'\right)\right) \\ V\left(k'\right) &= u\left(f\left(k'\right) + (1-\delta)k' - s\left(k'\right)\right) + \beta V\left(s\left(k'\right)\right) \\ &\geq u\left(f\left(k'\right) + (1-\delta)k' - s(k)\right) + \beta V(s(k)). \end{aligned}$$

Combining and rearranging these, we have

$$\begin{aligned} u(f(k) + (1-\delta)k - s(k)) - u\left(f(k) + (1-\delta)k - s\left(k'\right)\right) &\geq \beta\left[V\left(s\left(k'\right)\right) - V(s(k))\right] \\ &\geq u\left(f\left(k'\right) + (1-\delta)k' - s(k)\right) \\ &\quad - u\left(f\left(k'\right) + (1-\delta)k' - s\left(k'\right)\right). \end{aligned}$$

Or denoting $z \equiv f(k) + (1-\delta)k$ and $x \equiv s(k)$ and similarly for z' and x', we have

$$u\left(z - x'\right) - u(z - x) \leq u\left(z' - x'\right) - u\left(z' - x\right). \tag{6.36}$$

But clearly,

$$\left(z - x'\right) - (z - x) = \left(z' - x'\right) - \left(z' - x\right),$$

which combined with the fact that $z' > z$ (since $k' > k$) and $x > x'$ by hypothesis, and that u is strictly concave and increasing implies

$$u\left(z - x'\right) - u(z - x) > u\left(z' - x'\right) - u\left(z' - x\right),$$

contradicting (6.36). This establishes that $s(k)$ must be nondecreasing everywhere. □

In addition, Assumption 2 (the Inada conditions) imply that savings and consumption levels have to be interior, thus Theorem 6.6 applies and immediately establishes:

PROPOSITION 6.2. *Given Assumptions 1, 2 and 3′, the value function* $V(k)$ *defined above is differentiable.*

Consequently, from Theorem 6.10, we can look at the Euler equations. The Euler equation from (6.35) takes the simple form:

$$u'(c) = \beta V'(s)$$

where s denotes the next date's capital stock. Applying the envelope condition, we have

$$V'(k) = \left[f'(k) + (1-\delta)\right] u'(c).$$

Consequently, we have the familiar condition

$$u'(c(t)) = \beta\left[f'(k(t+1)) + (1-\delta)\right] u'(c(t+1)). \tag{6.37}$$

As before, a *steady state* is as an allocation in which the capital-labor ratio and consumption do not depend on time, so again denoting this by *, we have the steady state capital-labor ratio as

$$\beta\left[f'(k^*) + (1-\delta)\right] = 1, \tag{6.38}$$

which is a remarkable result, because it shows that the steady state capital-labor ratio does not depend on preferences, but simply on technology, depreciation and the discount factor. We will obtain an analog of this result in the continuous-time neoclassical model as well.

Moreover, since $f(\cdot)$ is strictly concave, k^* is uniquely defined. Thus we have

Proposition 6.3. *In the neoclassical optimal growth model specified in (6.33) and (6.34) with Assumptions 1, 2 and 3′, there exists a unique steady-state capital-labor ratio k^* given by (6.38), and starting from any initial $k(0) > 0$, the economy monotonically converges to this unique steady state, i.e., if $k(0) < k^*$, then the equilibrium capital stock sequence $k(t) \uparrow k^*$ and if $k(0) > k^*$, then the equilibrium capital stock sequence $k(t) \downarrow k^*$.*

Proof. Uniqueness and existence were established above. To establish monotone convergence, we start with arbitrary initial capital stock $k(0)$ and observe that $k(t+1) = s(k(t))$ for all $t \geq 0$, where $s(\cdot)$ was defined and shown to be nondecreasing in Proposition 6.1. It must be the case that either $k(1) = s(k(0)) \geq k(0)$ or $k(1) = s(k(0)) < k(0)$.

Consider the first case. Since $s(\cdot)$ is nondecreasing and $k(2) = s(k(1))$, we must have $k(2) \geq k(1)$. By induction, $k(t) = s(k(t-1)) \geq k(t-1) = s(k(t-2))$. Moreover, by definition $k(t) \in \left[0, \vec{k}\right]$. Therefore, in this case $\{k(t)\}_{t=0}^{\infty}$ is a nondecreasing sequence in a compact set starting with $k(0) > 0$, thus it necessarily converges to some limit $k(\infty) > 0$, which by definition satisfies $k(\infty) = s(k(\infty))$. Since k^* is the unique steady state (corresponding to positive capital-labor ratio), this implies that $k(\infty) = k^*$, and thus $k(t) \to k^*$. Moreover, since $\{k(t)\}_{t=0}^{\infty}$ is nondecreasing, it must be the case that $k(t) \uparrow k^*$, and thus this corresponds to the case where $k(0) \leq k^*$.

Next consider the case in which $k(1) = s(k(0)) < k(0)$. The same argument as above applied in reverse now establishes that $\{k(t)\}_{t=0}^{\infty}$ is a nonincreasing sequence in the compact set $\left[0, \vec{k}\right]$, thus it converges to a uniquely limit point $k(\infty)$. In this case, there are two candidate values for $k(\infty)$, $k(\infty) = 0$ or $k(\infty) = k^*$. The former is not possible, since, as Exercise 6.18 shows, Assumption 2 implies that $s(\varepsilon) > \varepsilon$ for ε sufficiently small. Thus $k(\infty) = k^*$. Since $\{k(t)\}_{t=0}^{\infty}$ is nonincreasing, in this case we must have $k(0) > k^*$ and thus $\{k(t)\}_{t=0}^{\infty} \downarrow k^*$, completing the proof. □

Consequently, in the optimal growth model there exists a unique steady state and the economy monotonically converges to the unique steady state, for example by accumulating more and more capital (if it starts with a too low capital-labor ratio).

We can also show that consumption also monotonically increases (or decreases) along the path of adjustments to the unique-steady state:

Proposition 6.4. *$c(k)$ defined in Proposition 6.1 is nondecreasing. Moreover, if $k_0 < k^*$, then the equilibrium consumption sequence $c(t) \uparrow c^*$ and if $k_0 > k^*$, then $c(t) \downarrow c^*$, where c^* is given by*

$$c^* = f(k^*) - \delta k^*.$$

Proof. See Exercise 6.16. □

This discussion illustrates that the optimal growth model is very tractable, and we can easily incorporate population growth and technological change as in the Solow growth model. There is no immediate counterpart of a saving rate, since this depends on the utility function. But interestingly and very differently from the Solow growth model, the steady state capital-labor ratio and steady state income level do not depend on the saving rate anyway.

We will return to all of these issues, and provide a more detailed discussion of the equilibrium growth in the context of the continuous time model. But for now, it is also useful to see how this optimal growth allocation can be decentralized, i.e., in this particular case we can use the Second Welfare Theorem to show that the optimal growth allocation is also a competitive equilibrium.

Finally, it is worth noting that results concerning the convergence behavior of the optimal growth model are sometimes referred to as the "*Turnpike Theorem*". This term is motivated by the study of finite-horizon versions of this model. In particular, suppose that the economy ends at some date $T > 0$. How do optimal growth and capital accumulation look like in this economy? The early literature on optimal growth showed that as $T \to \infty$, the optimal capital-labor ratio sequence $\{k(t)\}_{t=0}^{T}$ would become arbitrarily close to k^* as defined by (6.38), but then in the last few periods, it would sharply decline to zero to satisfy the transversality condition (recall the discussion of the finite-horizon transversality condition in Section 6.5).

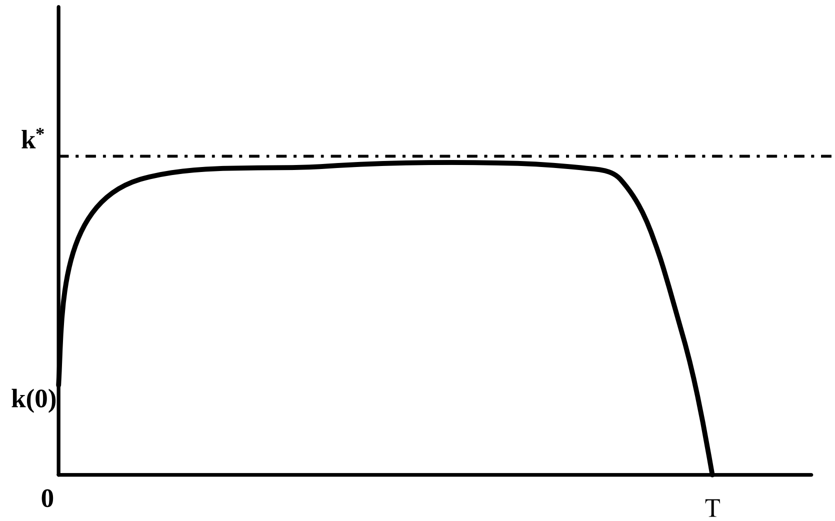

Figure 6.1. Turnpike dynamics in a finite-horizon (T-periods) neoclassical growth model starting with initial capital-labor ratio $k(0)$.

The path of the capital-labor ratio thus resembles a turnpike approaching a highway as shown in Figure 6.1 (see Exercise 6.17).

6.7. Competitive Equilibrium Growth

Our main interest is not optimal growth, but equilibrium growth. Nevertheless, the Second Welfare Theorem, Theorem 5.7 of the previous chapter, implies that the optimal growth path characterized in the previous section also corresponds to an equilibrium growth path (in the sense that, it can be decentralized as a competitive equilibrium). In fact, since we have focused on an economy admitting a representative household, the most straightforward competitive allocation would be a symmetric one, where all households, each with the instantaneous utility function $u(c)$, make the same decisions and receive the same allocations. We now discuss this symmetric competitive equilibrium briefly.

Suppose that each household starts with an endowment of capital stock K_0, meaning that the initial endowments are also symmetric (recall that there is a mass 1 of households and the total initial endowment of capital of the economy is K_0). The other side of the economy is populated by a large number of competitive firms, which are modeled using the aggregate production function.

The definition of a competitive equilibrium in this economy is standard. In particular, we have:

Definition 6.3. *A competitive equilibrium consists of paths of consumption, capital stock, wage rates and rental rates of capital,* $\{C(t), K(t), w(t), R(t)\}_{t=0}^{\infty}$*, such that the representative household maximizes its utility given initial capital stock* K_0 *and the time path of prices* $\{w(t), R(t)\}_{t=0}^{\infty}$*, and the time path of prices* $\{w(t), R(t)\}_{t=0}^{\infty}$ *is such that given the time path of capital stock and labor* $\{K(t), L(t)\}_{t=0}^{\infty}$ *all markets clear.*

Households rent their capital to firms. As in the basic Solow model, they will receive the competitive rental price of

$$R(t) = f'(k(t)),$$

and thus face a gross rate of return equal to

$$1 + r(t+1) = f'(k(t)) + (1-\delta) \tag{6.39}$$

for renting one unit of capital at time t in terms of date $t+1$ goods. Notice that the gross rate of return on assets is defined as $1+r$, since r often refers to the net interest rate. In fact in the continuous time model, this is exactly what the term r will correspond to. This notation should therefore minimize confusion.

In addition, to capital income, households in this economy will receive wage income for supplying their labor at the market wage of $w(t) = f(k(t)) - k(t) f'(k(t))$.

Now consider the maximization problem of the representative household:

$$\max_{\{c(t), a(t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c(t))$$

subject to the flow budget constraint

$$a(t+1) = (1 + r(t+1)) a(t) - c(t) + w(t), \tag{6.40}$$

where $a(t)$ denotes asset holdings at time t and as before, $w(t)$ is the wage income of the individual (since labor supply is normalized to 1). The timing underlying the flow budget constraint (6.40) is that the individual rents his capital or asset holdings, $a(t)$, to firms to be used as capital at time $t+1$. Out of the proceeds, he consumes and whatever is left, together with his wage earnings, $w(t)$, make up his asset holdings at the next date, $a(t+1)$.

In addition to this flow budget constraint, we have to impose a *no Ponzi game* constraint to ensure that the individual asset holdings do not tend to minus infinity. Since this constraint will be discussed in detail in Chapter 8, we do not introduce it here (though note that without this constraint, there are other, superfluous solutions to the consumer maximization problem).

For now, it suffices to look at the Euler equation for the consumer maximization problem:

$$u'(c(t)) = (1 + r(t+1)) \beta u'(c(t+1)). \tag{6.41}$$

Imposing steady state implies that $c(t) = c(t+1)$. Therefore, in steady state we must have

$$(1 + r(t+1)) \beta = 1.$$

Next, market clearing immediately implies that $1+r(t+1)$ is given by (6.39), so the capital-labor ratio of the competitive equilibrium is given by

$$\beta\left[f'(k(t+1))+(1-\delta)\right]=1,$$

The steady state is given by

$$\beta\left[f'(k^{*})+(1-\delta)\right]=1.$$

These two equations are identical to equations (6.38) and (6.39), which characterize the solution to the optimum growth problem. In fact, a similar argument establishes that the entire competitive equilibrium path is identical to the optimal growth path. Specifically, substituting for $1+r(t+1)$ from (6.39) into (6.41), we obtain

$$u'(c(t))=\beta\left[f'(k(t+1))+(1-\delta)\right]u'(c(t+1)), \tag{6.42}$$

which is identical to (6.37). This condition also implies that given the same initial condition, the trajectory of capital-labor ratio in the competitive equilibrium will be identical to the behavior of the capital-labor ratio in the optimal growth path (see Exercise 6.20). This is, of course, not surprising in view of the second (and first) welfare theorems we saw above.

We will discuss many of the implications of competitive equilibrium growth in the neoclassical model once we go through the continuous time version as well.

6.8. Taking Stock

This chapter has been concerned with basic dynamic programming techniques for discrete time infinite-dimensional problems. These techniques are not only essential for the study of economic growth, but are widely used in many diverse areas of macroeconomics and economics more generally. A good understanding of these techniques is essential for an appreciation of the mechanics of economic growth, i.e., how different models of economic growth work, how they can be improved and how they can be taken to the data. For this reason, this chapter is part of the main body of the text, rather than relegated to the Mathematical Appendices.

This chapter also presented a number of applications of dynamic programming, including a preliminary but detailed analysis of the one-sector optimal growth problem. The reader will have already noted the parallels between this model and the basic Solow model discussed in Chapter 2. These parallels will be developed further in Chapter 8. We have also briefly discussed the decentralization of the optimal growth path and the problem of utility maximization in a dynamic competitive equilibrium. Finally, we presented a model of searching for ideas or for better techniques. While this is not a topic typically covered in growth or introductory macro textbooks, it provides a tractable application of dynamic programming techniques and is also useful as an introduction to models in which ideas and technologies are endogenous objects.

It is important to emphasize that the treatment in this chapter has assumed away a number of difficult technical issues. First, the focus has been on discounted problems, which are simpler than undiscounted problems. In economics, very few situations call for modeling using undiscounted objective functions (i.e., $\beta = 1$ rather than $\beta \in (0, 1)$). More important, throughout we have assumed that payoffs are bounded and the state vector x belongs to a compact subset of the Euclidean space, X. This rules out many interesting problems, such as endogenous growth models, where the state vector grows over time. Almost all of the results presented here have equivalents for these cases, but these require somewhat more advanced treatments.

6.9. References and Literature

At some level the main idea of dynamic programming, the Principle of Optimality, is a straightforward concept. Nevertheless, it is also a powerful concept and this will be best appreciated once a number of its implications are derived. The basic ideas of dynamic programming, including the Principle of Optimality, were introduced by Richard Bellman, in his famous monograph, Bellman (1957). Most of the basic results about finite and infinite-dimensional dynamic programming problems are contained in this monograph. Interestingly, many of these ideas were anticipated by Shapley (1953) in his study of stochastic games. Shapley analyzed the characterization of equilibrium points of zero-sum stochastic games. His formulation of these games anticipated what later became known as Markov Decision Problems, which are closely related to dynamic programming problems. Moreover, Shapley used ideas similar to the Principle of Optimality and the Contraction Mapping Theorem to show the existence of a unique solution to these dynamic zero-sum games. A more detailed treatment of Markov Decision Problems can be found in Puterman (1994), who also discusses the relationship between Shapley's (1953) work, the general theory of Markov Decision Problems and dynamic programming.

To the best of my knowledge, Karlin (1955) was the first one to provide a simple formal proof of the Principle of Optimality, which is similar to the one presented here. Denardo (1967) developed the use of the contraction mappings in the theory of dynamic programming. Howard (1960) contains a more detailed analysis of discounted stochastic dynamic programming problems. Blackwell (1965) introduced the Blackwell's sufficient conditions for a contraction mapping and applied them in the context of stochastic discounted dynamic programming problems. The result on the differentiability of the value function was first proved in Benveniste and Scheinkman (1979).

The most complete treatment of discounted dynamic programming problems is in Stokey, Lucas and Prescott (1989). My treatment here is heavily influenced by theirs and borrows much from their insights. Relative to their treatment, some of the proofs have been simplified and we have limited the analysis to the case with compact sets and bounded payoff functions. The reader can find generalizations of Theorems 6.1-6.6 to certain problems with unbounded returns and choice sets in Stokey, Lucas and Prescott (1989), Chapter 4 for the deterministic case, and the equivalent theorems for stochastic dynamic programming problems in their Chapter 9.

A much simpler but insightful exposition of dynamic programming is in Sundaram (1996), which also has a proof of Proposition 6.1 similar to the one given here.

Some useful references on the Contraction Mapping Theorem and its applications include Denardo (1967), Kolmogorov and Fomin (1970), Kreyszig (1978) and the eminently readable Bryant (1985), which contains applications of the Contraction Mapping Theorem to prove existence and uniqueness of solutions to differential equations and the Implicit Function Theorem.

Another excellent reference for applications of dynamic programming to economics problems is Ljungqvist and Sargent (2005), which also gives a more informal introduction to the main results of dynamic programming.

6.10. Exercises

Exercise 6.1. Consider the formulation of the discrete time optimal growth model as in Example 6.1. Show that with this formulation and Assumptions 1 and 2 from Chapter 2, the discrete time optimal growth model satisfies Assumptions 6.1-6.5.

Exercise 6.2. * Prove that if for some $n \in \mathbb{Z}_+$ T^n is a contraction over a complete metric space (S, d), then T has a unique fixed point in S.

Exercise 6.3. * Suppose that T is a contraction over the metric space (S, d) with modulus $\beta \in (0, 1)$. Prove that for any $z, z' \in S$ and $n \in \mathbb{Z}_+$, we have

$$d\left(T^n z, z'\right) \leq \beta^n d\left(z, z'\right).$$

Discuss how this result can be useful in numerical computations.

Exercise 6.4. *

(1) Prove the claims made in Example 6.3 and that the differential equation in (6.7) has a unique continuous solution.
(2) Recall equation (6.8) from Example 6.3. Now apply the same argument to Tg and $T\tilde{g}$ and prove that

$$\left\|T^2 g - T^2 \tilde{g}\right\| \leq M^2 \times \frac{s^2}{2} \times \|g - \tilde{g}\|.$$

(3) Applying this argument recursively, prove that for any $n \in \mathbb{Z}_+$, we have

$$\|T^n g - T^n \tilde{g}\| \leq M^n \times \frac{s^n}{n!} \times \|g - \tilde{g}\|.$$

(4) Using the previous inequality, the fact that for any $B < \infty$, $B^n/n! \to 0$ as $n \to 0$ and the result in Exercise 6.2, prove that the differential equation has a unique continuous solution on the compact interval $[0, s]$ for any $s \in \mathbb{R}_+$.

Exercise 6.5. * Recall the Implicit Function Theorem, Theorem A.23 in Appendix Chapter A. Here is a slightly simplified version of it: consider the function $\phi(y, x)$ such that that $\phi : \mathbb{R}\times [a, b] \to \mathbb{R}$ is continuously differentiable with bounded first derivatives. In particular, there exists $0 < m < M < \infty$ such that

$$m \leq \frac{\partial \phi(y, x)}{\partial y} \leq M$$

for all x and y. Then the Implicit Function Theorem states that there exists a continuously differentiable function $y : [a, b] \to \mathbb{R}$ such that

$$\phi(y(x), x) = 0 \text{ for all } x \in [a, b].$$

Provide a proof for this theorem using the Contraction Mapping Theorem, Theorem 6.7 along the following lines:

(1) Let $\mathbf{C}^1([a, b])$ be the space of continuously differentiable functions defined on$[a, b]$. Then for every $y \in \mathbf{C}^1([a, b])$, construct the operator

$$Ty = y(x) - \frac{\phi(y(x), x)}{M} \text{ for } x \in [a, b].$$

Show that $T : \mathbf{C}^1([a, b]) \to \mathbf{C}^1([a, b])$ and is a contraction.

(2) Applying Theorem 6.7 derive the Implicit Function Theorem.

Exercise 6.6. * Prove that T defined in (6.15) is a contraction.

Exercise 6.7. Let us return to Example 6.4.

(1) Prove that the law of motion of capital stock given by 6.28 monotonically converges to a unique steady state value of k^* starting with any $k_0 > 0$. What happens to the level of consumption along the transition path?

(2) Now suppose that instead of (6.28), you hypothesize that

$$\pi(x) = ax^\alpha + bx + c.$$

Verify that the same steps will lead to the conclusion that $b = c = 0$ and $a = \beta a$.

(3) Now let us characterize the explicit solution by guessing and verifying the form of the value function. In particular, make the following guess: $V(x) = A \ln x$, and using this together with the first-order conditions derive the explicit form solution.

Exercise 6.8. Consider the following discrete time optimal growth model with full depreciation:

$$\max_{\{c(t),k(t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left(c(t) - \frac{a}{2}[c(t)]^2\right)$$

subject to

$$k(t+1) = Ak(t) - c(t)$$

and $k(0) = k_0$. Assume that $k(t) \in [0, \bar{k}]$ and $a < \bar{k}^{-1}$, so that the utility function is always increasing in consumption.

(1) Formulate this maximization problem as a dynamic programming problem.

(2) Argue without solving this problem that there will exist a unique value function $V(k)$ and a unique policy rule $c = \pi(k)$ determining the level of consumption as a function of the level of capital stock.

(3) Solve explicitly for $V(k)$ and $\pi(k)$ [Hint: guess the form of the value function $V(k)$, and use this together with the Bellman and Euler equations; verify that this guess satisfies these equations, and argue that this must be the unique solution].

Exercise 6.9. Consider Problem A1 or A2 with $x \in X \subset \mathbb{R}$ and suppose that Assumptions 6.1-6.3 and 6.5 hold. Prove that the optimal policy function $y = \pi(x)$ is nondecreasing if $\partial^2 U(x, y) / \partial x \partial y \geq 0$.

Exercise 6.10. Show that in Theorem 6.10, a sequence $\{x'(t)\}_{t=0}^{\infty}$ that satisfies the Euler equations, but not the transversality condition could yield a suboptimal plan.

Exercise 6.11. Consider the following modified version of Problem A1:

$$\begin{aligned} V^*(x(0)) &= \sup_{\{x(t+1)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(x(t), x(t+1)) \\ &\quad \text{subject to} \\ x(t+1) &\in G(x(t), t), \qquad \text{for all } t \geq 0 \\ &\quad x(0) \text{ given,} \end{aligned}$$

where the main difference from Problem A1 is that the constraint correspondence is time-varying. Suppose that $x(t) \in X \subset \mathbb{R}_+^K$ for all t, that $U : \mathbb{R}_+^{2K} \to \mathbb{R}_+^K$ is continuously differentiable, concave and strictly increasing an its first K arguments, and that the correspondence $G : \mathbb{R}_+^{K+1} \rightrightarrows \mathbb{R}_+^K$ is continuous and convex-valued. Show that a sequence $\{x^*(t+1)\}_{t=0}^{\infty}$ with $x^*(t+1) \in \text{Int}G(x^*(t), t),\ t = 0, 1, \ldots$, is an optimal solution to this problem given $x(0)$, if it satisfies (6.21) and (6.25).

Exercise 6.12. Let us return to the problem discussed in Example 6.5.

(1) Show that even when the sequence of labor earnings $\{w(t)\}_{t=0}^{\infty}$ is not constant over time, the result in Exercise 6.11 can be applied to obtain exactly the same result as in Example 6.5. In particular, show that the optimal consumption profile is still given by (6.31).

(2) Using the transversality condition together with $a(0)$ and $\{w(t)\}_{t=0}^{\infty}$, find an expression implicitly determining the initial level of consumption, $c(0)$. What happens to this level of consumption when $a(0)$ increases?

(3) Consider the special case where $u(c) = \ln c$. Provide a closed-form solution for $c(0)$.

(4) Next, returning to the general utility function $u(\cdot)$, consider a change in the earnings profile to a new sequence $\{\tilde{w}(t)\}_{t=0}^{\infty}$ such that for some $T < \infty$, $w(t) < \tilde{w}(t)$ for all $t < T$, $w(t) \geq \tilde{w}(t)$ for all $t \geq T$, and $\sum_{t=0}^{\infty} (1+r)^{-t} w(t) = \sum_{t=0}^{\infty} (1+r)^{-t} \tilde{w}(t)$. What is the effect of this on the initial consumption level and the consumption path? Provide a detailed economic intuition for this result.

EXERCISE 6.13. Consider again Example 6.5.

(1) Show that when $r \leq \beta - 1$, $a(t) \in (0, \bar{a})$ for all t, where recall that $\bar{a} \equiv a(0) + \sum_{t=0}^{\infty} (1+r)^{-t} w(t)$.

(2) Show that when $r > \beta - 1$, there does not exist $\bar{a} < \infty$ such that $a(t) \in (0, \bar{a})$ for all t.

EXERCISE 6.14. Consider the following discrete time optimal growth model

$$\max_{\{c(t), k(t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} u(c(t))$$

subject to

$$k(t+1) = k(t) - c(t)$$

and

$$k(0) = k_0 < \infty.$$

Assume that $u(\cdot)$ is a strictly increasing, strictly concave and bounded function. Prove that there exists no optimal solution to this problem. Explain why.

EXERCISE 6.15. Consider the following discrete time optimal growth model with full depreciation:

$$\max_{\{c(t), k(t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c(t))$$

subject to

$$k(t+1) = f(k(t)) - c(t)$$

and

$$k(0) = k_0.$$

Assume that $u(\cdot)$ is strictly concave and increasing, and $f(\cdot)$ is concave and increasing.

(1) Formulate this maximization problem as a dynamic programming problem.

(2) Prove that there exists unique value function $V(k)$ and a unique policy rule $c = \pi(k)$, and that $V(k)$ is continuous and strictly concave and $\pi(k)$ is continuous and increasing.

(3) When will $V(k)$ be differentiable?

(4) Assuming that $V(k)$ and all the other functions are differentiable, characterize the Euler equation that determines the optimal path of consumption and capital accumulation.

(5) Is this Euler equation enough to determine the path of k and c? If not, what other condition do we need to impose? Write down this condition and explain intuitively why it makes sense.

Exercise 6.16. Prove that, as claimed in Proposition 6.4, in the basic discrete-time optimal growth model, the optimal consumption plan $c(k)$ is nondecreasing, and when the economy starts with $k_0 < k^*$, the unique equilibrium involves $c(t) \uparrow c^*$.

Exercise 6.17. Consider the finite-horizon optimal growth model described at the end of Section 6.6. Let the optimal capital-labor ratio sequence of the economy with horizon T be denoted by $\left\{k^T(t)\right\}_{t=0}^{T}$ with $k^T(0) = k_0$. Show that for every $\varepsilon > 0$, there exists $T < \infty$ and $t' < T$ such that $\left|k^T(t') - k^*\right| < \varepsilon$. Show that $k^T(T) = 0$. Then assuming that k_0 is sufficiently small, show that the optimal capital-labor ratio sequence looks as in Figure 6.1.

Exercise 6.18. Prove that as claimed in the proof of Proposition 6.3, Assumption 2 implies that $s(\varepsilon) > \varepsilon$ for ε sufficiently small. Provide an intuition for this result.

Exercise 6.19. * Provide a proof of Proposition 6.1 without the differentiability assumption on the utility function $u(\cdot)$ imposed in Assumption 3′.

Exercise 6.20. Prove that the optimal growth path starting with capital-labor ratio k_0, which satisfies (6.37) is identical to the competitive equilibrium starting with capital-labor ratio and satisfying the same condition (or equivalently, equation (6.42)).

CHAPTER 7

Review of the Theory of Optimal Control

The previous chapter introduced the basic tools of dynamic optimization in discrete time. I will now review a number of basic results in dynamic optimization in continuous time—particularly the so-called *optimal control* approach. Both dynamic optimization in discrete time and in continuous time are useful tools for macroeconomics and other areas of dynamic economic analysis. One approach is not superior to another; instead, certain problems become simpler in discrete time while, certain others are naturally formulated in continuous time.

Continuous-time optimization introduces a number of new mathematical issues. The main reason is that even with a finite horizon, the maximization is with respect to an infinite-dimensional object (in fact an entire function, $y : [t_0, t_1] \to \mathbb{R}$). This requires a brief review of some basic ideas from the *calculus of variations* and from the theory of optimal control. Most of the tools and ideas that are necessary for this book are straightforward. Nevertheless, a reader who simply wishes to apply these tools may decide skim most of this chapter, focusing on the main theorems, especially Theorems 7.14 and 7.15, and their application to the canonical continuous-time optimal growth problem in Section 7.7.

In the rest of this chapter, I first review the finite-horizon continuous-time maximization problem and provide the simplest treatment of this problem (which is more similar to calculus of variations than to optimal control). I then present the more powerful theorems from the theory of optimal control as developed by Pontryagin and co-authors.

The canonical problem we are interested in can be written as

$$\max_{\mathbf{x}(t),\mathbf{y}(t)} W\left(\mathbf{x}\left(t\right), \mathbf{y}\left(t\right)\right) \equiv \int_0^{t_1} f\left(t, \mathbf{x}\left(t\right), \mathbf{y}\left(t\right)\right) dt$$

subject to

$$\dot{\mathbf{x}}\left(t\right) = g\left(t, \mathbf{x}\left(t\right), \mathbf{y}\left(t\right)\right)$$

and

$$\mathbf{y}\left(t\right) \in \mathcal{Y}\left(t\right) \text{ for all } t, \ \mathbf{x}\left(0\right) = \mathbf{x}_0,$$

where for each t, $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are finite-dimensional vectors (i.e., $\mathbf{x}(t) \in \mathbb{R}^{K_x}$ and $\mathbf{y}(t) \in \mathbb{R}^{K_y}$, where K_x and K_y are integers). We refer to $\mathbf{x}$ as the *state* variable. Its behavior is governed by a vector-valued differential equation (i.e., a set of differential equations) given the behavior of the *control* variables $\mathbf{y}(t)$. The end of the planning horizon t_1 can be equal to infinity. The function $W(\mathbf{x}(t), \mathbf{y}(t))$ denotes the value of the objective function when controls are given by $\mathbf{y}(t)$ and the resulting behavior of the state variable is summarized by $\mathbf{x}(t)$. We also refer to f as the *objective function* (or the payoff function) and to g as the *constraint function.*

This problem formulation is general enough to incorporate discounting, since both the instantaneous payoff function f and the constraint function g depend directly on time in an arbitrary fashion. We will start with the finite-horizon case and then treat the infinite-horizon maximization problem, focusing particularly on the case where there is exponential discounting.

7.1. Variational Arguments

Consider the following finite-horizon continuous time problem

$$\max_{x(t),y(t),x_1} W(x(t), y(t)) \equiv \int_0^{t_1} f(t, x(t), y(t))\,dt \tag{7.1}$$

subject to

$$\dot{x}(t) = g(t, x(t), y(t)) \tag{7.2}$$

and

$$y(t) \in \mathcal{Y}(t) \text{ for all } t,\ x(0) = x_0 \text{ and } x(t_1) = x_1. \tag{7.3}$$

Here the state variable $x(t) \in \mathbb{R}$ is one-dimensional and its behavior is governed by the differential equation (7.2). The control variable $y(t)$ must belong to the set $\mathcal{Y}(t) \subset \mathbb{R}$. Throughout, we assume that $\mathcal{Y}(t)$ is nonempty and convex. We refer to a pair of functions $(x(t), y(t))$ that jointly satisfy (7.2) and (7.3) as an *admissible pair*. Throughout, as in the previous chapter, we assume the value of the objective function is finite, that is, $W(x(t), y(t)) < \infty$ for any admissible pair $(x(t), y(t))$.

Let us first suppose that $t_1 < \infty$, so that we have a finite-horizon optimization problem. Notice that there is also a terminal value constraint $x(t_1) = x_1$, but x_1 is included as an additional choice variable. This implies that the terminal value of the state variable x is *free*. Below, we will see that in the context of finite-horizon economic problems, the formulation where x_1 is *not* a choice variable may be simpler (see Example 7.1), but the development in this section is more natural when the terminal value x_1 is free.

In addition, to simplify the exposition, throughout we assume that f and g are continuously differentiable functions.

The difficulty in characterizing the optimal solution to this problem lies in two features:

1. We are choosing a function $y : [0, t_1] \to \mathcal{Y}$ rather than a vector or a finite dimensional object.
2. The constraint takes the form of a differential equation, rather than a set of inequalities or equalities.

These features make it difficult for us to know what type of optimal policy to look for. For example, y may be a highly discontinuous function. It may also hit the boundary of the feasible set—thus corresponding to a "corner solution". Fortunately, in most economic problems there will be enough structure to make optimal solutions continuous functions. Moreover, in most macroeconomic and growth applications, the Inada conditions make sure that the optimal solutions to the relevant dynamic optimization problems lie in the interior of the feasible set. These features considerably simplify the characterization of the optimal solution. In fact, when y is a continuous function of time and lies in the interior of the feasible set, it can be characterized by using the variational arguments similar to those developed by Euler, Lagrange and others in the context of the theory of calculus of variations. Since these tools are not only simpler but also more intuitive, we start our treatment with these variational arguments.

The *variational principle* of the calculus of variations simplifies the above maximization problem by first assuming that a continuous solution (function) $\hat{y}$ that lies everywhere in the interior of the set $\mathcal{Y}$ exists, and then characterizes what features this solution must have in order to reach an optimum (for the relationship of the results here to the calculus of variations, see Exercise 7.3).

More formally let us assume that $(\hat{x}(t), \hat{y}(t))$ is an admissible pair such that $\hat{y}(\cdot)$ is continuous over $[0, t_1]$ and $\hat{y}(t) \in \text{Int}\mathcal{Y}(t)$, and we have

$$W(\hat{x}(t), \hat{y}(t)) \geq W(x(t), y(t))$$

for any other admissible pair $(x(t), y(t))$.

The important and stringent assumption here is that $(\hat{x}(t), \hat{y}(t))$ is an optimal solution that never hits the boundary and that does not involve any discontinuities. Even though this will be a feature of optimal controls in most economic applications, in purely mathematical terms this is a strong assumption. Recall, for example, that in the previous chapter, we did not make such an assumption and instead started with a result on the existence of solutions and then proceeded to characterizing the properties of this solution (such as continuity and differentiability of the value function). However, the problem of continuous time optimization is sufficiently difficult that proving existence of solutions is not a trivial matter. We will return to a further discussion of this issue below, but for now we follow the standard practice and assume that an interior continuous solution $\hat{y}(t) \in \text{Int}\mathcal{Y}(t)$, together with the corresponding law of motion of the state variable, $\hat{x}(t)$, exists. Note also that since the behavior of the state variable x is given by the differential equation (7.2), when $y(t)$ is continuous, $\dot{x}(t)$ will also be continuous, so that $x(t)$ is continuously differentiable. When $y(t)$ is piecewise continuous, $x(t)$ will be, correspondingly, piecewise smooth.

We now exploit these features to derive *necessary* conditions for an optimal path of this form. To do this, consider the following *variation*

$$y(t,\varepsilon) \equiv \hat{y}(t) + \varepsilon\eta(t),$$

where $\eta(t)$ is an arbitrary *fixed* continuous function and $\varepsilon \in \mathbb{R}$ is a scalar. We refer to this as a variation, because given $\eta(t)$, by varying ε, we obtain different sequences of controls. The problem, of course, is that some of these may be infeasible, i.e., $y(t,\varepsilon) \notin \mathcal{Y}(t)$ for some t. However, since $\hat{y}(t) \in \text{Int}\mathcal{Y}(t)$, and a continuous function over a compact set $[0,t_1]$ is bounded, for any fixed $\eta(\cdot)$ function, we can always find $\varepsilon_\eta > 0$ such that

$$y(t,\varepsilon) \equiv \hat{y}(t) + \varepsilon\eta(t) \in \text{Int}\mathcal{Y}(t)$$

for all $\varepsilon \in [-\varepsilon_\eta, \varepsilon_\eta]$, so that $y(t,\varepsilon)$ constitutes a *feasible variation*. Consequently, we can use variational arguments for sufficiently small ε's. The fact that we have to look at small ε's is not a drawback for deriving necessary conditions for optimality. In analogy with standard calculus, necessary conditions require that there should be no small change in controls that increase the value of the objective function, but this does not tell us that there are no non-infinitesimal changes that might lead to a higher value of the objective function.

To prepare for these arguments, let us fix an arbitrary $\eta(\cdot)$, and define $x(t,\varepsilon)$ as the path of the state variable corresponding to the path of control variable $y(t,\varepsilon)$. This implies that $x(t,\varepsilon)$ is given by:

$$\dot{x}(t,\varepsilon) \equiv g(t, x(t,\varepsilon), y(t,\varepsilon)) \text{ for all } t \in [0,t_1], \text{ with } x(0,\varepsilon) = x_0. \tag{7.4}$$

For $\varepsilon \in [-\varepsilon_\eta, \varepsilon_\eta]$, define:

$$\begin{aligned} \mathcal{W}(\varepsilon) &\equiv W(x(t,\varepsilon), y(t,\varepsilon)) \\ &= \int_0^{t_1} f(t, x(t,\varepsilon), y(t,\varepsilon))\,dt. \end{aligned} \tag{7.5}$$

By the fact that $\hat{y}(t)$ is optimal, and that for $\varepsilon \in [-\varepsilon_\eta, \varepsilon_\eta]$, $y(t,\varepsilon)$ and $x(t,\varepsilon)$ are feasible, we have that

$$\mathcal{W}(\varepsilon) \leq \mathcal{W}(0) \text{ for all } \varepsilon \in [-\varepsilon_\eta, \varepsilon_\eta].$$

Next, rewrite the equation (7.4), so that

$$g(t, x(t,\varepsilon), y(t,\varepsilon)) - \dot{x}(t,\varepsilon) \equiv 0$$

for all $t \in [0,t_1]$. This implies that for *any* function $\lambda : [0,t_1] \to \mathbb{R}$, we have

$$\int_0^{t_1} \lambda(t)\left[g(t, x(t,\varepsilon), y(t,\varepsilon)) - \dot{x}(t,\varepsilon)\right] dt = 0, \tag{7.6}$$

since the term in square brackets is identically equal to zero. In what follows, we suppose that the function $\lambda(\cdot)$ is continuously differentiable. This function, when chosen suitably, will be the *costate* variable, with a similar interpretation to the Lagrange multipliers in standard (constrained) optimization problems. As with Lagrange multipliers, this will not be true for any $\lambda(\cdot)$ function, but only for a $\lambda(\cdot)$ that is chosen appropriately to play the role of the costate variable.

Adding (7.6) to (7.5), we obtain

$$\mathcal{W}(\varepsilon) \equiv \int_0^{t_1} \{f(t, x(t,\varepsilon), y(t,\varepsilon)) + \lambda(t)[g(t, x(t,\varepsilon), y(t,\varepsilon)) - \dot{x}(t,\varepsilon)]\} dt. \tag{7.7}$$

To evaluate (7.7), let us first consider the integral $\int_0^{t_1} \lambda(t)\dot{x}(t,\varepsilon)dt$. Integrating this expression by parts (see Appendix Chapter B), we obtain

$$\int_0^{t_1} \lambda(t)\dot{x}(t,\varepsilon)dt = \lambda(t_1)x(t_1,\varepsilon) - \lambda(0)x_0 - \int_0^{t_1} \dot{\lambda}(t)x(t,\varepsilon)dt.$$

Substituting this expression back into (7.7), we obtain:

$$\begin{aligned} \mathcal{W}(\varepsilon) \equiv & \int_0^{t_1} \left[f(t, x(t,\varepsilon), y(t,\varepsilon)) + \lambda(t)g(t, x(t,\varepsilon), y(t,\varepsilon)) + \dot{\lambda}(t)x(t,\varepsilon)\right] dt \\ & -\lambda(t_1)x(t_1,\varepsilon) + \lambda(0)x_0. \end{aligned}$$

Recall that f and g are continuously differentiable, and $y(t,\varepsilon)$ is continuously differentiable in ε by construction, which also implies that $x(t,\varepsilon)$ is continuously differentiable in ε. Let us denote the partial derivatives of x and y by x_ε and y_ε, and the partial derivatives of f and g by f_t, f_x, f_y, etc.. Differentiating the previous expression with respect to ε (making use of Leibniz's rule, Theorem B.4 in Appendix Chapter B), we obtain

$$\begin{aligned} \mathcal{W}'(\varepsilon) \equiv & \int_0^{t_1} \left[f_x(t, x(t,\varepsilon), y(t,\varepsilon)) + \lambda(t)g_x(t, x(t,\varepsilon), y(t,\varepsilon)) + \dot{\lambda}(t)\right] x_\varepsilon(t,\varepsilon) dt \\ & + \int_0^{t_1} [f_y(t, x(t,\varepsilon), y(t,\varepsilon)) + \lambda(t)g_y(t, x(t,\varepsilon), y(t,\varepsilon))]\eta(t)dt \\ & -\lambda(t_1)x_\varepsilon(t_1,\varepsilon). \end{aligned}$$

Let us next evaluate this derivative at $\varepsilon = 0$ to obtain:

$$\begin{aligned} \mathcal{W}'(0) \equiv & \int_0^{t_1} \left[f_x(t, \hat{x}(t), \hat{y}(t)) + \lambda(t)g_x(t, \hat{x}(t), \hat{y}(t)) + \dot{\lambda}(t)\right] x_\varepsilon(t,0) dt \\ & + \int_0^{t_1} [f_y(t, \hat{x}(t), \hat{y}(t)) + \lambda(t)g_y(t, \hat{x}(t), \hat{y}(t))]\eta(t)dt \\ & -\lambda(t_1)x_\varepsilon(t_1,0). \end{aligned}$$

where, as above, $\hat{x}(t) = x(t, \varepsilon = 0)$ denotes the path of the state variable corresponding to the optimal plan, $\hat{y}(t)$. As with standard finite-dimensional optimization, if there exists some function $\eta(t)$ for which $\mathcal{W}'(0) \neq 0$, this means that $W(x(t), y(t))$ can be increased and thus the pair $(\hat{x}(t), \hat{y}(t))$ could not be an optimal solution. Consequently, optimality requires that

$$\mathcal{W}'(0) \equiv 0 \text{ for all } \eta(t). \tag{7.8}$$

Recall that the expression for $\mathcal{W}'(0)$ applies for any continuously differentiable $\lambda(t)$ function. Clearly, not all such functions $\lambda(\cdot)$ will play the role of a costate variable. Instead, as it is the case with Lagrange multipliers, the function $\lambda(\cdot)$ has to be chosen appropriately, and in this case, it must satisfy

$$f_y(t, \hat{x}(t), \hat{y}(t)) + \lambda(t) g_y(t, \hat{x}(t), \hat{y}(t)) \equiv 0 \text{ for all } t \in [0, t_1]. \tag{7.9}$$

This immediately implies that

$$\int_0^{t_1} [f_y(t, \hat{x}(t), \hat{y}(t)) + \lambda(t) g_y(t, \hat{x}(t), \hat{y}(t))] \eta(t) dt = 0 \text{ for all } \eta(t).$$

Since $\eta(t)$ is arbitrary, this implies that $x_\varepsilon(t,0)$ is also arbitrary. Thus the condition in (7.8) can hold only if the first and the third terms are also (individually) equal to zero. The first term, $[f_x(t, \hat{x}(t), \hat{y}(t)) + \lambda(t) g_x(t, \hat{x}(t), \hat{y}(t)) + \dot{\lambda}(t)]$, will be equal to zero for all $x_\varepsilon(t,0)$, if and only if

$$\dot{\lambda}(t) = -[f_x(t, \hat{x}(t), \hat{y}(t)) + \lambda(t) g_x(t, \hat{x}(t), \hat{y}(t))], \tag{7.10}$$

while the third term will be equal to zero for all values of $x_\varepsilon(t_1, 0)$, if and only if $\lambda(t_1) = 0$. The last two steps are further elaborated in Exercise 7.1. We have therefore obtained the result that the necessary conditions for an interior continuous solution to the problem of maximizing (7.1) subject to (7.2) and (7.3) are such that there should exist a continuously differentiable function $\lambda(\cdot)$ that satisfies (7.9), (7.10) and $\lambda(t_1) = 0$.

The condition that $\lambda(t_1) = 0$ is the *transversality condition* of continuous time optimization problems, which is naturally related to the transversality condition we encountered in the previous chapter. Intuitively, this condition captures the fact that after the planning horizon, there is no value to having more x.

This derivation, which builds on the standard arguments of calculus of variations, has therefore established the following theorem.[1]

Theorem 7.1. ***(Necessary Conditions)*** *Consider the problem of maximizing (7.1) subject to (7.2) and (7.3), with f and g continuously differentiable. Suppose that this problem has an interior continuous solution $\hat{y}(t) \in \text{Int}\mathcal{Y}(t)$ with corresponding path of state variable $\hat{x}(t)$. Then there exists a continuously differentiable costate function $\lambda(\cdot)$ defined over $t \in [0, t_1]$ such that (7.2), (7.9) and (7.10) hold, and moreover $\lambda(t_1) = 0$.*

As noted above, (7.9) looks similar to the first-order conditions of the constrained maximization problem, with $\lambda(t)$ playing the role of the Lagrange multiplier. We will return to this interpretation of the costate variable $\lambda(t)$ below.

Let us next consider a slightly different version of Theorem 7.1, where the terminal value of the state variable, x_1, is fixed, so that the maximization problem is

$$\max_{x(t), y(t)} W(x(t), y(t)) \equiv \int_0^{t_1} f(t, x(t), y(t)) dt, \tag{7.11}$$

[1] Below we present a more rigorous proof of Theorem 7.9, which generalizes the results in Theorem 7.2 in a number of dimensions.

subject to (7.2) and (7.3). The only difference is that there is no longer a choice over the terminal value of the state variable, x_1. In this case, we have:

Theorem 7.2. ***(Necessary Conditions II)*** *Consider the problem of maximizing (7.11) subject to (7.2) and (7.3), with f and g continuously differentiable. Suppose that this problem has an interior continuous solution $\hat{y}(t) \in \text{Int}\mathcal{Y}(t)$ with corresponding path of state variable $\hat{x}(t)$. Then there exists a continuously differentiable costate function $\lambda(\cdot)$ defined over $t \in [0, t_1]$ such that (7.2), (7.9) and (7.10) hold.*

Proof. The proof is similar to the arguments leading to Theorem 7.1, with the main change that now $x(t_1, \varepsilon)$ must equal x_1 for feasibility, so $x_\varepsilon(t_1, 0) = 0$ and $\lambda(t_1)$ is unrestricted. Exercise 7.5 asks you to complete the details. □

The new feature in this theorem is that the transversality condition $\lambda(t_1) = 0$ is no longer present, but we need to know what the terminal value of the state variable x should be.[2] We first start with an application of the necessary conditions in Theorem 7.2 to a simple economic problem. More interesting economic examples are provided later in the chapter and in the exercises.

Example 7.1. Consider a relatively common application of the techniques developed so far, which is the problem of utility-maximizing choice of consumption plan by an individual that lives between dates 0 and 1 (perhaps the most common application of these techniques is a physical one, that of finding the shortest curve between two points in the plane, see Exercise 7.4). The individual has an instantaneous utility function $u(c)$ and discounts the future exponentially at the rate $\rho > 0$. We assume that $u : [0, 1] \to \mathbb{R}$ is a strictly increasing, continuously differentiable and strictly concave function. The individual starts with a level of assets equal to $a(0) > 0$, earns an interest rate r on his asset holdings and also has a constant flow of labor earnings equal to w. Let us also suppose that the individual can never have negative asset position, so that $a(t) \geq 0$ for all t. Therefore, the problem of the individual can be written as

$$\max_{[c(t),a(t)]_{t=0}^{1}} \int_0^1 \exp(-\rho t)\, u(c(t))\, dt$$

subject to

$$\dot{a}(t) = r\left[a(t) + w - c(t)\right]$$

[2]It is also worth noting that the hypothesis that there exists an interior solution is more restrictive in this case than in Theorem 7.1. This is because the set of controls, the equivalent of $\mathcal{F}$ defined in (7.59) in Section 7.6 below,

$$\mathcal{F} = \left\{[y(t)]_{t=0}^{t_1} : \dot{x}(t) = g(t, x(t), y(t)) \text{ with } x(0) = x_0 \text{ satisfies } x(t_1) = x_1\right\},$$

may have an empty interior, making it impossible that an interior solution exists. See Exercise 7.17 for an example and Section 7.6 for a formal definition of the set $\mathcal{F}$.

and $a(t) \geq 0$, with an initial value of $a(0) > 0$. In this problem, consumption is the control variable, while the asset holdings of the individual are the state variable.

To be able to apply Theorem 7.2, we need a terminal condition for $a(t)$, i.e., some value a_1 such that $a(1) = a_1$. The economics of the problem makes it clear that the individual would not like to have any positive level of assets at the end of his planning horizon (since he could consume all of these at date $t = 1$ or slightly before, and $u(\cdot)$ is strictly increasing). Therefore, we must have $a(1) = 0$.

With this observation, Theorem 7.2 provides the following the necessary conditions for an interior continuous solution: there exists a continuously differentiable costate variable $\lambda(t)$ such that the optimal path of consumption and asset holdings, $(\hat{c}(t), \hat{a}(t))$, satisfy a consumption Euler equation similar to equation (6.29) in Example 6.5 in the previous chapter:

$$\exp(-\rho t)\, u'(\hat{c}(t)) = \lambda(t)\, r.$$

In particular, this equation can be rewritten as $u'(\hat{c}(t)) = \beta r \lambda(t)$, with $\beta = \exp(-\rho t)$, and would be almost identical to equation (6.29), except for the presence of $\lambda(t)$ instead of the derivative of the value function. But as we will see below, $\lambda(t)$ is exactly the derivative of the value function, so that the consumption Euler equations in discrete and continuous time are identical. This is of course not surprising, since they capture the same economic phenomenon, in slightly different mathematical formulations.

The next necessary condition determines the behavior of $\lambda(t)$ as

$$\dot{\lambda}(t) = -r.$$

Now using this condition and differentiating $u'(\hat{c}(t)) = \beta r \lambda(t)$, we can obtain a differential equation in consumption. This differential equation, derived in the next chapter in a somewhat more general context, will be the key consumption Euler equation in continuous time. Leaving the derivation of this equation to the next chapter, we can make progress here by simply integrating this condition to obtain

$$\lambda(t) = \lambda(0) \exp(-rt).$$

Combining this with the first-order condition for consumption yields a straightforward expression for the optimal consumption level at time t:

$$\hat{c}(t) = u'^{-1}\left[R\lambda(0) \exp((\rho - r)\, t)\right],$$

where $u'^{-1}[\cdot]$ is the inverse function of the marginal utility u'. It exists and is strictly decreasing in view of the fact that u is strictly concave. This equation therefore implies that when $\rho = r$, so that the discount factor and the rate of return on assets are equal, the individual will have a constant consumption profile. When $\rho > r$, the argument of u'^{-1} is increasing over time, so consumption must be declining. This reflects the fact that the individual discounts the future more heavily than the rate of return, thus wishes to have a front-loaded consumption profile. In contrast, when $\rho < r$, the opposite reasoning applies and the individual chooses a back-loaded consumption profile. These are of course identical to the conclusions we reached in the discrete time intertemporal consumer optimization problem in Example 6.5, in particular, equation (6.31).

The only variable to determine in order to completely characterize the consumption profile is the initial value of the costate variable. This comes from the budget constraint of the individual together with the observation that the individual will run down all his assets by the end of his planning horizon, thus $a(1) = 0$. Now using the consumption rule, we have

$$\dot{a}(t) = R\left\{a(t) + w - u'^{-1}\left[R\lambda(0)\exp\left((\rho - R)t\right)\right]\right\}.$$

The initial value of the costate variable, $\lambda(0)$, then has to be chosen such that $a(1) = 0$. You are asked to complete the details of this step in Exercise 7.6.

Example 7.1 applied the results of Theorem 7.2. It may at first appear that Theorem 7.1 is more convenient to use than Theorem 7.2, since it would enable us to directly formulate the problem as one of dynamic optimization rather than first argue about what the terminal value of the state variable, $a(1)$, should be (based on economic reasoning as we did in Example 7.1). However, as the continuation of the previous example illustrates, this is not necessarily the case:

Example 7.1 (continued). Let us try to apply Theorem 7.1 to the economic environment in Example 7.1. The first-order necessary conditions still give

$$\lambda(t) = \lambda(0)\exp(-Rt).$$

However, since $\lambda(1) = 0$, this is only possible if $\lambda(t) = 0$ for all $t \in [0, 1]$. But then the Euler equation

$$\exp(-\rho t)\, u'(\hat{c}(t)) = \lambda(t)\, R,$$

which still applies from the necessary conditions, cannot be satisfied, since $u' > 0$ by assumption. This implies that when the terminal value of the assets, $a(1)$, is a choice variable, there exists no solution (at least no solution with an interior continuous control). How is this possible?

The answer is that Theorem 7.1 cannot be applied to this problem, because there is an additional constraint that $a(t) \geq 0$. We would need to consider a version of Theorem 7.1 with inequality constraints. The necessary conditions with inequality constraints are messier and more difficult to work with. Using a little bit of economic reasoning to observe that the terminal value of the assets must be equal to zero and then applying Theorem 7.2 simplifies the analysis considerably.

7.2. The Maximum Principle: A First Look

7.2.1. The Hamiltonian and the Maximum Principle. By analogy with the Lagrangian, a much more economical way of expressing Theorem 7.2 is to construct the *Hamiltonian:*[3]

$$H(t,x,y,\lambda) \equiv f(t,x(t),y(t)) + \lambda(t)\, g(t,x(t),y(t)). \tag{7.12}$$

Since f and g are continuously differentiable, so is H. Denote the partial derivatives of the Hamiltonian with respect to $x(t)$, $y(t)$ and $\lambda(t)$, by H_x, H_y and H_λ. Theorem 7.2 then immediately leads to the following result:

Theorem 7.3. ***(Maximum Principle)*** *Consider the problem of maximizing (7.1) subject to (7.2) and (7.3), with f and g continuously differentiable. Suppose that this problem has an interior continuous solution $\hat{y}(t) \in \text{Int}\mathcal{Y}(t)$ with corresponding path of state variable $\hat{x}(t)$. Then there exists a continuously differentiable function $\lambda(t)$ such that the optimal control $\hat{y}(t)$ and the corresponding path of the state variable $\hat{x}(t)$ satisfy the following necessary conditions: $x(0) = x_0$,*

$$H_y(t,\hat{x}(t),\hat{y}(t),\lambda(t)) = 0 \text{ for all } t \in [0,t_1], \tag{7.13}$$

$$\dot{\lambda}(t) = -H_x(t,\hat{x}(t),\hat{y}(t),\lambda(t)) \text{ for all } t \in [0,t_1], \tag{7.14}$$

$$\dot{x}(t) = H_\lambda(t,\hat{x}(t),\hat{y}(t),\lambda(t)) \text{ for all } t \in [0,t_1], \tag{7.15}$$

and $\lambda(t_1) = 0$, with the Hamiltonian $H(t,x,y,\lambda)$ given by (7.12). Moreover, the Hamiltonian $H(t,x,y,\lambda)$ also satisfies the Maximum Principle *that*

$$H(t,\hat{x}(t),\hat{y}(t),\lambda(t)) \geq H(t,\hat{x}(t),y,\lambda(t)) \text{ for all } y \in \mathcal{Y}(t),$$

for all $t \in [0,t_1]$.

For notational simplicity, in equation (7.15), I wrote $\dot{x}(t)$ instead of $\dot{\hat{x}}(t)$ $(= \partial\hat{x}(t)/\partial t)$. The latter notation is rather cumbersome, and we will refrain from using it as long as the context makes it clear that $\dot{x}(t)$ stands for this expression.

Theorem 7.3 is a simplified version of the celebrated *Maximum Principle* of Pontryagin. The more general version of this Maximum Principle will be given below. For now, a couple of features are worth noting:

[3]More generally, the Hamiltonian should be written as

$$H(t,x,y,\lambda) \equiv \lambda_0 f(t,x(t),y(t)) + \lambda(t)\, g(t,x(t),y(t)).$$

for some $\lambda_0 \geq 0$. In some pathological cases λ_0 may be equal to 0. However, in all economic applications this will not be the case, and we will have $\lambda_0 > 0$. When $\lambda_0 > 0$, it can be normalized to 1 without loss of any generality. Thus the definition of the Hamiltonian in (7.12) is appropriate for all of our economic applications.

(1) As in the usual constrained maximization problems, we find the optimal solution by looking jointly for a set of "multipliers" $\lambda(t)$ and the optimal path of the control and state variables, $\hat{y}(t)$ and $\hat{x}(t)$. Here the multipliers are referred to as the *costate* variables.
(2) Again as with the Lagrange multipliers in the usual constrained maximization problems, the costate variable $\lambda(t)$ is informative about the value of relaxing the constraint (at time t). In particular, we will see that $\lambda(t)$ is the value of an infinitesimal increase in $x(t)$ at time t.
(3) With this interpretation, it makes sense that $\lambda(t_1) = 0$ is part of the necessary conditions. After the planning horizon, there is no value to having more x. This is therefore the finite-horizon equivalent of the *transversality condition* we encountered in the previous section.

While Theorem 7.3 gives necessary conditions, as in regular optimization problems, these may not be sufficient. First, these conditions may correspond to stationary points rather than maxima. Second, they may identify a local rather than a global maximum. Sufficiency is again guaranteed by imposing concavity. The following theorem, first proved by Mangasarian, shows that concavity of the Hamiltonian ensures that conditions (7.13)-(7.15) are not only necessary but also sufficient for a maximum.

Theorem 7.4. ***(Mangasarian's Sufficient Conditions)*** *Consider the problem of maximizing (7.1) subject to (7.2) and (7.3), with f and g continuously differentiable. Define $H(t,x,y,\lambda)$ as in (7.12), and suppose that an interior continuous solution $\hat{y}(t) \in \text{Int}\mathcal{Y}(t)$ and the corresponding path of state variable $\hat{x}(t)$ satisfy (7.13)-(7.15). Suppose also that given the resulting costate variable $\lambda(t)$, $H(t,x,y,\lambda)$ is jointly concave in (x,y) for all $t \in [0,t_1]$, then the $\hat{y}(t)$ and the corresponding $\hat{x}(t)$ achieve a global maximum of (7.1). Moreover, if $H(t,x,y,\lambda)$ is strictly jointly concave in (x,y) for all $t \in [0,t_1]$, then the pair $(\hat{x}(t), \hat{y}(t))$ achieves the unique global maximum of (7.1).*

The proof of Theorem 7.4 is similar to the proof of Theorem 7.5, which is provided below, and is therefore left as an exercise (see Exercise 7.7).

The condition that the Hamiltonian $H(t,x,y,\lambda)$ should be concave is rather demanding. The following theorem, first derived by Arrow, weakens these conditions. Before stating this result, let us define the maximized Hamiltonian as

$$M(t,x,\lambda) \equiv \max_{y \in \mathcal{Y}(t)} H(t,x,y,\lambda), \tag{7.16}$$

with $H(t,x,y,\lambda)$ itself defined as in (7.12). Clearly, the necessary conditions for an interior maximum in (7.16) is (7.13). Therefore, an interior pair of state and control variables $(\hat{x}(t), \hat{y}(t))$ satisfies (7.13)-(7.15), then $M(t,\hat{x},\lambda) \equiv H(t,\hat{x},\hat{y},\lambda)$.

Theorem 7.5. ***(Arrow's Sufficient Conditions)*** *Consider the problem of maximizing (7.1) subject to (7.2) and (7.3), with f and g continuously differentiable. Define $H(t,x,y,\lambda)$ as in (7.12), and suppose that an interior continuous solution $\hat{y}(t) \in \text{Int}\mathcal{Y}(t)$ and the corresponding path of state variable $\hat{x}(t)$ satisfy (7.13)-(7.15). Given the resulting costate variable $\lambda(t)$, define $M(t,\hat{x},\lambda)$ as the maximized Hamiltonian as in (7.16). If $M(t,\hat{x},\lambda)$ is concave in x for all $t \in [0,t_1]$, then $\hat{y}(t)$ and the corresponding $\hat{x}(t)$ achieve a global maximum of (7.1). Moreover, if $M(t,\hat{x},\lambda)$ is strictly concave in x for all $t \in [0,t_1]$, then the pair $(\hat{x}(t),\hat{y}(t))$ achieves the unique global maximum of (7.1) and $\hat{x}(t)$ is uniquely defined.*

Proof. Consider the pair of state and control variables $(\hat{x}(t),\hat{y}(t))$ that satisfy the necessary conditions (7.13)-(7.15) as well as (7.2) and (7.3). Consider also an arbitrary pair $(x(t),y(t))$ that satisfy (7.2) and (7.3) and define $M(t,x,\lambda) \equiv \max_y H(t,x,y,\lambda)$. Since f and g are differentiable, H and M are also differentiable in x. Denote the derivative of M with respect to x by M_x. Then concavity implies that

$$M(t,x(t),\lambda(t)) \leq M(t,\hat{x}(t),\lambda(t)) + M_x(t,\hat{x}(t),\lambda(t))(x(t)-\hat{x}(t)) \text{ for all } t \in [0,t_1].$$

Integrating both sides over $[0,t_1]$ yields

$$\int_0^{t_1} M(t,x(t),\lambda(t))\,dt \leq \int_0^{t_1} M(t,\hat{x}(t),\lambda(t))\,dt + \int_0^{t_1} M_x(t,\hat{x}(t),\lambda(t))(x(t)-\hat{x}(t))\,dt. \tag{7.17}$$

Moreover, we have

$$\begin{aligned} M_x(t,\hat{x}(t),\lambda(t)) &= H_x(t,\hat{x}(t),\hat{y}(t),\lambda(t)) \\ &= -\dot{\lambda}(t), \end{aligned} \tag{7.18}$$

where the first line follows by an Envelope Theorem type reasoning (since $H_y = 0$ from equation (7.13)), while the second line follows from (7.15). Next, exploiting the definition of the maximized Hamiltonian, we have

$$\int_0^{t_1} M(t,x(t),\lambda(t))\,dt = W(x(t),y(t)) + \int_0^{t_1} \lambda(t)\,g(t,x(t),y(t))\,dt,$$

and

$$\int_0^{t_1} M(t,\hat{x}(t),\lambda(t))\,dt = W(\hat{x}(t),\hat{y}(t)) + \int_0^{t_1} \lambda(t)\,g(t,\hat{x}(t),\hat{y}(t))\,dt.$$

Equation (7.17) together with (7.18) then implies

$$\begin{aligned} W(x(t),y(t)) \leq\ & W(\hat{x}(t),\hat{y}(t)) \\ &+ \int_0^{t_1} \lambda(t)\left[g(t,\hat{x}(t),\hat{y}(t)) - g(t,x(t),y(t))\right] dt \\ &- \int_0^{t_1} \dot{\lambda}(t)(x(t)-\hat{x}(t))\,dt. \end{aligned} \tag{7.19}$$

Integrating the last term by parts and using the fact that by feasibility $x(0) = \hat{x}(0) = x_0$ and by the transversality condition $\lambda(t_1) = 0$, we obtain

$$\int_0^{t_1} \dot{\lambda}(t)(x(t) - \hat{x}(t))\,dt = -\int_0^{t_1} \lambda(t)\left(\dot{x}(t) - \dot{\hat{x}}(t)\right)dt.$$

Substituting this into (7.19), we obtain

$$\begin{aligned} W(x(t), y(t)) \leq\ & W(\hat{x}(t), \hat{y}(t)) \\ & + \int_0^{t_1} \lambda(t)\left[g(t, \hat{x}(t), \hat{y}(t)) - g(t, x(t), y(t))\right]dt \\ & + \int_0^{t_1} \lambda(t)\left[\dot{x}(t) - \dot{\hat{x}}(t)\right]dt. \end{aligned} \tag{7.20}$$

Since by definition of the admissible pairs $(x(t), y(t))$ and $(\hat{x}(t), \hat{y}(t))$, we have $\dot{\hat{x}}(t) = g(t, \hat{x}(t), \hat{y}(t))$ and $\dot{x}(t) = g(t, x(t), y(t))$, (7.20) implies that $W(x(t), y(t)) \leq W(\hat{x}(t), \hat{y}(t))$ for any admissible pair $(x(t), y(t))$, establishing the first part of the theorem.

If M is strictly concave in x, then the inequality in (7.17) is strict, and therefore the same argument establishes $W(x(t), y(t)) < W(\hat{x}(t), \hat{y}(t))$, and no other $\hat{x}(t)$ could achieve the same value, establishing the second part. □

Theorems 7.4 and 7.5 play an important role in the applications of optimal control. They ensure that a pair $(\hat{x}(t), \hat{y}(t))$ that satisfies the necessary conditions specified in Theorem 7.3 and the sufficiency conditions in either Theorem 7.4 or Theorem 7.5 is indeed an optimal solution. This is important, since without Theorem 7.4 and Theorem 7.5, Theorem 7.3 does not tell us that there exists an interior continuous solution, thus an admissible pair that satisfies the conditions of Theorem 7.3 may not constitute an optimal solution.

Unfortunately, however, both Theorem 7.4 and Theorem 7.5 are not straightforward to check since neither concavity nor convexity of the $g(\cdot)$ function would guarantee the concavity of the Hamiltonian unless we know something about the sign of the costate variable $\lambda(t)$. Nevertheless, in many economically interesting situations, we can ascertain that the costate variable $\lambda(t)$ is everywhere positive. For example, a sufficient (but not necessary) condition for this would be $f_x(t, \hat{x}(t), \hat{y}(t), \lambda(t)) > 0$ (see Exercise 7.9). Below we will see that $\lambda(t)$ is related to the value of relaxing the constraint on the maximization problems, which also gives us another way of ascertaining that it is positive (or negative depending on the problem). Once we know that $\lambda(t)$ is positive, checking Theorem 7.4 is straightforward, especially when f and g are concave functions.

7.2.2. Generalizations. The above theorems can be immediately generalized to the case in which the state variable and the controls are vectors rather than scalars, and also to the case in which there are other constraints. The constrained case requires *constraint qualification conditions* as in the standard finite-dimensional optimization case (see, e.g., Simon and Blume, 1994). These are slightly more messy to express, and since we will make no use of the constrained maximization problems in this book, we will not state these theorems.

The vector-valued theorems are direct generalizations of the ones presented above and are useful in growth models with multiple capital goods. In particular, let

$$\max_{\mathbf{x}(t),\mathbf{y}(t)} W(\mathbf{x}(t),\mathbf{y}(t)) \equiv \int_0^{t_1} f(t,\mathbf{x}(t),\mathbf{y}(t))\,dt \tag{7.21}$$

subject to

$$\dot{\mathbf{x}}(t) = g(t,\mathbf{x}(t),\mathbf{y}(t)), \tag{7.22}$$

and

$$\mathbf{y}(t) \in \mathcal{Y}(t) \text{ for all } t,\ \mathbf{x}(0) = \mathbf{x}_0 \text{ and } \mathbf{x}(t_1) = \mathbf{x}_1. \tag{7.23}$$

Here $\mathbf{x}(t) \in \mathbb{R}^K$ for some $K \geq 1$ is the state variable and again $y(t) \in \mathcal{Y}(t) \subset \mathbb{R}^N$ for some $N \geq 1$ is the control variable. In addition, we again assume that f and g are continuously differentiable functions. We then have:

Theorem 7.6. ***(Maximum Principle for Multivariate Problems)*** *Consider the problem of maximizing (7.21) subject to (7.22) and (7.23), with f and g continuously differentiable, has an interior continuous solution $\hat{\mathbf{y}}(t) \in \mathrm{Int}\mathcal{Y}(t)$ with corresponding path of state variable $\hat{\mathbf{x}}(t)$. Let $H(t,\mathbf{x},\mathbf{y},\boldsymbol{\lambda})$ be given by*

$$H(t,\mathbf{x},\mathbf{y},\boldsymbol{\lambda}) \equiv f(t,\mathbf{x}(t),\mathbf{y}(t)) + \boldsymbol{\lambda}(t)\,g(t,\mathbf{x}(t),\mathbf{y}(t)), \tag{7.24}$$

where $\boldsymbol{\lambda}(t) \in \mathbb{R}^K$. Then the optimal control $\hat{\mathbf{y}}(t)$ and the corresponding path of the state variable $\mathbf{x}(t)$ satisfy the following necessary conditions:

$$D_{\mathbf{y}}H(t,\hat{\mathbf{x}}(t),\hat{\mathbf{y}}(t),\boldsymbol{\lambda}(t)) = 0 \text{ for all } t \in [0,t_1]. \tag{7.25}$$

$$\dot{\boldsymbol{\lambda}}(t) = -D_{\mathbf{x}}H(t,\hat{\mathbf{x}}(t),\hat{\mathbf{y}}(t),\boldsymbol{\lambda}(t)) \text{ for all } t \in [0,t_1]. \tag{7.26}$$

$$\dot{\mathbf{x}}(t) = D_{\boldsymbol{\lambda}}H(t,\hat{\mathbf{x}}(t),\hat{\mathbf{y}}(t),\boldsymbol{\lambda}(t)) \text{ for all } t \in [0,t_1],\ \mathbf{x}(0) = \mathbf{x}_0 \text{ and } \mathbf{x}(1) = \mathbf{x}_1. \tag{7.27}$$

Proof. See Exercise 7.10. □

Moreover, we have straightforward generalizations of the sufficiency conditions. The proofs of these theorems are very similar to those of Theorems 7.4 and 7.5 and are thus omitted.

Theorem 7.7. ***(Mangasarian's Sufficient Conditions)*** *Consider the problem of maximizing (7.21) subject to (7.22) and (7.23), with f and g continuously differentiable. Define $H(t,\mathbf{x},\mathbf{y},\boldsymbol{\lambda})$ as in (7.24), and suppose that an interior continuous solution $\hat{\mathbf{y}}(t) \in \mathrm{Int}\mathcal{Y}(t)$ and the corresponding path of state variable $\hat{\mathbf{x}}(t)$ satisfy (7.25)-(7.27). Suppose also that for the resulting costate variable $\boldsymbol{\lambda}(t)$, $H(t,\mathbf{x},\mathbf{y},\boldsymbol{\lambda})$ is jointly concave in $(\mathbf{x},\mathbf{y})$ for all $t \in [0,t_1]$,* *then* $\hat{\mathbf{y}}(t)$ *and the corresponding* $\hat{\mathbf{x}}(t)$ *achieves a global maximum of (7.21). Moreover, if* $H(t,\mathbf{x},\mathbf{y},\boldsymbol{\lambda})$ *is strictly jointly concave, then the pair* $(\hat{\mathbf{x}}(t),\hat{\mathbf{y}}(t))$ *achieves the unique global maximum of (7.21).*

Theorem 7.8. ***(Arrow's Sufficient Conditions)*** *Consider the problem of maximizing (7.21) subject to (7.22) and (7.23), with* f *and* g *continuously differentiable. Define* $H(t,\mathbf{x},\mathbf{y},\boldsymbol{\lambda})$ *as in (7.24), and suppose that an interior continuous solution* $\hat{\mathbf{y}}(t) \in \mathrm{Int}\mathcal{Y}(t)$ *and the corresponding path of state variable* $\hat{\mathbf{x}}(t)$ *satisfy (7.25)-(7.27). Suppose also that for the resulting costate variable* $\boldsymbol{\lambda}(t)$*, define* $M(t,\mathbf{x},\boldsymbol{\lambda}) \equiv \max_{\mathbf{y}(t)\in\mathcal{Y}(t)} H(t,\mathbf{x},\mathbf{y},\boldsymbol{\lambda})$*. If* $M(t,\mathbf{x},\boldsymbol{\lambda})$ *is concave in* $\mathbf{x}$ *for all* $t \in [0,t_1]$*, then* $\hat{\mathbf{y}}(t)$ *and the corresponding* $\hat{\mathbf{x}}(t)$ *achieve a global maximum of (7.21). Moreover, if* $M(t,\mathbf{x},\boldsymbol{\lambda})$ *is strictly concave in* $\mathbf{x}$*, then the pair* $(\hat{\mathbf{x}}(t),\hat{\mathbf{y}}(t))$ *achieves the unique global maximum of (7.21).*

The proofs of both of these Theorems are similar to that of Theorem 7.5 and are left to the reader.

7.2.3. Limitations. The limitations of what we have done so far are obvious. First, we have assumed that a continuous and interior solution to the optimal control problem exists. Second, and equally important for our purposes, we have so far looked at the finite horizon case, whereas analysis of growth models requires us to solve infinite horizon problems. To deal with both of these issues, we need to look at the more modern theory of optimal control. This is done in the next section.

7.3. Infinite-Horizon Optimal Control

The results presented so far are most useful in developing an intuition for how dynamic optimization in continuous time works. While a number of problems in economics require finite-horizon optimal control, most economic problems—including almost all growth models—are more naturally formulated as infinite-horizon problems. This is obvious in the context of economic growth, but is also the case in repeated games, political economy or industrial organization, where even if individuals may have finite expected lifes, the end date of the game or of their lives may be uncertain. For this reason, the canonical model of optimization and economic problems is the infinite-horizon one.

7.3.1. The Basic Problem: Necessary and Sufficient Conditions. Let us focus on infinite-horizon control with a single control and a single state variable. Using the same notation as above, the problem is

$$\max_{x(t),y(t)} W(x(t),y(t)) \equiv \int_0^\infty f(t,x(t),y(t))\,dt \tag{7.28}$$

subject to

$$\dot{x}(t) = g(t, x(t), y(t)), \tag{7.29}$$

and

$$y(t) \in \mathbb{R} \text{ for all } t,\ x(0) = x_0 \text{ and } \lim_{t\to\infty} x(t) \geq x_1. \tag{7.30}$$

The main difference is that now time runs to infinity. Note also that this problem allows for an implicit choice over the endpoint x_1, since there is no terminal date. The last part of (7.30) imposes a lower bound on this endpoint. In addition, we have further simplified the problem by removing the feasibility requirement that the control $y(t)$ should always belong to the set $\mathcal{Y}$, instead simply requiring this function to be real-valued. Notice also that we have not assumed that the state variable $x(t)$ lies in a compact set, thus the results developed here can be easily applied to models with exogenous or endogenous growth.

For this problem, we call a pair $(x(t), y(t))$ *admissible* if $y(t)$ is a piecewise continuous function of time, meaning that it has at most a finite number of discontinuities.[4] Since $x(t)$ is given by a continuous differential equation, the piecewise continuity of $y(t)$ ensures that $x(t)$ is piecewise smooth. Allowing for piecewise continuous controls is a significant generalization of the above approach.

There are a number of technical difficulties when dealing with the infinite-horizon case, which are similar to those in the discrete time analysis. Primary among those is the fact that the value of the functional in (7.28) may not be finite. We will deal with some of these issues below.

The main theorem for the infinite-horizon optimal control problem is the following more general version of the *Maximum Principle*. Before stating this theorem, let us recall that the Hamiltonian is defined by (7.12), with the only difference that the horizon is now infinite. In addition, let us define the *value function*, which is the analog of the value function in discrete time dynamic programming introduced in the previous chapter:

$$\begin{aligned} V(t_0, x_0) &\equiv \max_{x(t)\in\mathbb{R}, y(t)\in\mathbb{R}} \int_{t_0}^{\infty} f(t, x(t), y(t))\, dt \\ \text{subject to } \dot{x}(t) &= g(t, x(t), y(t)),\ x(t_0) = x_0 \text{ and } \lim_{t\to\infty} x(t) \geq x_1. \end{aligned} \tag{7.31}$$

In words, $V(t_0, x_0)$ gives the optimal value of the dynamic maximization problem starting at time t_0 with state variable x_0. Clearly, we have that

$$V(t_0, x_0) \geq \int_{t_0}^{\infty} f(t, x(t), y(t))\, dt \text{ for any admissible pair } (x(t), y(t)). \tag{7.32}$$

Note that as in the previous chapter, there are issues related to whether the "max" is reached. When it is not reached, we should be using "sup" instead. However, recall that we have

[4] More generally, $y(t)$ could be allowed to have a countable number of discontinuities, but this added generality is not necessary for any economic application.

assumed that all admissible pairs give finite value, so that $V(t_0, x_0) < \infty$, and our focus throughout will be on admissible pairs $(\hat{x}(t), \hat{y}(t))$ that are optimal solutions to (7.28) subject to (7.29) and (7.30), and thus reach the value $V(t_0, x_0)$.

Our first result is a weaker version of the Principle of Optimality, which we encountered in the context of discrete time dynamic programming in the previous chapter:

Lemma 7.1. ***(Principle of Optimality)*** *Suppose that the pair $(\hat{x}(t), \hat{y}(t))$ is an optimal solution to (7.28) subject to (7.29) and (7.30), i.e., it reaches the maximum value $V(t_0, x_0)$. Then,*

$$V(t_0, x_0) = \int_{t_0}^{t_1} f(t, \hat{x}(t), \hat{y}(t))\, dt + V(t_1, \hat{x}(t_1)) \text{ for all } t_1 \geq t_0. \tag{7.33}$$

Proof. We have

$$\begin{aligned} V(t_0, x_0) &\equiv \int_{t_0}^{\infty} f(t, \hat{x}(t), \hat{y}(t))\, dt \\ &= \int_{t_0}^{t_1} f(t, \hat{x}(t), \hat{y}(t))\, dt + \int_{t_1}^{\infty} f(t, \hat{x}(t), \hat{y}(t))\, dt. \end{aligned}$$

The proof is completed if $V(t_1, \hat{x}(t_1)) = \int_{t_1}^{\infty} f(t, \hat{x}(t), \hat{y}(t))\, dt$. By definition $V(t_1, \hat{x}(t_1)) \geq \int_{t_1}^{\infty} f(t, x(t), y(t))\, dt$ for all admissible $(x(t), y(t))$. Thus this equality can only fail if $V(t_1, \hat{x}(t_1)) > \int_{t_1}^{\infty} f(t, \hat{x}(t), \hat{y}(t))\, dt$. To obtain a contradiction, suppose that this is the case. Then there must exist an admissible pair from t_1 onwards, $(\tilde{x}(t), \tilde{y}(t))$ with $\tilde{x}(t_1) = \hat{x}(t_1)$ such that $\int_{t_1}^{\infty} f(t, \tilde{x}(t), \tilde{y}(t))\, dt > \int_{t_1}^{\infty} f(t, \hat{x}(t), \hat{y}(t))\, dt$. Then construct the pair $(\vec{x}(t), \vec{y}(t))$ such that $(\vec{x}(t), \vec{y}(t)) = (\hat{x}(t), \hat{y}(t))$ for all $t \in [t_0, t_1]$ and $(\vec{x}(t), \vec{y}(t)) = (\tilde{x}(t), \tilde{y}(t))$ for all $t \geq t_1$. Since $(\tilde{x}(t), \tilde{y}(t))$ is admissible from t_1 onwards with $\tilde{x}(t_1) = \hat{x}(t_1)$, $(\vec{x}(t), \vec{y}(t))$ is admissible, and moreover,

$$\begin{aligned} \int_{t_0}^{\infty} f(t, \vec{x}(t), \vec{y}(t))\, dt &= \int_{t_0}^{t_1} f(t, \vec{x}(t), \vec{y}(t))\, dt + \int_{t_1}^{\infty} f(t, \vec{x}(t), \vec{y}(t))\, dt \\ &= \int_{t_0}^{t_1} f(t, \hat{x}(t), \hat{y}(t))\, dt + \int_{t_1}^{\infty} f(t, \tilde{x}(t), \tilde{y}(t))\, dt \\ &> \int_{t_0}^{t_1} f(t, \hat{x}(t), \hat{y}(t))\, dt + \int_{t_1}^{\infty} f(t, \hat{x}(t), \hat{y}(t))\, dt \\ &= \int_{t_0}^{\infty} f(t, \hat{x}(t), \hat{y}(t))\, dt \\ &= V(t_0, x_0), \end{aligned}$$

which contradicts (7.32) establishing that $V(t_1, \hat{x}(t_1)) = \int_{t_1}^{\infty} f(t, \hat{x}(t), \hat{y}(t))\, dt$ and thus (7.33). □

Two features in this version of the Principle of Optimality are noteworthy. First, in contrast to the similar equation in the previous chapter, it may appear that there is no discounting in (7.33). This is not the case, since the discounting is embedded in the instantaneous payoff function f, and is thus implicit in $V(t_1, \hat{x}(t_1))$. Second, this lemma may appear to contradict our discussion of "time consistency" in the previous chapter, since the lemma is stated without additional assumptions that ensure time consistency. The important point here is that in the time consistency discussion, the decision-maker considered updating his or her plan, with the payoff function being potentially different after date t_1 (at least because bygones were bygones). In contrast, here the payoff function remains constant. The issue of time consistency is discussed further in Exercise 7.21. We now state one of the main results of this chapter.

THEOREM 7.9. ***(Infinite-Horizon Maximum Principle)*** *Suppose that problem of maximizing (7.28) subject to (7.29) and (7.30), with f and g continuously differentiable, has a piecewise continuous solution $\hat{y}(t)$ with corresponding path of state variable $\hat{x}(t)$. Let $H(t, x, y, \lambda)$ be given by (7.12). Then the optimal control $\hat{y}(t)$ and the corresponding path of the state variable $\hat{x}(t)$ are such that the Hamiltonian $H(t, x, y, \lambda)$ satisfies the Maximum Principle, that*

$$H(t, \hat{x}(t), \hat{y}(t), \lambda(t)) \geq H(t, \hat{x}(t), y, \lambda(t)) \text{ for all } y(t),$$

for all $t \in \mathbb{R}$. Moreover, whenever $\hat{y}(t)$ is continuous, the following necessary conditions are satisfied:

$$H_y(t, \hat{x}(t), \hat{y}(t), \lambda(t)) = 0, \tag{7.34}$$

$$\dot{\lambda}(t) = -H_x(t, \hat{x}(t), \hat{y}(t), \lambda(t)), \tag{7.35}$$

$$\dot{x}(t) = H_\lambda(t, \hat{x}(t), \hat{y}(t), \lambda(t)), \text{ with } x(0) = x_0 \text{ and } \lim_{t\to\infty} x(t) \geq x_1, \tag{7.36}$$

for all $t \in \mathbb{R}_+$.

The proof of this theorem is relatively long and will be provided later in this section.[5] Notice that whenever an optimal solution of the specified form exists, it satisfies the Maximum Principle. Thus in some ways Theorem 7.9 can be viewed as stronger than the theorems presented in the previous chapter, especially since it does not impose compactness type conditions. Nevertheless, this theorem only applies when the maximization problem has a piecewise continuous solution $\hat{y}(t)$. Sufficient conditions to ensure that such a solution exist are somewhat involved and are discussed further in Appendix Chapter A. In addition,

[5]The reader may also wonder when an optimal piecewise continuous solution will exist as hypothesized in the theorem. Theorem 7.17 below will provide the conditions to ensure that a solution exists, though checking the conditions of this theorem is not always a trivial task. Ensuring the existence of a solution that is piecewise continuous is considerably harder. Nevertheless, in most economic problems there will be enough structure to ensure the existence of an interior solution and this structure will also often guarantee that the solution is also piecewise continuous and in fact fully continuous.

Theorem 7.9 states that if the optimal control, $\hat{y}(t)$, is a continuous function of time, the conditions (7.34)-(7.36) are also satisfied. This qualification is necessary, since we now allow $\hat{y}(t)$ to be a piecewise continuous function of time. The fact that $\hat{y}(t)$ is a piecewise continuous function implies that the optimal control may include discontinuities, but these will be relatively "rare"—in particular, it will be continuous "most of the time". More important, the added generality of allowing discontinuities is somewhat superfluous in most economic applications, because economic problems often have enough structure to ensure that $\hat{y}(t)$ is indeed a continuous function of time. Consequently, in most economic problems (and in all of the models studied in this book) it will be sufficient to focus on the necessary conditions (7.34)-(7.36).

It is also useful to have a different version of the necessary conditions in Theorem 7.9, which are directly comparable to the necessary conditions generated by dynamic programming in the discrete time dynamic optimization problems studied in the previous chapter. In particular, the necessary conditions can also be expressed in the form of the so-called *Hamilton-Jacobi-Bellman* (HJB) equation.

THEOREM 7.10. ***(Hamilton-Jacobi-Bellman Equations)*** *Let $V(t,x)$ be as defined in (7.31) and suppose that the hypotheses in Theorem 7.9 hold. Then whenever $V(t,x)$ is differentiable in (t,x), the optimal pair $(\hat{x}(t), \hat{y}(t))$ satisfies the HJB equation:*

$$f(t,\hat{x}(t),\hat{y}(t)) + \frac{\partial V(t,\hat{x}(t))}{\partial t} + \frac{\partial V(t,\hat{x}(t))}{\partial x} g(t,\hat{x}(t),\hat{y}(t)) = 0 \text{ for all } t \in \mathbb{R}. \tag{7.37}$$

PROOF. From Lemma 7.1, we have that for the optimal pair $(\hat{x}(t), \hat{y}(t))$,

$$V(t_0, x_0) = \int_{t_0}^{t} f(s,\hat{x}(s),\hat{y}(s))\,ds + V(t,\hat{x}(t)) \text{ for all } t.$$

Differentiating this with respect to t and using the differentiability of V and Leibniz's rule (Theorem B.4 in Appendix Chapter B), we obtain

$$f(t,\hat{x}(t),\hat{y}(t)) + \frac{\partial V(t,\hat{x}(t))}{\partial t} + \frac{\partial V(t,\hat{x}(t))}{\partial x}\dot{x}(t) = 0 \text{ for all } t.$$

Setting $\dot{x}(t) = g(t,\hat{x}(t),\hat{y}(t))$ gives (7.37). □

The HJB equation will be useful in providing an intuition for the Maximum Principle, in the proof of Theorem 7.9 and also in many of the endogenous technology models studied below. For now it suffices to note a few important features. First, given that the continuous differentiability of f and g, the assumption that $V(t,x)$ is differentiable is not very restrictive, since the optimal control $\hat{y}(t)$ is piecewise continuous. From the definition (7.31), at all t where $\hat{y}(t)$ is continuous, $V(t,x)$ will also be differentiable in t. Moreover, an envelope theorem type argument also implies that when $\hat{y}(t)$ is continuous, $V(t,x)$ should also be differentiable in x (though the exact conditions to ensure differentiability in x are somewhat involved). Second, (7.37) is a partial differential equation, since it features the derivative of

V with respect to both time and the state variable x. Third, this partial differential equation also has a similarity to the Euler equation derived in the context of discrete time dynamic programming. In particular, the simplest Euler equation (6.22) required the current gain from increasing the control variable to be equal to the discounted loss of value. The current equation has a similar interpretation, with the first term corresponding to the current gain and the last term to the potential discounted loss of value. The second term results from the fact that the maximized value can also change over time.

Since in Theorem 7.9 there is no boundary condition similar to $x(t_1) = x_1$, we may expect that there should be a transversality condition similar to the condition that $\lambda(t_1) = 0$ in Theorem 7.1. One might be tempted to impose a transversality condition of the form

$$\lim_{t \to \infty} \lambda(t) = 0, \tag{7.38}$$

which would be generalizing the condition that $\lambda(t_1) = 0$ in Theorem 7.1. But this is not in general the case. We will see an example where this does not apply soon. A milder transversality condition of the form

$$\lim_{t \to \infty} H(t, x, y, \lambda) = 0 \tag{7.39}$$

always applies, but is not easy to check. Stronger transversality conditions apply when we put more structure on the problem. We will discuss these issues in Section 7.4 below. Before presenting these results, there are immediate generalizations of the sufficiency theorems to this case.

Theorem 7.11. ***(Mangasarian's Sufficient Conditions for Infinite Horizon)*** *Consider the problem of maximizing (7.28) subject to (7.29) and (7.30), with f and g continuously differentiable. Define $H(t, x, y, \lambda)$ as in (7.12), and suppose that a piecewise continuous solution $\hat{y}(t)$ and the corresponding path of state variable $\hat{x}(t)$ satisfy (7.34)-(7.36). Suppose also that for the resulting costate variable $\lambda(t)$, $H(t, x, y, \lambda)$ is jointly concave in (x, y) for all $t \in \mathbb{R}_+$ and that $\lim_{t \to \infty} \lambda(t)(\hat{x}(t) - \tilde{x}(t)) \leq 0$ for all $\tilde{x}(t)$ implied by an admissible control path $\tilde{y}(t)$, then $\hat{y}(t)$ and the corresponding $\hat{x}(t)$ achieve the unique global maximum of (7.28).*

Theorem 7.12. ***(Arrow's Sufficient Conditions for Infinite Horizon)*** *Consider the problem of maximizing (7.28) subject to (7.29) and (7.30), with f and g continuously differentiable. Define $H(t, x, y, \lambda)$ as in (7.12), and suppose that a piecewise continuous solution $\hat{y}(t)$ and the corresponding path of state variable $\hat{x}(t)$ satisfy (7.34)-(7.36). Given the resulting costate variable $\lambda(t)$, define $M(t, x, \lambda) \equiv \max_{y(t) \in \mathcal{Y}(t)} H(t, x, y, \lambda)$. If $M(t, x, \lambda)$ is concave in x and $\lim_{t \to \infty} \lambda(t)(\hat{x}(t) - \tilde{x}(t)) \leq 0$ for all $\tilde{x}(t)$ implied by an admissible control path $\tilde{y}(t)$, then the pair $(\hat{x}(t), \hat{y}(t))$ achieves the unique global maximum of (7.28).*

The proofs of both of these theorems are similar to that of Theorem 7.5 and are left for the reader (See Exercise 7.11). Since $x(t)$ can potentially grow without bounds and we require only concavity (not strict concavity), Theorems 7.11 and 7.12 can be applied to models with constant returns and endogenous growth, thus will be particularly useful in later chapters. Notice that both of these sufficiency theorems involve the difficult to check condition that $\lim_{t\to\infty} \lambda(t)(x(t) - \tilde{x}(t)) \leq 0$ for all $\tilde{x}(t)$ implied by an admissible control path $\tilde{y}(t)$. This condition will disappear when we can impose a proper transversality condition.

7.3.2. Economic Intuition. The Maximum Principle is not only a powerful mathematical tool, but from an economic point of view, it is *the right tool*, because it captures the essential economic intuition of dynamic economic problems. In this subsection, we provide two different and complementary economic intuitions for the Maximum Principle. One of them is based on the original form as stated in Theorem 7.3 or Theorem 7.9, while the other is based on the dynamic programming (HJB) version provided in Theorem 7.10.

To obtain the first intuition consider the problem of maximizing

$$\int_0^{t_1} H(t, \hat{x}(t), y(t), \lambda(t))\, dt = \int_0^{t_1} [f(t, \hat{x}(t), y(t)) + \lambda(t)\, g(t, \hat{x}(t), y(t))]\, dt \tag{7.40}$$

with respect to the entire function $y(t)$ for given $\lambda(t)$ and $\hat{x}(t)$, where t_1 can be finite or equal to $+\infty$. The condition $H_y(t, \hat{x}(t), y(t), \lambda(t)) = 0$ would then be a necessary condition for this alternative maximization problem. Therefore, the Maximum Principle is implicitly maximizing the sum the original maximand $\int_0^{t_1} f(t, \hat{x}(t), y(t))\, dt$ plus an additional term $\int_0^{t_1} \lambda(t)\, g(t, \hat{x}(t), y(t))\, dt$. Understanding why this is true provides much of the intuition for the Maximum Principle.

First recall that $V(t, \hat{x}(t))$ is defined in equation (7.33) as the value of starting at time t with state variable $\hat{x}(t)$ and pursuing the optimal policy from then on. We will see in the next subsection, in particular in equation (7.44), that

$$\lambda(t) = \frac{\partial V(t, \hat{x}(t))}{\partial x}.$$

Therefore, similar to the Lagrange multipliers in the theory of constraint optimization, $\lambda(t)$ measures the impact of a small increase in x on the optimal value of the program. Consequently, $\lambda(t)$ is the (shadow) value of relaxing the constraint (7.29) by increasing the value of $x(t)$ at time t.[6] Moreover, recall that $\dot{x}(t) = g(t, \hat{x}(t), y(t))$, so that the second term in the Hamiltonian is equivalent to $\int_0^{t_1} \lambda(t)\, \dot{x}(t)\, dt$. This is clearly the shadow value of $x(t)$ at time t and the increase in the stock of $x(t)$ at this point. Moreover, recall that $x(t)$ is the state variable, thus we can think of it as a "stock" variable in contrast to the control $y(t)$, which corresponds to a "flow" variable.

[6]Here I am using the language of "relaxing the constraint" implicitly presuming that a high value of $x(t)$ contributes to increasing the value of the objective function. This simplifies terminology, but is not necessary for any of the arguments, since $\lambda(t)$ can be negative.

Therefore, maximizing (7.40) is equivalent to maximizing instantaneous returns as given by the function $f(t, \hat{x}(t), y(t))$, plus the value of stock of $x(t)$, as given by $\lambda(t)$, times the increase in the stock, $\dot{x}(t)$. This implies that the essence of the Maximum Principle is to maximize the flow return plus the value of the current stock of the state variable. This stock-flow type maximization has a clear economic logic.

Let us next turn to the interpreting the costate equation,

$$\begin{aligned} \dot{\lambda}(t) &= -H_x(t, \hat{x}(t), \hat{y}(t), \lambda(t)) \\ &= -f_x(t, \hat{x}(t), \hat{y}(t)) - \lambda(t) g_x(t, \hat{x}(t), \hat{y}(t)). \end{aligned}$$

This equation is also intuitive. Since $\lambda(t)$ is the value of the stock of the state variable, $x(t)$, $\dot{\lambda}(t)$ is the appreciation in this stock variable. A small increase in x will change the current flow return plus the value of the stock by the amount H_x, but it will also affect the value of the stock by the amount $\dot{\lambda}(t)$. The Maximum Principle states that this gain should be equal to the depreciation in the value of the stock, $-\dot{\lambda}(t)$, since, otherwise, it would be possible to change the $x(t)$ and increase the value of $H(t, x(t), y(t))$.

The second and complementary intuition for the Maximum Principle comes from the HJB equation (7.37) in Theorem 7.10. In particular, let us consider an exponentially discounted problem like those discussed in greater detail in Section 7.5 below. In particular, suppose that the payoff function is exponentially discounted, i.e., $f(t, x(t), y(t)) = \exp(-\rho t) f(x(t), y(t))$, and the law of motion of the state variable is given by an autonomous differential equation, i.e., $g(t, x(t), y(t)) = g(x(t), y(t))$. In this case, one can easily verify that if an admissible pair $(\hat{x}(t), \hat{y}(t))_{t \geq 0}$ is optimal starting at $t = 0$ with initial condition $x(0) = x_0$, then it is also optimal starting at $s > 0$, starting with the same initial condition, that is, $(\hat{x}(t), \hat{y}(t))_{t \geq s}$ is optimal for the problem with initial condition $x(s) = x_0$ (see Exercise 7.15). In view of this, let us define $V(x) \equiv V(0, x)$, that is, the value of pursuing the optimal plan $(\hat{x}(t), \hat{y}(t))$ starting with initial condition x, evaluated at $t = 0$. Since $(\hat{x}(t), \hat{y}(t))$ is an optimal plan irrespective of the starting date, we have that $V(t, x(t)) \equiv \exp(-\rho t) V(x(t))$. Then, by definition,

$$\frac{\partial V(t, x(t))}{\partial t} = -\exp(-\rho t) \rho V(x(t)).$$

Moreover, let $\dot{V}(x(t)) \equiv (\partial V(t, x(t)) / \partial x) \dot{x}(t)$ the change in the function V over time, which results from the change in the unique state variable $x(t)$ over time. Now substituting these expressions into (7.37) and noting that $\dot{x}(t) = g(\hat{x}(t), \hat{y}(t))$, we obtain the "stationary" form of the Hamilton-Jacobi-Bellman equation takes

$$\rho V(x(t)) = f(\hat{x}(t), \hat{y}(t)) + \dot{V}(x(t)). \tag{7.41}$$

This is a very important and widely used equation in dynamic economic analysis and can be interpreted as a "no-arbitrage asset value equation," and given its importance, an alternative derivation is provided in Exercise 7.16. Intuitively, we can think of V as the value of an asset traded in the stock market and ρ as the required rate of return for (a large number of) investors. When will investors be happy to hold this asset? Loosely speaking, they will do so when the asset pays out at least the required rate of return. In contrast, if the asset pays out more than the required rate of return, there would be excess demand for it from the investors until its value adjusts so that its rate of return becomes equal to the required rate of return. Therefore, we can think of the return on this asset in "equilibrium" being equal to the required rate of return, ρ. The return on the assets come from two sources: first, "dividends," that is current returns paid out to investors. In the current context, this corresponds to the flow payoff $f(\hat{x}(t), \hat{y}(t))$. If this dividend were constant and equal to d, and there were no other returns, then we would naturally have that $V = d/\rho$ or

$$\rho V = d.$$

However, in general the returns to the holding an asset come not only from dividends but also from capital gains or losses (appreciation or depreciation of the asset). In the current context, this is equal to $\dot{V}$. Therefore, instead of $\rho V = d$, we have

$$\rho V(x(t)) = d + \dot{V}(x(t)).$$

Thus, at an intuitive level, the Maximum Principle amounts to requiring that the maximized value of dynamic maximization program, $V(x(t))$, and its rate of change, $\dot{V}(x(t))$, should be consistent with this no-arbitrage condition.

7.3.3. Proof of Theorem 7.9*. In this subsection, we provide a sketch of the proof of Theorems 7.9. A fully rigorous proof of Theorem 7.9 is quite long and involved. It can be found in a number of sources mentioned in the references below. The version provided here contains all the basic ideas, but is stated under the assumption that $V(t, x)$ is twice differentiable in t and x. As discussed above, the assumption that $V(t, x)$ is differentiable in t and x is not particularly restrictive, though the additional assumption that it is twice differentiable is quite stringent.

The main idea of the proof is due to Pontryagin and co-authors. Instead of smooth variations from the optimal pair $(\hat{x}(t), \hat{y}(t))$, the method of proof considers "needle-like" variations, that is, piecewise continuous paths for the control variable that can deviate from the optimal control path by an arbitrary amount for a small interval of time.

Sketch Proof of Theorem 7.9: Suppose that the admissible pair $(\hat{x}(t), \hat{y}(t))$ is a solution and attains the maximal value $V(0, x_0)$. Take an arbitrary $t_0 \in \mathbb{R}_+$. Construct the following perturbation: $y_\delta(t) = \hat{y}(t)$ for all $t \in [0, t_0)$ and for some sufficiently small Δt and $\delta \in \mathbb{R}$, $y_\delta(t) = \delta$ for $t \in [t_0, t_0 + \Delta t]$ for all $t \in [t_0, t_0 + \Delta t]$. Moreover, let $y_\delta(t)$ for $t \geq t_0 + \Delta t$ be the optimal control for $V(t_0 + \Delta t, x_\delta(t_0 + \Delta t))$, where $x_\delta(t)$ is the value of the state variable resulting from the perturbed control y_δ, with $x_\delta(t_0+\Delta t)$ being the value at time $t_0+\Delta t$. Note by construction $x_\delta(t_0)=\hat{x}(t_0)$ (since $y_\delta(t)=\hat{y}(t)$ for all $t\in[0,t_0]$).

Since the pair $(\hat{x}(t),\hat{y}(t))$ is optimal, we have that

$$\begin{aligned} V(t_0,\hat{x}(t_0)) &= \int_{t_0}^{\infty} f(t,\hat{x}(t),\hat{y}(t))\,dt \\ &\geq \int_{t_0}^{\infty} f(t,x_\delta(t),y_\delta(t))\,dt \\ &= \int_{t_0}^{t_0+\Delta t} f(t,x_\delta(t),y_\delta(t))\,dt + V(t_0+\Delta t,x_\delta(t_0+\Delta t)), \end{aligned}$$

where the last equality uses the fact that the admissible pair $(x_\delta(t),y_\delta(t))$ is optimal starting with state variable $x_\delta(t_0+\Delta t)$ at time $t_0+\Delta t$. Rearranging terms and dividing by Δt yields

$$\frac{V(t_0+\Delta t,x_\delta(t_0+\Delta t))-V(t_0,\hat{x}(t_0))}{\Delta t} \leq -\frac{\int_{t_0}^{t_0+\Delta t} f(t,x_\delta(t),y_\delta(t))\,dt}{\Delta t} \text{ for all } \Delta t\geq 0.$$

Now take limits as $\Delta t\to 0$ and note that $x_\delta(t_0)=\hat{x}(t_0)$ and that

$$\lim_{\Delta t\to 0}\frac{\int_{t_0}^{t_0+\Delta t} f(t,x_\delta(t),y_\delta(t))\,dt}{\Delta t} = f(t,x_\delta(t),y_\delta(t)).$$

Moreover, let $\mathcal{T}\subset\mathbb{R}_+$ be the set of points where the optimal control $\hat{y}(t)$ is a continuous function of time. Note that $\mathcal{T}$ is a dense subset of $\mathbb{R}_+$ since $\hat{y}(t)$ is a piecewise continuous function. Let us now take V to be a differentiable function of time at all $t\in\mathcal{T}$, so that

$$\begin{aligned} \lim_{\Delta t\to 0}\frac{V(t_0+\Delta t,x_\delta(t_0+\Delta t))-V(t_0,\hat{x}(t_0))}{\Delta t} &= \frac{\partial V(t,x_\delta(t))}{\partial t}+\frac{\partial V(t,x_\delta(t))}{\partial x}\dot{x}_\delta(t), \\ &= \frac{\partial V(t,x_\delta(t))}{\partial t}+\frac{\partial V(t,x_\delta(t))}{\partial x}g(t,x_\delta(t),y_\delta(t)), \end{aligned}$$

where $\dot{x}_\delta(t)=g(t,x_\delta(t),y_\delta(t))$ is the law of motion of the state variable given by (7.29) together with the control y_δ. Putting all these together, we obtain that

$$f(t_0,x_\delta(t_0),y_\delta(t_0))+\frac{\partial V(t_0,x_\delta(t_0))}{\partial t}+\frac{\partial V(t_0,x_\delta(t_0))}{\partial x}g(t_0,x_\delta(t_0),y_\delta(t_0))\leq 0$$

for all $t_0\in\mathcal{T}$ (which correspond to points of continuity of $\hat{y}(t)$) and for all admissible perturbation pairs $(x_\delta(t),y_\delta(t))$. Moreover, from Theorem 7.10, which applies at all $t_0\in\mathcal{T}$,

$$f(t_0,\hat{x}(t_0),\hat{y}(t_0))+\frac{\partial V(t_0,\hat{x}(t_0))}{\partial t}+\frac{\partial V(t_0,\hat{x}(t_0))}{\partial x}g(t_0,\hat{x}(t_0),\hat{y}(t_0))=0. \tag{7.42}$$

Once more using the fact that $x_\delta(t_0)=\hat{x}(t_0)$, this implies that

$$\begin{aligned} & f(t_0,\hat{x}(t_0),\hat{y}(t_0))+\frac{\partial V(t_0,\hat{x}(t_0))}{\partial x}g(t_0,\hat{x}(t_0),\hat{y}(t_0))\geq \\ & f(t_0,x_\delta(t_0),y_\delta(t_0))+\frac{\partial V(t_0,\hat{x}(t_0))}{\partial x}g(t_0,x_\delta(t_0),y_\delta(t_0)) \end{aligned} \tag{7.43}$$

for all $t_0\in\mathcal{T}$ and for all admissible perturbation pairs $(x_\delta(t),y_\delta(t))$. Now defining

$$\lambda(t_0)\equiv\frac{\partial V(t_0,\hat{x}(t_0))}{\partial x}, \tag{7.44}$$

inequality (7.43) can be written as

$$\begin{aligned} f\left(t_0, \hat{x}\left(t_0\right), \hat{y}\left(t_0\right)\right) + \lambda\left(t_0\right) g\left(t_0, \hat{x}\left(t_0\right), \hat{y}\left(t_0\right)\right) &\geq f\left(t_0, x_\delta\left(t_0\right), y_\delta\left(t_0\right)\right) \\ &\quad + \lambda\left(t_0\right) g\left(t_0, x_\delta\left(t_0\right), y_\delta\left(t_0\right)\right), \end{aligned}$$

or equivalently,

$$\begin{aligned} H\left(t_0, \hat{x}\left(t_0\right), \hat{y}\left(t_0\right)\right) &\geq H\left(t_0, x_\delta\left(t_0\right), y_\delta\left(t_0\right)\right) \\ &\quad \text{for all admissible } \left(x_\delta\left(t_0\right), y_\delta\left(t_0\right)\right). \end{aligned}$$

Therefore,

$$H\left(t, \hat{x}\left(t\right), \hat{y}\left(t\right)\right) \geq \max_y H\left(t, \hat{x}\left(t\right), y\right).$$

This establishes the Maximum Principle.

The necessary condition (7.34) directly follows from the Maximum Principle together with the fact that H is differentiable in x and y (a consequence of the fact that f and g are differentiable in x and y). Condition (7.36) holds by definition. Finally, (7.35) follows from differentiating (7.42) with respect to x at all points of continuity of $\hat{y}(t)$, which gives

$$\begin{aligned} & \frac{\partial f\left(t, \hat{x}\left(t\right), \hat{y}\left(t\right)\right)}{\partial x} + \frac{\partial^2 V\left(t, \hat{x}\left(t\right)\right)}{\partial t \partial x} \\ + & \frac{\partial^2 V\left(t, \hat{x}\left(t\right)\right)}{\partial x^2} g\left(t, \hat{x}\left(t\right), \hat{y}\left(t\right)\right) + \frac{\partial V\left(t, \hat{x}\left(t\right)\right)}{\partial x} \frac{\partial g\left(t, \hat{x}\left(t\right), \hat{y}\left(t\right)\right)}{\partial x} = 0, \end{aligned}$$

for all for all $t \in \mathcal{T}$. Using the definition of the Hamiltonian, this gives (7.35). □

7.4. More on Transversality Conditions

We next turn to a study of the boundary conditions at infinity in infinite-horizon maximization problems. As in the discrete time optimization problems, these limiting boundary conditions are referred to as "transversality conditions". As mentioned above, a natural conjecture might be that, as in the finite-horizon case, the transversality condition should be similar to that in Theorem 7.1, with t_1 replaced with the limit of $t \to \infty$, that is, $\lim_{t\to\infty} \lambda(t) = 0$. The following example, which is very close to the original Ramsey model, illustrates that this is not the case; without further assumptions, the valid transversality condition is given by the weaker condition (7.39).

Example 7.2. Consider the following problem:

$$\max \int_0^\infty \left[\log\left(c\left(t\right)\right) - \log c^*\right] dt$$

subject to

$$\dot{k}\left(t\right) = \left[k\left(t\right)\right]^\alpha - c\left(t\right) - \delta k\left(t\right)$$

$$k\left(0\right) = 1$$

and

$$\lim_{t\to\infty} k\left(t\right) \geq 0$$

where $c^* \equiv [k^*]^\alpha - \delta k^*$ and $k^* \equiv (\alpha/\delta)^{1/(1-\alpha)}$. In other words, c^* is the maximum level of consumption that can be achieved in the steady state of this model and k^* is the corresponding steady-state level of capital. This way of writing the objective function makes sure that the integral converges and takes a finite value (since $c(t)$ cannot exceed c^* forever).

The Hamiltonian is straightforward to construct; it does not explicitly depend on time and takes the form

$$H(k, c, \lambda) = [\log c(t) - \log c^*] + \lambda [k(t)^\alpha - c(t) - \delta k(t)],$$

and implies the following necessary conditions (dropping time dependence to simplify the notation):

$$\begin{aligned} H_c(k, c, \lambda) &= \frac{1}{c(t)} - \lambda(t) = 0 \\ H_k(k, c, \lambda) &= \lambda(t)\left(\alpha k(t)^{\alpha-1} - \delta\right) = -\dot{\lambda}(t). \end{aligned}$$

It can be verified that any optimal path must feature $c(t) \to c^*$ as $t \to \infty$. This, however, implies that

$$\lim_{t\to\infty} \lambda(t) = \frac{1}{c^*} > 0 \text{ and } \lim_{t\to\infty} k(t) = k^*.$$

Therefore, the equivalent of the standard finite-horizon transversality conditions do not hold. It can be verified, however, that along the optimal path we have

$$\lim_{t\to\infty} H(k(t), c(t), \lambda(t)) = 0.$$

We will next see that this is indeed the relevant transversality condition.

Theorem 7.13. ***(Transversality Condition for Infinite-Horizon Problems)*** *Suppose that problem of maximizing (7.28) subject to (7.29) and (7.30), with f and g continuously differentiable, has an interior piecewise continuous solution $\hat{y}(t)$ with corresponding path of state variable $\hat{x}(t)$. Suppose moreover that $\lim_{t\to\infty} V(t, \hat{x}(t))$ exists (where $V(t, x(t))$ is defined in (7.33)). Let $H(t, x, y, \lambda)$ be given by (7.12). Then the optimal control $\hat{y}(t)$ and the corresponding path of the state variable $\hat{x}(t)$ satisfy the necessary conditions (7.34)-(7.36) and the transversality condition*

$$\lim_{t\to\infty} H(t, \hat{x}(t), \hat{y}(t), \lambda(t)) = 0. \tag{7.45}$$

Proof. Let us focus on points where $V(t, x)$ is differentiable in t and x so that the Hamilton-Jacobi-Bellman equation, (7.37) holds. Noting that $\partial V(t, \hat{x}(t))/\partial x = \lambda(t)$, this equation can be written as

$$\begin{aligned} \frac{\partial V(t, \hat{x}(t))}{\partial t} + f(t, \hat{x}(t), \hat{y}(t)) + \lambda(t) g(t, \hat{x}(t), \hat{y}(t)) &= 0 \text{ for all } t \\ \frac{\partial V(t, \hat{x}(t))}{\partial t} + H(t, \hat{x}(t), \hat{y}(t), \lambda(t)) &= 0 \text{ for all } t. \end{aligned} \tag{7.46}$$

Now take the limit as $t \to \infty$. Since $\lim_{t\to\infty} V(t, \hat{x}(t))$ exists, we have that either $\lim_{t\to\infty} \partial V(t, \hat{x}(t))/\partial t > 0$, so that $\lim_{t\to\infty} V(t, \hat{x}(t)) = +\infty$, or $\lim_{t\to\infty} \partial V(t, \hat{x}(t))/\partial t < 0$ everywhere, so that $\lim_{t\to\infty} V(t, \hat{x}(t)) = -\infty$ or $\lim_{t\to\infty} \partial V(t, \hat{x}(t))/\partial t = 0$. The first two possibilities are ruled out by the hypothesis that an optimal solution that reaches the maximum exists. Thus we must have $\lim_{t\to\infty} \partial V(t, \hat{x}(t))/\partial t = 0$. (7.46) then implies (7.45). □

The transversality condition (7.45) is not particularly convenient to work with. In the next section, we will see that as we consider discounted infinite-horizon problems stronger and more useful versions of this transversality condition can be developed.

7.5. Discounted Infinite-Horizon Optimal Control

Part of the difficulty, especially regarding the absence of a transversality condition, comes from the fact that we did not impose enough structure on the functions f and g. As discussed above, our interest is with the growth models where the utility is discounted exponentially. Consequently, economically interesting problems often take the following more specific form:

$$\max_{x(t),y(t)} W(x(t), y(t)) \equiv \int_0^\infty \exp(-\rho t) f(x(t), y(t))\, dt \text{ with } \rho > 0, \tag{7.47}$$

subject to

$$\dot{x}(t) = g(x(t), y(t)), \tag{7.48}$$

and

$$y(t) \in \mathbb{R} \text{ for all } t,\ x(0) = x_0 \text{ and } \lim_{t\to\infty} x(t) \geq x_1. \tag{7.49}$$

Notice that throughout we assume $\rho > 0$, so that there is indeed *discounting*.

The special feature of this problem is that the objective function, f, depends on time only through exponential discounting, while the constraint equation, g, is not a function of time directly. The Hamiltonian in this case would be:

$$\begin{aligned} H(t, x(t), y(t), \lambda(t)) &= \exp(-\rho t) f(x(t), y(t)) + \lambda(t) g(x(t), y(t)) \\ &= \exp(-\rho t)\left[f(x(t), y(t)) + \mu(t) g(x(t), y(t))\right], \end{aligned}$$

where the second line defines

$$\mu(t) \equiv \exp(\rho t) \lambda(t). \tag{7.50}$$

This equation makes it clear that the Hamiltonian depends on time explicitly only through the $\exp(-\rho t)$ term.

In fact, in this case, rather than working with the standard Hamiltonian, we can work with *the current-value Hamiltonian*, defined as

$$\hat{H}(x(t), y(t), \mu(t)) \equiv f(x(t), y(t)) + \mu(t) g(x(t), y(t)) \tag{7.51}$$

which is "autonomous" in the sense that it does not directly depend on time.

The following result establishes the necessity of a stronger transversality condition under some additional assumptions, which are typically met in economic applications. In preparation for this result, let us refer to the functions $f(x, y)$ and $g(x, y)$ as weakly monotone, if each one is monotone in each of its arguments (for example, nondecreasing in x and nonincreasing in y). Furthermore, let us simplify the statement of this theorem by assuming that the optimal control $\hat{y}(t)$ is everywhere a continuous function of time (though this is not necessary for any of the results).

Theorem 7.14. ***(Maximum Principle for Discounted Infinite-Horizon Problems)*** *Suppose that problem of maximizing (7.47) subject to (7.48) and (7.49), with f and g continuously differentiable, has a solution $\hat{y}(t)$ with corresponding path of state variable $\hat{x}(t)$. Suppose moreover that $\lim_{t\to\infty} V(t, \hat{x}(t))$ exists (where $V(t, x(t))$ is defined in (7.33)). Let $\hat{H}(\hat{x}, \hat{y}, \mu)$ be the current-value Hamiltonian given by (7.51). Then the optimal control $\hat{y}(t)$ and the corresponding path of the state variable $\hat{x}(t)$ satisfy the following necessary conditions:*

$$\hat{H}_y(\hat{x}(t), \hat{y}(t), \mu(t)) = 0 \text{ for all } t \in \mathbb{R}_+, \tag{7.52}$$

$$\rho\mu(t) - \dot{\mu}(t) = \hat{H}_x(\hat{x}(t), \hat{y}(t), \mu(t)) \text{ for all } t \in \mathbb{R}_+, \tag{7.53}$$

$$\dot{x}(t) = \hat{H}_\mu(\hat{x}(t), \hat{y}(t), \mu(t)) \text{ for all } t \in \mathbb{R}_+,\ x(0) = x_0 \text{ and } \lim_{t\to\infty} x(t) \geq x_1, \tag{7.54}$$

and the transversality condition

$$\lim_{t\to\infty} \exp(-\rho t)\, \hat{H}(\hat{x}(t), \hat{y}(t), \mu(t)) = 0. \tag{7.55}$$

Moreover, if f and g are weakly monotone, the transversality condition can be strengthened to:

$$\lim_{t\to\infty} [\exp(-\rho t)\, \mu(t)\, \hat{x}(t)] = 0. \tag{7.56}$$

Proof. The derivation of the necessary conditions (7.52)-(7.54) and the transversality condition (7.55) follows by using the definition of the current-value Hamiltonian and from Theorem 7.13. They are left for as an exercise (see Exercise 7.13).

We therefore only give the proof for the stronger transversality condition (7.56). The weaker transversality condition (7.55) can be written as

$$\lim_{t\to\infty} \exp(-\rho t)\, f(\hat{x}(t), \hat{y}(t)) + \lim_{t\to\infty} \exp(-\rho t)\, \mu(t)\, g(\hat{x}(t), \hat{y}(t)) = 0.$$

The first term must be equal to zero, since otherwise $\lim_{t\to\infty} V(t, \hat{x}(t)) = \infty$ or $-\infty$, and the pair $(\hat{x}(t), \hat{y}(t))$ cannot be reaching the optimal solution. Therefore

$$\lim_{t\to\infty} \exp(-\rho t)\, \mu(t)\, g(\hat{x}(t), \hat{y}(t)) = \lim_{t\to\infty} \exp(-\rho t)\, \mu(t)\, \dot{x}(t) = 0. \tag{7.57}$$

Since $\lim_{t\to\infty} V(t, \hat{x}(t))$ exists and f and g are weakly monotone, $\lim_{t\to\infty} \hat{y}(t)$ and $\lim_{t\to\infty} \hat{x}(t)$ must exist, though they may be infinite (otherwise the limit of $V(t, \hat{x}(t))$ would fail to exist). The latter fact also implies that $\lim_{t\to\infty} \dot{x}(t)$ exists (though it may also be infinite). Moreover, $\lim_{t\to\infty} \dot{x}(t)$ is nonnegative, since otherwise the condition $\lim_{t\to\infty} x(t) \geq x_1$ would be violated. From (7.53), (7.55) implies that as $t \to \infty$, $\lambda(t) \equiv \exp(-\rho t)\mu(t) \to \kappa$ for some $\kappa \in \mathbb{R}_+$.

Suppose first that $\lim_{t\to\infty} \dot{x}(t) = 0$. Then $\lim_{t\to\infty} \hat{x}(t) = \hat{x}^* \in \mathbb{R}$ (i.e., a finite value). This also implies that $f(\hat{x}(t), \hat{y}(t))$, $g(\hat{x}(t), \hat{y}(t))$ and therefore $f_y(\hat{x}(t), \hat{y}(t))$ and $g_y(\hat{x}(t), \hat{y}(t))$ limit to constant values. Then from (7.52), we have that as $t \to \infty$, $\mu(t) \to \mu^* \in \mathbb{R}$ (i.e., a finite value). This implies that $\kappa = 0$ and

$$\lim_{t\to\infty} \exp(-\rho t)\mu(t) = 0, \tag{7.58}$$

and moreover since $\lim_{t\to\infty} \hat{x}(t) = \hat{x}^* \in \mathbb{R}$, (7.56) also follows.

Suppose now that $\lim_{t\to\infty} \dot{x}(t) = g\hat{x}(t)$, where $g \in \mathbb{R}_+$, so that $\hat{x}(t)$ grows *at an exponential rate*. Then substituting this into (7.57) we obtain (7.56).

Next, suppose that $0 < \lim_{t\to\infty} \dot{x}(t) < g\hat{x}(t)$, for any $g > 0$, so that $\hat{x}(t)$ grows at *less than an exponential rate*. In this case, since $\dot{x}(t)$ is increasing over time, (7.57) implies that (7.58) must hold and thus again we must have that as $t \to \infty$, $\lambda(t) \equiv \exp(-\rho t)\mu(t) \to 0$, i.e., $\kappa = 0$ (otherwise $\lim_{t\to\infty} \exp(-\rho t)\mu(t)\dot{x}(t) = \lim_{t\to\infty} \dot{x}(t) > 0$, violating (7.57)) and thus $\lim_{t\to\infty} \dot{\mu}(t)/\mu(t) < \rho$. Since $\hat{x}(t)$ grows at less than an exponential rate, we also have $\lim_{t\to\infty} \exp(-gt)\hat{x}(t) = 0$ for any $g > 0$, and in particular for $g = \rho - \lim_{t\to\infty} \dot{\mu}(t)/\mu(t)$. Consequently, asymptotically $\mu(t)\hat{x}(t)$ grows at a rate lower than ρ and we again obtain (7.56).

Finally, suppose that $\lim_{t\to\infty} \dot{x}(t) > g\hat{x}(t)$ for any $g < \infty$, i.e., $\hat{x}(t)$ grows at *more than an exponential rate*. In this case, for any $g > 0$, we have that

$$\lim_{t\to\infty} \exp(-\rho t)\mu(t)\dot{x}(t) \geq g \lim_{t\to\infty} \exp(-\rho t)\mu(t)\hat{x}(t) \geq g\kappa \lim_{t\to\infty} \hat{x}(t) \geq 0,$$

where the first inequality exploits the fact that $\lim_{t\to\infty} \dot{x}(t) > g\hat{x}(t)$ and the second, the fact that $\lambda(t) \equiv \exp(-\rho t)\mu(t) \to \lambda$ and that $\hat{x}(t)$ is increasing. But from (7.57), $\lim_{t\to\infty} \exp(-\rho t)\mu(t)\dot{x}(t) = 0$, so that all the inequalities in this expression must hold as equality, and thus (7.56) must be satisfied, completing the proof of the theorem. □

The proof of Theorem 7.14 also clarifies the importance of discounting. Without discounting the key equation, (7.57), is not necessarily true, and the rest of the proof does not go through.

Theorem 7.14 is the most important result of this chapter and will be used in almost all continuous time optimizations problems in this book. Throughout, when we refer to a discounted infinite-horizon optimal control problem, we mean a problem that satisfies all the assumptions in Theorem 7.14, including the weak monotonicity assumptions on f and g. Consequently, for our canonical infinite-horizon optimal control problems the stronger transversality condition (7.56) will be necessary. Notice that compared to the transversality condition in the finite-horizon case (e.g., Theorem 7.1), there is the additional term $\exp(-\rho t)$. This is because the transversality condition applies to the original costate variable $\lambda(t)$, i.e., $\lim_{t\to\infty}[x(t)\lambda(t)] = 0$, and as shown above, the current-value costate variable $\mu(t)$ is given by $\mu(t) = \exp(\rho t)\lambda(t)$. Note also that the stronger transversality condition takes the form $\lim_{t\to\infty}[\exp(-\rho t)\mu(t)\hat{x}(t)] = 0$, not simply $\lim_{t\to\infty}[\exp(-\rho t)\mu(t)] = 0$. Exercise 7.19 illustrates why this is.

The sufficiency theorems can also be strengthened now by incorporating the transversality condition (7.56) and expressing the conditions in terms of the current-value Hamiltonian:

Theorem 7.15. ***(Mangasarian Sufficient Conditions for Discounted Infinite-Horizon Problems)*** *Consider the problem of maximizing (7.47) subject to (7.48) and (7.49), with f and g continuously differentiable and weakly monotone. Define $\hat{H}(x, y, \mu)$ as the current-value Hamiltonian as in (7.51), and suppose that a solution $\hat{y}(t)$ and the corresponding path of state variable $\hat{x}(t)$ satisfy (7.52)-(7.54) and (7.56). Suppose also that $\lim_{t\to\infty} V(t, \hat{x}(t))$ exists and that for the resulting current-value costate variable $\mu(t)$, $\hat{H}(x, y, \mu)$ is jointly concave in (x, y) for all $t \in \mathbb{R}_+$, then $\hat{y}(t)$ and the corresponding $\hat{x}(t)$ achieve the unique global maximum of (7.47).*

Theorem 7.16. ***(Arrow Sufficient Conditions for Discounted Infinite-Horizon Problems)*** *Consider the problem of maximizing (7.47) subject to (7.48) and (7.49), with f and g continuously differentiable and weakly monotone. Define $\hat{H}(x, y, \mu)$ as the current-value Hamiltonian as in (7.51), and suppose that a solution $\hat{y}(t)$ and the corresponding path of state variable $\hat{x}(t)$ satisfy (7.52)-(7.54) and which leads to (7.56). Given the resulting current-value costate variable $\mu(t)$, define $M(t, x, \mu) \equiv \max_{y(t)\in\mathcal{Y}(t)} \hat{H}(x, y, \mu)$. Suppose that $\lim_{t\to\infty} V(t, \hat{x}(t))$ exists (where $V(t, x(t))$ is defined in (7.33)) and that $M(t, x, \mu)$ is concave in x. Then $\hat{y}(t)$ and the corresponding $\hat{x}(t)$ achieve the unique global maximum of (7.47).*

The proofs of these two theorems are again omitted and left as exercises (see Exercise 7.12).

We next provide a simple example of discounted infinite-horizon optimal control.

Example 7.3. One of the most common examples of this type of dynamic optimization problem is that of the optimal time path of consuming a non-renewable resource. In particular, imagine the problem of an infinitely-lived individual that has access to a non-renewable or exhaustible resource of size 1. The instantaneous utility of consuming a flow of resources y is $u(y)$, where $u : [0, 1] \to \mathbb{R}$ is a strictly increasing, continuously differentiable and strictly concave function. The individual discounts the future exponentially with discount rate $\rho > 0$, so that his objective function at time $t = 0$ is to maximize

$$\int_0^\infty \exp(-\rho t) u(y(t)) dt.$$

The constraint is that the remaining size of the resource at time t, $x(t)$ evolves according to

$$\dot{x}(t) = -y(t),$$

which captures the fact that the resource is not renewable and becomes depleted as more of it is consumed. Naturally, we also need that $x(t) \geq 0$.

The current-value Hamiltonian takes the form

$$\hat{H}(x(t), y(t), \mu(t)) = u(y(t)) - \mu(t) y(t).$$

Theorem 7.14 implies the following necessary condition for an interior continuously differentiable solution $(\hat{x}(t), \hat{y}(t))$ to this problem. There should exist a continuously differentiable function $\mu(t)$ such that

$$u'(\hat{y}(t)) = \mu(t),$$

and

$$\dot{\mu}(t) = \rho\mu(t).$$

The second condition follows since neither the constraint nor the objective function depend on $x(t)$. This is the famous *Hotelling rule* for the exploitation of exhaustible resources. It charts a path for the shadow value of the exhaustible resource. In particular, integrating both sides of this equation and using the boundary condition, we obtain that

$$\mu(t) = \mu(0)\exp(\rho t).$$

Now combining this with the first-order condition for $y(t)$, we obtain

$$\hat{y}(t) = u'^{-1}[\mu(0)\exp(\rho t)],$$

where $u'^{-1}[\cdot]$ is the inverse function of u', which exists and is strictly decreasing by virtue of the fact that u is strictly concave. This equation immediately implies that the amount of the resource consumed is monotonically decreasing over time. This is economically intuitive: because of discounting, there is preference for early consumption, whereas delayed consumption has no return (there is no production or interest payments on the stock). Nevertheless, the entire resource is not consumed immediately, since there is also a preference for smooth consumption arising from the fact that $u(\cdot)$ is strictly concave.

Combining the previous equation with the resource constraint gives

$$\dot{x}(t) = -u'^{-1}[\mu(0)\exp(\rho t)].$$

Integrating this equation and using the boundary condition that $x(0) = 1$, we obtain

$$\hat{x}(t) = 1 - \int_0^t u'^{-1}[\mu(0)\exp(\rho s)] ds.$$

Since along any optimal path we must have $\lim_{t\to\infty}\hat{x}(t)=0$, we have that

$$\int_0^\infty u'^{-1}\left[\mu(0)\exp(\rho s)\right]ds=1.$$

Therefore, the initial value of the costate variable $\mu(0)$ must be chosen so as to satisfy this equation.

Notice also that in this problem both the objective function, $u(y(t))$, and the constraint function, $-y(t)$, are weakly monotone in the state and the control variables, so the stronger form of the transversality condition, (7.56), holds. You are asked to verify that this condition is satisfied in Exercise 7.22.

7.6. Existence of Solutions*

The theorems presented so far characterize the properties of a (piecewise continuous) solution to a continuous-time maximization problem. The question of when a solution exists arises naturally. I provide a brief discussion on this topic in this section. Let us focus on discounted infinite-horizon problems, in particular, the problem of maximizing (7.47) subject to (7.48) and (7.49). To state the simplest result on existence of solutions, suppose that $y(t)\in\mathcal{Y}\subset\mathbb{R}$ for all t, $x(t)\in\mathcal{X}\subset\mathbb{R}$, and define

(7.59)
$$\mathcal{F}=\left\{[y(t)]_{t=0}^\infty\in\mathcal{L}(\mathbb{R}_+):\dot{x}(t)=g(x(t),y(t))\text{ with }x(0)=x_0\text{ and }\lim_{t\to\infty}x(t)\geq x_1\right\},$$

where $\mathcal{L}(\mathbb{R}_+)$ denotes the set of functions $y:\mathbb{R}_+\to\mathcal{Y}$ that are Lebesgue integrable. Therefore, $\mathcal{F}$ denotes the set of Lebesgue integrable controls that are "feasible". The basic result is the following:

Theorem 7.17. ***(Existence of Solutions)*** *Consider the maximization of (7.47) subject to (7.48) and (7.49). Suppose that f and g are continuous in all of their arguments, $\rho>0$, $\mathcal{Y}$ and $\mathcal{X}$ are compact and $\mathcal{F}$ is nonempty. Then a solution to the maximization problem exists.*

Proof. The proof follows from the results developed in Appendix Chapter A. In particular, from Part 2 of Theorem A.11, the objective function (7.47) is continuous in the product topology (since the instantaneous payoff function f is the same at each date and is defined over the compact set $\mathcal{Y}\times\mathcal{X}$, thus is uniformly bounded). The constraint set $\mathcal{F}$ is also bounded, since $\mathcal{Y}$ is compact and is defined by a continuous function, and is therefore, closed and hence also compact in the product topology (see Exercise A.21 in Appendix Chapter A). It is also nonempty by hypothesis. Then by Weierstrass's Theorem, Theorem A.9, a solution exists. □

Unfortunately, providing sufficient conditions for the solution to be continuous or piecewise continuous is much harder. Nevertheless, most economic problems possess enough structure to ensure this. For example, in most of the problems we will encounter Inada-type conditions ensure that optimal controls remain within the interior of the feasible set and consumption-smoothing and no-arbitrage arguments rule out discontinuous controls. In these cases, continuous solutions can be shown to exist. Moreover, using Theorem 7.15, we can often establish that these solutions are unique. Throughout the rest of the book, I will follow the standard practice and assume that it continues solution to this type of maximization problem exists.

7.7. A First Look at Optimal Growth in Continuous Time

In this section, we briefly show that the main theorems developed so far apply to the problem of optimal growth, which was introduced in Chapter 5 and then analyzed in discrete time in the previous chapter. We will not provide a full treatment of this model here, since this is the topic of the next chapter.

Consider the neoclassical economy without any population growth and without any technological progress. In this case, the optimal growth problem in continuous time can be written as:

$$\max_{[k(t),c(t)]_{t=0}^{\infty}} \int_0^{\infty} \exp(-\rho t)\, u(c(t))\, dt,$$

subject to

$$\dot{k}(t) = f(k(t)) - \delta k(t) - c(t)$$

and $k(0) > 0$. Recall that $u : \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing, continuously differentiable and strictly concave, while $f(\cdot)$ satisfies our basic assumptions, Assumptions 1 and 2. Clearly, the objective function $u(c)$ is weakly monotone. The constraint function, $f(k) - \delta k - c$, is decreasing in c, but may be nonmonotone in k. However, without loss of any generality we can restrict attention to $k(t) \in \left[0, \bar{k}\right]$, where $\bar{k}$ is defined such that $f'\left(\bar{k}\right) = \delta$. Increasing the capital stock above this level would reduce output and thus consumption both today and in the future. When $k(t) \in \left[0, \bar{k}\right]$, the constraint function is also weakly monotone in k and we can apply Theorem 7.14.

Let us first set up the current-value Hamiltonian, which, in this case, takes the form

$$\hat{H}(k, c, \mu) = u(c(t)) + \mu(t)\left[f(k(t)) - \delta k(t) - c(t)\right], \tag{7.60}$$

with state variable k, control variable c and current-value costate variable μ.

From Theorem 7.14, the following are the necessary conditions:

$$\begin{aligned}
\hat{H}_c(k, c, \mu) &= u'(c(t)) - \mu(t) = 0 \\
\hat{H}_k(k, c, \mu) &= \mu(t)\left(f'(k(t)) - \delta\right) = \rho\mu(t) - \dot{\mu}(t) \\
\lim_{t\to\infty}\left[\exp(-\rho t)\,\mu(t)\, k(t)\right] &= 0.
\end{aligned}$$

Moreover, the first necessary condition immediately implies that $\mu(t) > 0$ (since $u' > 0$ everywhere). Consequently, the current-value Hamiltonian given in (7.60) consists of the sum of two strictly concave functions and is itself strictly concave and thus satisfies the conditions of Theorem 7.15. Therefore, a solution that satisfies these necessary conditions in fact gives a global maximum. Characterizing the solution of these necessary conditions also establishes the existence of a solution in this case.

Since an analysis of optimal growth in the neoclassical model is more relevant in the context of the next chapter, we do not provide further details here.

7.8. The q-Theory of Investment

As another application of the methods developed in this chapter, we consider the canonical model of investment under adjustment costs, also known as the q-theory of investment. This problem is not only useful as an application of optimal control techniques, but it is one of the basic models of standard macroeconomic theory.

The economic problem is that of a price-taking firm trying to maximize the present discounted value of its profits. The only twist relative to the problems we have studied so far is that this firm is subject to "adjustment" costs when it changes its capital stock. In particular, let the capital stock of the firm be $k(t)$ and suppose that the firm has access to a production function $f(k(t))$ that satisfies Assumptions 1 and 2. For simplicity, let us normalize the price of the output of the firm to 1 in terms of the final good at all dates. The firm is subject to adjustment costs captured by the function $\phi(i)$, which is strictly increasing, continuously differentiable and strictly convex, and satisfies $\phi(0) = \phi'(0) = 0$. This implies that in addition to the cost of purchasing investment goods (which given the normalization of price is equal to i for an amount of investment i), the firm incurs a cost of adjusting its production structure given by the convex function $\phi(i)$. In some models, the adjustment cost is taken to be a function of investment relative to capital, i.e., $\phi(i/k)$ instead of $\phi(i)$, but this makes no difference for our main focus. We also assume that installed capital depreciates at an exponential rate δ and that the firm maximizes its net present discounted earnings with a discount rate equal to the interest rate r, which is assumed to be constant.

The firm's problem can be written as

$$\max_{k(t),i(t)} \int_0^\infty \exp(-rt)\left[f(k(t)) - i(t) - \phi(i(t))\right] dt$$

subject to

$$\dot{k}(t) = i(t) - \delta k(t) \qquad (7.61)$$

and $k(t) \geq 0$, with $k(0) > 0$ given. Clearly, both the objective function and the constraint function are weakly monotone, thus we can apply Theorem 7.14.

Notice that $\phi(i)$ does not contribute to capital accumulation; it is simply a cost. Moreover, since ϕ is strictly convex, it implies that it is not optimal for the firm to make "large" adjustments. Therefore it will act as a force towards a smoother time path of investment.

To characterize the optimal investment plan of the firm, let us write the current-value Hamiltonian:

$$\hat{H}(k,i,q) \equiv [f(k(t)) - i(t) - \phi(i(t))] + q(t)[i(t) - \delta k(t)],$$

where we used $q(t)$ instead of the familiar $\mu(t)$ for the costate variable, for reasons that will be apparent soon.

The necessary conditions for this problem are standard (suppressing the "^" to denote the optimal values in order to reduce notation):

$$\begin{aligned}
\hat{H}_i(k,i,q) &= -1 - \phi'(i(t)) + q(t) = 0 \\
\hat{H}_k(k,i,q) &= f'(k(t)) - \delta q(t) = rq(t) - \dot{q}(t) \\
\lim_{t\to\infty} \exp(-rt)\, q(t)\, k(t) &= 0.
\end{aligned}$$

The first necessary condition implies that

$$q(t) = 1 + \phi'(i(t)) \text{ for all } t. \tag{7.62}$$

Differentiating this equation with respect to time, we obtain

$$\dot{q}(t) = \phi''(i(t))\,\dot{i}(t). \tag{7.63}$$

Substituting this into the second necessary condition, we obtain the following law of motion for investment:

$$\dot{i}(t) = \frac{1}{\phi''(i(t))}\left[(r+\delta)\left(1+\phi'(i(t))\right) - f'(k(t))\right]. \tag{7.64}$$

A number of interesting economic features emerge from this equation. First, as $\phi''(i)$ tends to zero, it can be verified that $\dot{i}(t)$ diverges, meaning that investment jumps to a particular value. In other words, it can be shown that this value is such that the capital stock immediately reaches its state-state value (see Exercise 7.24). This is intuitive. As $\phi''(i)$ tends to zero, $\phi'(i)$ becomes linear. In this case, adjustment costs simply increase the cost of investment linearly and do not create any need for smoothing. In contrast, when $\phi''(i(t)) > 0$, there will be a motive for smoothing, $\dot{i}(t)$ will take a finite value, and investment will adjust slowly. Therefore, as claimed above, adjustment costs lead to a smoother path of investment.

We can now analyze the behavior of investment and capital stock using the differential equations (7.61) and (7.64). First, it can be verified easily that there exists a unique steady-state solution with $k > 0$. This solution involves a level of capital stock k^* for the firm and investment just enough to replenish the depreciated capital, $i^* = \delta k^*$. This steady-state level of capital satisfies the first-order condition (corresponding to the right-hand side of (7.64) being equal to zero):

$$f'(k^*) = (r+\delta)\left(1+\phi'(\delta k^*)\right).$$

This first-order condition differs from the standard "modified golden rule" condition, which requires the marginal product of capital to be equal to the interest rate plus the depreciation rate, because an additional cost of having a higher capital stock is that there will have to be more investment to replenish depreciated capital. This is captured by the term $\phi'(\delta k^*)$. Since ϕ is strictly convex and f is strictly concave and satisfies the Inada conditions (from Assumption 2), there exists a unique value of k^* that satisfies this condition.

The analysis of dynamics in this case requires somewhat different ideas than those used in the basic Solow growth model (cf., Theorems 2.4 and 2.5). In particular, instead of global stability in the k-i space, the correct concept is one of *saddle-path stability.* The reason for this is that instead of an initial value constraint, $i(0)$ is pinned down by a boundary condition at "infinity," that is, to satisfy the transversality condition,

$$\lim_{t\to\infty} \exp(-rt)\, q(t)\, k(t) = 0.$$

This implies that in the context of the current theory, with one state and one control variable, we should have a one-dimensional manifold (a curve) along which capital-investment pairs tend towards the steady state. This manifold is also referred to as the "stable arm". The initial value of investment, $i(0)$, will then be determined so that the economy starts along this manifold. In fact, if any capital-investment pair (rather than only pairs along this one dimensional manifold) were to lead to the steady state, we would not know how to determine $i(0)$; in other words, there would be an "indeterminacy" of equilibria. Mathematically, rather than requiring all eigenvalues of the linearized system to be negative, what we require now is saddle-path stability, which involves the number of negative eigenvalues to be the same as the number of state variables.

This notion of saddle path stability will be central in most of growth models we will study. Let this now make these notions more precise by considering the following generalizations of Theorems 2.4 and 2.5 (see Appendix Chapter B):

Theorem 7.18. *Consider the following linear differential equation system*

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b} \tag{7.65}$$

with initial value $\mathbf{x}(0)$, where $\mathbf{x}(t) \in \mathbb{R}^n$ for all t and $\mathbf{A}$ is an $n\times n$ matrix. Let $\mathbf{x}^*$ be the steady state of the system given by $\mathbf{A}\mathbf{x}^* + \mathbf{b} = 0$. Suppose that $m \leq n$ of the eigenvalues of $\mathbf{A}$ have negative real parts. Then there exists an m-dimensional subspace M of $\mathbb{R}^n$ such that starting from any $\mathbf{x}(0) \in M$, the differential equation (7.65) has a unique solution with $\mathbf{x}(t) \to \mathbf{x}^*$.

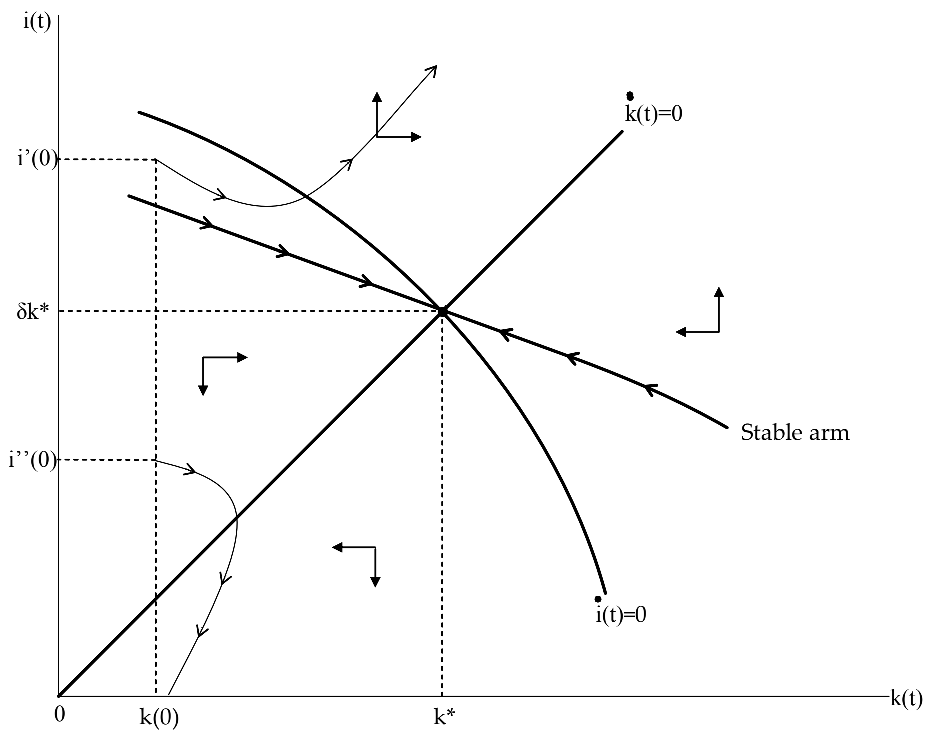

Figure 7.1. Dynamics of capital and investment in the q-theory.

Theorem 7.19. *Consider the following nonlinear autonomous differential equation*

$$\dot{\mathbf{x}}(t) = \mathbf{G}[\mathbf{x}(t)] \tag{7.66}$$

where $\mathbf{G} : \mathbb{R}^n \to \mathbb{R}^n$ *and suppose that* $\mathbf{G}$ *is continuously differentiable, with initial value* $\mathbf{x}(0)$*. Let* $\mathbf{x}^*$ *be a steady-state of this system, given by* $\mathbf{F}(\mathbf{x}^*) = 0$*. Define*

$$\mathbf{A} = D\mathbf{G}(\mathbf{x}^*),$$

and suppose that $m \leq n$ *of the eigenvalues of* $\mathbf{A}$ *have negative real parts and the rest have positive real parts. Then there exists an open neighborhood of* $\mathbf{x}^*$*,* $\mathbf{B}(\mathbf{x}^*) \subset \mathbb{R}^n$ *and an* m*-dimensional manifold* $M \subset \mathbf{B}(\mathbf{x}^*)$ *such that starting from any* $\mathbf{x}(0) \in M$*, the differential equation (7.66) has a unique solution with* $\mathbf{x}(t) \to \mathbf{x}^*$*.*

Put differently, these two theorems state that when only a subset of the eigenvalues have negative real parts, a lower-dimensional subset of the original space leads to stable solutions. Fortunately, in this context this is exactly what we require, since $i(0)$ should adjust in order to place us on exactly such a lower-dimensional subset of the original space.

Armed with these theorems, we can now investigate the transitional dynamics in the q-theory of investment. To see that the equilibrium will tend to this steady-state level of capital stock it suffices to plot (7.61) and (7.64) in the k-i space. This is done in Figure 7.1.

The curve corresponding to $\dot{k} = 0$, (7.61), is upward sloping, since a greater level of capital stock requires more investment to replenish the depreciated capital. When we are above this curve, there is more investment than necessary for replenishment, so that $\dot{k} > 0$. When we are below this curve, then $\dot{k} < 0$. On the other hand, the curve corresponding to $\dot{i} = 0$, (7.64), can be nonmonotonic. Nevertheless, it is straightforward to verify that in the neighborhood of the steady-state it is downward sloping (see Exercise 7.24). When we are to the right of this curve, $f'(k)$ is lower, thus $\dot{i} > 0$. When we are to its left, $\dot{i} < 0$. The resulting phase diagram, together with the one-dimensional stable manifold, is shown in Figure 7.1 (see again Exercise 7.24 for a different proof).

Starting with an arbitrary level of capital stock, $k(0) > 0$, the unique optimal solution involves an initial level of investment $i(0) > 0$, followed by a smooth and monotonic approach to the steady-state investment level of δk^*. In particular, it can be shown easily that when $k(0) < k^*$, $i(0) > i^*$ and it monotonically decreases towards i^* (see Exercise 7.24). This is intuitive. Adjustment costs discourage large values of investment, thus the firm cannot adjust its capital stock to its steady-state level immediately. However, because of diminishing returns, the benefit of increasing the capital stock is greater when the level of capital stock is low. Therefore, at the beginning the firm is willing to incur greater adjustment costs in order to increase its capital stock and $i(0)$ is high. As capital accumulates and $k(t) > k(0)$, the benefit of boosting the capital stock declines and the firm also reduces investment towards the steady-state investment level.

It is also informative to see why a level of initial investment other than $i(0)$ would violate either the transversality condition or the first-order necessary conditions. Consider, for example, $i'(0) > i(0)$ as the initial level. The phase diagram in Figure 7.1 makes it clear that starting from such a level of investment, $i(t)$ and $k(t)$ would tend to infinity. It can be verified that in this case $q(t)k(t)$ would tend to infinity at a rate faster than r, thus violating the transversality condition, $\lim_{t\to\infty} \exp(-rt)\, q(t)\, k(t) = 0$. To see this more explicitly, note that since along a trajectory starting at $i'(0)$, $\dot{k}(t)/k(t) > 0$, and thus we have

$$\begin{aligned} \frac{d(q(t)k(t))/dt}{q(t)k(t)} &\geq \frac{\dot{q}(t)}{q(t)} \\ &= \frac{\dot{i}(t)\phi''(i(t))}{1+\phi'(i(t))} \\ &= r + \delta - f'(k(t))/(1+\phi'(i(t))), \end{aligned}$$

where the second line uses (7.62) and (7.63), while the third line substitutes from (7.64). As $k(t) \to \infty$, we have that $f'(k(t)) \to 0$, implying that

$$\lim_{t\to\infty} \exp(-rt)\, q(t)\, k(t) \geq \lim_{t\to\infty} \exp(-rt) \exp((r+\delta)t) = \lim_{t\to\infty} \exp(\delta t) > 0,$$

violating the transversality condition. In contrast, if we start with $i''(0) < i(0)$ as the initial level, $i(t)$ would tend to 0 in finite time (as shown by the fact that the trajectories hit the horizontal axis) and $k(t)$ would also tend towards zero (though not reaching it in finite time). After the time where $i(t) = 0$, we also have $q(t) = 1$ and thus $\dot{q}(t) = 0$ (from (7.62)). Moreover, by the Inada conditions, as $k(t) \to 0$, $f'(k(t)) \to \infty$. Consequently, after $i(t)$ reaches 0, the necessary condition $\dot{q}(t) = (r+\delta) q(t) - f'(k(t))$ is necessarily violated. This proves that the unique optimal path involves investment starting at $i(0)$.

We next turn to the "q-theory" aspects. James Tobin argued that the value of an extra unit of capital to the firm divided by its replacement cost is a measure of the "value of investment to the firm". In particular, when this ratio is high, the firm would like to invest more. In steady state, the firm will settle where this ratio is 1 or close to 1. In our formulation, the costate variable $q(t)$ measures Tobin's q. To see this, let us denote the current (maximized) value of the firm when it starts with a capital stock of $k(t)$ by $V(k(t))$. The same arguments as above imply that

$$V'(k(t)) = q(t), \tag{7.67}$$

so that $q(t)$ measures exactly by how much one dollar increase in capital will raise the value of the firm.

In steady state, we have $\dot{q}(t) = 0$, so that $q^* = f'(k^*)/(r+\delta)$, which is approximately equal to 1 when $\phi'(\delta k^*)$ is small. Nevertheless, out of steady state, $q(t)$ can be significantly greater than this amount, signaling that there is need for greater investments. Therefore, in this model Tobin's q, or alternatively the costate variable $q(t)$, will play the role of signaling when investment demand is high.

The q-theory of investment is one of the workhorse models of macroeconomics and finance, since proxies for Tobin's q can be constructed using stock market prices and book values of firms. When stock market prices are greater than book values, this corresponds to periods in which the firm in question has a high Tobin's q—meaning that the value of installed capital is greater than its replacement cost, which appears on the books. Nevertheless, whether this is a good approach in practice is intensely debated, in part because Tobin's q does not contain all the relevant information when there are irreversibilities or fixed costs of investment, and also perhaps more importantly, what is relevant is the "marginal q," which corresponds to the marginal increase in value (as suggested by equation (7.67)), whereas we can typically only measure "average q". The discrepancy between these two concepts can be large.

7.9. Taking Stock

This chapter has reviewed the basic tools of dynamic optimization in continuous time. By its nature, this has been a technical (and unfortunately somewhat dry) chapter. The material covered here may have been less familiar than the discrete time optimization methods presented in the previous chapter. Part of the difficulty arises from the fact that optimization here is with respect to functions, even when the horizon is finite (rather than with respect to vectors or infinite sequences as in the discrete time case). This introduces a range of complications and some technical difficulties, which are not of great interest in the context of economic applications. As a result, this chapter has provided an overview of the main results, with an emphasis on those that are most useful in economic applications, together with some of the proofs. These proofs are included to provide the readers with a sense of where the results come from and to enable them to develop a better feel for their intuition.

While the basic ideas of optimal control may be a little less familiar than those of discrete time dynamic programming, these methods are used in much of growth theory and in other areas of macroeconomics. Moreover, while some problems naturally lend themselves to analysis in discrete time, other problems become easier in continuous time. Some argue that this is indeed the case for growth theory. Irrespective of whether one agrees with this assessment, it is important to have a good command of both discrete time and continuous time models in macroeconomics, since it should be the context and economic questions that dictate which type of model one should write down, not the force of habit. This motivated our choice of giving roughly equal weight to the two sets of techniques.

There is another reason for studying optimal control. The most powerful theorem in optimal control, Pontryagin's Maximum Principle, is as much an economic result as a mathematical result. As discussed above, the Maximum Principle has a very natural interpretation both in terms of maximizing flow returns plus the value of the stock, and also in terms of an asset value equation for the value of the maximization problem. These economic intuitions are not only useful in illustrating the essence of this mathematical technique, but they also provide a useful perspective on a large set of questions that involve the use of dynamic optimization techniques in macroeconomics, labor economics, finance and other fields.

Finally, to avoid having the current chapter just on techniques, we also introduced a number of economically substantive applications of optimal control. These include the intertemporal problem of a consumer, the problem of finding the optimal consumption path of a non-renewable resource and the q-theory of investment. We also used the q-theory of investment to illustrate how transitional dynamics can be analyzed in economic problems involving dynamic optimization (and corresponding boundary conditions at infinity). A detailed analysis of optimal and equilibrium growth is left for the next chapter.

This chapter also concludes our exposition of the "foundations" of growth theory, which comprised general equilibrium foundations of aggregative models as well as an introduction to mathematical tools necessary for dynamic economic analysis. We next turn to economically more substantive issues.

7.10. References and Literature

The main material covered in this chapter is the topic of many excellent applied mathematics and engineering books. The purpose here has been to provide a review of the results that are most relevant for economists, together with simplified versions of the most important proofs. The first part of the chapter is closer to the calculus of variations theory, because it makes use of variational arguments combined with continuity properties. Nevertheless, most economists do not need to study the calculus of variations in detail, both because it has been superseded by optimal control theory and also because most of the natural applications of the calculus of variations are in physics and other natural sciences. The interested reader can look at Gelfand and Fomin (2000). Chiang (1992) provides a readable and simple introduction to the calculus of variations with economic examples.

The theory of optimal control was originally developed by Pontryagin et al. (1962). For this reason, the main necessary condition is also referred to as the Pontryagin's (Maximum) Principle. The type of problem considered here (and in economics more generally) is referred to as the Lagrange problem in optimal control theory. The Maximum Principle is generally stated either for the somewhat simpler Meyer problem or the more general Bolza problem, though all of these problems are essentially equivalent, and when the problem is formulated in vector form, one can easily go back and forth between these different problems by simple transformations.

There are several books with varying levels of difficulty dealing with optimal control. Many of these books are not easy to read, but are also not entirely rigorous in their proofs. An excellent source that provides an advanced and complete treatment is Fleming and Rishel (1975). The first part of this book provides a complete (but rather different) proof of Pontryagin's Maximum Principle and various applications. It also provides a number of theorems on existence and continuity of optimal controls. A deeper understanding of sufficient conditions for existence of solution and the structure of necessary conditions can be gained from the excellent (but abstract and difficult) book by Luenberger (1969). The results in this book are general enough to cover both discrete time and continuous time dynamic optimization. This book also gives a very good sense of why maximization in function spaces is different from finite-dimensional maximization, and when such infinite-dimensional maximization problems may fail to have solutions.

For economists, books that develop the theory of optimal control with economic applications may be more appropriate. Here the best reference is Seierstad and Sydsaeter (1987). While not as rigorous as Fleming and Rishel (1975), this book also has a complete proof of the main results and is also easier and more interesting to read for economists. It also shows how the results can be applied to economic problems. Other references in economics are

Kamien and Schwartz (1991) and Leonard and Van Long (1992). Another classic is Arrow and Kurz's (1970) book, which covers the same material and also presents rich economic insight on growth theory and related problems. This book also states and provides a proof of Arrow's sufficiency theorem, which also appears in Arrow (1968).

Two recent books on applications of optimal control in economics, Caputo (2005) and Weitzman (2003), might be more readable. My treatment of the sufficiency results here is very similar to Caputo (2005). Weitzman (2003) provides a lively discussion of the applications of the Maximum Principle, especially in the context of environmental economics and depletion of natural resources.

There is some confusion in the literature over the role of the transversality condition. As commented in the previous chapter, in general there need not be a single transversality condition, since the transversality condition represents the necessary conditions obtained from specific types of variations. The example provided in Section 7.4 shows that the stronger transversality condition, which is very useful in many applications, does not always hold. This example is a variant of the famous example by Halkin (1974). The interested reader should look at Michel (1982), which contains the original result on the transversality condition of (7.45) for discounted infinite horizon optimal control problems and also a discussion of when the stronger condition (7.56) holds. The results presented here are closely related to Michel's (1982) results, but are stated under assumptions that are more relevant in economic situations. Michel assumes that the objective function is nonnegative, which is violated by many of the common utility functions used in economic growth models, and also imposes an additional technical assumption that is not easy to verify; instead the results here are stated under the assumption of weak monotonicity, which is satisfied in almost all economic applications.

The original economic interpretation of the Maximum Principle appeared in Dorfman (1969). The interpretation here builds on the discussion by Dorfman, but extends this based on the no-arbitrage interpretation of asset values in the Hamilton-Jacobi-Bellman equation. This interpretation of Hamilton-Jacobi-Bellman is well known in many areas of macroeconomics and labor economics, but is not often used to provide a general economic interpretation for the Maximum Principle. Weitzman (2003) also provides an economic interpretation for the Maximum Principle related to the Hamilton-Jacobi-Bellman equation.

The classic reference for exploitation of a non-renewable resource is Hotelling (1931). Weitzman (2003) provides a detailed treatment and a very insightful discussion. Dasgupta and Heal (1979) and Conrad (1999) are also useful references for applications of similar ideas to sustainability and environmental economics. Classic references on investment with costs of adjustment and the q-theory of investment include Eisner and Strotz (1963), Lucas (1967), Tobin (1969) and Hayashi (1982). Detailed treatments of the q-theory of investment can be found in any graduate-level economics textbook, for example, Blanchard and Fisher (1989) or Romer (1996), as well as in Dixit and Pindyck's (1994) book on investment under uncertainty and Caballero's (1999) survey. Caballero (1999) also includes a critique of the q-theory.

7.11. Exercises

Exercise 7.1. Consider the problem of maximizing (7.1) subject to (7.2) and (7.3) as in Section 7.1. Suppose that for the pair $(\hat{x}(t), \hat{y}(t))$ there exists a time interval (t', t'') with $t' < t''$ such that

$$\dot{\lambda}(t) \neq -\left[f_x(t, \hat{x}(t), \hat{y}(t)) + \lambda(t) g_x(t, \hat{x}(t), \hat{y}(t))\right] \text{ for all } t \in \left(t', t''\right).$$

Prove that the pair $(\hat{x}(t), \hat{y}(t))$ could not attain the optimal value of (7.1).

Exercise 7.2. * Prove that, given in optimal solution $\hat{x}(t), \hat{y}(t)$ to (7.1), the maximized Hamiltonian defined in (7.16) and evaluated at $\hat{x}(t)$, $M(t, \hat{x}(t), \lambda(t))$, is differentiable in x and satisfies $\dot{\lambda}(t) = -M_x(t, \hat{x}(t), \lambda(t))$ for all $t \in [0, t_1]$.

Exercise 7.3. The key equation of the calculus of variations is the Euler-Legrange equation, which characterizes the solution to the following problem (under similar regularity conditions to those of Theorem 7.2):

$$\max_{x(t)} \int_0^{t_1} F(t, x(t), \dot{x}(t))\, dx$$

subject to $x(t) = 0$. Suppose that F is differentiable in all of its arguments and an interior continuously differentiable solution exists. The so-called Euler-Legrange equation, which provides the necessary conditions for an optimal solution, is

$$\frac{\partial F(t, x(t), \dot{x}(t))}{\partial x(t)} - \frac{\partial^2 F(t, x(t), \dot{x}(t))}{\partial \dot{x}(t)\, \partial t} = 0.$$

Derive this equation from Theorem 7.2. [Hint: define $y(t) \equiv \dot{x}(t)$].

Exercise 7.4. This exercise asks you to use the Euler-Legrange equation derived in Exercise 7.3 to solve the canonical problem that motivated Euler and Legrange, that of finding the shortest distance between two points in a plane. In particular, consider a two dimensional plane and two points on this plane with coordinates (z_0, u_0) and (z_1, u_1). We would like to find the curve that has the shortest length that connects these two points. Such a curve can be represented by a function $x : \mathbb{R} \to \mathbb{R}$ such that $u = x(z)$, together with initial and terminal conditions $u_0 = x(z_0)$ and $u_1 = x(z_1)$. It is also natural to impose that this curve $u = x(z)$ be smooth, which corresponds to requiring that the solution be continuously differentiable so that $x'(z)$ exists.

To solve this problem, observe that the (arc) length along the curve x can be represented as

$$A[x(z)] \equiv \int_{z_1}^{z_2} \sqrt{1 + [x'(z)]^2}\, dz.$$

The problem is to minimize this object by choosing $x(z)$.

Now, without loss of any generality let us take $(z_0, u_0) = (0, 0)$ and let $t = z$ to transform the problem into a more familiar form, which becomes that of maximizing

$$-\int_0^{t_1} \sqrt{1 + [x'(t)]^2} dt.$$

Prove that the solution to this problem requires

$$\frac{d\left[x'(t)\left(1 + (x'(t))^2\right)\right]}{dt} = 0.$$

Show that this is only possible if $x''(t) = 0$, so that the shortest path between two points is a straight-line.

Exercise 7.5. Prove Theorem 7.2, in particular, paying attention to constructing feasible variations that ensure $x(t_1, \varepsilon) = x_1$ for all ε in some neighborhood of 0. What happens if there are no such feasible variations?

Exercise 7.6.

(1) Provide an expression for the initial level of consumption $c(0)$ as a function of $a(0)$, w, r and β in Example 7.1.
(2) What is the effect of an increase in $a(0)$ on the initial level of consumption $c(0)$? What is the effect on the consumption path?
(3) How would the consumption path change if instead of a constant level of labor earnings, w, the individual faced a time-varying labor income profile given by $[w(t)]_{t=0}^{1}$? Explain the reasoning for the answer in detail.

Exercise 7.7. Prove Theorem 7.4.

Exercise 7.8. * Prove a version of Theorem 7.5 corresponding to Theorem 7.2. [Hint: instead of $\lambda(t_1) = 0$, the proof should exploit the fact that $x(1) = \hat{x}(1) = x_1$].

Exercise 7.9. * Prove that in the finite-horizon problem of maximizing (7.1) or (7.11) subject to (7.2) and (7.3), $f_x(t, \hat{x}(t), \hat{y}(t), \lambda(t)) > 0$ for all $t \in [0, t_1]$ implies that $\lambda(t) > 0$ for all $t \in [0, t_1]$.

Exercise 7.10. * Prove Theorem 7.6.

Exercise 7.11. Prove Theorem 7.11.

Exercise 7.12. Provide a proof of Theorem 7.15.

Exercise 7.13. Prove that in the discounted infinite-horizon optimal control problem considered in Theorem 7.14 conditions (7.52)-(7.54) are necessary.

Exercise 7.14. Consider a finite horizon continuous time maximization problem, where the objective function is

$$W(x(t), y(t)) = \int_0^{t_1} f(t, x(t), y(t)) \, dt$$

with $x(0) = x_0$ and $t_1 < \infty$, and the constraint equation is

$$\dot{x}(t) = g(t, x(t), y(t)).$$

Imagine that t_1 is also a choice variable.

(1) Show that $W(x(t), y(t))$ can be written as

$$W(x(t), y(t)) = \int_0^{t_1} \left[H(t, x(t), y(t)) + \dot{\lambda}(t) x(t) \right] dt - \lambda(t_1) x(t_1) + \lambda(0) x_0,$$

where $H(t, x, y) \equiv f(t, x(t), y(t)) + \lambda(t) g(t, x(t), y(t))$ is the Hamiltonian and $\lambda(t)$ is the costate variable.

(2) Now suppose that the pair $(\hat{x}(t), \hat{y}(t))$ together with terminal date $\hat{t}_1$ constitutes an optimal solution for this problem. Consider the following class of variations:

$$\begin{aligned} y(t, \varepsilon) &= \hat{y}(t) + \varepsilon \eta(t) \text{ for } t \in \left[0, \hat{t}_1\right] \text{ and } y(t, \varepsilon) = \hat{y}\left(\hat{t}_1\right) + \varepsilon \eta(t) \text{ for } t \in \left[\hat{t}_1, \hat{t}_1 + \varepsilon \Delta t\right], \\ t_1 &= \hat{t}_1 + \varepsilon \Delta t. \end{aligned}$$

Denote the corresponding path of the state variable by $x(t, \varepsilon)$. Evaluate $W(x(t, \varepsilon), y(t, \varepsilon))$ at this variation. Explain why this variation is feasible for ε small enough.

(3) Show that for a feasible variation,

$$\begin{aligned} \left. \frac{dW(x(t, \varepsilon), y(t, \varepsilon))}{d\varepsilon} \right|_{\varepsilon=0} &= \int_0^{\hat{t}_1} \left[H_x(t, \hat{x}(t), \hat{y}(t)) + \dot{\lambda}(t) \right] \frac{\partial x(t, \varepsilon)}{\partial \varepsilon} dt \\ &\quad + \int_0^{\hat{t}_1} H_y(t, \hat{x}(t), \hat{y}(t)) \eta(t) dt \\ &\quad + H\left(\hat{t}_1, \hat{x}\left(\hat{t}_1\right), \hat{y}\left(\hat{t}_1\right)\right) \Delta t - \lambda\left(\hat{t}_1\right) \frac{\partial x\left(\hat{t}_1, \varepsilon\right)}{\partial \varepsilon}. \end{aligned}$$

(4) Explain why the previous expression has to be equal to 0.

(5) Now taking the limit as $\hat{t}_1 \to \infty$, derive the weaker form of the transversality condition (7.45).

(6) What are the advantages and disadvantages of this method of derivation relative to that used in the proof of Theorem 7.13.

Exercise 7.15. Consider the discounted infinite-horizon problem, with $f(t, x(t), y(t)) = \exp(-\rho t) f(x(t), y(t))$, and $g(t, x(t), y(t)) = g(x(t), y(t))$. Prove that if an admissible pair $(\hat{x}(t), \hat{y}(t))_{t \geq 0}$ is optimal starting at $t = 0$ with initial condition $x(0) = x_0$, then $(\hat{x}(t), \hat{y}(t))_{t \geq s}$ is also admissible and optimal for the problem starting at $t = s$ with initial condition $x(s) = x_0$.

Exercise 7.16. This exercise contains an alternative derivation of the stationary form of HJB equation, (7.41). Consider the discounted infinite-horizon problem, with $f(t, x(t), y(t)) = \exp(-\rho t) f(x(t), y(t))$, and $g(t, x(t), y(t)) = g(x(t), y(t))$. Suppose that the admissible pair $(\hat{x}(t), \hat{y}(t))$ is optimal starting at $t = 0$ with initial condition $x(0) = x_0$. Let

$$V(x(0)) = \int_0^{\infty} \exp(-\rho t) f((\hat{x}(t), \hat{y}(t))) dt,$$

which is well defined in view of Exercise 7.15. Now write

$$V(x(0)) = f((\hat{x}(t), \hat{y}(t)))\Delta t + o(\Delta t) + \int_{\Delta t}^{\infty} \exp(-\rho t) f((\hat{x}(t), \hat{y}(t)))\, dt,$$

where $o(\Delta t)$ denotes second-order terms that satisfy $\lim_{\Delta t \to 0} o(\Delta t)/\Delta t = 0$. Explain why this equation can be written as

$$V(x(0)) = f((\hat{x}(t), \hat{y}(t)))\Delta t + o(\Delta t) + \exp(-\rho \Delta t) V(x(\Delta t))$$

[Hint: again use Exercise 7.15]. Now subtract $V(x(\Delta t))$ from both sides and divide both sides by Δt to obtain

$$\frac{V(x(0)) - V(x(\Delta t))}{\Delta t} = f((\hat{x}(t), \hat{y}(t))) + \frac{o(\Delta t)}{\Delta t} + \frac{\exp(-\rho \Delta t) - 1}{\Delta t} V(x(\Delta t)).$$

Show that taking the limit as $\Delta t \to 0$ gives (7.41).

EXERCISE 7.17. * Consider the following maximization problem:

$$\max_{x(t), y(t)} \int_0^1 f(x(t), y(t))\, dt$$

subject to

$$\dot{x}(t) = y(t)^2$$

$x(0) = 0$ and $x(1) = 1$, where $y(t) \in \mathbb{R}$ and f is an arbitrary continuously differentiable function. Show that the unique solution to this maximization problem does not satisfy the necessary conditions in Theorem 7.2. Explain why this is.

EXERCISE 7.18. * Consider the following maximization problem:

$$\max_{x(t), y(t)} - \int_0^1 x(t)^2\, dt$$

subject to

$$\dot{x}(t) = y(t)^2$$

$x(0) = 0$ and $x(1) = 1$, where $y(t) \in \mathbb{R}$. Show that there does not exist a continuously differentiable solution to this problem.

EXERCISE 7.19. Consider the following discounted infinite-horizon maximization problem

$$\max \int_0^{\infty} \exp(-\rho t) \left[2y(t)^{1/2} + \frac{1}{2} x(t)^2 \right] dt$$

subject to

$$\dot{x}(t) = -\rho x(t) y(t)$$

and $x(0) = 1$.

(1) Show that this problem satisfies all the assumptions of Theorem 7.14.
(2) Set up at the current-value Hamiltonian and derive the necessary conditions, with the costate variable $\mu(t)$.
(3) Show that the following is an optimal solution $y(t) = 1$, $x(t) = \exp(-\rho t)$, and $\mu(t) = \exp(\rho t)$ for all t.

(4) Show that this optimal solution violates the condition that $\lim_{t\to\infty} \exp(-\rho t)\,\mu(t)$, but satisfies (7.56).

EXERCISE 7.20. Consider the following optimal growth model without discounting:

$$\max \int_0^\infty \left[u\left(c\left(t\right)\right) - u\left(c^*\right)\right] dt$$

subject to

$$\dot{k}(t) = f(k(t)) - c(t) - \delta k(t)$$

with initial condition $k(0) > 0$, and c^* defined as the golden rule consumption level

$$c^* = f(k^*) - \delta k^*$$

where k^* is the golden rule capital-labor ratio given by $f'(k^*) = \delta$.

(1) Set up the Hamiltonian for this problem with costate variable $\lambda(t)$.
(2) Characterize the solution to this optimal growth program.
(3) Show that the standard transversality condition that $\lim_{t\to\infty} \lambda(t)\,k(t) = 0$ is not satisfied at the optimal solution. Explain why this is the case.

EXERCISE 7.21. Consider the infinite-horizon optimal control problem given by the maximization of (7.28) subject to (7.29) and (7.30). Suppose that the problem has a quasi-stationary structure, so that

$$\begin{aligned} f(t,x,y) &\equiv \beta(t)\,f(x,y) \\ g(t,x,y) &\equiv g(x,y), \end{aligned}$$

where $\beta(t)$ is the discount factor that applies to returns that are an interval of time t away from the present.

(1) Set up Hamiltonian and characterize the necessary conditions for this problem.
(2) Prove that the solution to this problem is time consistent (meaning that the solution chosen at some date s cannot be improved upon at some future date s' by changing the continuation plans after this date) if and only if $\beta(t) = \exp(-\rho t)$ for some $\rho \geq 0$.
(3) Interpret this result and explain in what way the conclusion is different from that of Lemma 7.1.

EXERCISE 7.22. Consider the problem of consuming a non-renewable resource in Example 7.3. Show that the solution outlined their satisfies the stronger transversality condition (7.56).

EXERCISE 7.23. Consider the following continuous time discounted infinite horizon problem:

$$\max \int_0^\infty \exp(-\rho t)\,u(c(t))\,dt$$

subject to

$$\dot{x}(t) = g(x(t)) - c(t)$$

with initial condition $x(0) > 0$.

Suppose that $u(\cdot)$ is strictly increasing and strictly concave, with $\lim_{c\to\infty} u'(c) = 0$ and $\lim_{c\to 0} u'(c) = \infty$, and $g(\cdot)$ is increasing and strictly concave with $\lim_{x\to\infty} g'(x) = 0$ and $\lim_{x\to 0} g'(x) = \infty$.

(1) Set up the current value Hamiltonian and derive the Euler equations for an optimal path.
(2) Show that the standard transversality condition and the Euler equations are necessary and sufficient for a solution.
(3) Characterize the optimal path of solutions and their limiting behavior.

Exercise 7.24. (1) In the q-theory of investment, prove that when $\phi''(i) = 0$ (for all i), investment jumps so that the capital stock reaches its steady-state value k^* immediately.
(2) Prove that as shown in Figure 7.1, the curve for (7.64) is downward sloping in the neighborhood of the steady state.
(3) As an alternative to the diagrammatic analysis of Figure 7.1, linearize (7.61) and (7.64), and show that in the neighborhood of the steady state this system has one positive and one negative eigenvalue. Explain why this implies that optimal investment plans will tend towards the stationary solution (steady state).
(4) Prove that when $k(0) < k^*$, $i(0) > i^*$ and $i(t) \downarrow i^*$.
(5) Derive the equations for the q-theory of investment when the adjustment cost takes the form $\phi(i/k)$. How does this affect the steady-state marginal product of capital?
(6) Derive the optimal equation path when investment is irreversible, in the sense that we have the additional constraint $\dot{i} \geq 0$.

Part 3

Neoclassical Growth

This part of the book covers the basic workhorse models of the theory of economic growth. We start with the infinite-horizon neoclassical growth model, which we have already encountered in the previous chapters. A closely related model is the baseline overlapping-generations model of Samuelson and Diamond, and this is the topic of Chapter 9. Despite the similarities between the two models, they have quite different normative and positive implications, and each model may be appropriate for different sets of issues. It is therefore important to have a detailed discussion of both.

This part of the book also presents the basic economic growth model with human capital investments. The inclusion of this model is motivated both because of the increasingly important role of human capital in economic growth and macroeconomics, and also as a way of linking macroeconomic approaches to microdata on schooling and returns to schooling.

Finally, Chapter 11 introduces the first models of sustained economic growth. These are contained in this part of the book rather than the next, because they are models of sustained growth *without* technological change. Despite their simplicity, these models lead to a number of important economic insights and provide a good introduction to the set of issues we will encounter in the next part of the book.

CHAPTER 8

The Neoclassical Growth Model

We are now ready to start our analysis of the standard neoclassical growth model (also known as the Ramsey or Cass-Koopmans model). This model differs from the Solow model only in one crucial respect: it explicitly models the consumer side and endogenizes savings. In other words, it allows consumer optimization. Beyond its use as a basic growth model, this model has become a workhorse for many areas of macroeconomics, including the analysis of fiscal policy, taxation, business cycles, and even monetary policy.

Since both the basic equilibrium and optimal growth models in discrete time were already presented as applications of dynamic programming in Chapter 6, this chapter focuses on the continuous time neoclassical growth model (returning to discrete time examples in exercises).

8.1. Preferences, Technology and Demographics

Consider an infinite-horizon economy in continuous time. We assume that the economy admits a representative household with instantaneous utility function

$$u\left(c\left(t\right)\right), \tag{8.1}$$

and we make the following standard assumptions on this utility function:

Assumption 3. $u(c)$ *is strictly increasing, concave, twice continuously differentiable with derivatives* u' *and* u''*, and satisfies the following Inada type assumptions:*

$$\lim_{c\to 0} u'(c) = \infty \text{ and } \lim_{c\to\infty} u'(c) = 0.$$

More explicitly, let us suppose that this representative household represents a set of identical households (with measure normalized to 1). Each household has an instantaneous utility function given by (8.1). Population within each household grows at the rate n, starting with $L(0) = 1$, so that total population is

$$L(t) = \exp(nt). \tag{8.2}$$

All members of the household supply their labor inelastically.

Our baseline assumption is that the household is fully altruistic towards all of its future members, and always makes the allocations of consumption (among household members) cooperatively. This implies that the objective function of each household at time $t = 0$, $U(0)$, can be written as

$$U(0) \equiv \int_0^\infty \exp(-(\rho - n)t)\, u(c(t))\, dt, \tag{8.3}$$

where $c(t)$ is consumption per capita at time t, ρ is the subjective discount rate, and the effective discount rate is $\rho - n$, since it is assumed that the household derives utility from the consumption per capita of its additional members in the future as well (see Exercise 8.1).

It is useful to be a little more explicit about where the objective function (8.3) is coming from. First, given the strict concavity of $u(\cdot)$ and the assumption that within-household allocation decisions are cooperative, each household member will have an equal consumption (Exercise 8.1). This implies that each member will consume

$$c(t) \equiv \frac{C(t)}{L(t)}$$

at date t, where $C(t)$ is total consumption and $L(t)$ is the size of the representative household (equal to total population, since the measure of households is normalized to 1). This implies that the household will receive a utility of $u(c(t))$ per household member at time t, or a total utility of $L(t)\, u(c(t)) = \exp(nt)\, u(c(t))$. Since utility at time t is discounted back to time 0 with a discount rate of $\exp(-\rho t)$, we obtain the expression in (8.3).

We also assume throughout that

Assumption 4′.

$$\rho > n.$$

This assumption ensures that there is in fact discounting of future utility streams. Otherwise, (8.3) would have infinite value, and standard optimization techniques would not be useful in characterizing optimal plans. Assumption 4′ makes sure that in the model without growth, discounted utility is finite. When there is growth, we will strengthen this assumption and introduce Assumption 4.

We start with an economy without any technological progress. Factor and product markets are competitive, and the production possibilities set of the economy is represented by the aggregate production function

$$Y(t) = F[K(t), L(t)],$$

which is a simplified version of the production function (2.1) used in the Solow growth model in Chapter 2. In particular, there is now no technology term (labor-augmenting technological change will be introduced below). As in the Solow model, we impose the standard constant returns to scale and Inada assumptions embedded in Assumptions 1 and 2. The constant returns to scale feature enables us to work with the per capita production function $f(\cdot)$ such that, output per capita is given by

$$\begin{aligned} y(t) &\equiv \frac{Y(t)}{L(t)} \\ &= F\left[\frac{K(t)}{L(t)}, 1\right] \\ &\equiv f(k(t)), \end{aligned}$$

where, as before,

$$k(t) \equiv \frac{K(t)}{L(t)}. \tag{8.4}$$

Competitive factor markets then imply that, at all points in time, the rental rate of capital and the wage rate are given by:

$$R(t) = F_K[K(t), L(t)] = f'(k(t)). \tag{8.5}$$

and

$$w(t) = F_L[K(t), L(t)] = f(k(t)) - k(t) f'(k(t)). \tag{8.6}$$

The household optimization side is more complicated, since each household will solve a continuous time optimization problem in deciding how to use their assets and allocate consumption over time. To prepare for this, let us denote the asset holdings of the representative household at time t by $\mathcal{A}(t)$. Then we have the following law of motion for the total assets of the household

$$\dot{\mathcal{A}}(t) = r(t) \mathcal{A}(t) + w(t) L(t) - c(t) L(t)$$

where $c(t)$ is consumption per capita of the household, $r(t)$ is the risk-free market flow rate of return on assets, and $w(t) L(t)$ is the flow of labor income earnings of the household. Defining per capita assets as

$$a(t) \equiv \frac{\mathcal{A}(t)}{L(t)},$$

we obtain:

$$\dot{a}(t) = (r(t) - n) a(t) + w(t) - c(t). \tag{8.7}$$

In practice, household assets can consist of capital stock, $K(t)$, which they rent to firms and government bonds, $B(t)$. In models with uncertainty, households would have a portfolio choice between the capital stock of the corporate sector and riskless bonds. Government bonds play an important role in models with incomplete markets, allowing households to smooth idiosyncratic shocks. But in representative household models without government, their only use is in pricing assets (for example riskless bonds versus equity), since they have to be in zero net supply, i.e., total supply of bonds has to be $B(t) = 0$. Consequently, assets per capita will be equal to the capital stock per capita (or the capital-labor ratio in the economy), that is,

$$a(t) = k(t).$$

Moreover, since there is no uncertainty here and a depreciation rate of δ, the market rate of return on assets will be given by

$$r(t) = R(t) - \delta. \tag{8.8}$$

The equation (8.7) is only a flow constraint. As already noted above, it is not sufficient as a proper budget constraint on the individual (unless we impose a lower bound on assets, such as $a(t) \geq 0$ for all t). To see this, let us write the single budget constraint of the form:

$$\begin{aligned} &\int_0^T c(t)\, L(t) \exp\left(\int_t^T r(s)\, ds\right) dt + \mathcal{A}(T) \\ = &\int_0^T w(t)\, L(t) \exp\left(\int_t^T r(s)\, ds\right) dt + \mathcal{A}(0) \exp\left(\int_0^T r(s)\, ds\right), \end{aligned} \tag{8.9}$$

for some arbitrary $T > 0$. This constraint states that the household's asset position at time T is given by his total income plus initial assets minus expenditures, all carried forward to date T units. Differentiating this expression with respect to T and dividing $L(t)$ gives (8.7) (see Exercise 8.2).

Now imagine that (8.9) applies to a finite-horizon economy ending at date T. In this case, it becomes clear that the flow budget constraint (8.7) by itself does not guarantee that $\mathcal{A}(T) \geq 0$. Therefore, in the finite-horizon, we would simply impose this lifetime budget constraint as a boundary condition.

In the infinite-horizon case, we need a similar boundary condition. This is generally referred to as the no-Ponzi-game condition, and takes the form

$$\lim_{t\to\infty} a(t) \exp\left(-\int_0^t (r(s) - n)\, ds\right) \geq 0. \tag{8.10}$$

This condition is stated as an inequality, to ensure that the individual does not asymptotically tend to a negative wealth. Exercise 8.3 shows why this no-Ponzi-game condition is necessary. Furthermore, the transversality condition ensures that the individual would never want to have positive wealth asymptotically, so the no-Ponzi-game condition can be alternatively stated as:

$$\lim_{t\to\infty} a(t) \exp\left(-\int_0^t (r(s) - n)\, ds\right) = 0. \tag{8.11}$$

In what follows we will use (8.10), and then derive (8.11) using the transversality condition explicitly.

The name no-Ponzi-game condition comes from the chain-letter or pyramid schemes, which are sometimes called Ponzi games, where an individual can continuously borrow from a competitive financial market (or more often, from unsuspecting souls that become part of the chain-letter scheme) and pay his or her previous debts using current borrowings. The consequence of this scheme would be that the asset holding of the individual would tend to $-\infty$ as time goes by, clearly violating feasibility at the economy level.

To understand where this form of the no-Ponzi-game condition comes from, multiply both sides of (8.9) by $\exp\left(-\int_0^T r(s)\,ds\right)$ to obtain

$$\begin{aligned}
&\int_0^T c(t)\,L(t)\exp\left(-\int_0^t r(s)\,ds\right)dt+\exp\left(-\int_0^T r(s)\,ds\right)\mathcal{A}(T) \\
= &\int_0^T w(t)\,L(t)\exp\left(-\int_0^t r(s)\,ds\right)dt+\mathcal{A}(0),
\end{aligned}$$

then divide everything by $L(0)$ and note that $L(t)$ grows at the rate n, to obtain

$$\begin{aligned}
&\int_0^T c(t)\exp\left(-\int_0^t (r(s)-n)\,ds\right)dt+\exp\left(-\int_0^T (r(s)-n)\,ds\right)a(T) \\
= &\int_0^T w(t)\exp\left(-\int_0^t (r(s)-n)\,ds\right)dt+a(0).
\end{aligned}$$

Now take the limit as $T\to\infty$ and use the no-Ponzi-game condition (8.11) to obtain

$$\int_0^\infty c(t)\exp\left(-\int_0^t (r(s)-n)\,ds\right)dt=a(0)+\int_0^\infty w(t)\exp\left(-\int_0^t (r(s)-n)\,ds\right)dt,$$

which requires the discounted sum of expenditures to be equal to initial income plus the discounted sum of labor income. Therefore this equation is a direct extension of (8.9) to infinite horizon. This derivation makes it clear that the no-Ponzi-game condition (8.11) essentially ensures that the individual's lifetime budget constraint holds in infinite horizon.

8.2. Characterization of Equilibrium

8.2.1. Definition of Equilibrium. We are now in a position to define an equilibrium in this dynamic economy. We will provide two definitions, the first is somewhat more formal, while the second definition will be more useful in characterizing the equilibrium below.

Definition 8.1. *A competitive equilibrium of the Ramsey economy consists of paths of consumption, capital stock, wage rates and rental rates of capital,* $[C(t),K(t),w(t),R(t)]_{t=0}^\infty$, *such that the representative household maximizes its utility given initial capital stock* $K(0)$ *and the time path of prices* $[w(t),R(t)]_{t=0}^\infty$, *and all markets clear.*

Notice that in equilibrium we need to determine the entire time path of real quantities and the associated prices. This is an important point to bear in mind. In dynamic models whenever we talk of "equilibrium", this refers to the entire path of quantities and prices. In some models, we will focus on the steady-state equilibrium, but equilibrium always refers to the entire path.

Since everything can be equivalently defined in terms of per capita variables, we can state an alternative and more convenient definition of equilibrium:

Definition 8.2. *A competitive equilibrium of the Ramsey economy consists of paths of per capita consumption, capital-labor ratio, wage rates and rental rates of capital, $[c(t), k(t), w(t), R(t)]_{t=0}^{\infty}$, such that the representative household maximizes (8.3) subject to (8.7) and (8.10) given initial capital-labor ratio $k(0)$, factor prices $[w(t), R(t)]_{t=0}^{\infty}$ as in (8.5) and (8.6), and the rate of return on assets $r(t)$ given by (8.8).*

8.2.2. Household Maximization. Let us start with the problem of the representative household. From the definition of equilibrium we know that this is to maximize (8.3) subject to (8.7) and (8.11). Let us first ignore (8.11) and set up the current-value Hamiltonian:

$$\hat{H}(a, c, \mu) = u(c(t)) + \mu(t)[w(t) + (r(t) - n)a(t) - c(t)],$$

with state variable a, control variable c and current-value costate variable μ. This problem is closely related to the intertemporal utility maximization examples studied in the previous two chapters, with the main difference being that the rate of return on assets is also time varying. It can be verified that this problem satisfies all the assumptions of Theorem 7.14, including weak monotonicity.

Thus applying Theorem 7.14, we obtain the following necessary conditions:

$$\begin{aligned} \hat{H}_c(a, c, \mu) &= u'(c(t)) - \mu(t) = 0 \\ \hat{H}_a(a, c, \mu) &= \mu(t)(r(t) - n) = -\dot{\mu}(t) + (\rho - n)\mu(t) \\ \lim_{t\to\infty}[\exp(-(\rho - n)t)\mu(t)a(t)] &= 0. \end{aligned}$$

and the transition equation (8.7).

Notice that the transversality condition is written in terms of the current-value costate variable, which is more convenient given the rest of the necessary conditions.

Moreover, as discussed in the previous chapter, for any $\mu(t) > 0$, $\hat{H}(a, c, \mu)$ is a concave function of (a, c). The first necessary condition (and equation (8.13) below), in turn, imply that $\mu(t) > 0$ for all t. Therefore, Theorem 7.15 implies that these conditions are sufficient for a solution.

We can next rearrange the second condition to obtain:

(8.12) $$\frac{\dot{\mu}(t)}{\mu(t)} = -(r(t) - \rho),$$

which states that the multiplier changes depending on whether the rate of return on assets is currently greater than or less than the discount rate of the household.

Next, the first necessary condition above implies that

(8.13) $$u'(c(t)) = \mu(t).$$

To make more progress, let us differentiate this with respect to time and divide by $\mu(t)$, which yields

$$\frac{u''(c(t))\,c(t)}{u'(c(t))}\frac{\dot{c}(t)}{c(t)} = \frac{\dot{\mu}(t)}{\mu(t)}.$$

Substituting this into (8.12), we obtain another form of the famous consumer Euler equation:

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))}(r(t) - \rho) \tag{8.14}$$

where

$$\varepsilon_u(c(t)) \equiv -\frac{u''(c(t))\,c(t)}{u'(c(t))} \tag{8.15}$$

is the elasticity of the marginal utility $u'(c(t))$. This equation is closely related to the consumer Euler equation we derived in the context of the discrete time problem, equation (6.30), as well as to the consumer Euler equation in continuous time with constant interest rates in Example 7.1 in the previous chapter. As with equation (6.30), it states that consumption will grow over time when the discount rate is less than the rate of return on assets. It also specifies the speed at which consumption will grow in response to a gap between this rate of return and the discount rate, which is related to the elasticity of marginal utility of consumption, $\varepsilon_u(c(t))$.

Notice that $\varepsilon_u(c(t))$ is not only the elasticity of marginal utility, but even more importantly, it is the inverse of the *intertemporal elasticity of substitution*, which plays a crucial role in most macro models. The intertemporal elasticity of substitution regulates the willingness of individuals to substitute consumption (or labor or any other attribute that yields utility) over time. The elasticity between the dates t and $s > t$ is defined as

$$\sigma_u(t,s) = -\frac{d\log(c(s)/c(t))}{d\log(u'(c(s))/u'(c(t)))}.$$

As $s \downarrow t$, we have

$$\sigma_u(t,s) \to \sigma_u(t) = -\frac{u'(c(t))}{u''(c(t))\,c(t)} = \frac{1}{\varepsilon_u(c(t))}. \tag{8.16}$$

This is not surprising, since the concavity of the utility function $u(\cdot)$—or equivalently, the elasticity of marginal utility—determines how willing individuals are to substitute consumption over time.

Next, integrating (8.12), we have

$$\begin{aligned}\mu(t) &= \mu(0)\exp\left(-\int_0^t (r(s) - \rho)\,ds\right)\\ &= u'(c(0))\exp\left(-\int_0^t (r(s) - \rho)\,ds\right),\end{aligned}$$

where the second line uses the first optimality condition of the current-value Hamiltonian at time $t = 0$. Now substituting into the transversality condition, we have

$$\begin{aligned} \lim_{t\to\infty}\left[\exp\left(-\left(\rho-n\right)t\right)a\left(t\right)u'\left(c\left(0\right)\right)\exp\left(-\int_0^t\left(r\left(s\right)-\rho\right)ds\right)\right] &= 0, \\ \lim_{t\to\infty}\left[a\left(t\right)\exp\left(-\int_0^t\left(r\left(s\right)-n\right)ds\right)\right] &= 0, \end{aligned}$$

which implies that the strict no-Ponzi condition, (8.11) has to hold. Also, for future reference, notes that, since $a(t) = k(t)$, the transversality condition is also equivalent to

$$\lim_{t\to\infty}\left[\exp\left(-\int_0^t\left(r\left(s\right)-n\right)ds\right)k\left(t\right)\right]=0,$$

which requires that the discounted market value of the capital stock in the very far future is equal to 0. This "market value" version of the transversality condition is sometimes more convenient to work with.

We can derive further results on the consumption behavior of households. In particular, notice that the term $\exp\left(-\int_0^t r\left(s\right)ds\right)$ is a present-value factor that converts a unit of income at time t to a unit of income at time 0. In the special case where $r(s) = r$, this factor would be exactly equal to $\exp(-rt)$. But more generally, we can define an average interest rate between dates 0 and t as

$$\bar{r}\left(t\right)=\frac{1}{t}\int_0^t r\left(s\right)ds. \tag{8.17}$$

In that case, we can express the conversion factor between dates 0 and t as

$$\exp\left(-\bar{r}\left(t\right)t\right),$$

and the transversality condition can be written as

$$\lim_{t\to\infty}\left[\exp\left(-\left(\bar{r}\left(t\right)-n\right)t\right)a\left(t\right)\right]=0. \tag{8.18}$$

Now recalling that the solution to the differential equation

$$\dot{y}\left(t\right)=b\left(t\right)y\left(t\right)$$

is

$$y\left(t\right)=y\left(0\right)\exp\left(\int_0^t b\left(s\right)ds\right),$$

we can integrate (8.14), to obtain

$$c\left(t\right)=c\left(0\right)\exp\left(\int_0^t\frac{r\left(s\right)-\rho}{\varepsilon_u\left(c\left(s\right)\right)}ds\right)$$

as the consumption function. Once we determine $c(0)$, the initial level of consumption, the path of consumption can be exactly solved out. In the special case where $\varepsilon_u(c(s))$ is constant, for example, $\varepsilon_u(c(s)) = \theta$, this equation simplifies to

$$c(t) = c(0)\exp\left(\left(\frac{\bar{r}(t) - \rho}{\theta}\right)t\right),$$

and moreover, the lifetime budget constraint simplifies to

$$\int_0^\infty c(t)\exp(-(\bar{r}(t) - n)t)\,dt = a(0) + \int_0^\infty w(t)\exp(-(\bar{r}(t) - n)t)\,dt,$$

and substituting for $c(t)$ into this lifetime budget constraint in this iso-elastic case, we obtain

(8.19)

$$c(0) = \int_0^\infty \exp\left(-\left(\frac{(1-\theta)\bar{r}(t)}{\theta} - \frac{\rho}{\theta} + n\right)t\right)dt\left[a(0) + \int_0^\infty w(t)\exp(-(\bar{r}(t) - n)t)\,dt\right]$$

as the initial value of consumption.

8.2.3. Equilibrium Prices. Equilibrium prices are straightforward and are given by (8.5) and (8.6). This implies that the market rate of return for consumers, $r(t)$, is given by (8.8), i.e.,

$$r(t) = f'(k(t)) - \delta.$$

Substituting this into the consumer's problem, we have

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))}\left(f'(k(t)) - \delta - \rho\right) \tag{8.20}$$

as the equilibrium version of the consumption growth equation, (8.14). Equation (8.19) similarly generalizes for the case of iso-elastic utility function.

8.3. Optimal Growth

Before characterizing the equilibrium further, it is useful to look at the optimal growth problem, defined as the capital and consumption path chosen by a benevolent social planner trying to achieve a Pareto optimal outcome. In particular, recall that in an economy that admits a representative household, the optimal growth problem simply involves the maximization of the utility of the representative household subject to technology and feasibility constraints. That is,

$$\max_{[k(t),c(t)]_{t=0}^\infty} \int_0^\infty \exp(-(\rho - n)t)\,u(c(t))\,dt,$$

subject to

$$\dot{k}(t) = f(k(t)) - (n+\delta)k(t) - c(t),$$

and $k(0) > 0$.[1] As noted in Chapter 5, versions of the First and Second Welfare Theorems for economies with a continuum of commodities would imply that the solution to this problem should be the same as the equilibrium growth problem of the previous section. However, we

[1]In the case where the infinite-horizon problem represents dynastic utilities as discussed in Chapter 5, this specification presumes that the social planner gives the same weight to different generations as does the current dynastic decision-maker.

do not need to appeal to these theorems since in this together case it is straightforward to show the equivalence of the two problems.

To do this, let us once again set up the current-value Hamiltonian, which in this case takes the form

$$\hat{H}(k, c, \mu) = u(c(t)) + \mu(t)[f(k(t)) - (n+\delta)k(t) - c(t)],$$

with state variable k, control variable c and current-value costate variable μ. As noted in the previous chapter, in the relevant range for the capital stock, this problem satisfies all the assumptions of Theorem 7.14. Consequently, the necessary conditions for an optimal path are:

$$\begin{aligned}
\hat{H}_c(k, c, \mu) &= 0 = u'(c(t)) - \mu(t), \\
\hat{H}_k(k, c, \mu) &= -\dot{\mu}(t) + (\rho - n)\mu(t) = \mu(t)\left(f'(k(t)) - \delta - n\right), \\
\lim_{t\to\infty}[\exp(-(\rho - n)t)\mu(t)k(t)] &= 0.
\end{aligned}$$

Repeatingthe same steps as before, it is straightforward to see that these optimality conditions imply

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))}\left(f'(k(t)) - \delta - \rho\right),$$

which is identical to (8.20), and the transversality condition

$$\lim_{t\to\infty}\left[k(t)\exp\left(-\int_0^t \left(f'(k(s)) - \delta - n\right) ds\right)\right] = 0,$$

which is, in turn, identical to (8.11).

This establishes that the competitive equilibrium is a Pareto optimum and that the Pareto allocation can be decentralized as a competitive equilibrium. This result is stated in the next proposition:

Proposition 8.1. *In the neoclassical growth model described above, with Assumptions 1, 2, 3 and 4′, the equilibrium is Pareto optimal and coincides with the optimal growth path maximizing the utility of the representative household.*

8.4. Steady-State Equilibrium

Now let us characterize the steady-state equilibrium and optimal allocations. A steady-state equilibrium is defined as in Chapter 2 as an equilibrium path in which capital-labor ratio, consumption and output are constant. Therefore,

$$\dot{c}(t) = 0.$$

From (8.20), this implies that as long as $f(k^*) > 0$, *irrespective* of the exact utility function, we must have a capital-labor ratio k^* such that

$$f'(k^*) = \rho + \delta, \tag{8.21}$$

which is the equivalent of the steady-state relationship in the discrete-time optimal growth model.[2] This equation pins down the steady-state capital-labor ratio only as a function of the production function, the discount rate and the depreciation rate. This corresponds to the *modified golden rule*, rather than the golden rule we saw in the Solow model (see Exercise 8.8). The modified golden rule involves a level of the capital stock that does not maximize steady-state consumption, because earlier consumption is preferred to later consumption. This is because of discounting, which means that the objective is not to maximize steady-state consumption, but involves giving a higher weight to earlier consumption.

Given k^*, the steady-state consumption level is straightforward to determine as:

$$c^* = f(k^*) - (n+\delta)k^*, \tag{8.22}$$

which is similar to the consumption level in the basic Solow model. Moreover, given Assumption 4′, a steady state where the capital-labor ratio and thus output are constant necessarily satisfies the transversality condition.

This analysis therefore establishes:

PROPOSITION 8.2. *In the neoclassical growth model described above, with Assumptions 1, 2, 3 and 4′, the steady-state equilibrium capital-labor ratio, k^*, is uniquely determined by (8.21) and is independent of the utility function. The steady-state consumption per capita, c^*, is given by (8.22).*

As with the basic Solow growth model, there are also a number of straightforward comparative static results that show how the steady-state values of capital-labor ratio and consumption per capita change with the underlying parameters. For this reason, let us again parameterize the production function as follows

$$f(k) = a\tilde{f}(k),$$

where $a > 0$, so that a is again a shift parameter, with greater values corresponding to greater productivity of factors. Since $f(k)$ satisfies the regularity conditions imposed above, so does $\tilde{f}(k)$.

PROPOSITION 8.3. *Consider the neoclassical growth model described above, with Assumptions 1, 2, 3 and 4′, and suppose that $f(k) = a\tilde{f}(k)$. Denote the steady-state level of the capital-labor ratio by $k^*(a, \rho, n, \delta)$ and the steady-state level of consumption per capita by*

[2] In addition, if $f(0) = 0$, there exists another, economically uninteresting steady state at $k = 0$. As in Chapter 2, we ignore this steady state throughout. Moreover, we will see below that starting with any $k(0) > 0$, the economy will always tend to the steady-state capital-labor ratio k^* given by (8.21).

$c^*(a,\rho,n,\delta)$ *when the underlying parameters are* a, ρ, n *and* δ. *Then we have*

$$\frac{\partial k^*(a,\rho,n,\delta)}{\partial a} > 0,\ \frac{\partial k^*(a,\rho,n,\delta)}{\partial \rho} < 0,\ \frac{\partial k^*(a,\rho,n,\delta)}{\partial n} = 0 \text{ and } \frac{\partial k^*(a,\rho,n,\delta)}{\partial \delta} < 0$$
$$\frac{\partial c^*(a,\rho,n,\delta)}{\partial a} > 0,\ \frac{\partial c^*(a,\rho,n,\delta)}{\partial \rho} < 0,\ \frac{\partial c^*(a,\rho,n,\delta)}{\partial n} < 0 \text{ and } \frac{\partial c^*(a,\rho,n,\delta)}{\partial \delta} < 0.$$

Proof. See Exercise 8.5. □

The new results here relative to the basic Solow model concern the comparative statics with respect the discount factor. In particular, instead of the saving rate, it is now the discount factor that affects the rate of capital accumulation. There is a close link between the discount rate in the neoclassical growth model and the saving rate in the Solow model. Loosely speaking, a lower discount rate implies greater patience and thus greater savings. In the model without technological progress, the steady-state saving rate can be computed as

$$s^* = \frac{\delta k^*}{f(k^*)}. \tag{8.23}$$

Exercise 8.7 looks at the relationship between the discount rate, the saving rate and the steady-state per capita consumption level.

Another interesting result is that the rate of population growth has no impact on the steady state capital-labor ratio, which contrasts with the basic Solow model. We will see in Exercise 8.4 that this result depends on the way in which intertemporal discounting takes place. Another important result, which is more general, is that k^* and thus c^* do *not* depend on the instantaneous utility function $u(\cdot)$. The form of the utility function only affects the transitional dynamics (which we will study next), but has no impact on steady states. This is because the steady state is determined by the modified golden rule. This result will not be true when there is technological change, however.

8.5. Transitional Dynamics

Next, we can determine the transitional dynamics of this model. Recall that transitional dynamics in the basic Solow model were given by a single differential equation with an initial condition. This is no longer the case, since the equilibrium is determined by two differential equations, repeated here for convenience:

$$\dot{k}(t) = f(k(t)) - (n+\delta)k(t) - c(t) \tag{8.24}$$

and

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))}\left(f'(k(t)) - \delta - \rho\right). \tag{8.25}$$

Moreover, we have an initial condition $k(0) > 0$, also a boundary condition at infinity, of the form

$$\lim_{t\to\infty}\left[k(t)\exp\left(-\int_0^t \left(f'(k(s)) - \delta - n\right)ds\right)\right] = 0.$$

As we already discussed in the context of the q-theory of investment, this combination of an initial condition and a transversality condition is quite typical for economic optimal control problems where we are trying to pin down the behavior of both state and control variables. This means that we will again use the notion of saddle-path stability introduced in Theorems 7.18 and 7.19 instead of those in Theorems 2.4, 2.5 and 2.6. In particular, the consumption level (or equivalently the costate variable μ) is the control variable, and its initial value $c(0)$ (or equivalently $\mu(0)$) is free. It has to adjust so as to satisfy the transversality condition (the boundary condition at infinity). Since $c(0)$ or $\mu(0)$ can jump to any value, we again need that there exists a one-dimensional curve (manifold) tending to the steady state. In fact, as in the q-theory of investment, if there were more than one paths tending to the steady state, the equilibrium would be indeterminate, since there would be multiple values of $c(0)$ that could be consistent with equilibrium. Therefore, the correct notion of stability in models with state and control variables is one in which the dimension of the stable curve (manifold) is the same as that of the state variables, requiring the control variables jump on to this curve.

Fortunately, the economic forces are such that the correct notion of stability is guaranteed and indeed there will be a one-dimensional manifold of stable solutions tending to the unique steady state. There are two ways of seeing this. The first one simply involves studying the above system diagrammatically. This is done in Figure 8.1.

The vertical line is the locus of points where $\dot{c} = 0$. The reason why the $\dot{c} = 0$ locus is just a vertical line is that in view of the consumer Euler equation (8.25), only the unique level of k^* given by (8.21) can keep per capita consumption constant. The inverse U-shaped curve is the locus of points where $\dot{k} = 0$ in (8.24). The intersection of these two loci defined the steady state. The shape of the $\dot{k} = 0$ locus can be understood by analogy to the diagram where we discussed the golden rule in Chapter 2. If the capital stock is too low, steady-state consumption is low, and if the capital stock is too high, then the steady-state consumption is again low. There exists a unique level, k_{gold} that maximizes the state-state consumption per capita. The $\dot{c} = 0$ locus intersects the $\dot{k} = 0$ locus always to the left of k_{gold} (see Exercise 8.8). Once these two loci are drawn, the rest of the diagram can be completed by looking at the direction of motion according to the differential equations. Given this direction of movements, it is clear that there exists a unique stable arm, the one-dimensional manifold tending to the steady state. All points away from this stable arm diverge, and eventually reach zero consumption or zero capital stock as shown in the figure. To see this, note that if initial consumption, $c(0)$, started above this stable arm, say at $c'(0)$, the capital stock would reach 0 in finite time, while consumption would remain positive. But this would violate feasibility. Therefore, initial values of consumption above this stable arm cannot be part of the equilibrium (or the optimal growth solution). If the initial level of consumption were below it, for example, at $c''(0)$, consumption would reach zero, thus capital would

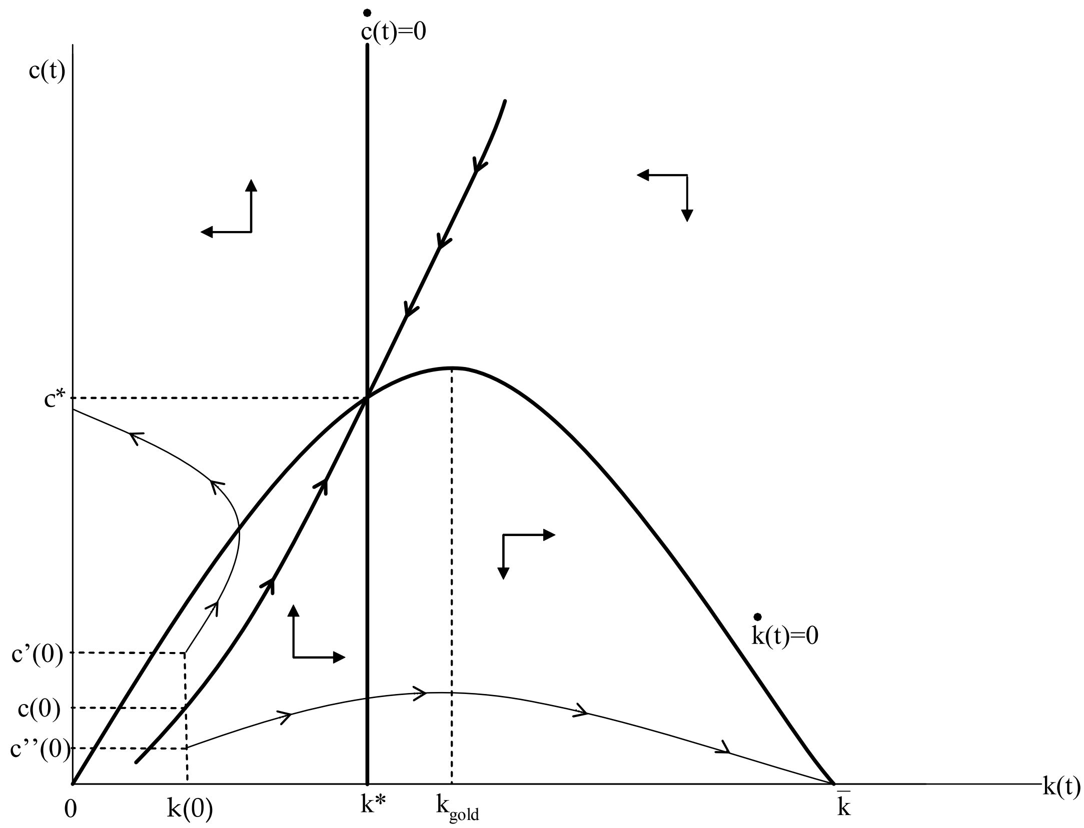

Figure 8.1. Transitional dynamics in the baseline neoclassical growth model.

accumulate continuously until the maximum level of capital (reached with zero consumption) $\bar{k} > k_{gold}$. Continuous capital accumulation towards $\bar{k}$ with no consumption would violate the transversality condition. This establishes that the transitional dynamics in the neoclassical growth model will take the following simple form: $c(0)$ will "jump" to the stable arm, and then (k, c) will monotonically travel along this arm towards the steady state. This establishes:

Proposition 8.4. *In the neoclassical growth model described above, with Assumptions 1, 2, 3 and 4′, there exists a unique equilibrium path starting from any* $k(0) > 0$ *and converging to the unique steady-state* (k^*, c^*) *with* k^* *given by (8.21). Moreover, if* $k(0) < k^*$*, then* $k(t) \uparrow k^*$ *and* $c(t) \uparrow c^*$*, whereas if* $k(0) > k^*$*, then* $k(t) \downarrow k^*$ *and* $c(t) \downarrow c^*$ *.*

An alternative way of establishing the same result is by linearizing the set of differential equations, and looking at their eigenvalues. Recall the two differential equations determining the equilibrium path:

$$\dot{k}(t) = f(k(t)) - (n + \delta)k(t) - c(t)$$

and

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))}\left(f'(k(t)) - \delta - \rho\right).$$

Linearizing these equations around the steady state (k^*, c^*), we have (suppressing time dependence)

$$\begin{aligned}\dot{k} &= \text{constant} + \left(f'(k^*) - n - \delta\right)(k - k^*) - c \\ \dot{c} &= \text{constant} + \frac{c^* f''(k^*)}{\varepsilon_u(c^*)}(k - k^*).\end{aligned}$$

Moreover, from (8.21), $f'(k^*) - \delta = \rho$, so the eigenvalues of this two-equation system are given by the values of ξ that solve the following quadratic form:

$$\det\begin{pmatrix} \rho - n - \xi & -1 \\ \frac{c^* f''(k^*)}{\varepsilon_u(c^*)} & 0 - \xi \end{pmatrix} = 0.$$

It is straightforward to verify that, since $c^* f''(k^*)/\varepsilon_u(c^*) < 0$, there are two real eigenvalues, one negative and one positive. This implies that there exists a one-dimensional stable manifold converging to the steady state, exactly as the stable arm in the above figure (see Exercise 8.11). Therefore, the local analysis also leads to the same conclusion. However, the local analysis can only establish local stability, whereas the above analysis established global stability.

8.6. Technological Change and the Canonical Neoclassical Model

The above analysis was for the neoclassical growth model without any technological change. As with the basic Solow model, the neoclassical growth model would not be able to account for the long-run growth experience of the world economy without some type of exogenous technological change. Therefore, the more interesting version of this model is the one that incorporates technological change. We now analyze the neoclassical model with exogenous technological change.

We extend the production function to:

$$Y(t) = F[K(t), A(t)L(t)], \tag{8.26}$$

where

$$A(t) = \exp(gt)A(0).$$

Notice that the production function (8.26) imposes purely labor-augmenting—Harrod-neutral—technological change. This is a consequence of Theorem 2.7 above, which was proved in the context of the constant saving rate model, but equally applies in this context. Only purely labor-augmenting technological change is consistent with balanced growth.

We continue to adopt all the other assumptions, in particular Assumptions 1, 2 and 3. Assumption 4$'$ will be strengthened further in order to ensure finite discounted utility in the presence of sustained economic growth.

The constant returns to scale feature again enables us to work with normalized variables. Now let us define

$$\begin{aligned}\hat{y}(t) &\equiv \frac{Y(t)}{A(t)L(t)}\\ &= F\left[\frac{K(t)}{A(t)L(t)},1\right]\\ &\equiv f(k(t)),\end{aligned}$$

where

$$k(t)\equiv\frac{K(t)}{A(t)L(t)}. \tag{8.27}$$

is the capital to effective capital-labor ratio, which is defined taking into account that effective labor is increasing because of labor-augmenting technological change. Naturally, this is similar to the way that the effective capital-labor ratio was defined in the basic Solow growth model.

In addition to the assumptions on technology, we also need to impose a further assumption on preferences in order to ensure balanced growth. Again as in the basic Solow model, we define balanced growth as a pattern of growth consistent with the *Kaldor facts* of constant capital-output ratio and capital share in national income. These two observations together also imply that the rental rate of return on capital, $R(t)$, has to be constant, which, from (8.8), implies that $r(t)$ has to be constant. Be again refer to an equilibrium path that satisfies these conditions as a balanced growth path (BGP). Balanced growth also requires that consumption and output grow at a constant rate. The Euler equation implies that

$$\frac{\dot{c}(t)}{c(t)}=\frac{1}{\varepsilon_u(c(t))}(r(t)-\rho).$$

If $r(t)\to r^*$, then $\dot{c}(t)/c(t)\to g_c$ is only possible if $\varepsilon_u(c(t))\to\varepsilon_u$, i.e., if the elasticity of marginal utility of consumption is asymptotically constant. Therefore, balanced growth is only consistent with utility functions that have asymptotically constant elasticity of marginal utility of consumption. Since this result is important, we state it as a proposition:

Proposition 8.5. *Balanced growth in the neoclassical model requires that asymptotically (as $t\to\infty$) all technological change is purely labor-augmenting and the elasticity of intertemporal substitution, $\varepsilon_u(c(t))$, tends to a constant ε_u.*

The next example shows the family of utility functions with constant intertemporal elasticity of substitution, which are also those with a constant coefficient of relative risk aversion.

Example 8.1. **(CRRA Utility)** Recall that the Arrow-Pratt coefficient of relative risk aversion for a twice-continuously differentiable concave utility function $U(c)$ is

$$\mathcal{R}=-\frac{U''(c)\,c}{U'(c)}.$$

Constant relative risk aversion (CRRA) utility function satisfies the property that $\mathcal{R}$ is constant. Now integrating both sides of the previous equation, setting $\mathcal{R}$ to a constant, implies that the family of CRRA utility functions is given by

$$U(c) = \begin{cases} \frac{c^{1-\theta}-1}{1-\theta} & \text{if } \theta \neq 1 \text{ and } \theta \geq 0 \\ \ln c & \text{if } \theta = 1 \end{cases},$$

with the coefficient of relative risk aversion given by θ. In writing this expression, we separated the case where $\theta = 1$, since $\left(c^{1-\theta}-1\right)/(1-\theta)$ is undefined at $\theta = 1$. However, it can be shown that $\ln c$ is indeed the right limit when $\theta \to 1$ (see Exercise 5.4).

With time separable utility functions, the inverse of the elasticity of intertemporal substitution (defined in equation (8.16)) and the coefficient of relative risk aversion are identical. Therefore, the family of CRRA utility functions are also those with constant elasticity of intertemporal substitution.

Now to link this utility function to the Gorman preferences discussed in Chapter 5, let us consider a slightly different problem in which an individual has preferences defined over the consumption of N commodities $\{c_1, ..., c_N\}$ given by

$$U(\{c_1, ..., c_N\}) = \begin{cases} \sum_{j=1}^{N} \frac{c_j^{1-\theta}}{1-\theta} & \text{if } \theta \neq 1 \text{ and } \theta \geq 0 \\ \sum_{j=1}^{N} \ln c_j & \text{if } \theta = 1 \end{cases}. \tag{8.28}$$

Suppose also that this individual faces a price vector $\mathbf{p} = (p_1, ..., p_N)$ and has income y, so that his budget constraint can be expressed as

$$\sum_{j=1}^{N} p_j c_j \leq y. \tag{8.29}$$

Maximizing utility subject to this budget constraint leads of the following indirect utility function

$$v(p,y) = \frac{y^{\frac{\sigma-1}{\sigma}}}{\left[\sum_{j=1}^{N} p_j^{1-\sigma}\right]^{1/\sigma}}$$

(see Exercise 5.6). Although this indirect utility function does not satisfy the Gorman form in Theorem 5.2, a monotonic transformation thereof does (that is, we simply raise it to the power $\sigma/(\sigma-1)$).

This establishes that CRRA utility functions are within the Gorman class, and if all individuals have CRRA utility functions, then we can aggregate their preferences and represent them as if it belonged to a single individual.

Now consider a dynamic version of these preferences (defined over infinite horizon):

$$U = \begin{cases} \sum_{t=0}^{\infty} \beta^t \frac{c(t)^{1-\theta}-1}{1-\theta} & \text{if } \theta \neq 1 \text{ and } \theta \geq 0 \\ \sum_{t=0}^{\infty} \beta^t \ln c(t) & \text{if } \theta = 1 \end{cases}.$$

The important feature of these preferences for us is not that the coefficient of relative risk aversion constant per se, but that the intertemporal elasticity of substitution is constant. This is the case because most of the models we focus on in this book do not feature uncertainty, so that attitudes towards risk are not important. However, as noted before and illustrated in Exercise 5.2 in Chapter 5, with time-separable utility functions the coefficient of relative risk aversion in the inverse of the intertemporal elasticity of substitution are identical. The intertemporal elasticity of substitution is particularly important in growth models because it will regulate how willing individuals are to substitute consumption over time, thus their savings and consumption behavior. In view of this, it may be more appropriate to refer to CRRA preferences as "constant intertemporal elasticity of substitution" preferences. Nevertheless, we follow the standard convention in the literature and stick to the term CRRA.

Given the restriction that balanced growth is only possible with preferences featuring a constant elasticity of intertemporal substitution, we might as well start with a utility function that has this feature throughout. As noted above, the unique time-separable utility function with this feature is the CRRA preferences, given by

$$u(c(t)) = \begin{cases} \frac{c(t)^{1-\theta}-1}{1-\theta} & \text{if } \theta \neq 1 \text{ and } \theta \geq 0 \\ \ln c(t) & \text{if } \theta = 1 \end{cases},$$

where the elasticity of marginal utility of consumption, ε_u, is given by the constant θ. When $\theta = 0$, these represent linear preferences, whereas when $\theta = 1$, we have log preferences. As $\theta \to \infty$, these preferences become infinitely risk-averse, and infinitely unwilling to substitute consumption over time.

More specifically, we now assume that the economy admits a representative household with CRRA preferences

$$\int_0^\infty \exp\left(-(\rho - n)t\right) \frac{\tilde{c}(t)^{1-\theta} - 1}{1-\theta} dt, \tag{8.30}$$

where $\tilde{c}(t) \equiv C(t)/L(t)$ is per capita consumption. We used to notation $\tilde{c}(t)$ in order to preserve $c(t)$ for a further normalization.

We refer to this model, with labor-augmenting technological change and CRRA preference as given by (8.30) as the *canonical model*, since it is the model used in almost all applications of the neoclassical growth model. The Euler equation in this case takes the simpler form:

$$\frac{\dot{\tilde{c}}(t)}{\tilde{c}(t)} = \frac{1}{\theta}\left(r(t) - \rho\right). \tag{8.31}$$

Let us first characterize the steady-state equilibrium in this model with technological progress. Since with technological progress there will be growth in per capita income, $\tilde{c}(t)$ will grow. Instead, in analogy with $k(t)$, let us define

$$\begin{aligned} c(t) &\equiv \frac{C(t)}{A(t)L(t)} \\ &\equiv \frac{\tilde{c}(t)}{A(t)}. \end{aligned}$$

We will see that this normalized consumption level will remain constant along the BGP. In particular, we have

$$\begin{aligned}\frac{\dot{c}(t)}{c(t)} &\equiv \frac{\dot{\tilde{c}}(t)}{\tilde{c}(t)} - g \\ &= \frac{1}{\theta}\left(r(t) - \rho - \theta g\right).\end{aligned}$$

Moreover, for the accumulation of capital stock, we have

$$\dot{k}(t) = f(k(t)) - c(t) - (n+g+\delta)k(t),$$

where recall that $k(t) \equiv K(t)/A(t)L(t)$.

The transversality condition, in turn, can be expressed as

$$\lim_{t\to\infty}\left\{k(t)\exp\left(-\int_0^t\left[f'(k(s)) - g - \delta - n\right]ds\right)\right\} = 0. \tag{8.32}$$

In addition, the equilibrium interest rate, $r(t)$, is still given by (8.8), so

$$r(t) = f'(k(t)) - \delta$$

Since in steady state $c(t)$ must remain constant, we also have

$$r(t) = \rho + \theta g$$

or

$$f'(k^*) = \rho + \delta + \theta g, \tag{8.33}$$

which pins down the steady-state value of the normalized capital ratio k^* uniquely, in a way similar to the model without technological progress. The level of normalized consumption is then given by

$$c^* = f(k^*) - (n+g+\delta)k^*, \tag{8.34}$$

while per capita consumption grows at the rate g.

The only additional condition in this case is that because there is growth, we have to make sure that the transversality condition is in fact satisfied. Substituting (8.33) into (8.32), we have

$$\lim_{t\to\infty}\left\{k(t)\exp\left(-\int_0^t\left[\rho - (1-\theta)g - n\right]ds\right)\right\} = 0,$$

which can only hold if the integral within the exponent goes to zero, i.e., if $\rho - (1-\theta)g - n > 0$, or alternatively if the following assumption is satisfied:

Assumption 4.

$$\rho - n > (1-\theta)g.$$

Note that this assumption strengthens Assumption 4′ when $\theta < 1$. Alternatively, recall that in steady state we have $r = \rho + \theta g$ and the growth rate of output is $g + n$. Therefore, Assumption 4 is equivalent to requiring that $r > g + n$. We will encounter conditions like this all throughout, and they will also be related to issues of "dynamic efficiency" as we will see below.

The following is an immediate generalization of Proposition 8.2:

Proposition 8.6. *Consider the neoclassical growth model with labor-augmenting technological progress at the rate g and preferences given by (8.30). Suppose that Assumptions 1, 2, 3 and 4 hold. Then there exists a unique balanced growth path with a normalized capital to effective labor ratio of* k^**, given by (8.33), and output per capita and consumption per capita grow at the rate* g*.*

As noted above, the result that the steady-state capital-labor ratio was independent of preferences is no longer the case, since now k^* given by (8.33) depends on the elasticity of marginal utility (or the inverse of the intertemporal elasticity of substitution), θ. The reason for this is that there is now positive growth in output per capita, and thus in consumption per capita. Since individuals face an upward-sloping consumption profile, their willingness to substitute consumption today for consumption tomorrow determines how much they will accumulate and thus the equilibrium effective capital-labor ratio.

Perhaps the most important implication of Proposition 8.6 is that, while the steady-state effective capital-labor ratio, k^*, is determined endogenously, the steady-state growth rate of the economy is given exogenously and is equal to the rate of labor-augmenting technological progress, g. Therefore, the neoclassical growth model, like the basic Solow growth model, endogenizes the capital-labor ratio, but not the growth rate of the economy. The advantage of the neoclassical growth model is that the capital-labor ratio and the equilibrium level of (normalized) output and consumption are determined by the preferences of the individuals rather than an exogenously fixed saving rate. This also enables us to compare equilibrium and optimal growth (and in this case conclude that the competitive equilibrium is Pareto optimal and any Pareto optimum can be decentralized). But the determination of the rate of growth of the economy is still outside the scope of analysis.

A similar analysis to before also leads to a generalization of Proposition 8.4.

Proposition 8.7. *Consider the neoclassical growth model with labor-augmenting technological progress at the rate g and preferences given by (8.30). Suppose that Assumptions 1, 2, 3 and 4 hold. Then there exists a unique equilibrium path of normalized capital and consumption,* $(k(t), c(t))$ *converging to the unique steady-state* (k^*, c^*) *with* k^* *given by (8.33). Moreover, if* $k(0) < k^*$*, then* $k(t) \uparrow k^*$ *and* $c(t) \uparrow c^*$*, whereas if* $k(0) > k^*$*, then* $c(t) \downarrow k^*$ *and* $c(t) \downarrow c^*$*.*

Proof. See Exercise 8.9. □

It is also useful to briefly look at an example with Cobb-Douglas technology.

Example 8.2. Consider the model with CRRA utility and labor-augmenting technological progress at the rate g. Assume that the production function is given by $F(K, AL) = K^{\alpha}(AL)^{1-\alpha}$, so that

$$f(k) = k^{\alpha},$$

and thus $r = \alpha k^{\alpha-1} - \delta$. In this case, suppressing time dependence to simplify notation, the Euler equation becomes:

$$\frac{\dot{c}}{c} = \frac{1}{\theta}\left(\alpha k^{\alpha-1} - \delta - \rho - \theta g\right),$$

and the accumulation equation can be written as

$$\frac{\dot{k}}{k} = k^{\alpha-1} - \delta - g - n - \frac{c}{k}.$$

Now define $z \equiv c/k$ and $x \equiv k^{\alpha-1}$, which implies that $\dot{x}/x = (\alpha - 1)\dot{k}/k$. Therefore, these two equations can be written as

$$\frac{\dot{x}}{x} = -(1-\alpha)(x - \delta - g - n - z) \tag{8.35}$$

$$\frac{\dot{z}}{z} = \frac{\dot{c}}{c} - \frac{\dot{k}}{k},$$

thus

$$\begin{aligned}\frac{\dot{z}}{z} &= \frac{1}{\theta}(\alpha x - \delta - \rho - \theta g) - x + \delta + g + n + z \\ &= \frac{1}{\theta}((\alpha-\theta)x - (1-\theta)\delta + \theta n) - \frac{\rho}{\theta} + z. \end{aligned} \tag{8.36}$$

The two differential equations (8.35) and (8.36) together with the initial condition $x(0)$ and the transversality condition completely determine the dynamics of the system. In Exercise 8.12, you are asked to complete this example for the special case in which $\theta \to 1$ (i.e., log preferences).

8.7. Comparative Dynamics

We now briefly discuss how comparative dynamics are different in the neoclassical growth model than those in the basic Solow model. Recall that while comparative statics refer to changes in steady state in response to changes in parameters, comparative dynamics look at how the entire equilibrium path of variables changes in response to a change in policy or parameters. Since the purpose here is to give a sense of how these results are different, we will only look at the effect of a change in a single parameter, the discount rate ρ. Imagine a neoclassical growth economy with population growth at the rate n, labor-augmenting technological progress at the rate g and a discount rate ρ that has settled into a steady state represented by (k^*, c^*). Now imagine that the discount rate declines to $\rho' < \rho$. How does the equilibrium path change?

We know from Propositions 8.6 and 8.7 that at the new discount rate $\rho' > 0$, there exists a unique steady state equilibrium that is saddle path stable. Let this steady state be denoted by (k^{**}, c^{**}). Therefore, the equilibrium will ultimately tend to this new steady-state equilibrium. Moreover, since $\rho' < \rho$, we know that the new steady-state effective capital-labor ratio has to be greater than k^*, that is, $k^{**} > k^*$ (while the equilibrium growth rate will remain unchanged). Figure 8.2 shows diagrammatically how the comparative dynamics work out. This figure is drawn under the assumption that the change in the discount rate (corresponding to the change in the preferences of the representative household in the economy) is unanticipated and occurs at some date T. At this point, the curve corresponding to $\dot{c}/c = 0$ shifts to the right and together with this, the laws of motion represented by the phase diagram change (in the figure, the arrows represents the dynamics of the economy after the change). It can be seen that following this decline in the discount factor, the previous steady-state level of consumption, c^*, is above the stable arm of the new dynamical system. Therefore, consumption must drop immediately to reach the news stable arm, so that capital can accumulate towards its new steady-state level. This is shown in the figure with the arc representing the jump in consumption immediately following the decline in the discount rate. Following this initial reaction, consumption slowly increases along the stable arm to a higher level of (normalized) consumption. Therefore, a decline in the discount rate lead to a temporary decline in consumption, associated with a long-run increase in consumption. We know that the overall level of normalized consumption will necessarily increase, since the intersection between the curve for $\dot{c}/c = 0$ and the inverse U-shaped curve for $\dot{k}/k = 0$ will necessarily be to the left side of k_{gold}.

Comparative dynamics in response to changes in other parameters, including the rate of labor-augmenting technological progress g, the rate of population growth n, and other aspects of the utility function, can also be analyzed similarly. Exercise 8.13 asks you to work through the comparative dynamics in response to a change in the rate of labor-augmenting technological progress, g, and in response to an anticipated future change in ρ.

8.8. The Role of Policy

In the above model, the rate of growth of per capita consumption and output per worker (per capita) are determined exogenously, by the growth rate of labor-augmenting technological progress. The level of income, on the other hand, depends on preferences, in particular, on the intertemporal elasticity of substitution, $1/\theta$, the discount rate, ρ, the depreciation rate, δ, the population growth rate, n, and naturally the form of the production function $f(\cdot)$.

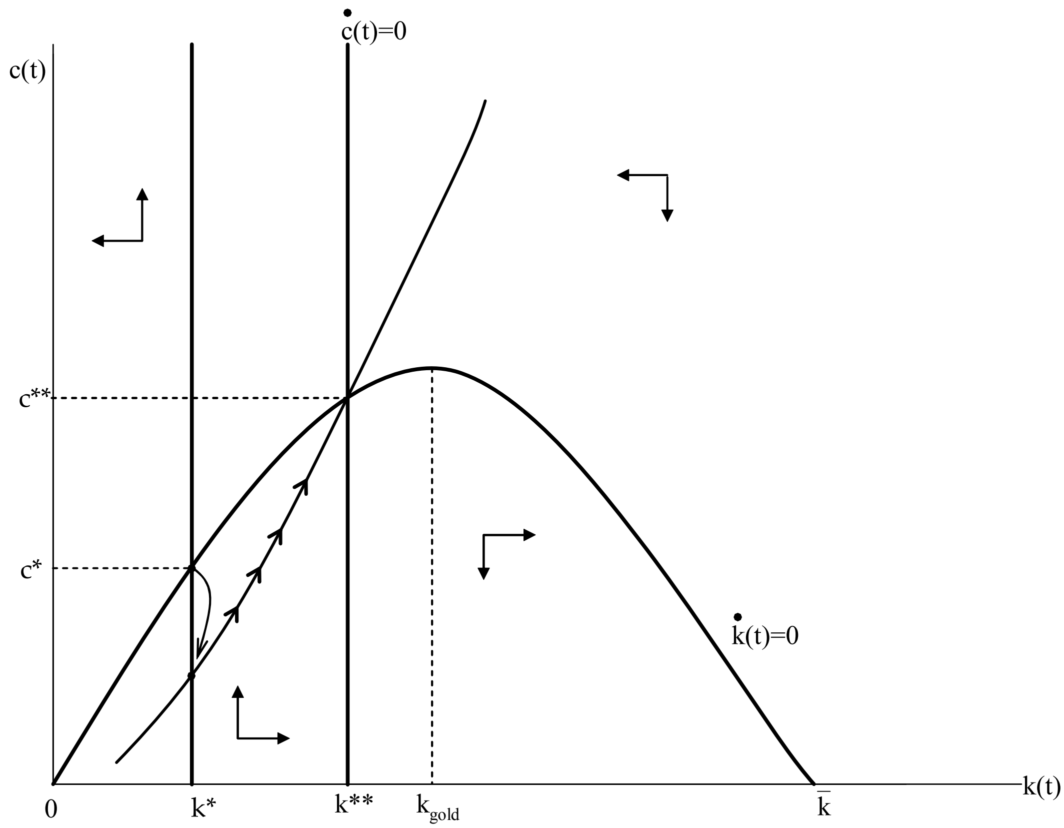

Figure 8.2. The dynamic response of capital and consumption to a decline in the discount rate from ρ to $\rho' < \rho$.

If we were to go back to the proximate causes of differences in income per capita of economic growth across countries, this model would give us a way of understanding those differences only in terms of preference and technology parameters. However, as already discussed in Chapter 4, and we would also like to link the proximate causes of economic growth to potential fundamental causes. The intertemporal elasticity of substitution and the discount rate can be viewed as potential determinants of economic growth related to cultural or geographic factors. However, an explanation for cross-country and over-time differences in economic growth based on differences or changes in preferences is unlikely to be satisfactory. A more appealing direction may be to link the incentives to accumulate physical capital (and later to accumulate human capital and technology) to the institutional environment of an economy. We will discuss how institutions might affect various investment decisions in detail in Part 8 of the book. For now, it is useful to focus on a particularly simple way in which institutional differences might affect investment decisions, which is through differences in policies. To do this, let us extend the above framework in a simple way and introduce linear tax policy. Suppose that returns on capital net of depreciation are taxed at the rate τ and the proceeds of this are redistributed back to the consumers. In that case, the capital accumulation equation, in terms of normalized capital, still remains as above:

$$\dot{k}(t) = f(k(t)) - c(t) - (n+g+\delta)k(t),$$

but the net interest rate faced by households now changes to:

$$r(t) = (1-\tau)\left(f'(k(t)) - \delta\right),$$

because of the taxation of capital returns. The growth rate of normalized consumption is then obtained from the consumer Euler equation, (8.31), as

$$\begin{aligned}\frac{\dot{c}(t)}{c(t)} &= \frac{1}{\theta}(r(t) - \rho - \theta g). \\ &= \frac{1}{\theta}\left((1-\tau)\left(f'(k(t)) - \delta\right) - \rho - \theta g\right).\end{aligned}$$

An identical argument to that we have used above immediately implies that the steady-state capital to effective labor ratio is given by

$$f'(k^*) = \delta + \frac{\rho + \theta g}{1-\tau}. \tag{8.37}$$

This equation shows the effects of taxes on steady-state capital tp effective labor ratio and output per capita. A higher tax rate τ increases the right-hand side of (8.37), and since from Assumption 1, $f'(\cdot)$ is decreasing, it reduces k^*. Therefore, higher taxes on capital have the effect of depressing capital accumulation and reducing income per capita. This shows one channel through which policy (thus institutional) differences might affect economic performance. We can also note that similar results would be obtained if instead of taxes being imposed on returns from capital, they were imposed on the amount of investment (see next section). Naturally, we have not so far offered a reason why some countries may tax capital at a higher rate than others, which is again a topic that will be discussed later. Before doing this, in the next section we will also discuss how large these effects can be and whether they could account for the differences in cross-country incomes.

8.9. A Quantitative Evaluation

As a final exercise, let us investigate whether the quantitative implications of the neoclassical growth model are reasonable. For this purpose, consider a world consisting of a collection $\mathcal{J}$ of closed neoclassical economies (with all the caveats of ignoring technological, trade and financial linkages across countries, already discussed in Chapter 3; see also Chapter 19). Suppose that each country $j \in \mathcal{J}$ admits a representative household with identical preferences, given by

$$\int_0^\infty \exp(-\rho t)\frac{C_j^{1-\theta} - 1}{1-\theta}dt. \tag{8.38}$$

Let us assume that there is no population growth, so that c_j is both total or per capita consumption. Equation (8.38) imposes that all countries have the same discount rate ρ (see Exercise 8.16).

All countries also have access to the same production technology given by the Cobb-Douglas production function

$$Y_j = K_j^{1-\alpha} (AH_j)^{\alpha}, \tag{8.39}$$

with H_j representing the exogenously given stock of effective labor (human capital). The accumulation equation is

$$\dot{K}_j = I_j - \delta K_j.$$

The only difference across countries is in the budget constraint for the representative household, which takes the form

$$(1+\tau_j) I_j + C_j \leq Y_j, \tag{8.40}$$

where τ_j is the tax on investment. This tax varies across countries, for example because of policies or differences in institutions/property rights enforcement. Notice that $1+\tau_j$ is also the relative price of investment goods (relative to consumption goods): one unit of consumption goods can only be transformed into $1/(1+\tau_j)$ units of investment goods.

Note that the right-hand side variable of (8.40) is still Y_j, which implicitly assumes that $\tau_j I_j$ is wasted, rather than simply redistributed to some other agents in the economy. This is without any major consequence, since, as noted in Theorem 5.2 above, CRRA preferences as in (8.38) have the nice feature that they can be exactly aggregated across individuals, so we do not have to worry about the distribution of income in the economy.

The competitive equilibrium can be characterized as the solution to the maximization of (8.38) subject to (8.40) and the capital accumulation equation.

With the same steps as above, the Euler equation of the representative household is

$$\frac{\dot{C}_j}{C_j} = \frac{1}{\theta}\left(\frac{(1-\alpha)}{(1+\tau_j)}\left(\frac{AH_j}{K_j}\right)^{\alpha} - \delta - \rho\right).$$

Consider the steady state. Because A is assumed to be constant, the steady state corresponds to $\dot{C}_j/C_j = 0$. This immediately implies that

$$K_j = \frac{(1-\alpha)^{1/\alpha} AH_j}{[(1+\tau_j)(\rho+\delta)]^{1/\alpha}}$$

So countries with higher taxes on investment will have a lower capital stock in steady state. Equivalently, they will also have lower capital per worker, or a lower capital output ratio (using (8.39) the capital output ratio is simply $K/Y = (K/AH)^{\alpha}$).

Now substituting this into (8.39), and comparing two countries with different taxes (but the same human capital), we obtain the relative incomes as

$$\frac{Y(\tau)}{Y(\tau')} = \left(\frac{1+\tau'}{1+\tau}\right)^{\frac{1-\alpha}{\alpha}} \tag{8.41}$$

So countries that tax investment, either directly or indirectly, at a higher rate will be poorer. The advantage of using the neoclassical growth model for quantitative evaluation relative to the Solow growth model is that the extent to which different types of distortions (here captured by the tax rates on investment) will affect income and capital accumulation is determined endogenously. In contrast, in the Solow growth model, what matters is the saving rate, so we would need other evidence to link taxes or distortions to savings (or to other determinants of income per capita such as technology).

How large will be the effects of tax distortions captured by equation (8.41)? Put differently, can the neoclassical growth model combined with policy differences account for quantitatively large cross-country income differences?

The advantage of the current approach is its parsimony. As equation (8.41) shows, only differences in τ across countries and the value of the parameter α matter for this comparison. Recall also that a plausible value for α is $2/3$, since this is the share of labor income in national product which, with Cobb-Douglas production function, is equal to α, so this parameter can be easily mapped to data.

Where can we obtain estimates of differences in τ's across countries? There is no obvious answer to this question. A popular approach in the literature is to obtain estimates of τ from the relative price of investment goods (as compared to consumption goods), motivated by the fact that in equation (8.40) τ corresponds to a distortion directly affecting investment expenditures. Therefore, we may expect τ to have an effect on the market price of investment goods. Data from the Penn World tables suggest that there is a large amount of variation in the relative price of investment goods. For example, countries with the highest relative price of investment goods have relative prices almost eight times as high as countries with the lowest relative price.

Then, using $\alpha = 2/3$, equation (8.41) implies that the income gap between two such countries should be approximately threefolds:

$$\frac{Y(\tau)}{Y(\tau')} \approx 8^{1/2} \approx 3.$$

Therefore, differences in capital-output ratios or capital-labor ratios caused by taxes or tax-type distortions, even by very large differences in taxes or distortions, are unlikely to account for the large differences in income per capita that we observe in practice. This is of course not surprising and parallels our discussion of the Mankiw-Romer-Weil approach in Chapter 3. In particular, recall that the discussion in Chapter 3 showed that differences in income per capita across countries are unlikely to be accounted for by differences in capital per worker alone. Instead, to explain such large differences in income per capita across countries, we need sizable differences in the efficiency with which these factors are used. Such differences do not feature in this model. Therefore, the simplest neoclassical model does not generate sufficient differences in capital-labor ratios to explain cross-country income differences.

Nevertheless, many economists have tried (and still try) to use versions of the neoclassical model to go further. The motivation is simple. If instead of using $\alpha = 2/3$, we take $\alpha = 1/3$ from the same ratio of distortions, we obtain

$$\frac{Y(\tau)}{Y(\tau')} \approx 8^2 \approx 64.$$

Therefore, if there is any way of increasing the responsiveness of capital or other factors to distortions, the predicted differences across countries can be made much larger. How could we have a model in which $\alpha = 1/3$? Such a model must have additional accumulated factors, while still keeping the share of labor income in national product roughly around 2/3. One possibility might be to include human capital (see Chapter 10 below). Nevertheless, the discussion in Chapter 3 showed that human capital differences appear to be insufficient to explain much of the income per capita differences across countries. Another possibility is to introduce other types of capital or perhaps technology that responds to distortions in the same way as capital. While these are all logically possible, a serious analysis of these issues requires models of endogenous technology, which will be our focus in the next part of the book.

8.10. Extensions

There are many empirically and theoretically relevant extensions of the neoclassical growth model. We will not present these here for the sake of brevity. But the most important ones are presented as exercises. In particular, Exercise 8.17 endogenizes the labor supply decisions on individuals by introducing leisure in the utility function. The model presented in this exercise is particularly important, since it corresponds to the version of the neoclassical growth model most often employed in short-run and medium-run macroeconomic analyses. This exercise also shows that further restrictions on the form of the utility function need to be imposed in order to preserve the balanced growth structure. Exercise 8.18 introduces government expenditures and taxation into the basic model. Exercise 8.20 looks at the behavior of the basic neoclassical growth model with a free capital account, representing borrowing and lending opportunities for the economy at some exogenously given international interest rate r^*. Exercise 8.21 combines the costs of adjustments in investment as in the q-theory with the basic neoclassical model. Finally, Exercise 8.22 looks at a version of the neoclassical model with multiple sectors.

8.11. Taking Stock

This chapter presented arguably the most important model in macroeconomics; the one-sector neoclassical growth model. Recall that our study of the basic models of economic growth started in Chapter 2, with the Solow growth model. We saw that while this model gives a number of important insights, it treats much of the mechanics of economic growth as a "black box". Growth can only be generated by technological progress (unless we are in the special AK model without diminishing returns to capital), but technological progress is outside the model. The next important element in determining cross-country differences in income is the saving rate, but in the Solow growth model this was also taken as exogenous. The major contribution of the current chapter has been to open the black box of capital accumulation by specifying the preferences of consumers. Consequently, we can link the saving rates to the instantaneous utility function, to the discount rate and also to technology and prices in the economy. Moreover, as Exercise 8.23 shows the implications of policy on equilibrium quantities are different in the neoclassical model than in the Solow growth model with exogenously given saving rates.

Another major advantage of the neoclassical growth model is that by specifying individual preferences we can explicitly compare equilibrium and optimal growth.

Perhaps the most important contribution of this model is that it paves the way for further analysis of capital accumulation, human capital and endogenous technological progress, which will be our topic in the next few chapters (starting with human capital in Chapter 10). Therefore, this chapter is the first, and perhaps conceptually most important, step towards opening the black box of economic growth and providing us with the tools and perspectives for modeling capital accumulation, human capital accumulation and technological change endogenously.

Did our study of the neoclassical growth model generate new insights about the sources of cross-country income differences and economic growth relative to the Solow growth model? The answer here is largely no. While the current model is an important milestone in our study of the mechanics of economic growth, as with the Solow growth model, the focus is on the proximate causes of these differences—we are still looking at differences in saving rates, investments in human capital and technology, perhaps as determined by preferences and other dimensions of technology (such as the rate of labor-augmenting technological change). It is therefore important to bear in mind that this model, by itself, does not enable us to answer questions about the fundamental causes of economic growth. What it does, however, is to clarify the nature of the economic decisions so that we are in a better position to ask such questions.

8.12. References and Literature

The neoclassical growth model goes back to Frank Ramsey's (1928) classic treatment and for that reason is often referred to as the "Ramsey model". Ramsey's model is very similar to standard neoclassical growth model, except that it did not feature discounting. Another early optimal growth model was presented by John von Neumann (1935), focusing more on the limiting behavior of the dynamics in a linear model. The current version of the neoclassical growth model is most closely related to the analysis of optimal growth by David Cass (1965) and Tjalling Koopmans (1960, 1965). An excellent discussion of equilibrium and optimal growth is provided in Arrow and Kurz's (1970) volume.

All growth and macroeconomic textbooks cover the neoclassical growth model. Barro and Sala-i-Martin (2004, Chapter 2) provides a detailed treatment focusing on the continuous time economy. Blanchard and Fisher (1989, Chapter 2) and Romer (2006, Chapter 2) also present the continuous time version of the neoclassical growth model. Sargent and Ljungqvist (2004, Chapter 14) provides an introductory treatment of the neoclassical growth model in discrete time.

Ricardian Equivalence discussed in Exercise 8.19 was first proposed by Barro (1974). It is further discussed in Chapter 9.

A systematic quantitative evaluation of the effects of policy differences is provided in Chari, Kehoe and McGrattan (1997). These authors follow Jones (1995) in emphasizing differences in the relative prices of investment goods (compared to consumption goods) in the Penn Worlds tables and interpret these as due to taxes and other distortions. This interpretation is not without any problems. In particular, in the presence of international trade, these relative price differences will reflect other technological factors or possible factor proportion differences (see Chapter 19, and also Acemoglu and Ventura (2002) and Hsieh and Klenow (2006)). Parente and Prescott (1994) use an extended version of the neoclassical growth model (where the "stock of technology," which is costly to adopt from the world frontier, is interpreted as a capital good) to perform similar quantitative exercises. Other authors have introduced yet other accumulable factors in order to increase the elasticity of output to distortions (that is, to reduce the α parameter above). Pete Klenow has dubbed these various accumulable factors introduced in the models to increase this elasticity the "mystery capital" to emphasize the fact that while they may help the quantitative match of the neoclassical-type models, they are not directly observable in the data.

8.13. Exercises

Exercise 8.1. Consider the consumption allocation decision of an infinitely-lived household with (a continuum of) $L(t)$ members at time t, with $L(0) = 1$. Suppose that the household has total consumption $C(t)$ to allocate at time t. The household has "utilitarian" preferences with instantaneous utility function $u(c)$ and discount the future at the rate $\rho > 0$.

(1) Show that the problem of the household can be written as

$$\max \int_0^\infty \exp(-\rho t) \left[\int_0^{L(t)} u(c_i(t))\, di \right] dt,$$

subject to

$$\int_0^{L(t)} c_i(t)\, di \leq C(t),$$

and subject to the budget constraint of the household,

$$\dot{\mathcal{A}}(t) = r(t)\mathcal{A}(t) + W(t) - C(t),$$

where i denotes a generic member of the household, $\mathcal{A}(t)$ is the total asset holding of the household, $r(t)$ is the rate of return on assets and $W(t)$ is total labor income.

(2) Show that as long as $u(\cdot)$ is strictly concave, this problem becomes

$$\max \int_0^\infty \exp(-(\rho - n)t)\, u(c(t))\, dt,$$

subject to

$$\dot{a}(t) = (r(t) - n)\, a(t) + w(t) - c(t),$$

where $w(t) \equiv W(t)/L(t)$ and $a(t) \equiv \mathcal{A}(t)/L(t)$. Provide an intuition for this transformed problem.

Exercise 8.2. Derive (8.7) from (8.9).

Exercise 8.3. Suppose that the consumer problem is formulated without the no-Ponzi game condition. Construct a sequence of feasible consumption decisions that provides strictly greater utility than those characterized in the text.

Exercise 8.4. Consider a variant of the neoclassical model (with constant population growth at the rate n)in which preferences are given by

$$\max \int_0^\infty \exp(-\rho t)\, u(c(t))\, dt,$$

and there is population growth at the constant rate n. How does this affect the equilibrium? How does the transversality condition need to be modified? What is the relationship between the rate of population growth, n, and the steady-state capital labor ratio k^*?

Exercise 8.5. Prove Proposition 8.3.

Exercise 8.6. Explain why the steady state capital-labor ratio k^* does not depend on the form of the utility function without technological progress but depends on the intertemporal elasticity of substitution when there is positive technological progress.

Exercise 8.7. (1) Show that the steady-state saving rate s^* defined in (8.23) is decreasing in ρ, so that lower discount rates lead to higher steady-state savings.

(2) Show that in contrast to the Solow model, the saving rate s^* can never be so high that a decline in savings (or an increase in ρ) can raise the steady-state level of consumption per capita.

EXERCISE 8.8. In the dynamics of the basic neoclassical growth model, depicted in Figure 8.1, prove that the $\dot{c} = 0$ locus intersects the $\dot{k} = 0$ locus always to the left of k_{gold}. Based on this analysis, explain why the modified golden rule capital-labor ratio, k^*, given by (8.21) differs from k_{gold}.

EXERCISE 8.9. Prove that, as stated in Proposition 8.7, in the neoclassical model with labor-augmenting technological change and the standard assumptions, starting with $k(0) > 0$, there exists a unique equilibrium path where normalized consumption and capital-labor ratio monotonically converge to the balanced growth path. [Hint: use Figure 8.1].

EXERCISE 8.10. Consider a neoclassical economy, with a representative household with preferences at time $t = 0$:

$$U(0) = \int_0^\infty \exp(-\rho t) \frac{c(t)^{1-\theta} - 1}{1 - \theta}.$$

There is no population growth and labor is supplied inelastically. Assume that the aggregate production function is given by $Y(t) = F[A(t) K(t), L(t)]$ where F satisfies the standard assumptions (constant returns to scale, differentiability, Inada conditions).

(1) Define a competitive equilibrium for this economy.
(2) Suppose that $A(t) = A(0)$ and characterize the steady-state equilibrium. Explain why the steady-state capital-labor ratio is independent of θ.
(3) Now assume that $A(t) = \exp(gt) A(0)$, and show that a balanced growth path (with constant capital share in national income and constant and equal rates of growth of output, capital and consumption) exists only if F takes the Cobb-Douglas form, $Y(t) = (A(t) K(t))^\gamma (L(t))^{1-\gamma}$.
(4) Characterize the balanced growth path in the Cobb-Douglas case. Derive the common growth rate of output, capital and consumption. Explain why the (normalized) steady-state capital-labor ratio now depends on θ.

EXERCISE 8.11. Consider the baseline neoclassical model with no technological progress.

(1) Show that in the neighborhood of the steady state k^*, the law of motion of $k(t) \equiv K(t)/L(t)$ can be written as

$$\log[k(t)] = \log[k^*] + \eta_1 \exp(\xi_1 t) + \eta_2 \exp(\xi_2 t),$$

where ξ_1 and ξ_2 are the eigenvalues of the linearized system.
(2) Compute these eigenvalues show that one of them, say ξ_2, is positive.
(3) What does this imply about the value of η_2?
(4) How is the value of η_1 determined?

(5) What determines the speed of adjustment of $k(t)$ towards its steady-state value k^*?

Exercise 8.12. Derive closed-form equations for the solution to the differential equations of transitional dynamics presented in Example 8.2 with log preferences.

Exercise 8.13. (1) Analyze the comparative dynamics of the basic model in response to unanticipated increase in the rate of labor-augmenting technological progress will increase to $g' > g$. Does consumption increase or decrease upon impact?

(2) Analyze the comparative dynamics in response to the announcement at time T that at some future date $T' > T$ the discount rate will decline to $\rho' < \rho$. Does consumption increase or decrease at time T. Explain.

Exercise 8.14. Consider the basic neoclassical growth model with technological change and CRRA preferences (8.30). Explain why $\theta > 1$ ensures that the transversality condition is always satisfied.

Exercise 8.15. Consider a variant of the neoclassical economy with preferences given by

$$U(0) = \int_0^\infty \exp(-\rho t) \frac{(c(t) - \gamma)^{1-\theta} - 1}{1 - \theta}$$

with $\gamma > 0$. There is no population growth. Assume that the production function is given by $Y(t) = F[K(t), A(t) L(t)]$, which satisfies all the standard assumptions and $A(t) = \exp(gt) A(0)$.

(1) Interpret the utility function.
(2) Define the competitive equilibrium for this economy.
(3) Characterize the equilibrium of this economy. Does a balanced growth path with positive growth in consumption exist? Why or why not?
(4) Derive a parameter restriction ensuring that the standard transversality condition is satisfied.
(5) Characterize the transitional dynamics of the economy.

Exercise 8.16. Consider a world consisting of a collection of closed neoclassical economies $\mathcal{J}$. Each $j \in \mathcal{J}$ has access to the same neoclassical production technology and admits a representative household with preferences $(1-\theta)^{-1} \int_0^\infty \exp(-\rho_j t) \left(c_j^{1-\theta} - 1\right) dt$. Characterize the cross-country differences in income per capita in this economy. What is the effect of the 10% difference in discount factor (e.g., a difference between a discount rate of 0.02 versus 0.022) on steady-state per capita income differences? [Hint: use the fact that the capital share of income is about 1/3].

Exercise 8.17. Consider the standard neoclassical growth model augmented with labor supply decisions. In particular, there is a total population normalized to 1, and all individuals have utility function

$$U(0) = \int_0^\infty \exp(-\rho t) u(c(t), 1 - l(t)),$$

where $l(t) \in (0,1)$ is labor supply. In a symmetric equilibrium, employment $L(t)$ is equal to $l(t)$. Assume that the production function is given by $Y(t) = F[K(t), A(t) L(t)]$, which satisfies all the standard assumptions and $A(t) = \exp(gt) A(0)$.

(1) Define a competitive equilibrium.
(2) Set up the current value Hamiltonian that each household solves taking wages and interest rates as given, and determine first-order conditions for the allocation of consumption over time and leisure-labor trade off.
(3) Set up the current value Hamiltonian for a planner maximizing the utility of the representative household, and derive the necessary conditions for an optimal solution.
(4) Show that the two problems are equivalent given competitive markets.
(5) Show that unless the utility function is asymptotically equal to

$$u(c(t), 1-l(t)) = \begin{cases} \frac{Ac(t)^{1-\theta}}{1-\theta} h(1-l(t)) & \text{for } \theta \neq 1, \\ A \log c(t) + Bh(1-l(t)) & \text{for } \theta = 1, \end{cases}$$

for some $h(\cdot)$ with $h'(\cdot) > 0$, there will not exist a balanced growth path with constant and equal rates of consumption and output growth, and a constant level of labor supply.

EXERCISE 8.18. Consider the standard neoclassical growth model with a representative household with preferences

$$U(0) = \int_0^\infty \exp(-\rho t) \left[\frac{c(t)^{1-\theta} - 1}{1-\theta} + G(t) \right],$$

where $G(t)$ is a public good financed by government spending. Assume that the production function is given by $Y(t) = F[K(t), L(t)]$, which satisfies all the standard assumptions, and the budget set of the representative household is $C(t) + I(t) \leq Y(t)$, where $I(t)$ is private investment. Assume that $G(t)$ is financed by taxes on investment. In particular, the capital accumulation equation is

$$\dot{K}(t) = (1 - \tau(t)) I(t) - \delta K(t),$$

and the fraction $\tau(t)$ of the private investment $I(t)$ is used to finance the public good, i.e., $G(t) = \tau(t) I(t)$.

Take the sequence of tax rates $[\tau(t)]_{t=0}^{\infty}$ as given.

(1) Define a competitive equilibrium.
(2) Set up the individual maximization problem and characterize consumption and investment behavior.
(3) Assuming that $\lim_{t\to\infty} \tau(t) = \tau$, characterize the steady state.

(4) What value of τ maximizes the level of utility at the steady state. Starting away from the state state, is this also the tax rate that would maximize the initial utility level? Why or why not?

Exercise 8.19. Consider the neoclassical growth model with a government that needs to finance a flow expenditure of G. Suppose that government spending does not affect utility and that the government can finance this expenditure by using lump-sum taxes (that is, some amount $\mathcal{T}(t)$ imposed on each household at time t irrespective of their income level and capital holdings) and debt, so that the government budget constraint takes the form

$$\dot{b}(t) = r(t) b(t) + g - \mathcal{T}(t),$$

where $b(t)$ denotes its debt level. The no-Ponzi-game condition for the government is

$$\lim_{t\to\infty}\left[b(t)\exp\left(-\int_0^t (r(s)-n)\,ds\right)\right] = 0.$$

Prove the following *Ricardian equivalence* result: for any sequence of lump-sum taxes $[\mathcal{T}(t)]_{t=0}^{\infty}$ that satisfy the government's budget constraint (together with the no-Ponzi-game condition) leads of the same equilibrium sequence of capital-labor ratio and consumption. Interpret this result.

Exercise 8.20. Consider the baseline neoclassical growth model with no population growth and no technological change, and preferences given by the standard CRRA utility function (8.30). Assume, however, that the representative household can borrow and lend at the exogenously given international interest rate r^*. Characterize the steady state equilibrium and transitional dynamics in this economy. Show that if the economy starts with less capital than its steady state level it will immediately jump to the steady state level by borrowing internationally. How will the economy repay this debt?

Exercise 8.21. Modify the neoclassical economy (without technological change) by introducing cost of adjustment in investment as in the q-theory of investment studied in the previous chapter. Characterize the steady-state equilibrium and the transitional dynamics. What are the differences between the implications of this model and those of the baseline neoclassical model?

Exercise 8.22. * Consider a version of the neoclassical model that admits a representative household with preferences given by (8.30), no population growth and no technological progress. The main difference from the standard model is that there are multiple capital goods. In particular, suppose that the production function of the economy is given by

$$Y(t) = F(K_1(t), ..., K_M(t), L(t)),$$

where K_m denotes the m^{th} type of capital and L is labor. F is homogeneous of degree 1 in all of its variables. Capital in each sector accumulates in the standard fashion, with

$$\dot{K}_m(t) = I_m(t) - \delta_m K_m(t),$$

for $m = 1, ..., M$. The resource constraint of the economy is

$$C(t) + \sum_{m=1}^{M} I_m(t) \leq Y(t)$$

for all t.

(1) Write budget constraint of the representative household in this economy. Show that this can be done in two alternative and equivalent ways; first, with M separate assets, and second with only a single asset that is a claim to all of the capital in the economy.
(2) Define an equilibrium.
(3) Characterize the equilibrium by specifying the profit-maximizing decision of firms in each sector and the dynamic optimization problem of consumers.
(4) Write down the optimal growth problem in the form of a multi-dimensional current-value Hamiltonian and show that the optimum growth problem coincides with the equilibrium growth problem. Interpret this result.
(5) Characterize the transitional dynamics in this economy. Define and discuss the appropriate notion of saddle-path stability and show that the equilibrium is always saddle-path stable and the equilibrium dynamics can be reduced to those in the one-sector neoclassical growth model.
(6) Characterize the transitional dynamics under the additional assumption that investment is irreversible in each sector, i.e., $I_m(t) \geq 0$ for all t and each $m = 1, ..., M$.

Exercise 8.23. Contrast the effects of taxing capital income at the rate τ in the Solow growth model and the neoclassical growth model. Show that capital income taxes have no effect in the former, while they depress the effective capital-labor ratio in the latter. Explain why there is such a difference.

Exercise 8.24. Let us return to the discrete time version of the neoclassical growth model. Suppose that the economy admits a representative household with log preferences (i.e., $\theta = 1$ in terms of (8.30)) and the production function is Cobb-Douglas. Assume also that $\delta = 1$, so that there is full depreciation. Characterize the steady-state equilibrium and derive a difference equation that explicitly characterizes the behavior of capital stock away from the steady state.

Exercise 8.25. Again in the discrete time version of the neoclassical growth model, suppose that there is labor-augmenting technological progress at the rate g, i.e.,

$$A(t+1) = (1+g)A(t).$$

For simplicity, suppose that there is no population growth.

(1) Prove that balanced growth requires preferences to take the CRRA form
$$U(0) = \begin{cases} \sum_{t=0}^{\infty} \beta^t \frac{[c(t)]^{1-\theta}-1}{1-\theta} & \text{if } \theta \neq 1 \text{ and } \theta \geq 0 \\ \sum_{t=0}^{\infty} \beta^t \ln c(t) & \text{if } \theta = 1 \end{cases}.$$

(2) Assuming this form of preferences, prove that there exists a unique steady-state equilibrium in which effective capital-labor ratio remains constant.

(3) Prove that this steady-state equilibrium is globally stable and convergence to this steady-state starting from a non-steady-state level of effective capital-labor ratio is monotonic.

CHAPTER 9

Growth with Overlapping Generations

A key feature of the neoclassical growth model analyzed in the previous chapter is that it admits a representative household. This model is useful as it provides us with a tractable framework for the analysis of capital accumulation. Moreover, it enables us to appeal to the First and Second Welfare Theorems to establish the equivalence between equilibrium and optimum growth problems. In many situations, however, the assumption of a representative household is not appropriate. One important set of circumstances where we may wish to depart from this assumption is in the analysis of an economy in which new households arrive (or are born) over time. The arrival of new households in the economy is not only a realistic feature, but it also introduces a range of new economic interactions. In particular, decisions made by older "generations" will affect the prices faced by younger "generations". These economic interactions have no counterpart in the neoclassical growth model. They are most succinctly captured in the *overlapping generations models* introduced and studied by Paul Samuelson and then later Peter Diamond.

These models are useful for a number of reasons; first, they capture the potential interaction of different generations of individuals in the marketplace; second they provide a tractable alternative to the infinite-horizon representative agent models; third, as we will see, some of their key implications are different from those of the neoclassical growth model; fourth, the dynamics of capital accumulation and consumption in some special cases of these models will be quite similar to the basic Solow model rather than the neoclassical model; and finally these models generate new insights about the role of national debt and Social Security in the economy.

We start with an illustration of why the First Welfare Theorem cannot be applied in overlapping generations models. We then discuss the baseline overlapping generations model and and a number of applications of this framework. Finally, we will discuss the overlapping generations model in continuous time. This latter model, originally developed by Menahem Yaari and Olivier Blanchard and also referred to as the *perpetual youth* model, is a tractable alternative to the basic overlapping generations model and also has a number of different implications. It will also be used in the context of human capital investments in the next chapter.

9.1. Problems of Infinity

Let us discuss the following abstract general equilibrium economy introduced by Karl Shell. We will see that the baseline overlapping generations model of Samuelson and Diamond is very closely related to this abstract economy.

Consider the following static economy with a countably infinite number of households, each denoted by $i \in \mathbb{N}$, and a countably infinite number of commodities, denoted by $j \in \mathbb{N}$. Assume that all households behave competitively (alternatively, we can assume that there are M households of each type, and M is a large number). Household i has preferences given by:

$$u_i = c_i^i + c_{i+1}^i,$$

where c_j^i denotes the consumption of the jth type of commodity by household i. These preferences imply that household i enjoys the consumption of the commodity with the same index as its own index and the next indexed commodity (i.e., if an individual's index is 3, she only derives utility from the consumption of goods indexed 3 and 4, etc.).

The endowment vector $\boldsymbol{\omega}$ of the economy is as follows: each household has one unit endowment of the commodity with the same index as its index. Let us choose the price of the first commodity as the numeraire, i.e., $p_0 = 1$.

The following proposition characterizes a competitive equilibrium. Exercise 9.1 asks you to prove that this is the unique competitive equilibrium in this economy.

Proposition 9.1. *In the above-described economy, the price vector $\bar{\mathbf{p}}$ such that $\bar{p}_j = 1$ for all $j \in \mathbb{N}$ is a competitive equilibrium price vector and induces an equilibrium with no trade, denoted by $\bar{\mathbf{x}}$.*

Proof. At $\bar{\mathbf{p}}$, each household has income equal to 1. Therefore, the budget constraint of household i can be written as

$$c_i^i + c_{i+1}^i \leq 1.$$

This implies that consuming own endowment is optimal for each household, establishing that the price vector $\bar{\mathbf{p}}$ and no trade, $\bar{\mathbf{x}}$, constitute a competitive equilibrium. □

However, the competitive equilibrium in Proposition 9.1 is not Pareto optimal. To see this, consider the following alternative allocation, $\tilde{\mathbf{x}}$. Household $i = 0$ consumes one unit of good $j = 0$ and one unit of good $j = 1$, and household $i > 0$ consumes one unit of good $i + 1$. In other words, household $i = 0$ consumes its own endowment and that of household 1, while all other households, indexed $i > 0$, consume the endowment of than neighboring household, $i + 1$. In this allocation, all households with $i > 0$ are as well off as in the competitive equilibrium $(\bar{\mathbf{p}}, \bar{\mathbf{x}})$, and individual $i = 0$ is strictly better-off. This establishes:

Proposition 9.2. *In the above-described economy, the competitive equilibrium at* $(\bar{\mathbf{p}}, \bar{\mathbf{x}})$ *is not Pareto optimal.*

So why does the First Welfare Theorem not apply in this economy? Recall that the first version of this theorem, Theorem 5.5, was stated under the assumption of a finite number of commodities, whereas we have an infinite number of commodities here. Clearly, the source of the problem must be related to the infinite number of commodities. The extended version of the First Welfare Theorem, Theorem 5.6, covers the case with an infinite number of commodities, but only under the assumption that $\sum_{j=0}^{\infty} p_j^* < \infty$, where p_j^* refers to the price of commodity j in the competitive equilibrium in question. It can be immediately verified that this assumption is not satisfied in the current example, since the competitive equilibrium in question features $p_j^* = 1$ for all $j \in \mathbb{N}$, so that $\sum_{j=0}^{\infty} p_j^* = \infty$. As discussed in Chapter 5, when the sum of prices is equal to infinity, there can be feasible allocations for the economy as a whole that Pareto dominate the competitive equilibrium. The economy discussed here gives a simple example where this happens.

If the failure of the First Welfare Theorem were a specific feature of this abstract (perhaps artificial) economy, it would not have been of great interest to us. However, this abstract economy is very similar (in fact "isomorphic") to the baseline overlapping generations model. Therefore, the source of inefficiency (Pareto suboptimality of the competitive equilibrium) in this economy will be the source of potential inefficiencies in the baseline overlapping generations model.

It is also useful to recall that, in contrast to Theorem 5.6, the Second Welfare Theorem, Theorem 5.7, did not make use of the assumption that $\sum_{j=0}^{\infty} p_j^* < \infty$ even when the number of commodities was infinite. Instead, this theorem made use of the convexity of preferences, consumption sets and production possibilities sets. So one might conjecture that in this model, which is clearly an exchange economy with convex preferences and convex consumption sets, Pareto optima must be decentralizable by some redistribution of endowments (even though competitive equilibrium may be Pareto suboptimal). This is in fact true, and the following proposition shows how the Pareto optimal allocation $\tilde{\mathbf{x}}$ described above can be decentralized as a competitive equilibrium:

Proposition 9.3. *In the above-described economy, there exists a reallocation of the endowment vector* $\boldsymbol{\omega}$ *to* $\tilde{\boldsymbol{\omega}}$*, and an associated competitive equilibrium* $(\bar{\mathbf{p}}, \tilde{\mathbf{x}})$ *that is Pareto optimal where* $\tilde{\mathbf{x}}$ *is as described above, and* $\bar{\mathbf{p}}$ *is such that* $\bar{p}_j = 1$ *for all* $j \in \mathbb{N}$.

Proof. Consider the following reallocation of the endowment vector $\boldsymbol{\omega}$: the endowment of household $i \geq 1$ is given to household $i-1$. Consequently, at the new endowment vector $\tilde{\boldsymbol{\omega}}$, household $i = 0$ has one unit of good $j = 0$ and one unit of good $j = 1$, while all other households i have one unit of good $i+1$. At the price vector $\bar{\mathbf{p}}$, household 0 has a budget set

$$c_0^0 + c_1^1 \leq 2,$$

thus chooses $c_0^0 = c_1^0 = 1$. All other households have budget sets given by

$$c_i^i + c_{i+1}^i \leq 1,$$

thus it is optimal for each household $i > 0$ to consume one unit of the good c_{i+1}^i, which is within its budget set and gives as high utility as any other allocation within his budget set, establishing that $\tilde{\mathbf{x}}$ is a competitive equilibrium. □

9.2. The Baseline Overlapping Generations Model

We now discuss the baseline two-period overlapping generation economy.

9.2.1. Demographics, Preferences and Technology. In this economy, time is discrete and runs to infinity. Each individual lives for two periods. For example, all individuals born at time t live for dates t and $t+1$. For now let us assume a general (separable) utility function for individuals born at date t, of the form

$$U(t) = u(c_1(t)) + \beta u(c_2(t+1)), \tag{9.1}$$

where $u : \mathbb{R}_+ \to \mathbb{R}$ satisfies the conditions in Assumption 3, $c_1(t)$ denotes the consumption of the individual born at time t when young (which is at date t), and $c_2(t+1)$ is the individual's consumption when old (at date $t+1$). Also $\beta \in (0,1)$ is the discount factor.

Factor markets are competitive. Individuals can only work in the first period of their lives and supply one unit of labor inelastically, earning the equilibrium wage rate $w(t)$.

Let us also assume that there is exponential population growth, so that total population is

$$L(t) = (1+n)^t L(0). \tag{9.2}$$

The production side of the economy is the same as before, characterized by a set of competitive firms, and is represented by a standard constant returns to scale aggregate production function, satisfying Assumptions 1 and 2;

$$Y(t) = F(K(t), L(t)).$$

To simplify the analysis let us assume that $\delta = 1$, so that capital fully depreciates after use (see Exercise 9.3). Thus, again defining $k \equiv K/L$, the (gross) rate of return to saving, which equals the rental rate of capital, is given by

$$1 + r(t) = R(t) = f'(k(t)), \tag{9.3}$$

where $f(k) \equiv F(k,1)$ is the standard per capita production function. As usual, the wage rate is

$$w(t) = f(k(t)) - k(t) f'(k(t)). \tag{9.4}$$

9.2.2. Consumption Decisions. Let us start with the individual consumption decisions. Savings by an individual of generation t, $s(t)$, is determined as a solution to the following maximization problem

$$\max_{c_1(t), c_2(t+1), s(t)} u(c_1(t)) + \beta u(c_2(t+1))$$

subject to

$$c_1(t) + s(t) \leq w(t)$$

and

$$c_2(t+1) \leq R(t+1) s(t),$$

where we are using the convention that old individuals rent their savings of time t as capital to firms at time $t+1$, so they receive the gross rate of return $R(t+1) = 1 + r(t+1)$ (we use R instead of $1+r$ throughout to simplify notation). The second constraint incorporates the notion that individuals will only spend money on their own end of life consumption (since there is no altruism or bequest motive). There is no need to introduce the additional constraint that $s(t) \geq 0$, since negative savings would violate the second-period budget constraint (given that $c_2(t+1) \geq 0$).

Since the utility function $u(\cdot)$ is strictly increasing (Assumption 3), both constraints will hold as equalities. Therefore, the first-order condition for a maximum can be written in the familiar form of the consumption Euler equation (for the discrete time problem, recall Chapter 6),

$$u'(c_1(t)) = \beta R(t+1) u'(c_2(t+1)). \tag{9.5}$$

Moreover, since the problem of each individual is strictly concave, this Euler equation is sufficient to characterize an optimal consumption path given market prices.

Solving this equations for consumption and thus for savings, we obtain the following implicit function that determines savings per person as

$$s(t) = s(w(t), R(t+1)), \tag{9.6}$$

where $s : \mathbb{R}^2_+ \to \mathbb{R}$ is strictly increasing in its first argument and may be increasing or decreasing in its second argument (see Exercise 9.4). Total savings in the economy will be equal to

$$S(t) = s(t) L(t),$$

where $L(t)$ denotes the size of generation t, who are saving for time $t+1$. Since capital depreciates fully after use and all new savings are invested in the only productive asset of the economy, capital, the law of motion of the capital stock is given by

$$K(t+1) = L(t)\, s(w(t), R(t+1)). \tag{9.7}$$

9.2.3. Equilibrium. A competitive equilibrium in the overlapping generations economy can be defined as follows:

Definition 9.1. *A competitive equilibrium can be represented by a sequence of aggregate capital stocks, individual consumption and factor prices,* $\{K(t), c_1(t), c_2(t), R(t), w(t)\}_{t=0}^{\infty}$*, such that the factor price sequence* $\{R(t), w(t)\}_{t=0}^{\infty}$ *is given by (9.3) and (9.4), individual consumption decisions* $\{c_1(t), c_2(t)\}_{t=0}^{\infty}$ *are given by (9.5) and (9.6), and the aggregate capital stock,* $\{K(t)\}_{t=0}^{\infty}$*, evolves according to (9.7).*

A steady-state equilibrium is defined in the usual fashion, as an equilibrium in which the capital-labor ratio $k \equiv K/L$ is constant.

To characterize the equilibrium, divide (9.7) by labor supply at time $t+1$, $L(t+1) = (1+n)\,L(t)$, to obtain the capital-labor ratio as

$$k(t+1) = \frac{s(w(t), R(t+1))}{1+n}.$$

Now substituting for $R(t+1)$ and $w(t)$ from (9.3) and (9.4), we obtain

$$k(t+1) = \frac{s\left(f(k(t)) - k(t) f'(k(t)), f'(k(t+1))\right)}{1+n} \tag{9.8}$$

as the fundamental law of motion of the overlapping generations economy. A steady state is given by a solution to this equation such that $k(t+1) = k(t) = k^*$, i.e.,

$$k^* = \frac{s\left(f(k^*) - k^* f'(k^*), f'(k^*)\right)}{1+n} \tag{9.9}$$

Since the savings function $s(\cdot,\cdot)$ can take any form, the difference equation (9.8) can lead to quite complicated dynamics, and multiple steady states are possible. The next figure shows some potential forms that equation (9.8) can take. The figure illustrates that the overlapping generations model can lead to a unique stable equilibrium, to multiple equilibria, or to an equilibrium with zero capital stock. In other words, without putting more structure on utility and/or production functions, the model makes few predictions.

9.2.4. Restrictions on Utility and Production Functions. In this subsection, we characterize the steady-state equilibrium and transition dynamics when further assumptions are imposed on the utility and production functions. In particular, let us suppose that the utility functions take the familiar CRRA form:

$$U(t) = \frac{c_1(t)^{1-\theta} - 1}{1-\theta} + \beta\left(\frac{c_2(t+1)^{1-\theta} - 1}{1-\theta}\right), \tag{9.10}$$

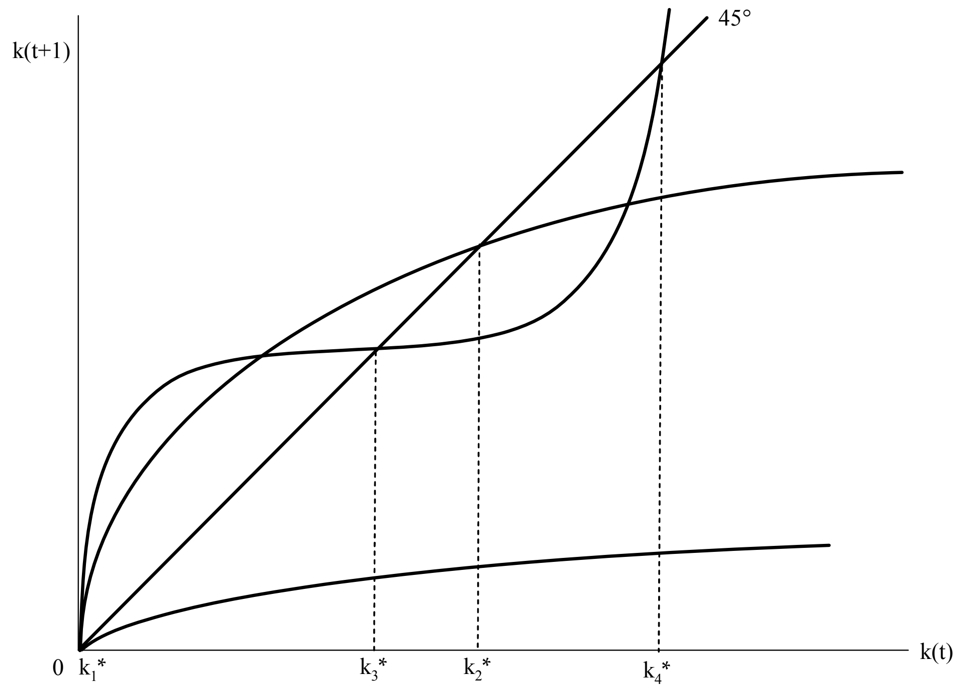

Figure 9.1. Various types of steady-state equilibria in the baseline overlapping generations model.

where $\theta > 0$ and $\beta \in (0,1)$. Furthermore, assume that technology is Cobb-Douglas, so that

$$f(k) = k^{\alpha}$$

The rest of the environment is as described above. The CRRA utility simplifies the first-order condition for consumer optimization and implies

$$\frac{c_2(t+1)}{c_1(t)} = (\beta R(t+1))^{1/\theta}.$$

Once again, this expression is the discrete-time consumption Euler equation from Chapter 6, now for the CRRA utility function. This Euler equation can be alternatively expressed in terms of savings as

$$s(t)^{-\theta} \beta R(t+1)^{1-\theta} = (w(t) - s(t))^{-\theta}, \tag{9.11}$$

which gives the following equation for the saving rate:

$$s(t) = \frac{w(t)}{\psi(t+1)}, \tag{9.12}$$

where

$$\psi(t+1) \equiv [1 + \beta^{-1/\theta} R(t+1)^{-(1-\theta)/\theta}] > 1,$$

which ensures that savings are always less than earnings. The impact of factor prices on savings is summarized by the following and derivatives:

$$s_w \equiv \frac{\partial s(t)}{\partial w(t)} = \frac{1}{\psi(t+1)} \in (0,1),$$
$$s_r \equiv \frac{\partial s(t)}{\partial R(t+1)} = \left(\frac{1-\theta}{\theta}\right)(\beta R(t+1))^{-1/\theta}\frac{s(t)}{\psi(t+1)}.$$

Since $\psi(t+1) > 1$, we also have that $0 < s_w < 1$. Moreover, in this case $s_r > 0$ if $\theta < 1$, $s_r < 0$ if $\theta > 1$, and $s_r = 0$ if $\theta = 1$. The relationship between the rate of return on savings and the level of savings reflects the counteracting influences of income and substitution effects. The case of $\theta = 1$ (log preferences) is of special importance and is often used in many applied models. This special case is sufficiently common that it may deserve to be called the *canonical overlapping generations model* and will be analyzed separately in the next section.

In the current somewhat more general context, equation (9.8) implies

$$(9.13)\qquad k(t+1) = \frac{s(t)}{(1+n)} = \frac{w(t)}{(1+n)\psi(t+1)},$$

or more explicitly,

$$(9.14)\qquad k(t+1) = \frac{f(k(t)) - k(t)f'(k(t))}{(1+n)\left[1+\beta^{-1/\theta}f'(k(t+1))^{-(1-\theta)/\theta}\right]}$$

The steady state then involves a solution to the following implicit equation:

$$k^* = \frac{f(k^*) - k^*f'(k^*)}{(1+n)\left[1+\beta^{-1/\theta}f'(k^*)^{-(1-\theta)/\theta}\right]}.$$

Now using the Cobb-Douglas formula, we have that the steady state is the solution to the equation

$$(9.15)\qquad (1+n)\left[1+\beta^{-1/\theta}\left(\alpha(k^*)^{\alpha-1}\right)^{(\theta-1)/\theta}\right] = (1-\alpha)(k^*)^{\alpha-1}.$$

For simplicity, define $R^* \equiv \alpha(k^*)^{\alpha-1}$ as the marginal product of capital in steady-state, in which case, equation (9.15) can be rewritten as

$$(9.16)\qquad (1+n)\left[1+\beta^{-1/\theta}(R^*)^{(\theta-1)/\theta}\right] = \frac{1-\alpha}{\alpha}R^*.$$

The steady-state value of R^*, and thus k^*, can now be determined from equation (9.16), which always has a unique solution. We can next investigate the stability of this steady state. To do this, substitute for the Cobb-Douglas production function in (9.14) to obtain

$$(9.17)\qquad k(t+1) = \frac{(1-\alpha)k(t)^{\alpha}}{(1+n)\left[1+\beta^{-1/\theta}\left(\alpha k(t+1)^{\alpha-1}\right)^{-(1-\theta)/\theta}\right]}.$$

Using (9.17), we can establish the following proposition can be proved:[1]

[1] In this proposition and throughout the rest of this chapter, we again ignore the trivial steady state with $k = 0$.

Proposition 9.4. *In the overlapping-generations model with two-period lived households, Cobb-Douglas technology and CRRA preferences, there exists a unique steady-state equilibrium with the capital-labor ratio k^* given by (9.15) and as long as $\theta \geq 1$, this steady-state equilibrium is globally stable for all $k(0) > 0$.*

Proof. See Exercise 9.5 □

In this particular (well-behaved) case, equilibrium dynamics are very similar to the basic Solow model, and are shown in Figure 9.2, which is specifically drawn for the canonical overlapping generations model of the next section. The figure shows that convergence to the unique steady-state capital-labor ratio, k^*, is monotonic. In particular, starting with an initial capital-labor ratio $k(0) < k^*$, the overlapping generations economy steadily accumulates more capital and converge to k^*. Starting with $k'(0) > k^*$, the equilibrium involves lower and lower levels of capital-labor ratio, ultimately converging to k^*.

9.3. The Canonical Overlapping Generations Model

Even the model with CRRA utility and Cobb-Douglas production function is relatively messy. For this reason, many of the applications of the overlapping generations model use an even more specific utility function, log preferences (or equivalently $\theta = 1$ in terms of the CRRA preferences of the last section). Log preferences are particularly useful in this context, since they ensure that income and substitution effects exactly cancel each other out, so that changes in the interest rate (and therefore changes in the capital-labor ratio of the economy) have no effect on the saving rate.

Since this version of the model is sufficiently common, it may deserve to be called the canonical overlapping generations model and will be the focus of this section. Another interesting feature of this model is that the structure of the equilibrium is essentially identical to the basic Solow model we studied in Chapter 2.

Suppose that the utility of the household and generation t is given by

$$U(t) = \log c_1(t) + \beta \log c_2(t+1), \tag{9.18}$$

where as before $\beta \in (0,1)$ (even though $\beta \geq 1$ could be allowed here without any change in the analysis). The aggregate production technology is again Cobb-Douglas, that is, $f(k) = k^\alpha$. The consumption Euler equation now becomes even simpler:

$$\frac{c_2(t+1)}{c_1(t)} = \beta R(t+1)$$

and implies that savings should satisfy the equation

$$s(t) = \frac{\beta}{1+\beta} w(t), \tag{9.19}$$

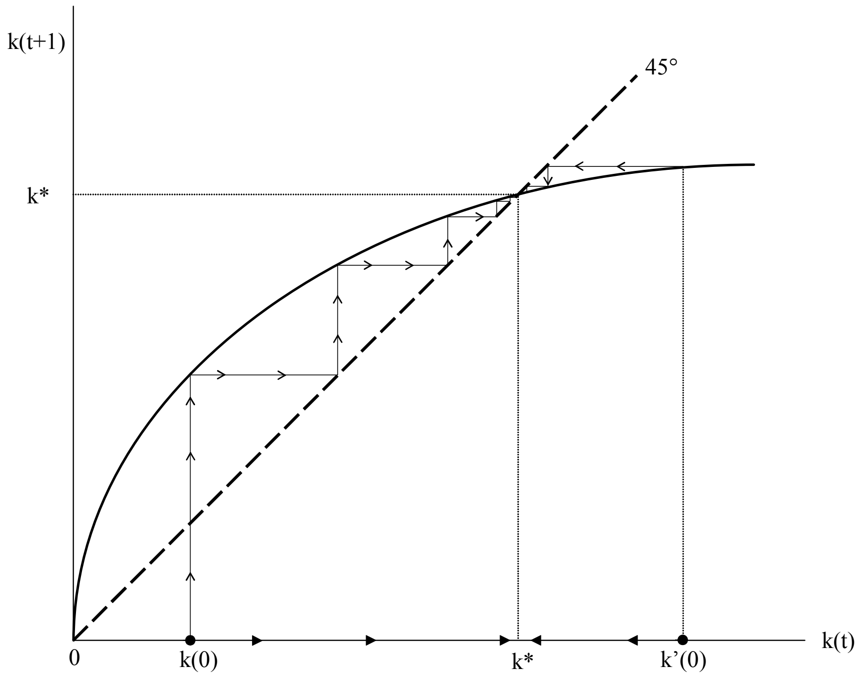

Figure 9.2. Equilibrium dynamics in the canonical overlapping generations model.

which corresponds to a constant saving rate, equal to $\beta/(1+\beta)$, out of labor income for each individual. This constant saving rate makes this model very similar to the baseline Solow growth model of Chapter 2.

Now combining this with the capital accumulation equation (9.8), we obtain

$$\begin{aligned} k(t+1) &= \frac{s(t)}{(1+n)} \\ &= \frac{\beta w(t)}{(1+n)(1+\beta)} \\ &= \frac{\beta(1-\alpha)[k(t)]^{\alpha}}{(1+n)(1+\beta)}, \end{aligned}$$

where the second line uses (9.19) and the last line uses the fact that, given competitive factor markets, the wage rate is equal to $w(t) = (1-\alpha)[k(t)]^{\alpha}$.

It is straightforward to verify that there exists a unique steady state with capital-labor ratio given by

$$k^* = \left[\frac{\beta(1-\alpha)}{(1+n)(1+\beta)}\right]^{\frac{1}{1-\alpha}}. \tag{9.20}$$

Moreover, starting with any $k(0) > 0$, equilibrium dynamics are identical to those of the basic Solow model and monotonically converge to k^*. This is illustrated in Figure 9.2 and stated in the next proposition:

Proposition 9.5. *In the canonical overlapping generations model with log preferences and Cobb-Douglas technology, there exists a unique steady state, with capital-labor ratio* k^* *given by (9.20). Starting with any* $k(0) \in (0, k^*)$*, equilibrium dynamics are such that* $k(t) \uparrow k^*$*, and starting with any* $k'(0) > k^*$*, equilibrium dynamics involve* $k(t) \downarrow k^*$*.*

Exercise 9.6 asks you to introduce technological progress into this canonical model and to conduct a range of comparative static exercises. Exercise 9.7, on the other hand, asks you to analyze the same economy without the Cobb-Douglas technology assumption.

9.4. Overaccumulation and Pareto Optimality of Competitive Equilibrium in the Overlapping Generations Model

Let us now return to the general problem, and compare the overlapping-generations equilibrium to the choice of a social planner wishing to maximize a weighted average of all generations' utilities. In particular, suppose that the social planner maximizes

$$\sum_{t=0}^{\infty} \beta_S^t U(t)$$

where β_S is the discount factor of the social planner, which reflects how she values the utilities of different generations. Substituting from (9.1), this implies:

$$\sum_{t=0}^{\infty} \beta_S^t \left(u(c_1(t)) + \beta u(c_2(t+1))\right)$$

subject to the resource constraint

$$F(K(t), L(t)) = K(t+1) + L(t) c_1(t) + L(t-1) c_2(t).$$

Dividing this by $L(t)$ and using (9.2), the resource constraint can be written in per capita terms as

$$f(k(t)) = (1+n) k(t+1) + c_1(t) + \frac{c_2(t)}{1+n}.$$

The social planner's maximization problem then implies the following first-order necessary condition:

$$u'(c_1(t)) = \beta f'(k(t+1)) u'(c_2(t+1)).$$

Since $R(t+1) = f'(k(t+1))$, this is identical to (9.5). This result is not surprising; the social planner prefers to allocate consumption of a given individual in exactly the same way as the individual himself would do; there are no "market failures" in the over-time allocation of consumption at given prices.

However, the social planner's and the competitive economy's allocations across generations will differ, since the social planner is giving different weights to different generations as captured by the parameter β_S. In particular, it can be shown that the socially planned economy will converge to a steady state with capital-labor ratio k^S such that

$$\beta_S f'\left(k^S\right) = 1 + n,$$

which is similar to the modified golden rule we saw in the context of the Ramsey growth model in discrete time (cf., Chapter 6). In particular, the steady-state level of capital-labor ratio k^S chosen by the social planner does not depend on preferences (i.e., on the utility function $u(\cdot)$) and does not even depend on the individual rate of time preference, β. Clearly, k^S will typically differ from the steady-state value of the competitive economy, k^*, given by (9.9).

More interesting is the question of whether the competitive equilibrium is Pareto optimal. The example in Section 9.1 suggests that it may not be. Exactly as in that example, we cannot use the First Welfare Theorem, Theorem 5.6, because there is an infinite number of commodities and the sum of their prices is not necessarily less than infinity.

In fact, the competitive equilibrium is not in general Pareto optimal. The simplest way of seeing this is that the steady state level of capital stock, k^*, given by (9.9), can be so high that it is in fact greater than k_{gold}. Recall that k_{gold} is the golden rule level of capital-labor ratio that maximizes the steady-state level of consumption (recall, for example, Figure 8.1 in Chapter 8 for the discussion in Chapter 2). When $k^* > k_{gold}$, reducing savings can increase consumption for every generation.

More specifically, note that in steady state we have

$$\begin{aligned} f(k^*) - (1+n)k^* &= c_1^* + (1+n)^{-1} c_2^* \\ &\equiv c^*, \end{aligned}$$

where the first line follows by national income accounting, and the second defines c^* as the total steady-state consumption. Therefore

$$\frac{\partial c^*}{\partial k^*} = f'(k^*) - (1+n)$$

and k_{gold} is defined as

$$f'(k_{gold}) = 1 + n.$$

Now if $k^* > k_{gold}$, then $\partial c^* / \partial k^* < 0$, so reducing savings can increase (total) consumption for everybody. If this is the case, the economy is referred to as *dynamically inefficient*—it involves overaccumulation. Another way of expressing dynamic inefficiency is that

$$r^* < n,$$

that is, the steady-state (net) interest rate $r^* = R^* - 1$ is less than the rate of population growth. Recall that in the infinite-horizon Ramsey economy, the transversality condition (which follows from individual optimization) required that $r > g + n$, therefore, dynamic inefficiency could never arise in this Ramsey economy. Dynamic inefficiency arises because of the heterogeneity inherent in the overlapping generations model, which removes the transversality condition.

In particular, suppose we start from steady state at time T with $k^* > k_{gold}$. Consider the following variation where the capital stock for next period is reduced by a small amount. In particular, change next period's capital stock by $-\Delta k$, where $\Delta k > 0$, and from then on, imagine that we immediately move to a new steady state (which is clearly feasible). This implies the following changes in consumption levels:

$$\begin{aligned}\Delta c(T) &= (1+n)\,\Delta k > 0 \\ \Delta c(t) &= -\left(f'(k^* - \Delta k) - (1+n)\right)\Delta k \text{ for all } t > T\end{aligned}$$

The first expression reflects the direct increase in consumption due to the decrease in savings. In addition, since $k^* > k_{gold}$, for small enough Δk, $f'(k^* - \Delta k) - (1+n) < 0$, thus $\Delta c(t) > 0$ for all $t \geq T$, and explains the second expression. The increase in consumption for each generation can be allocated equally during the two periods of their lives, thus necessarily increasing the utility of all generations. This variation clearly creates a Pareto improvement in which all generations are better-off. This establishes:

Proposition 9.6. *In the baseline overlapping-generations economy, the competitive equilibrium is not necessarily Pareto optimal. More specifically, whenever $r^* < n$ and the economy is dynamically inefficient, it is possible to reduce the capital stock starting from the competitive steady state and increase the consumption level of all generations.*

As the above derivation makes it clear, Pareto inefficiency of the competitive equilibrium is intimately linked with *dynamic inefficiency.* Dynamic inefficiency, the rate of interest being less than the rate of population growth, is not a theoretical curiosity. Exercise 9.8 shows that dynamic inefficiency can arise under reasonable circumstances.

Loosely speaking, the intuition for dynamic inefficiency can be given as follows. Individuals who live at time t face prices determined by the capital stock with which they are working. This capital stock is the outcome of actions taken by previous generations. Therefore, there is a pecuniary externality from the actions of previous generations affecting the welfare of the current generation. Pecuniary externalities are typically second-order and do not matter for welfare (in a sense this could be viewed as the essence of the First Welfare Theorem). This ceases to be the case, however, when there are an infinite stream of newborn agents joining the economy. These agents are affected by the pecuniary externalities created by previous generations, and it is possible to rearrange accumulation decisions and consumption plans in such a way that these pecuniary externalities can be exploited.

A complementary intuition for dynamic inefficiency, which will be particularly useful in the next section, is as follows. Dynamic inefficiency arises from overaccumulation, which, in turn, is a result of the fact that the current young generation needs to save for old age. However, the more they save, the lower is the rate of return to capital and this may encourage them to save even more. Once again, the effect of the savings by the current generation on the future rate of return to capital is a pecuniary externality on the next generation. We may reason that this pecuniary externality should not lead to Pareto suboptimal allocations, as in the equilibria of standard competitive economies with a finite number of commodities and households. But this reasoning is no longer correct when there are an infinite number of commodities and an infinite number of households. This second intuition also suggests that if, somehow, alternative ways of providing consumption to individuals in old age were introduced, the overaccumulation problem could be solved or at least ameliorated. This is the topic of the next section.

9.5. Role of Social Security in Capital Accumulation

We now briefly discuss how Social Security can be introduced as a way of dealing with overaccumulation in the overlapping-generations model. We first consider a fully-funded system, in which the young make contributions to the Social Security system and their contributions are paid back to them in their old age. The alternative is an unfunded system or a *pay-as-you-go* Social Security system, where transfers from the young directly go to the current old. We will see that, as is typically presumed, pay-as-you-go (unfunded) Social Security discourages aggregate savings. However, when there is dynamic inefficiency, discouraging savings may lead to a Pareto improvement.

9.5.1. Fully Funded Social Security. In a fully funded Social Security system, the government at date t raises some amount $d(t)$ from the young, for example, by compulsory contributions to their Social Security accounts. These funds are invested in the only productive asset of the economy, the capital stock, and pays the workers when they are old an amount $R(t+1)\, d(t)$. This implies that the individual maximization problem under a fully funded social security system becomes

$$\max_{c_1(t), c_2(t+1), s(t)} u(c_1(t)) + \beta u(c_2(t+1))$$

subject to

$$c_1(t) + s(t) + d(t) \leq w(t)$$

and

$$c_2(t+1) \leq R(t+1)\,(s(t) + d(t)),$$

for a given choice of $d(t)$ by the government. Notice that now the total amount invested in capital accumulation is $s(t) + d(t) = (1+n)\, k(t+1)$.

It is also no longer the case that individuals will always choose $s(t) > 0$, since they have the income from Social Security. Therefore this economy can be analyzed under two alternative assumptions, with the constraint that $s(t) \geq 0$ and without.

It is clear that as long as $s(t)$ is free, whatever the sequence of feasible Social Security payments $\{d(t)\}_{t=0}^{\infty}$, the competitive equilibrium applies. When $s(t) \geq 0$ is imposed as a constraint, then the competitive equilibrium applies if given the sequence $\{d(t)\}_{t=0}^{\infty}$, the privately-optimal saving sequence $\{s(t)\}_{t=0}^{\infty}$ is such that $s(t) > 0$ for all t. Consequently, we have the following straightforward results:

Proposition 9.7. *Consider a fully funded Social Security system in the above-described environment whereby the government collects $d(t)$ from young individuals at date t.*

(1) *Suppose that $s(t) \geq 0$ for all t. If given the feasible sequence $\{d(t)\}_{t=0}^{\infty}$ of Social Security payments, the utility-maximizing sequence of savings $\{s(t)\}_{t=0}^{\infty}$ is such that $s(t) > 0$ for all t, then the set of competitive equilibria without Social Security are the set of competitive equilibria with Social Security.*

(2) *Without the constraint $s(t) \geq 0$, given any feasible sequence $\{d(t)\}_{t=0}^{\infty}$ of Social Security payments, the set of competitive equilibria without Social Security are the set of competitive equilibria with Social Security.*

Proof. See Exercise 9.10. □

This is very intuitive: the $d(t)$ taken out by the government is fully offset by a decrease in $s(t)$ as long as individuals were performing enough savings (or always when there are no constraints to force positive savings privately). Exercise 9.11 shows that even when there is the restriction that $s(t) \geq 0$, a funded Social Security program cannot lead to the Pareto improvement.

9.5.2. Unfunded Social Security. The situation is different with unfunded Social Security. Now we have that the government collects $d(t)$ from the young at time t and distributes this to the current old with per capita transfer $b(t) = (1+n)\,d(t)$ (which takes into account that there are more young than old because of population growth). Therefore, the individual maximization problem becomes

$$\max_{c_1(t), c_2(t+1), s(t)} u(c_1(t)) + \beta u(c_2(t+1))$$

subject to

$$c_1(t) + s(t) + d(t) \leq w(t)$$

and

$$c_2(t+1) \leq R(t+1)\,s(t) + (1+n)\,d(t+1),$$

for a given feasible sequence of Social Security payment levels $\{d(t)\}_{t=0}^{\infty}$.

What this implies is that the rate of return on Social Security payments is n rather than $r(t+1) = R(t+1) - 1$, because unfunded Social Security is a pure transfer system. Only $s(t)$—rather than $s(t)$ plus $d(t)$ as in the funded scheme—goes into capital accumulation. This is the basis of the claim that unfunded Social Security systems discourage aggregate savings. Of course, it is possible that $s(t)$ will change in order to compensate this effect, but such an offsetting change does not typically take place. Consequently, unfunded Social Security reduces capital accumulation. Discouraging capital accumulation can have negative consequences for growth and welfare. In fact, the empirical evidence we have seen in Chapters 1-4 suggests that there are many societies in which the level of capital accumulation is suboptimally low. In contrast, in the current model reducing aggregate savings and capital accumulation may be a good thing when the economy exhibits dynamic inefficiency (and overaccumaltion). This leads to the following proposition.

Proposition 9.8. *Consider the above-described overlapping generations economy and suppose that the decentralized competitive equilibrium is dynamically inefficient. Then there exists a feasible sequence of unfunded Social Security payments* $\{d(t)\}_{t=0}^{\infty}$ *which will lead to a competitive equilibrium starting from any date* t *that Pareto dominates the competitive equilibrium without Social Security.*

Proof. See Exercise 9.13. □

Unfunded Social Security reduces the overaccumulation and improves the allocation of resources. The similarity between the way in which unfunded Social Security achieves a Pareto improvement in this proposition and the way in which the Pareto optimal allocation was decentralized in the example economy of Section 9.1 is apparent. In essence, unfunded Social Security is transferring resources from future generations to initial old generation, and when designed appropriately, it can do so without hurting the future generations. Once again, this depends on dynamic inefficiency; when there is *no* dynamic inefficiency, any transfer of resources (and any unfunded Social Security program) would make some future generation worse-off. You are asked to prove this result in Exercise 9.14.

9.6. Overlapping Generations with Impure Altruism

Section 5.3 in Chapter 5 demonstrate that altruism within families (for example of parents towards their offspring) can lead to a structure of preferences identical to those of the representative household in the neoclassical growth model. In contrast, in this section we have so far ignored altruistic preferences in order to emphasize the effect of finite lives and the economic implications of the arrival of new agents in the economy. As briefly noted in Section 5.3, the exact form of altruism within a family matters for whether the representative household would provide a good approximation. In particular, a potentially empirically relevant form of altruism is one in which parents care about certain dimensions of the consumption vector of their offspring instead of their total utility. These types of preferences are often referred to as "impure altruism" to distinguish it from the pure altruism discussed in Section 5.3. A particular type of impure altruism, commonly referred to as "warm glow preferences", plays an important role in many growth models because of its tractability. Warm glow preferences assume that parents derive utility from (the warm glow of) their bequest, rather than the utility or the consumption of their offspring. This class of preferences turn out to constitute another very tractable alternative to the neoclassical growth and the baseline overlapping generations models. It has some clear parallels to the canonical overlapping generations model of last section, since it will also lead to equilibrium dynamics very similar to that of the Solow growth model. Given the importance of this class of preferences in many applied growth models, it is useful to review them briefly. These preferences will also be used in the next chapter and again in Chapter 21.

Suppose that the production side of the economy is given by the standard neoclassical production function, satisfying Assumptions 1 and 2. We write this in per capita form as $f(k)$.

The economy is populated by a continuum of individuals of measure 1. Each individual lives for two periods, childhood and adulthood. In second period of his life, each individual begets an offspring, works and then his life comes to an end. For simplicity, let us assume that there is no consumption in childhood (or that this is incorporated in the parent's consumption). There are no new households, so population is constant at 1. Each individual supplies 1 unit of labor inelastically during is adulthood.

Let us assume that preferences of individual (i, t), who reaches adulthood at time t, are as follows

$$\log(c_i(t)) + \beta \log(b_i(t)), \tag{9.21}$$

where $c_i(t)$ denotes the consumption of this individual and $b_i(t)$ is bequest to his offspring. Log preferences are assumed to simplify the analysis (see Exercise **??**). The offspring starts the following period with the bequest, rents this out as capital to firms, supplies labor, begets his own offspring, and makes consumption and bequests decisions. We also assume that capital fully depreciates after use.

This formulation implies that the maximization problem of a typical individual can be written as

$$\max_{c_i(t), b_i(t)} \log(c_i(t)) + \beta \log(b_i(t)), \tag{9.22}$$

subject to

$$c_i(t) + b_i(t) \le y_i(t) \equiv w(t) + R(t) b_i(t-1), \tag{9.23}$$

where $y_i(t)$ denotes the income of this individual,

$$w(t) = f(k(t)) - k(t) f'(k(t)) \tag{9.24}$$

is the equilibrium wage rate,

$$R(t) = f'(k(t)) \tag{9.25}$$

is the rate of return on capital and $b_i(t-1)$ is the bequest received by this individual from his own parent.

The total capital-labor ratio at time $t+1$ is given by aggregating the bequests of all adults at time t:

$$k(t+1) = \int_0^1 b_i(t)\, di, \tag{9.26}$$

which exploits the fact that the total measure of workers is 1, so that the capital stock and capital-labor ratio are identical.

An equilibrium in this economy is a somewhat more complicated object than before, because we may want to keep track of the consumption and bequest levels of all individuals. Let us denote the distribution of consumption and bequests across households at time t by $[c_i(t)]_{i\in[0,1]}$ and $[b_i(t)]_{i\in[0,1]}$, and let us assume that the economy starts with the distribution of wealth (bequests) at time t given by $[b_i(0)]_{i\in[0,1]}$, which satisfies $\int_0^1 b_i(0)\, di > 0$.

Definition 9.2. *An equilibrium in this overlapping generations economy with warm glow preferences is a sequence of consumption and bequest levels for each household,* $\left\{[c_i(t)]_{i\in[0,1]}, [b_i(t)]_{i\in[0,1]}\right\}_{t=0}^{\infty}$, *that solve (9.22) subject to (9.23), a sequence of capital-labor ratios,* $\{k(t)\}_{t=0}^{\infty}$, *given by (9.26) with some initial distribution of bequests* $[b_i(0)]_{i\in[0,1]}$, *and sequences of factor prices,* $\{w(t), R(t)\}_{t=0}^{\infty}$, *that satisfy (9.24) and (9.25).*

The solution of (9.22) subject to (9.23) is straightforward because of the log preferences, and gives

$$\begin{aligned} b_i(t) &= \frac{\beta}{1+\beta} y_i(t) \\ &= \frac{\beta}{1+\beta} \left[w(t) + R(t) b_i(t-1)\right], \end{aligned} \tag{9.27}$$

for all i and t. This equation shows that individual bequest levels will follow non-trivial dynamics. Since $b_i(t)$ determines the asset holdings of individual i of generation t, it can alternatively be interpreted as his "wealth" level. Consequently, this economy will feature a distribution of wealth that will evolve endogenously over time. This evolution will depend on factor prices. To obtain factor prices, let us aggregate bequests to obtain the capital-labor ratio of the economy via equation (9.26). Integrating (9.27) across all individuals, we obtain

$$\begin{aligned}
k(t+1) &= \int_0^1 b_i(t)\, di \\
&= \frac{\beta}{1+\beta}\int_0^1 [w(t) + R(t)\, b_i(t-1)]\, di \\
&= \frac{\beta}{1+\beta} f(k(t)). \qquad (9.28)
\end{aligned}$$

The last equality follows from the fact that $\int_0^1 b_i(t-1)\, di = k(t)$ and because by Euler's Theorem, Theorem 2.1, $w(t) + R(t)\, k(t) = f(k(t))$.

Consequently, aggregate equilibrium dynamics in this economy are straightforward and again closely resemble those in the baseline Solow growth model. Moreover, it is worth noting that these aggregate dynamics do *not* depend on the distribution of bequests or income across households (we will see that this is no longer true when there are other imperfections in the economy as in Chapter 21).

Now, solving for the steady-state equilibrium capital-labor ratio from (9.28), we obtain

$$k^* = \frac{\beta}{1+\beta} f(k^*), \qquad (9.29)$$

which is uniquely defined and strictly positive in view of Assumptions 1 and 2. Moreover, equilibrium dynamics are again given by Figure 9.2 and involve monotonic convergence to this unique steady state.

A complete characterization of the equilibrium can now be obtained by looking at the dynamics of bequests. It turns out that different types of bequests dynamics are possible along the transition path. More can be said regarding the limiting distribution of wealth and bequests. In particular, we know that $k(t) \to k^*$, so the ultimate bequest dynamics are given by steady-state factor prices. Let these be denoted by $w^* = f(k^*) - k^* f'(k^*)$ and $R^* = f'(k^*)$. Then once the economy is in the neighborhood of the steady-state capital-labor ratio, k^*, individual bequest dynamics are given by

$$b_i(t) = \frac{\beta}{1+\beta}\left[w^* + R^* b_i(t-1)\right].$$

When $R^* < (1+\beta)/\beta$, starting from any level $b_i(t)$ will converge to a unique bequest (wealth) level given by

$$b^* = \frac{\beta w^*}{1+\beta(1-R^*)}. \qquad (9.30)$$

Moreover, it can be verified that the steady-state equilibrium must involve $R^* < (1+\beta)/\beta$. This follows from the fact that in steady state

$$\begin{aligned} R^* &= f'(k^*) \\ &< \frac{f(k^*)}{k^*} \\ &= \frac{1+\beta}{\beta}, \end{aligned}$$

where the second line exploits the strict concavity of $f(\cdot)$ and the last line uses the definition of the steady-state capital-labor ratio, k^*, from (9.29).

The following proposition summarizes this analysis:

Proposition 9.9. *Consider the overlapping generations economy with warm glow preferences described above. In this economy, there exists a unique competitive equilibrium. In this equilibrium the aggregate capital-labor ratio is given by (9.28) and monotonically converges to the unique steady-state capital-labor ratio k^* given by (9.29). The distribution of bequests and wealth ultimately converges towards full equality, with each individual having a bequest (wealth) level of b^* given by (9.30) with $w^* = f(k^*) - k^* f'(k^*)$ and $R^* = f'(k^*)$.*

9.7. Overlapping Generations with Perpetual Youth

A key feature of the baseline overlapping generation model is that individuals have finite lives and know exactly when their lives will come to an end. An alternative way of modeling finite lives is along the lines of the "Poisson death model" or the *perpetual youth model* introduced in Section 5.3 of Chapter 5. Let us start with the discrete time version of that model. Recall that in that model each individual is potentially infinitely lived, but faces a probability $\nu \in (0,1)$ that his life will come to an end at every date (and these probabilities are independent). Recall from equation (5.9) that the expected utility of an individual with a "pure" discount factor β is given by

$$\sum_{t=0}^{\infty} (\beta(1-\nu))^t u(c(t)),$$

where $u(\cdot)$ is as standard instantaneous utility function, satisfying Assumption 3, with the additional normalization that $u(0) = 0$. Since the probability of death is ν and is independent across periods, the expected lifetime of an individual in this model can be written as (see Exercise 9.15):

$$\text{Expected life} = \nu + 2(1-\nu)\nu + 3(1-\nu)^2\nu + ... = \frac{1}{\nu} < \infty. \tag{9.31}$$

This equation captures the fact that with probability ν the individual will have a total life of length 1, with probability $(1-\nu)\nu$, he will have a life of length 2, and so on. This model is referred to as the perpetual youth model, since even though each individual has a finite expected life, all individuals who have survived up to a certain date have exactly the same expectation of further life. Therefore, individuals who survive in this economy are "perpetually young"; their age has no effect on their future longevity and has no predictive power on how many more years they will live for.

Individual i's flow budget constraint can be written as

$$a_i(t+1) = (1 + r(t+1))\, a_i(t) - c_i(t) + w(t) + z_i(t), \tag{9.32}$$

which is similar to the standard flow budget constraint, for example (6.40) in Chapter 6. Recall that the gross rate of return on savings is $1 + r(t+1)$, with the timing convention reflecting that assets at time t are rented out as capital at time $t+1$. The only difference from the standard budget constraint is the additional term, $z_i(t)$, which reflects transfers to the individual. The reason why these transfers are introduced is as follows: since individuals face an uncertain time of death, there may be "accidental bequests". In particular, individuals will typically come to the end of their lives while their asset positions are positive. When this happens, one possibility is that the accidental bequests might be collected by the government and redistributed equally across all households in the economy. In this case, $z_i(t)$ would represent these receipts for individual i. However, this would require that we impose a constraint of the form $a_i(t) \geq 0$, in order to prevent individuals from accumulating debts by the time their life comes to an end.

An alternative, which avoids this additional constraint and makes the model more tractable, has been proposed and studied by Menahem Yaari and Olivier Blanchard. This alternative involves introducing life-insurance or annuity markets, where competitive life insurance firms make payments to individuals (as a function of their asset levels) in return for receiving their positive assets when they die. The term $z(t)$ captures these annuity payments. In particular, imagine the following type of life insurance contract: a company would make a payment equal to $z(a(t))$ to an individual as a function of his asset holdings during every period in which he is alive.[2] When the individual dies, all his assets go to the insurance company. The fact that the payment level $z(a(t))$ depends only on the asset holdings of the individual and not on his age is a consequence of the perpetual youth assumption—conditional expectation of further life is independent of when the individual was born and in fact, it is independent of everything else in the model. The profits of a particular insurance company contracting with an individual with asset holding equal to $a(t)$, at time t will be

$$\pi(a,t) = -(1-\nu)\, z(a) + \nu a.$$

[2] The reader might note that this is the opposite of the most common type of life insurance contract where individuals make payments in order for their families to receive payments after their death. These types of insurance contracts are not useful in the current model, since individuals do not have offsprings or are not altruistic towards them.

With free entry, insurance companies should make zero expected profits (in terms of net present discounted value), which requires that $\pi(a(t), t) = 0$ for all t and a, thus

$$z(a(t)) = \frac{\nu}{1-\nu} a(t). \tag{9.33}$$

The other important element of the model is the evolution of the demographics. Since each agent faces a probability of death equal to ν at every date, there is a natural force towards decreasing population. We assume, however, that there are also new agents who are born at every date. Differently from the basic neoclassical growth model, we assume that these new agents are not born into a dynasty; instead, they become separate households themselves. We assume that when the population at time t is $L(t)$, there are $nL(t)$ new households born. Consequently, the evolution of total population is given by

$$L(t+1) = (1+n-\nu) L(t), \tag{9.34}$$

with the boundary condition $L(0) = 1$, where we assume that

$$n > \nu,$$

so that there is positive population growth. Throughout this section, we ignore technological progress.

Perpetual youth, together with exponential population growth, leads to a simple pattern of demographics in this economy. In particular, it is easy to verify that at some point in time $t > 0$, there will be $n(1+n-\nu)^{t-1}$ one-year-olds, $n(1+n-\nu)^{t-2}(1-\nu)$ two-year-olds, $n(1+n-\nu)^{t-3}(1-\nu)^2$ three-year-olds, etc. (See Exercise 9.21).

The production side of the economy is standard and it is represented by an aggregate production function satisfying Assumptions 1 and 2, $F(K(t), L(t))$. Suppose that capital depreciates at the rate δ. Factor markets are competitive and factor prices are determined in the usual fashion. The rental return of capital at time t is again given by $R(t) = f'(k(t))$, so that the net return on saving is $r(t+1) = f'(k(t)) - \delta$, and the wage rate is $w(t) = f(k(t)) - k(t) f'(k(t))$.

An allocation in this economy is similar to an allocation in the neoclassical growth model and involves time paths for the aggregate capital stock, wage rates and rental rates of capital, $\{K(t), w(t), R(t)\}_{t=0}^{\infty}$. However, it is no longer sufficient to specify aggregate consumption, since the level of consumption is not the same for all individuals. Instead, individuals born at different times will have accumulated different amounts of assets and will consume different amounts. Let us denote the consumption at date t of a household born at date $\tau \leq t$ by $c(t \mid \tau)$. An allocation must now specify the entire sequence $\{c(t \mid \tau)\}_{t=0,\tau\leq t}^{\infty}$. Using this notation and the life insurance contracts introduced by (9.33), the flow budget constraint of an individual of generation τ can be written as:

$$a(t+1 \mid \tau) = \left(1 + r(t+1) + \frac{\nu}{1-\nu}\right) a(t \mid \tau) - c(t \mid \tau) + w(t). \tag{9.35}$$

A competitive equilibrium in this economy can then be defined as follows:

Definition 9.3. *A competitive equilibrium consists of paths of capital stock, wage rates and rental rates of capital, $\{K(t), w(t), R(t)\}_{t=0}^{\infty}$, and paths of consumption for each generation, $\{c(t \mid \tau)\}_{t=0,\tau\leq t}^{\infty}$, such that each individual maximizes utility and the time path of factor prices, $\{w(t), R(t)\}_{t=0}^{\infty}$, is such that given the time path of capital stock and labor $\{K(t), L(t)\}_{t=0}^{\infty}$, all markets clear.*

In addition to the competitive factor prices, the key equation is the consumer Euler equation for an individual of generation τ at time t. Taking into account that the gross rate of return on savings is $1 + r(t+1) + \nu/(1-\nu)$ and that the effective discount factor of the individual is $\beta(1-\nu)$, this Euler equation can be written as

$$u'(c(t \mid \tau)) = \beta\left[(1 + r(t+1))(1-\nu) + \nu\right] u'(c(t+1 \mid \tau)). \tag{9.36}$$

This equation looks similar to be standard consumption Euler equation, for example as in Chapter 6. It only differs from the equation there because it applies separately to each generation τ and because the term ν, the probability of death facing each individual, features in this equation. Note, however, that when both r and ν are small

$$(1+r)(1-\nu) + \nu \approx 1 + r,$$

and the terms involving ν disappear. In fact, the reason why these terms are present is because of the discrete time nature of the current model. In the next section, we will analyze the continuous time version of the perpetual youth model, where the approximation in the previous equation is exact. Moreover, the continuous time model will allow us to obtain closed-form solutions for aggregate consumption and capital stock dynamics. Therefore, this model gives one example of a situation in which continuous time methods turn out to be more appropriate than discrete time methods (whereas the baseline overlapping generations model required discrete time).

Recall that in the neoclassical model without technological progress, the consumer Euler equation admitted a simple solution because consumption had to be equal across dates for the representative household. This is no longer the case in the perpetual youth model, since different generations will have different levels of assets and may satisfy equation (9.36) with different growth rates of consumption depending on the form of the utility function $u(\cdot)$.

To simplify the analysis, let us now suppose that the utility function takes the logarithmic form,

$$u(c) = \log c.$$

In that case, (9.36) simplifies to

$$\frac{c(t+1 \mid \tau)}{c(t \mid \tau)} = \beta\left[(1+r(t+1))(1-\nu)+\nu\right], \tag{9.37}$$

and implies that the growth rate of consumption must be equal for all generations. Using this observation, it is possible to characterize the behavior of the aggregate capital stock, though this turns out to be much simpler in continuous time. For this reason, we now turn to the continuous time version of this model (details on the discrete time model are covered in Exercise 9.22).

9.8. Overlapping Generations in Continuous Time

9.8.1. Demographics, Technology and Preferences. We now turn to a continuous time version of the perpetual youth model. Suppose that each individual faces a Poisson death rate of $\nu \in (0,\infty)$. Suppose also that individuals have logarithmic preferences and a pure discount rate of $\rho > 0$. As demonstrated in Exercise 5.7 in Chapter 5, this implies that individual i will maximize the objective function

$$\int_0^\infty \exp\left(-(\rho+\nu)t\right)\log c_i(t)\,dt. \tag{9.38}$$

Demographics in this economy are similar to those in the discrete time perpetual youth model of the previous section. In particular, expected further life of an individual is independent of when he was born, and is equal to

$$\frac{1}{\nu} < \infty.$$

This is both the life expectancy at birth and the expected further life of an individual who has survived up to a certain point. Next, let population at time t be $L(t)$. Then the Poisson death rate implies that a total flow of $\nu L(t)$ individuals will die at time t. Once again we assume that there is arrival of new households at the exponential rate $n > \nu$, so that aggregate population dynamics are given by

$$\dot{L}(t) = (n-\nu)L(t), \tag{9.39}$$

again with initial condition $L(0) = 1$. It can also be computed that at time t the mass of individuals of cohort born at time $\tau < t$ is given by

$$L(t \mid \tau) = \exp\left(-\nu(t-\tau)+(n-\nu)\tau\right). \tag{9.40}$$

In this equation and throughout the section, we assume that at $t = 0$, the economy starts with a population of $L(0) = 1$ who are all newborn at that point. Equation (9.40) is derived in Exercise 9.23.

As in the previous section, it is sufficient to specify the consumption behavior and the budget constraints for each cohort. In particular, the flow budget constraint for cohort τ at time t is

$$\dot{a}(t \mid \tau) = r(t)\, a(t \mid \tau) - c(t \mid \tau) + w(t) + z(a(t \mid \tau) \mid t, \tau),$$

where again $z(a(t \mid \tau) \mid t, \tau)$ is the transfer payment or annuity payment at time t to an individual born at time τ holding assets $a(t \mid \tau)$. We follow Yaari and Blanchard and again assume complete annuity markets, with free entry. Now the instantaneous profits of a life insurance company providing such annuities at time t for an individual born at time τ with assets $a(t \mid \tau)$ is

$$\pi(a(t \mid \tau) \mid t, \tau) = \nu a(t \mid \tau) - z(a(t \mid \tau) \mid t, \tau),$$

since the individual will die and leave his assets to the life insurance company at the flow rate ν. Zero profits now implies that

$$z(a(t \mid \tau) \mid t, \tau) = \nu a(t \mid \tau).$$

Substituting this into the flow budget constraint above, we obtain the more useful expression

$$\dot{a}(t \mid \tau) = (r(t) + \nu)\, a(t \mid \tau) - c(t \mid \tau) + w(t). \tag{9.41}$$

Let us assume that the production side is given by the per capita aggregate production function $f(k)$ satisfying Assumptions 1 and 2, where k is the aggregate capital-labor ratio. Capital is assumed to depreciate at the rate δ. Factor prices are given by the usual expressions

$$R(t) = f'(k(t)) \text{ and } w(t) = f(k(t)) - k(t)\, f'(k(t)), \tag{9.42}$$

and as usual $r(t) = R(t) - \delta$. The law of motion of capital-labor ratio is given by

$$\dot{k}(t) = f(k(t)) - (n - \nu + \delta)k(t) - c(t), \tag{9.43}$$

where $c(t)$ is average consumption per capita, given by

$$\begin{aligned} c(t) &= \frac{\int_{-\infty}^{t} c(t \mid \tau)\, L(t \mid \tau)\, d\tau}{\int_{-\infty}^{t} L(t \mid \tau)\, d\tau} \\ &= \frac{\int_{-\infty}^{t} c(t \mid \tau)\, L(t \mid \tau)\, d\tau}{L(t)}, \end{aligned}$$

where recall that $L(t \mid \tau)$ is the size of the cohort born at τ at time t.

9.8.2. Equilibrium.

Definition 9.4. *A competitive equilibrium consists of paths of capital stock, wage rates and rental rates of capital,* $[K(t), w(t), R(t)]_{t=0}^{\infty}$ *and paths of consumption for each generation,* $[c(t \mid \tau)]_{t=0,\tau\leq t}^{\infty}$*, such that each individual maximizes (9.38) subject to (9.41), and the time path of prices,* $\{w(t), R(t)\}_{t=0}^{\infty}$*, are given by (9.42), and the capital-labor ratio evolves according to (9.43).*

Let us start with consumer optimization. The maximization of (9.38) subject to (9.41) gives the usual Euler equation

$$\frac{\dot{c}(t \mid \tau)}{c(t \mid \tau)} = r(t) - \rho, \tag{9.44}$$

where $\dot{c}(t \mid \tau) \equiv \partial c(t \mid \tau)/\partial t$. Notice that, in contrast to the discrete time version of this equation, (9.37), the probability (flow rate) of death, ν, does not feature here, since it exactly cancels out (the rate of return on assets is $r(t) + \nu$ and the effective discount factor is $\rho + \nu$, so that their difference is equal to $r(t) - \rho$).

The transversality condition for an individual of cohort τ can be written as

$$\lim_{t \to \infty} \exp\left(-\left(\bar{r}(t,\tau) + \nu\right)\right) a(t \mid \tau) = 0, \tag{9.45}$$

where

$$\bar{r}(t,\tau) \equiv \frac{1}{t-\tau} \int_{\tau}^{t} r(s)\, ds$$

is the average interest rate between dates τ and t as in equation (8.17) in Chapter 8, and the transversality condition here is the analog of equation (8.18) there. The transversality condition, (9.45), requires the net present discounted value of the assets in the very far future of an individual born at the time τ discounted back to this time to be equal to 0.

Combining (9.44) together with (9.41) and (9.45) gives the following consumption "function" for an individual of cohort τ (see Exercise 9.24):

$$c(t \mid \tau) = (\rho + \nu)\left[a(t \mid \tau) + \omega(t)\right]. \tag{9.46}$$

This linear form of the consumption function is a particularly attractive feature of logarithmic preferences and is the reason why we specified logarithmic preferences in this model in the first place. The term in square brackets is the asset and human wealth of the individual, with the second term defined as

$$\omega(t) = \int_{t}^{\infty} \exp\left(-\left(\bar{r}(s,t) + \nu\right)\right) w(s)\, ds.$$

This term clearly represents the net present discounted value of future wage earnings of an individual discounted to time t. It is independent of τ, since the future expected earnings of all individuals are the same irrespective of when they are born. The additional discounting with ν in this term arises because individuals will die at this rate and thus lose future earnings from then on.

Equation (9.46) implies that each individual consumes a fraction of this wealth equal to his effective discount rate, $\rho + \nu$. Now integrating this across cohorts, using the fact that the size of the cohort τ at time t is $\exp(-\nu(t-\tau) + (n-\nu)\tau)$, we obtain per capita consumption as

$$c(t) = (\rho + \nu)\left(a(t) + \omega(t)\right), \tag{9.47}$$

where $a(t)$ is average assets per capita. Since the only productive assets in this economy is capital, we also have that $a(t) = k(t)$. Finally, differentiating (9.47), we obtain

$$\dot{c}(t) = (\rho + \nu)(\dot{a}(t) + \dot{\omega}(t)). \tag{9.48}$$

The law of motion of assets per capita can be written as

$$\dot{a}(t) = (r(t) - (n - \nu))\, a(t) + w(t) - c(t).$$

This equation is intuitive. Aggregate wealth $(a(t)\, L(t))$ increases because of the returns to capital at the rate $r(t)$ and also because of the wage income, $w(t)\, L(t)$. Out of this, total consumption of $c(t)\, L(t)$ needs to be subtracted. Finally, since $L(t)$ grows at the rate $n - \nu$, this reduces the rate of growth of assets per capita. Human wealth per capita, on the other hand, satisfies

$$(r(t) + \nu)\, \omega(t) = \dot{\omega}(t) + w(t).$$

The intuition for this equation comes from the Hamilton-Jacobi-Bellman equations discussed in Chapter 7. We can think of $\omega(t)$ as the value of an asset with a claim to the future earnings of a typical individual. The required rate of return on this is $r(t) + \nu$, which takes into account that the individual will lose his future earnings stream at the rate ν when he dies. The return on this asset is equal to its capital gains, $\dot{\omega}(t)$, and dividends, $w(t)$. Now substituting for $\dot{a}(t)$ and $\dot{\omega}(t)$ from these two equations into (9.48), we obtain:

$$\begin{aligned}
\dot{c}(t) &= (\rho + \nu)\left((r(t) - (n - \nu))\, a(t) + w(t) - c(t) + (r(t) + \nu)\, \omega(t) - w(t)\right) \\
&= (\rho + \nu)\left((r(t) + \nu)(a(t) + \omega(t)) - na(t) - c(t)\right) \\
&= (\rho + \nu)\left(\frac{(r(t) + \nu)}{\rho + \nu} c(t) - na(t) - c(t)\right) \\
&= (r(t) - \rho)\, c(t) - (\rho + \nu)\, na(t),
\end{aligned}$$

where the third line uses (9.47). Dividing both sides by $c(t)$, using the fact that $a(t) = k(t)$, and substituting $r(t) = f'(k(t)) - \delta$, we obtain

$$\frac{\dot{c}(t)}{c(t)} = f'(k(t)) - \delta - \rho - (\rho + \nu)\, n \frac{k(t)}{c(t)}. \tag{9.49}$$

This is similar to the standard Euler equation (under logarithmic preferences), except for the last term. This last term reflects the fact that consumption growth per capita is slowed down by the arrival of new individuals at each instance, who have less wealth than the average individual. Their lower wealth implies lower consumption and reduces average consumption growth in the economy. This intuitively explains why the last term depends on n (the rate of arrival of new individuals) and on k/c (the size of average asset holdings relative to consumption).

The equilibrium path of the economy is completely characterized by the two differential equations, (9.43) and (9.49)—together with an initial condition for $k(0) > 0$ and the transversality condition (9.45) applied to average assets, thus to the capital-labor ratio, $k(t)$. First, a steady-state equilibrium is obtained when both $\dot{k}(t)/k(t)$ and $\dot{c}(t)/c(t)$ are equal to zero, and thus satisfies the following two equations:

$$\frac{c^*}{k^*} = \frac{(\rho+\nu)\,n}{f'(k^*) - \delta - \rho} \tag{9.50}$$

$$\frac{f(k^*)}{k^*} - (n - \nu + \delta) - \frac{(\rho+\nu)\,n}{f'(k^*) - \delta - \rho} = 0. \tag{9.51}$$

The second equation pins down a unique positive level of steady-state capital-labor ratio, k^*, ratio (since both $f(k)/k$ and $f'(k)$ are decreasing). Given k^* the first equation pins down a unique level of average consumption per capita, c^*. It can also be verified that at k^*,

$$f'(k^*) > \rho + \delta,$$

so that the capital-labor ratio is lower than the level consistent with the modified golden rule k_{mgr}, given by $f'(k_{mgr}) = \rho + \delta$. Recall that optimal steady-state capital-labor ratio of the neoclassical growth model satisfied the modified golden rule. In comparison, in this economy there is always *underaccumulation.* This contrasts with the baseline overlapping generations model, which potentially led to dynamic inefficiency and overaccumulation. We will momentarily return to a further discussion of this issue. Before doing this, let us analyze equilibrium dynamics.

Figure 9.3 plots (9.43) and (9.49), together with the arrows indicating how average consumption per capita and capital-labor ratio change in different regions. Both (9.43) and (9.49) are upward sloping and start at the origin. It is also straightforward to verify that while (9.43) is concave in the k-c space, (9.49) is convex. Thus they have a unique intersection. We also know from the preceding discussion that this unique intersection is at a capital-labor ratio less than that satisfying the modified golden rule, which is marked as k_{mgr} in the figure. Naturally, k_{mgr} itself is less than k_{gold}. The phase diagram also makes it clear that there exists a unique stable arm that is upward sloping in the k-c space. The shape of the stable arm is the same as in the basic neoclassical growth model. If the initial level of consumption is above this stable arm, feasibility is violated, while if it is below, the economy tends towards zero consumption and violates the transversality condition. Consequently, the steady-state equilibrium is globally saddle-path stable; consumption starts along the stable arm, and consumption and the capital-labor ratio monotonically converge to the steady state. Exercise 9.26 asks you to show local saddle-path stability by linearizing (9.43) and (9.49) around the steady state.

The following proposition summarizes this analysis.

Proposition 9.10. *In the continuous time perpetual youth model, there exists a unique steady state (k^*, c^*) given by (9.50) and (9.51). The level of capital-labor ratio is less than*

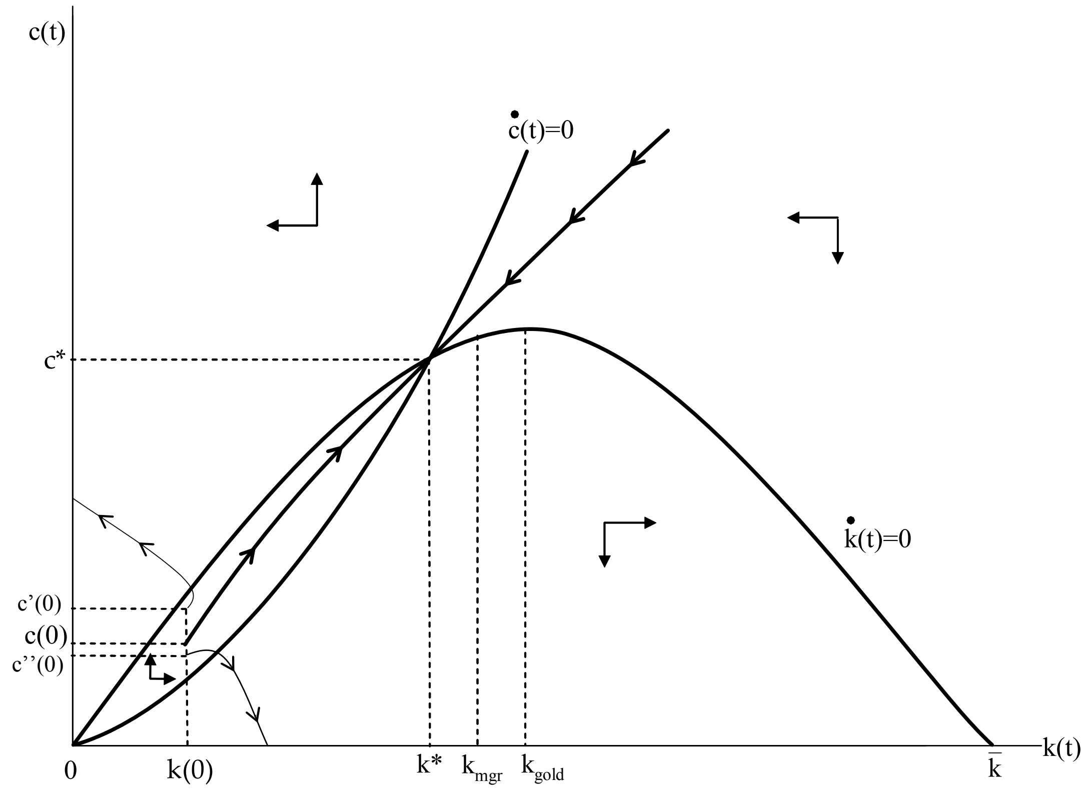

Figure 9.3. Steady state and transitional dynamics in the overlapping generations model in continuous time.

the level of capital-labor ratio that satisfies the modified golden rule, k_{mgr}. Starting with any $k(0) > 0$, equilibrium dynamics monotonically converge to this unique steady state.

Perhaps the most interesting feature of this equilibrium is that, despite finite lives and overlapping generations, there is no overaccumulation. The reason for this is that individuals have constant stream of labor income throughout their lives and thus do not need to save excessively in order to ensure smooth consumption. Is it possible to obtain overaccumulation in the continuous time perpetual youth model? The answer is yes and is demonstrated by Blanchard (1985). He assumes that each individual starts life with one unit of effective labor and then his effective labor units decline at some positive exponential rate $\zeta > 0$ throughout his life, so that the labor earnings of an individual on generation τ at time t is $\exp(-\zeta(t-\tau))\, w(t)$, where $w(t)$ is the market wage per unit of effective labor at time t. Consequently, individual consumption function changes from (9.46) to

$$c(t \mid \tau) = (\rho + \nu)\left[a(t \mid \tau) + \omega(t \mid \tau)\right],$$

where now $\omega(t \mid \tau)$ is the human wealth of an individual of generation τ at time t, given by (see Exercise 9.28):

$$\omega(t \mid \tau) = \int_t^\infty \exp\left(-\left(\bar{r}(t-s) + \nu\right)\right) \exp\left(-\zeta(s-\tau)\right) w(s)\, ds, \tag{9.52}$$

where $\exp(-\zeta(s-\tau))$ is the correction factor taking into account the decline in effective labor units. Repeating the same steps as before with this new expression for human wealth, we obtain

$$\frac{\dot{c}(t)}{c(t)} = f'(k(t)) - \delta - \rho - \zeta - (\rho + \nu)(n + \zeta)\frac{k(t)}{c(t)}, \tag{9.53}$$

while the behavior of capital-labor ratio is still given by (9.43). It can now be shown that for ζ sufficiently large, the steady-state capital-labor ratio k^* can exceed both the modified golden rule level k_{mgr} and the golden rule level, k_{gold} (see Exercise 9.28).

This discussion therefore illustrates that overaccumulation results when there are overlapping generations and a strong motive for saving for the future. Interestingly, it can be shown that what is important is not finite lives per se, but overlapping generations indeed. In particular, Exercise 9.30 shows that when $n = 0$, overaccumulation is not possible so that finite lives is not sufficient for overaccumulation. However, $k^* > k_{gold}$ is possible when $n > 0$ and $\nu = 0$, so that the overlapping generations model with infinite lives can generate overaccumulation.

9.9. Taking Stock

This chapter has continued our investigation of the mechanics of capital accumulation in dynamic equilibrium models. The main departure from the baseline neoclassical growth model of the last section has been the relaxation of the representative household assumption. The simplest way of accomplishing this is to introduce two-period lived overlapping generations (without pure altruism). In the baseline overlapping generations model of Samuelson and Diamond, each individual lives for two periods, but can only supply labor during the first period of his life. We have also investigated alternative non-representative-household models, in particular, overlapping generations with impure altruism and models of perpetual youth. In models of overlapping generations with impure altruism, individuals transfer resources to their offspring, but they do not care directly about the utility of their offspring and instead derive utility from the act of giving or from some subcomponent of the consumption vector on their descendent. In models of perpetual youth, the economy features expected finite life and overlapping generations, but each individual still has an infinite planning horizon, because the time of death is uncertain.

All of these models fall outside the scope of the First Welfare Theorem. As a result, there is no guarantee that the resulting equilibrium path will be Pareto optimal. In fact, the extensive study of the baseline overlapping generations models were partly motivated by the possibility of Pareto suboptimal allocations in such models. We have seen that these equilibria may be "dynamically inefficient" and feature overaccumulation—a steady-state capital-labor ratio greater than the golden rule capital-labor ratio. We have also seen how an unfunded Social Security system can reduce aggregate savings and thus ameliorate the overaccumulation problem. The important role that unfunded Social Security (or national debt) plays in the overlapping generations model has made this model a workhorse for analysis of transfer programs, fiscal policies and generational accounting.

Our analysis of perpetual youth models, especially Yaari and Blanchard's continuous time perpetual youth model, further clarified the roles of the path of labor income, finite horizons and arrival of new individuals in in generating the overaccumulation result. In particular, this model shows that the declining path of labor income is important for the overaccumulation result (in the Samuelson-Diamond model there is an extreme form of this, since there is no labor income in the second period of the life of the individual). But perhaps the more important insight generated by these models is that what matters is not finite horizons per se, but the arrival of new individuals. Overaccumulation and Pareto suboptimality arise because of the pecuniary externalities created on individuals that are not yet in the marketplace.

While overaccumulation and dynamic inefficiency have dominated much of the discussion of overlapping generations models in the literature, one should not overemphasize the importance of dynamic inefficiency. As we discussed in Chapter 1, the major question of economic growth is why so many countries have so little capital for their workers and why the process of economic growth and capital accumulation started only over the past 200 years. It is highly doubtful that overaccumulation is a major problem for most countries in the world.

The models presented in this chapter are very useful for another reason, however. They significantly enrich our arsenal in the study of the mechanics of economic growth and capital accumulation. All three of the major models presented in this chapter, the baseline overlapping generations model, the overlapping generations model with impure altruism, and the perpetual youth model, are tractable and useful vehicles for the study of economic growth in a variety of circumstances. For example, the first two lead to equilibrium dynamics similar to the baseline Solow growth model, but without explicitly imposing an exogenously constant saving rate. The latter model, on the other hand, allows an analysis of equilibrium dynamics similar to the basic neoclassical growth model, but also incorporates finite lives and overlapping generations, which will be essential in many problems, for example in human capital investments studied in the next chapter.

In summary, this chapter has provided us with new modeling tools and new perspectives on the question of capital accumulation, aggregate saving and economic growth. It has not, however, offered new answers to questions of why countries grow (for example, technological progress) and why some countries are much poorer than others (related to the fundamental cause of income differences). Of course, its purpose was not to provide such answers in the first place.

9.10. References and Literature

The baseline overlapping generations model with two-period lived agents is due to Samuelson (1958) and Diamond (1965). A related model appears in French in Maurice Allais' work. Blanchard and Fischer (1989, Chapter 3) provide an excellent textbook treatment of the baseline overlapping generations model. Some textbooks use this setup as the main workhorse macroeconomic model, for example, Azariadis (1993), McCandless and Wallace (1991) and De La Croix and Michel (2002).

The economy studied in Section 9.1 is due to Shell (1974). The source of inefficiency in the overlapping generations model is much discussed in the literature. Shell's (1974) example economy in Section 9.1 provides the clearest intuitive explanation for why the First Welfare Theorem does not apply. A lucid discussion is contained in Bewley (2006).

The model of overlapping generations with impure altruism is due to Andreoni (1989). This model has been used extensively in the economic growth and economic development literatures, especially for the analysis of equilibrium dynamics in the presence of imperfect capital markets. Well-known examples include the models by Aghion and Bolton (1996), Banerjee and Newman (1989, 1994), Galor and Zeira (1993) and Piketty (1996), which we will study in Chapter 21. I am not aware of an analysis of wealth inequality dynamics with perfect markets in this economy along the lines of the model presented in Section 9.6, even though the analysis is quite straightforward. A similar analysis of wealth inequality dynamics is included in Stiglitz's (1979) model, but he assumes that each household can only use its savings in its own diminishing return technology (thus creating a strong force towards convergence of incomes).

The continuous time perpetual youth model is due to Yaari (1965) and Blanchard (1985). The discrete time version of this model was presented to facilitate the transition to the continuous time version. Our treatment of the continuous time version closely followed Blanchard (1985). Blanchard and Fischer (1989, Chapter 3) and Barro and Sala-i-Martin (2004, Chapter 3) provide clear textbook treatments. The importance of the path of labor income is emphasized and analyzed in Blanchard (1985). The importance of new arrivals in the market is emphasized and explained in Weil (1989).

Models with overlapping generations and finite lives are used extensively in the analysis of Ricardian Equivalence, introduced in Exercise 8.19 in Chapter 8, is a good approximation to reality. Blanchard (1985) and Bernheim (1987) include extensive discussions of this issue, while Barro (1974) is the reference for the original statement of the Ricardian Equivalence hypothesis. Another important application of overlapping generations models is to generational accounting, for example, as in the work by Auerbach and Kotlikoff (1987).

9.11. Exercises

Exercise 9.1. Prove that the allocation characterized in Proposition 9.1 is the unique competitive equilibrium. [Hint: first, show that there cannot be any equilibrium with $p_j > p_{j-1}$ for any j. Second, show that even if $p_0 > p_1$, household $i = 0$ must consume only commodity $j = 0$; then inductively, show that this is true for any household].

Exercise 9.2. Consider the following variant of economy with infinite number of commodities and infinite number of individuals presented in Section 9.1. The utility of individual indexed $i = j$ is

$$u\left(c\left(j\right)\right) + \beta u\left(c\left(j+1\right)\right)$$

where $\beta \in (0, 1)$, and each individual has one unit of the good with the same index as his own.

(1) Define a competitive equilibrium for this economy.
(2) Characterize the set of competitive equilibria in this economy.
(3) Characterize the set of Pareto optima in this economy.
(4) Can all Pareto optima be decentralized without changing endowments? Can they be decentralized by changing endowments?

Exercise 9.3. Show that in the model of Section 9.2 the dynamics of capital stock are identical to those derived in the text even when $\delta < 1$.

Exercise 9.4. In the baseline overlapping generations model, verify that savings $s\left(w, R\right)$, given by (9.6), are increasing in the first argument, w. Provide conditions on the utility function $u\left(\cdot\right)$ such that they are also increasing in the second argument, the interest rate R.

Exercise 9.5. Prove Proposition 9.4

Exercise 9.6. Consider the canonical overlapping generations model with log preferences

$$\log\left(c_1\left(t\right)\right) + \beta \log\left(c_2\left(t\right)\right)$$

for each household. Suppose that there is population growth at the rate n. Individuals work only when they are young, and supply one unit of labor inelastically. Production technology is given by

$$Y\left(t\right) = A\left(t\right) K\left(t\right)^{\alpha} L\left(t\right)^{1-\alpha},$$

where $A\left(t+1\right) = \left(1+g\right) A\left(t\right)$, with $A\left(0\right) > 0$ and $g > 0$.

(1) Define a competitive equilibrium and the steady-state equilibrium.
(2) Characterize the steady-state equilibrium and show that it is globally stable.
(3) What is the effect of an increase in g on the equilibrium?

(4) What is the effect of an increase in β on the equilibrium? Provide an intuition for this result.

Exercise 9.7. Consider the canonical model with log preferences, $\log(c_1(t)) + \beta \log(c_2(t))$, and the general neoclassical technology $F(K, L)$ satisfying Assumptions 1 and 2. Show that multiple steady-state equilibria are possible in this economy.

Exercise 9.8. Consider again the canonical overlapping generations model with log preferences and Cobb-Douglas production function.

(1) Define a competitive equilibrium.
(2) Characterize the competitive equilibrium and derive explicit conditions under which the steady-state equilibrium is dynamically inefficient.
(3) Using plausible numbers argue whether or not dynamic inefficiency can arise in "realistic" economies.
(4) Show that when there is dynamic inefficiency, it is possible to construct an unfunded Social Security system which creates a Pareto improvement relative to the competitive allocation.

Exercise 9.9. Consider again the canonical overlapping generations model with log preferences and Cobb-Douglas production function, but assume that individuals now work in both periods of their lives.

(1) Define a competitive equilibrium and the steady-state equilibrium.
(2) Characterize the steady-state equilibrium and the transitional dynamics in this economy.
(3) Can this economy generate overaccumulation?

Exercise 9.10. Prove Proposition 9.7.

Exercise 9.11. Consider the overlapping generations model with fully funded Social Security. Prove that even when the restriction $s(t) \geq 0$ for all t is imposed, no fully funded Social Security program can lead to a Pareto improvement.

Exercise 9.12. Consider an overlapping generations economy with a dynamically inefficient steady-state equilibrium. Show that the government can improve the allocation of resources by introducing national debt. [Hint: suppose that the government borrows from the current young and redistributes to the current old, paying back the current young the following period with another round of borrowing]. Contrast this result with the Ricardian equivalence result in Exercise 8.19 in Chapter 8.

Exercise 9.13. Prove Proposition 9.8.

Exercise 9.14. Consider the baseline overlapping generations model and suppose that the equilibrium is dynamically efficient, i.e., $r^* > n$. Show that any unfunded Social Security system will increase the welfare of the current old generation and reduce the welfare of some future generation.

Exercise 9.15. Derive equation (9.31).

Exercise 9.16. Consider the overlapping generations model with warm glow preferences in Section 9.6, and suppose that preferences are given by

$$c(t)^{\eta} b(t+1)^{1-\eta},$$

with $\eta \in (0,1)$, instead of equation (9.21). The production side is the same as in Section 9.6. Characterize the dynamic equilibrium of this economy.

Exercise 9.17. Consider the overlapping generations model with warm glow preferences in Section 9.6, and suppose that preferences are given by $u_1(c_i(t)) + u_2(b_i(t))$, where u_1 and u_2 are strictly increasing and concave functions. The production side is the same as in the text. Characterize a dynamic equilibrium of this economy.

Exercise 9.18. Characterize the aggregate equilibrium dynamics and the dynamics of wealth distribution in the overlapping generations model with warm glow preferences as in Section 9.6, when the per capita production function is given by the Cobb-Douglas form $f(k) = Ak^{\alpha}$. Show that away from the steady state, there can be periods during which wealth inequality increases. Explain why this may be the case.

Exercise 9.19. Generalize the results in Section 9.6 to an environment in which the preferences of an individual of generation t are given by

$$u(c(t)) + v(b(t)),$$

where $c(t)$ denotes own consumption, $b(t)$ is bequests, and $u(\cdot)$ and $v(\cdot)$ are strictly increasing, continuously differentiable and strictly concave utility functions. Determine conditions on $u(\cdot)$ and $v(\cdot)$ such that aggregate dynamics are globally stable. Provide conditions on $u(\cdot)$ and $v(\cdot)$ to ensure that asymptotically all individuals tend to the same wealth level.

Exercise 9.20. Show that the steady-state capital labor ratio in the overlapping generations model with impure altruism (of Section 9.6) can lead to overaccumulation, i.e., $k^* > k_{gold}$.

Exercise 9.21. Prove that given the perpetual youth assumption and population dynamics in equation (9.34), at time $t > 0$, there will be $n(1+n-\nu)^{t-s}(1-\nu)^{s-1}$ s-year-olds for any $s \in \{1, 2, ..., t-1\}$

Exercise 9.22. * Consider the discrete time perpetual youth model discussed in Section 9.7 and assume that preferences are logarithmic. Characterize the steady-state equilibrium and the equilibrium dynamics of the capital-labor ratio.

Exercise 9.23. Consider the continuous time perpetual youth model of Section 9.8.

(1) Show that given $L(0) = 1$, the initial size of a cohort born at the time $\tau \geq 0$ is $\exp((n-\nu)\tau)$.

(2) Show that the probability that an individual born at the time τ is alive at time $t \geq \tau$ is $\exp(-\nu(t-\tau))$.
(3) Derive equation (9.40).
(4) Show that this equation would not apply at any finite time if the economy starts at $t=0$ with an arbitrary age distribution.

Exercise 9.24. Derive equation (9.46). [Hint: first integrate the flow budget constraint of the individual, (9.41) using the transversality condition (9.45) and then use the Euler equation (9.44)].

Exercise 9.25. Generalize the analysis of the continuous time perpetual youth model of Section 9.8 to an economy with labor-augmenting technological progress at the rate g. Prove that the steady-state equilibrium is unique and globally (saddle-path) stable. What is the impact of a higher rate of technological progress?

Exercise 9.26. Linearize the differential equations (9.43) and (9.49) around the steady state, (k^*, c^*), and show that the linearized system has one negative and one positive eigenvalue.

Exercise 9.27. Determine the effects of n and ν on the steady-state equilibrium (k^*, c^*) in the continuous time perpetual youth model of Section 9.8.

Exercise 9.28. (1) Derive equations (9.52) and (9.53).
(2) Show that for ζ sufficiently large, the steady-state equilibrium capital-labor ratio, k^*, can exceed k_{gold}, so that there is overaccumulation. Provide an intuition for this result.

Exercise 9.29. Consider the continuous time perpetual youth model with a constant flow of government spending G. Suppose that this does not affect consumer utility and that lump-sum taxes $[\mathcal{T}(t)]_{t=0}^{\infty}$ are allowed. Specify the government budget constraint as in Exercise 8.19 in Chapter 8. Prove that contrary to the Ricardian Equivalence result in Exercise 8.19, the sequence of taxes affects the equilibrium path of capital-labor ratio and consumption. Interpret this result and explain the difference between the overlapping generations model and the neoclassical growth model.

Exercise 9.30. * Consider the continuous time perpetual youth model with labor income declining at the rate $\zeta > 0$.

(1) Show that if $n = 0$, $k^* \leq k_{gold}$ for any $\zeta > 0$.
(2) Show that there exists $\zeta > 0$ sufficiently high such that if $n > 0$ and $\nu = 0$, $k^* > k_{gold}$.

Exercise 9.31. Consider an economy with aggregate production function

$$Y(t) = AK(t)^{1-\alpha} L(t)^{\alpha}.$$

All markets are competitive, the labor supply is normalized to 1, capital fully depreciates after use, and the government imposes a linear tax on capital income at the rate τ, and uses the proceeds for government consumption. Consider two specifications of preferences:

- All agents are infinitely lived, with preferences

$$\sum_{t=0}^{\infty} \beta^t \ln c\,(t)$$

- An overlapping generations model where agents work in the first period, and consume the capital income from their savings in the second period. The preferences of a generation born at time t, defined over consumption when young and old, are given by

$$\ln c^y\,(t) + \beta \ln c^0\,(t)$$

(1) Characterize the equilibria in these two economies, and show that in the first economy, taxation reduces output, while in the second, it does not.

(2) Interpret this result, and in the light of this result discuss the applicability of models which try to explain income differences across countries with differences in the rates of capital income taxation.

CHAPTER 10

Human Capital and Economic Growth

In this chapter, we will discuss human capital investments and the role of human capital in economic growth and in cross-country income differences. As already discussed in Chapter 3, human capital can play a major role in economic growth and cross-country income differences, and our main purpose is to understand which factors affect human capital investments and how these influence the process of economic growth and economic development. Human capital refers to all the attributes of workers that potentially increase their productivity in all or some productive tasks. The term is coined because much of these attributes are accumulated by workers through investments. Human capital theory, developed primarily by Becker (1965) and Mincer (1974), is about the role of human capital in the production process and about the incentives to invest in skills, including pre-labor market investments, in form of schooling, and on-the-job investments, in the form of training. It would not be an exaggeration to say that this theory is the basis of much of labor economics and plays an equally important role in macroeconomics. The literature on education and other types of human capital investments is vast, so only parts of this literature that are relevant to the main focus of this book will be covered here. There are a number of other important connections between human capital and economic growth, especially related to its effect on technological progress and its role in economic takeoff, which are not covered in this chapter, but will be discussed later in the book.

10.1. A Simple Separation Theorem

Let us start with the partial equilibrium schooling decisions and establish a simple general result, sometimes referred to as a "separation theorem" for human capital investments. We set up the basic model in continuous time for simplicity.

Consider the schooling decision of a single individual facing exogenously given prices for human capital. Throughout, we assume that there are perfect capital markets. The separation theorem referred to in the title of this section will show that, with perfect capital markets, schooling decisions will maximize the net present discounted value of the individual (we return to human capital investments with imperfect capital markets in Chapter 21). In particular, consider an individual with an instantaneous utility function $u(c)$ that satisfies Assumption 3 above. Suppose that the individual has a planning horizon of T (where $T = \infty$

is allowed), discounts the future at the rate $\rho > 0$ and faces a constant flow rate of death equal to $\nu \geq 0$ (as in the perpetual youth model studied in the previous chapter). Standard arguments imply that the objective function of this individual at time $t = 0$ is

$$\max \int_0^T \exp\left(-\left(\rho + \nu\right) t\right) u\left(c\left(t\right)\right) dt. \tag{10.1}$$

Now suppose that this individual is born with some human capital $h(0) \geq 0$. Suppose that his human capital evolves over time according to the differential equation

$$\dot{h}\left(t\right) = G\left(t, h\left(t\right), s\left(t\right)\right), \tag{10.2}$$

where $s(t) \in [0,1]$ is the fraction of time that the individual spends for investments in schooling, and $G : \mathbb{R}_+^2 \times [0,1] \to \mathbb{R}_+$ determines how human capital evolves as a function of time, the individual's stock of human capital and schooling decisions. In addition, we can impose a further restriction on schooling decisions, for example,

$$s\left(t\right) \in \mathcal{S}\left(t\right), \tag{10.3}$$

where $\mathcal{S}(t) \subset [0,1]$ and captures the fact that all schooling may have to be full-time, i.e., $s(t) \in \{0,1\}$, or that there may exist thisother restrictions on schooling decisions.

The individual is assumed to face an exogenous sequence of wage per unit of human capital given by $[w(t)]_{t=0}^T$, so that his labor earnings at time t are

$$W\left(t\right) = w\left(t\right)\left[1 - s\left(t\right)\right]\left[h\left(t\right) + \omega\left(t\right)\right],$$

where $1 - s(t)$ is the fraction of time spent supplying labor to the market and $\omega(t)$ is non-human capital labor that the individual may be supplying to the market at time t. The sequence of non-human capital labor that the individual can supply to the market, $[\omega(t)]_{t=0}^T$, is exogenous. This formulation assumes that the only margin of choice is between market work and schooling (i.e., there is no leisure).

Finally, let us assume that the individual faces a constant (flow) interest rate equal to r on his savings (potentially including annuity payments as discussed in the previous chapter). Using the equation for labor earnings, the lifetime budget constraint of the individual can be written as

$$\int_0^T \exp\left(-rt\right) c\left(t\right) dt \leq \int_0^T \exp\left(-rt\right) w\left(t\right)\left[1 - s\left(t\right)\right]\left[h\left(t\right) + \omega\left(t\right)\right] dt. \tag{10.4}$$

The Separation Theorem, which is the subject of this section, can be stated as follows:

Theorem 10.1. ***(Separation Theorem)*** *Suppose that the instantaneous utility function $u(\cdot)$ is strictly increasing. Then the sequence $\left[\hat{c}(t), \hat{s}(t), \hat{h}(t)\right]_{t=0}^T$ is a solution to the maximization of (10.1) subject to (10.2), (10.3) and (10.4) if and only if $\left[\hat{s}(t), \hat{h}(t)\right]_{t=0}^T$*

maximizes

$$\int_0^T \exp(-rt)\, w(t)\, [1 - s(t)]\, [h(t) + \omega(t)]\, dt \tag{10.5}$$

subject to (10.2) and (10.3), and $[\hat{c}(t)]_{t=0}^T$ *maximizes (10.1) subject to (10.4) given* $\left[\hat{s}(t), \hat{h}(t)\right]_{t=0}^T$. *That is, human capital accumulation and supply decisions can be* separated *from consumption decisions.*

PROOF. To prove the "only if" part, suppose that $\left[\hat{s}(t), \hat{h}(t)\right]_{t=0}^T$ does not maximize (10.5), but there exists $\hat{c}(t)$ such that $\left[\hat{c}(t), \hat{s}(t), \hat{h}(t)\right]_{t=0}^T$ is a solution to (10.1). Let the value of (10.5) generated by $\left[\hat{s}(t), \hat{h}(t)\right]_{t=0}^T$ be denoted Y. Since $\left[\hat{s}(t), \hat{h}(t)\right]_{t=0}^T$ does not maximize (10.5), there exists $[s(t), h(t)]_{t=0}^T$ reaching a value of (10.5), $Y' > Y$. Consider the sequence $[c(t), s(t), h(t)]_{t=0}^T$, where $c(t) = \hat{c}(t) + \varepsilon$. By the hypothesis that $\left[\hat{c}(t), \hat{s}(t), \hat{h}(t)\right]_{t=0}^T$ is a solution to (10.1), the budget constraint (10.4) implies

$$\int_0^T \exp(-rt)\, \hat{c}(t)\, dt \le Y.$$

Let $\varepsilon > 0$ and consider $c(t) = \hat{c}(t) + \varepsilon$ for all t. We have that

$$\begin{aligned} \int_0^T \exp(-rt)\, c(t)\, dt &= \int_0^T \exp(-rt)\, \hat{c}(t)\, dt + \frac{[1 - \exp(-rT)]}{r}\varepsilon. \\ &\le Y + \frac{[1 - \exp(-rT)]}{r}\varepsilon. \end{aligned}$$

Since $Y' > Y$, for ε sufficiently small, the previous inequality can be satisfied and thus $[c(t), s(t), h(t)]_{t=0}^T$ is feasible. Since $u(\cdot)$ is strictly increasing, $[c(t), s(t), h(t)]_{t=0}^T$ is strictly preferred to $\left[\hat{c}(t), \hat{s}(t), \hat{h}(t)\right]_{t=0}^T$, leading to a contradiction and proving the "only if" part.

The proof of the "if" part is similar. Suppose that $\left[\hat{s}(t), \hat{h}(t)\right]_{t=0}^T$ maximizes (10.5). Let the maximum value be denoted by Y. Consider the maximization of (10.1) subject to the constraint that $\int_0^T \exp(-rt)\, c(t)\, dt \le Y$. Let $[\hat{c}(t)]_{t=0}^T$ be a solution. This implies that if $[c'(t)]_{t=0}^T$ is a sequence that is strictly preferred to $[\hat{c}(t)]_{t=0}^T$, then $\int_0^T \exp(-rt)\, c'(t)\, dt > Y$. This implies that $\left[\hat{c}(t), \hat{s}(t), \hat{h}(t)\right]_{t=0}^T$ must be a solution to the original problem, because any other $[s(t), h(t)]_{t=0}^T$ leads to a value of (10.5) $Y' \le Y$, and if $[c'(t)]_{t=0}^T$ is strictly preferred to $[\hat{c}(t)]_{t=0}^T$, then $\int_0^T \exp(-rt)\, c'(t)\, dt > Y \ge Y'$ for any Y' associated with any feasible $[s(t), h(t)]_{t=0}^T$. □

The intuition for this theorem is straightforward: in the presence of perfect capital markets, the best human capital accumulation decisions are those that maximize the lifetime budget set of the individual. Exercise 10.2 shows that this theorem does not hold when there are imperfect capital markets and also does not generalize to the case where leisure is also an argument of the utility function.

10.2. Schooling Investments and Returns to Education

We now turn to the simplest model of schooling decisions in partial equilibrium, which will illustrate the main tradeoffs in human capital investments. The model presented here is a version of Mincer's (1974) seminal contribution. This model also enables a simple mapping from the theory of human capital investments to the large empirical literature on returns to schooling.

Let us first assume that $T = \infty$, which will simplify the expressions. The flow rate of death, ν, is positive, so that individuals have finite expected lives. Suppose that (10.2) is such that the individual has to spend an interval S with $s(t) = 1$—i.e., in full-time schooling, and $s(t) = 0$ thereafter. At the end of the schooling interval, the individual will have a schooling level of

$$h(S) = \eta(S),$$

where $\eta(\cdot)$ is an increasing, continuously differentiable and concave function. For $t \in [S, \infty)$, human capital accumulates over time (as the individual works) according to the differential equation

$$\dot{h}(t) = g_h h(t), \tag{10.6}$$

for some $g_h \geq 0$. Suppose also that wages grow exponentially,

$$\dot{w}(t) = g_w w(t), \tag{10.7}$$

with boundary condition $w(0) > 0$.

Suppose that

$$g_w + g_h < r + \nu,$$

so that the net present discounted value of the individual is finite. Now using Theorem 10.1, the optimal schooling decision must be a solution to the following maximization problem

$$\max_S \int_S^\infty \exp(-(r+\nu)t)\, w(t)\, h(t)\, dt. \tag{10.8}$$

Now using (10.6) and (10.7), this is equivalent to (see Exercise 10.3):

$$\max_S \frac{\eta(S)\, w(0) \exp(-(r+\nu-g_w)S)}{r+\nu-g_h-g_w}. \tag{10.9}$$

Since $\eta(S)$ is concave, the objective function in (10.9) is strictly concave. Therefore, the unique solution to this problem is characterized by the first-order condition

$$\frac{\eta'(S^*)}{\eta(S^*)} = r + \nu - g_w. \tag{10.10}$$

Equation (10.10) shows that higher interest rates and higher values of ν (corresponding to shorter planning horizons) reduce human capital investments, while higher values of g_w increase the value of human capital and thus encourage further investments.

Integrating both sides of this equation with respect to S, we obtain

$$\ln \eta\left(S^{*}\right)=\text{constant}+\left(r+\nu-g_{w}\right) S^{*}. \tag{10.11}$$

Now note that the wage earnings of the worker of age $\tau \geq S^{*}$ in the labor market at time t will be given by

$$W(S, t)=\exp \left(g_{w} t\right) \exp \left(g_{h}(t-S)\right) \eta(S).$$

Taking logs and using equation (10.11) implies that the earnings of the worker will be given by

$$\ln W\left(S^{*}, t\right)= \text{ constant}+\left(r+\nu-g_{w}\right) S^{*}+g_{w} t+g_{h}\left(t-S^{*}\right),$$

where $t-S$ can be thought of as worker experience (time after schooling). If we make a cross-sectional comparison across workers, the time trend term $g_{w} t$, will also go into the constant, so that we obtain the canonical Mincer equation where, in the cross section, log wage earnings are proportional to schooling and experience. Written differently, we have the following cross-sectional equation

$$\ln W_{j}= \text{ constant}+\gamma_{s} S_{j}+\gamma_{e} \text{experience}, \tag{10.12}$$

where j refers to individual j. Note however that we have not introduced any source of heterogeneity that can generate different levels of schooling across individuals. Nevertheless, equation (10.12) is important, since it is the typical empirical model for the relationship between wages and schooling estimated in labor economics.

The economic insight provided by this equation is quite important; it suggests that the functional form of the Mincerian wage equation is not just a mere coincidence, but has economic content: the opportunity cost of one more year of schooling is foregone earnings. This implies that the benefit has to be commensurate with these foregone earnings, thus should lead to a proportional increase in earnings in the future. In particular, this proportional increase should be at the rate $\left(r+\nu-g_{w}\right)$.

As already discussed in Chapter 3, empirical work using equations of the form (10.12) leads to estimates for γ in the range of 0.06 to 0.10. Equation (10.12) suggests that these returns to schooling are not unreasonable. For example, we can think of the annual interest rate r as approximately 0.10, ν as corresponding to 0.02 that gives an expected life of 50 years, and g_{w} corresponding to the rate of wage growth holding the human capital level of the individual constant, which should be approximately about 2%. Thus we should expect an estimate of γ around 0.10, which is consistent with the upper range of the empirical estimates.

10.3. The Ben-Porath Model

The baseline Ben-Porath model enriches the model studied in the previous section by allowing human capital investments and non-trivial labor supply decisions throughout the lifetime of the individual. In particular, we now let $s(t) \in [0,1]$ for all $t \geq 0$. Together with the Mincer equation (10.12) (and the model underlying this equation presented in the previous section), the Ben-Porath model is the basis of much of labor economics. Here it is sufficient to consider a simple version of this model where the human capital accumulation equation, (10.2), takes the form

$$\dot{h}(t) = \phi(s(t)h(t)) - \delta_h h(t), \tag{10.13}$$

where $\delta_h > 0$ captures "depreciation of human capital," for example because new machines and techniques are being introduced, eroding the existing human capital of the worker. The individual starts with an initial value of human capital $h(0) > 0$. The function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing, continuously differentiable and strictly concave. Furthermore, we simplify the analysis by assuming that this function satisfies the Inada-type conditions,

$$\lim_{x \to 0} \phi'(x) = \infty \text{ and } \lim_{x \to h(0)} \phi'(x) = 0.$$

The latter condition makes sure that we do not have to impose additional constraints to ensure $s(t) \in [0,1]$ (see Exercise 10.5).

Let us also suppose that there is no non-human capital component of labor, so that $\omega(t) = 0$ for all t, that $T = \infty$, and that there is a flow rate of death $\nu > 0$. Finally, we assume that the wage per unit of human capital is constant at w and the interest rate is constant and equal to r. We also normalize $w = 1$ without loss of any generality.

Again using Theorem 10.1, human capital investments can be determined as a solution to the following problem

$$\max \int_0^\infty \exp(-(r+\nu))(1-s(t))h(t)\,dt$$

subject to (10.13).

This problem can be solved by setting up the current-value Hamiltonian, which in this case takes the form

$$\mathcal{H}(h,s,\mu) = (1-s(t))h(t) + \mu(t)(\phi(s(t)h(t)) - \delta_h h(t)),$$

where we used $\mathcal{H}$ to denote the Hamiltonian to avoid confusion with human capital. The necessary conditions for this problem are

$$\begin{aligned}
\mathcal{H}_s(h,s,\mu) &= -h(t) + \mu(t)h(t)\phi'(s(t)h(t)) = 0 \\
\mathcal{H}_h(h,s,\mu) &= (1-s(t)) + \mu(t)\big(s(t)\phi'(s(t)h(t)) - \delta_h\big) \\
&= (r+\nu)\mu(t) - \dot{\mu}(t) \\
\lim_{t\to\infty} \exp(-(r+\nu)t)\mu(t)h(t) &= 0.
\end{aligned}$$

To solve for the optimal path of human capital investments, let us adopt the following transformation of variables:

$$x(t) \equiv s(t)\,h(t).$$

Instead of $s(t)$ (or $\mu(t)$) and $h(t)$, we will study the dynamics of the optimal path in $x(t)$ and $h(t)$.

The first necessary condition then implies that

$$1 = \mu(t)\,\phi'(x(t)), \tag{10.14}$$

while the second necessary condition can be expressed as

$$\frac{\dot{\mu}(t)}{\mu(t)} = r + \nu + \delta_h - s(t)\,\phi'(x(t)) - \frac{1 - s(t)}{\mu(t)}.$$

Substituting for $\mu(t)$ from (10.14), and simplifying, we obtain

$$\frac{\dot{\mu}(t)}{\mu(t)} = r + \nu + \delta_h - \phi'(x(t)). \tag{10.15}$$

The steady-state (stationary) solution of this optimal control problem involves $\dot{\mu}(t) = 0$ and $\dot{h}(t) = 0$, and thus implies that

$$x^* = \phi'^{-1}(r + \nu + \delta_h), \tag{10.16}$$

where $\phi'^{-1}(\cdot)$ is the inverse function of $\phi'(\cdot)$ (which exists and is strictly decreasing since $\phi(\cdot)$ is strictly concave). This equation shows that $x^* \equiv s^* h^*$ will be higher when the interest rate is low, when the life expectancy of the individual is high, and when the rate of depreciation of human capital is low.

To determine s^* and h^* separately, we set $\dot{h}(t) = 0$ in the human capital accumulation equation (10.13), which gives

$$\begin{aligned} h^* &= \frac{\phi(x^*)}{\delta_h} \\ &= \frac{\phi\left(\phi'^{-1}(r + \nu + \delta_h)\right)}{\delta_h}. \end{aligned} \tag{10.17}$$

Since $\phi'^{-1}(\cdot)$ is strictly decreasing and $\phi(\cdot)$ is strictly increasing, this equation implies that the steady-state solution for the human capital stock is uniquely determined and is decreasing in r, ν and δ_h.

More interesting than the stationary (steady-state) solution to the optimization problem is the time path of human capital investments in this model. To derive this, differentiate (10.14) with respect to time to obtain

$$\frac{\dot{\mu}(t)}{\mu(t)} = \varepsilon_{\phi'}(x)\,\frac{\dot{x}(t)}{x(t)},$$

where

$$\varepsilon_{\phi'}(x) = -\frac{x\phi''(x)}{\phi'(x)} > 0$$

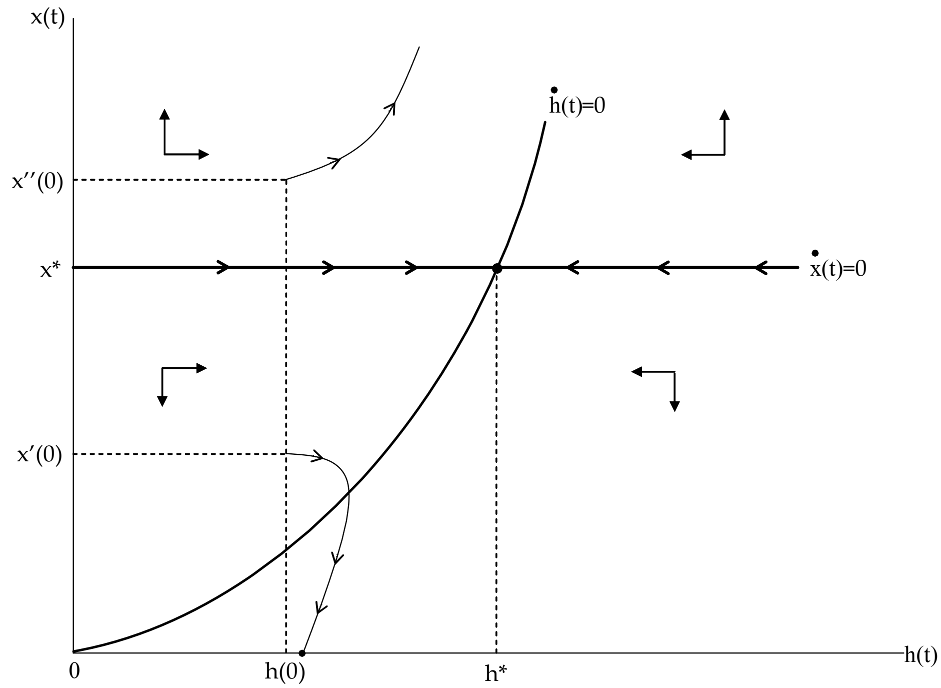

FIGURE 10.1. Steady state and equilibrium dynamics in the simplified Ben Porath model.

is the elasticity of the function $\phi'(\cdot)$ and is positive since $\phi'(\cdot)$ is strictly decreasing (thus $\phi''(\cdot)<0$). Combining this equation with (10.15), we obtain

$$\frac{\dot{x}(t)}{x(t)}=\frac{1}{\varepsilon_{\phi'}(x(t))}\left(r+\nu+\delta_h-\phi'(x(t))\right). \tag{10.18}$$

Figure 10.1 plots (10.13) and (10.18) in the h-x space. The upward-sloping curve corresponds to the locus for $\dot{h}(t)=0$, while (10.18) can only be zero at x^*, thus the locus for $\dot{x}(t)=0$ corresponds to the horizontal line in the figure. The arrows of motion are also plotted in this phase diagram and make it clear that the steady-state solution (h^*,x^*) is globally saddle-path stable, with the stable arm coinciding with the horizontal line for $\dot{x}(t)=0$. Starting with $h(0)\in(0,h^*)$, $s(0)$ jumps to the level necessary to ensure $s(0)h(0)=x^*$. From then on, $h(t)$ increases and $s(t)$ decreases so as to keep $s(t)h(t)=x^*$. Therefore, the pattern of human capital investments implied by the Ben-Porath model is one of high investment at the beginning of an individual's life followed by lower investments later on.

In our simplified version of the Ben-Porath model this all happens smoothly. In the original Ben-Porath model, which involves the use of other inputs in the production of human capital and finite horizons, the constraint for $s(t)\leq 1$ typically binds early on in the life of the individual, and the interval during which $s(t)=1$ can be interpreted as full-time

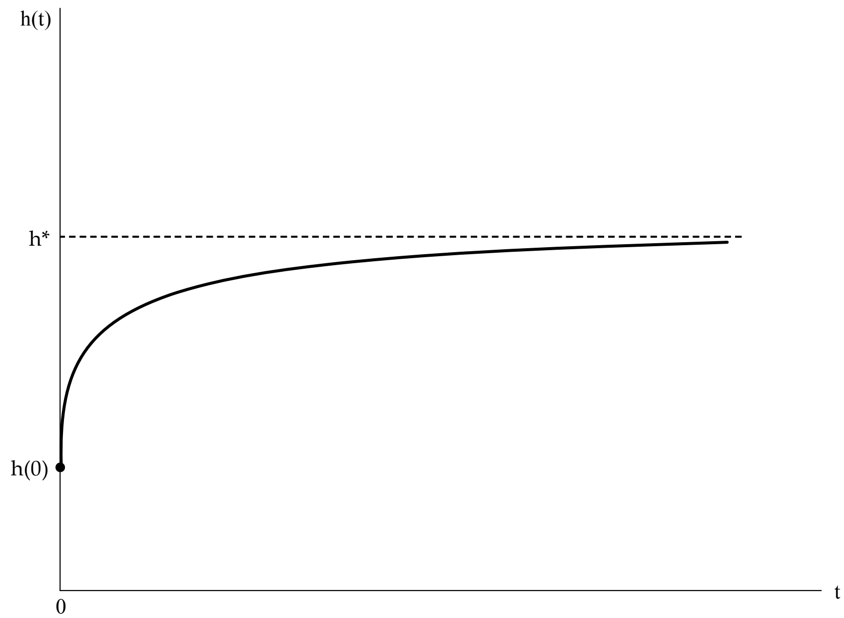

Figure 10.2. Time path of human capital investments in the simplified Ben Porath model.

schooling. After full-time schooling, the individual starts working (i.e., $s(t) < 1$). But even on-the-job, the individual continues to accumulate human capital (i.e., $s(t) > 0$), which can be interpreted as spending time in training programs or allocating some of his time on the job to learning rather than production. Moreover, because the horizon is finite, if the Inada conditions were relaxed, the individual could prefer to stop investing in human capital at some point. As a result, the time path of human capital generated by the standard Ben-Porath model may be hump-shaped, with a possibly declining portion at the end (see Exercise 10.6). Instead, the path of human capital (and the earning potential of the individual) in the current model is always increasing as shown in Figure 10.2.

The importance of the Ben-Porath model is twofold. First, it emphasizes that schooling is not the only way in which individuals can invest in human capital and there is a continuity between schooling investments and other investments in human capital. Second, it suggests that in societies where schooling investments are high we may also expect higher levels of on-the-job investments in human capital. Thus there may be systematic mismeasurement of the amount or the quality human capital across societies.

10.4. Neoclassical Growth with Physical and Human Capital

Our next task is to incorporate human capital investments into the baseline neoclassical growth model. This is useful both to investigate the interactions between physical and human capital, and also to generate a better sense of the impact of differential human capital investments on economic growth. Physical-human capital interactions could potentially be important, since a variety of evidence suggests that physical capital and human capital (capital and skills) are complementary, meaning that greater capital increases the productivity of high human capital workers more than that of low skill workers. This may play an important role in economic growth, for example, by inducing a "virtuous cycle" of investments in physical and human capital. These types of virtue cycles will be discussed in greater detail in Chapter 21. It is instructive to see to what extent these types of complementarities manifest themselves in the neoclassical growth model. The potential for complementarities also raises the issue of "imbalances". If physical and human capital are complementary, the society will achieve the highest productivity when there is a balance between these two different types of capital. However, whether the decentralized equilibrium will ensure such a balance is a question that needs to be investigated.

The impact of human capital on economic growth (and on cross-country income differences) has already been discussed in Chapter 3, in the context of an augmented Solow model, where the economy was assumed to accumulate physical and human capital with two exogenously given constant saving rates. In many ways, that model was less satisfactory than the baseline Solow growth model, since not only was the aggregate saving rate assumed exogenous, but the relative saving rates in human and physical capital were also taken as given. The neoclassical growth model with physical and human capital investments will enable us to investigate the same set of issues from a different perspective.

Consider the following continuous time economy admitting a representative household with preferences

$$\int_0^\infty \exp(-\rho t)\, u(c(t))\, dt, \tag{10.19}$$

where the instantaneous utility function $u(\cdot)$ satisfies Assumption 3 and $\rho > 0$. We ignore technological progress and population growth to simplify the discussion. Labor is again supplied inelastically.

We follow the specification in Chapter 3 and assume that the aggregate production possibilities frontier of the economy is represented by the following aggregate production function:

$$Y(t) = F(K(t), H(t), L(t)),$$

where $K(t)$ is the stock of physical capital, $L(t)$ is total employment, and $H(t)$ represents human capital. Since there is no population growth and labor is supplied inelastically, $L(t) =$

L for all t. This production function is assumed to satisfy Assumptions $1'$ and $2'$ in Chapter 3, which generalize Assumptions 1 and 2 to this production function with three inputs. As already discussed in that chapter, a production function in which "raw" labor and human capital are separate factors of production may be less natural than one in which human capital increases the effective units of labor of workers (as in the analysis of the previous two sections). Nevertheless, this production function allows a simple analysis of neoclassical growth with physical and human capital. As usual, it is more convenient to express all objects in per capita units, thus we write

$$\begin{aligned} y(t) &\equiv \frac{Y(t)}{L} \\ &= f(k(t), h(t)), \end{aligned}$$

where

$$k(t) \equiv \frac{K(t)}{L} \text{ and } h(t) \equiv \frac{H(t)}{L}$$

are the physical and human capital levels per capita. In view of Assumptions $1'$ and $2'$, $f(k,h)$ is strictly increasing, continuously differentiable and jointly strictly concave in both of its arguments. We denote its derivatives by f_k, f_h, f_{kh}, etc. Throughout, we assume that physical and human capital are complementary, that is, $f_{kh}(k,h) > 0$ for all $k, h > 0$.

We assume that physical and human capital per capita evolve according to the following two differential equations

$$\dot{k}(t) = i_k(t) - \delta_k k(t), \tag{10.20}$$

and

$$\dot{h}(t) = i_h(t) - \delta_h h(t) \tag{10.21}$$

where $i_k(t)$ and $i_h(t)$ are the investment levels in physical and human capital, while δ_k and δ_h are the depreciation rates of these two capital stocks. The resource constraint for the economy, expressed in per capita terms, is

$$c(t) + i_k(t) + i_h(t) \leq f(k(t), h(t)) \text{ for all } t. \tag{10.22}$$

Since the environment described here is very similar to the standard neoclassical growth model, equilibrium and optimal growth will coincide. For this reason, we focus on the optimal growth problem (the competitive equilibrium is discussed in Exercise 10.12). The optimal growth problem involves the maximization of (10.19) subject to (10.20), (10.21), and (10.22). The solution to this maximization problem can again be characterized by setting up the current-value Hamiltonian. To simplify the analysis, we first observe that since $u(c)$ is strictly increasing, (10.22) will always hold as equality. We can then substitute for $c(t)$ using this constraint and write the current-value Hamiltonian as

$$\begin{aligned}\mathcal{H}\left(k\left(t\right),h\left(t\right),i_{k}\left(t\right),i_{h}\left(t\right),\mu_{k}\left(t\right),\mu_{h}\left(t\right)\right) &= u\left(f\left(k\left(t\right),h\left(t\right)\right)-i_{h}\left(t\right)-i_{k}\left(t\right)\right)\\ &\quad+\mu_{h}\left(t\right)\left(i_{h}\left(t\right)-\delta_{h}h\left(t\right)\right)+\mu_{k}\left(t\right)\left(i_{k}\left(t\right)-\delta_{k}k\left(t\right)\right),\end{aligned} \tag{10.23}$$

where we now have two control variables, $i_k(t)$ and $i_h(t)$ and two state variables, $k(t)$ and $h(t)$, as well as two costate variables, $\mu_k(t)$ and $\mu_h(t)$, corresponding to the two constraints, (10.20) and (10.21). The necessary conditions for an optimal solution are

$$\begin{aligned}\mathcal{H}_{i_k}\left(k\left(t\right),h\left(t\right),i_{k}\left(t\right),i_{h}\left(t\right),\mu_{k}\left(t\right),\mu_{h}\left(t\right)\right) &= -u'\left(c\left(t\right)\right)+\mu_{k}\left(t\right)=0\\ \mathcal{H}_{i_h}\left(k\left(t\right),h\left(t\right),i_{k}\left(t\right),i_{h}\left(t\right),\mu_{k}\left(t\right),\mu_{h}\left(t\right)\right) &= -u'\left(c\left(t\right)\right)+\mu_{h}\left(t\right)=0\\ \mathcal{H}_{k}\left(k\left(t\right),h\left(t\right),i_{k}\left(t\right),i_{h}\left(t\right),\mu_{k}\left(t\right),\mu_{h}\left(t\right)\right) &= f_{k}\left(k\left(t\right),h\left(t\right)\right)u'\left(c\left(t\right)\right)-\mu_{k}\left(t\right)\delta_{k}\\ &= \rho\mu_{k}\left(t\right)-\dot{\mu}_{k}\left(t\right)\\ \mathcal{H}_{h}\left(k\left(t\right),h\left(t\right),i_{k}\left(t\right),i_{h}\left(t\right),\mu_{k}\left(t\right),\mu_{h}\left(t\right)\right) &= f_{h}\left(k\left(t\right),h\left(t\right)\right)u'\left(c\left(t\right)\right)-\mu_{h}\left(t\right)\delta_{h}\\ &= \rho\mu_{h}\left(t\right)-\dot{\mu}_{h}\left(t\right)\\ \lim_{t\to\infty}\exp\left(-\rho t\right)\mu_{k}\left(t\right)k\left(t\right) &= 0\\ \lim_{t\to\infty}\exp\left(-\rho t\right)\mu_{h}\left(t\right)h\left(t\right) &= 0.\end{aligned}$$

There are two necessary transversality conditions since there are two state variables (and two costate variables). Moreover, it can be shown that $\mathcal{H}\left(k\left(t\right),h\left(t\right),i_{k}\left(t\right),i_{h}\left(t\right),\mu_{k}\left(t\right),\mu_{h}\left(t\right)\right)$ is concave given the costate variables $\mu_k(t)$ and $\mu_h(t)$, so that a solution to the necessary conditions indeed gives an optimal path (see Exercise 10.9).

The first two necessary conditions immediately imply that

$$\mu_{k}\left(t\right)=\mu_{h}\left(t\right)=\mu\left(t\right).$$

Combining this with the next two conditions gives

$$f_{k}\left(k\left(t\right),h\left(t\right)\right)-f_{h}\left(k\left(t\right),h\left(t\right)\right)=\delta_{k}-\delta_{h}, \tag{10.24}$$

which (together with $f_{kh}>0$) implies that there is a one-to-one relationship between physical and human capital, of the form

$$h=\xi\left(k\right),$$

where $\xi(\cdot)$ is uniquely defined, strictly increasing and differentiable (with derivative denoted by $\xi'(\cdot)$, see Exercise 10.10).

This observation makes it clear that the model can be reduced to the neoclassical growth model and has exactly the same dynamics as the neoclassical growth model, and thus establishes the following proposition:

Proposition 10.1. *In the neoclassical growth model with physical and human capital investments described above, the optimal path of physical capital and consumption are given as in the one-sector neoclassical growth model, and satisfy the following two differential equations*

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))}\left[f_k(k(t), \xi(k(t))) - \delta_k - \rho\right],$$

$$\dot{k}(t) = \frac{1}{1+\xi'(k)}\left[f(k(t), \xi(k(t))) - \delta_h \xi(k(t)) - \delta_k k(t) - c(t)\right],$$

where $\varepsilon_u(c(t)) = -u''(c(t))c(t)/u'(c(t))$, *together with the transversality condition* $\lim_{t\to\infty}\left[k(t)\exp\left(-\int_0^t f_k(k(s), \xi(k(s)))\,ds\right)\right] = 0$, *while the level of human capital at time* t *is given by* $h(t) = \xi(k(t))$.

Proof. see Exercise 10.11 □

What is perhaps more surprising, at first, is that equation (10.24) implies that human and physical capital are always in "balance". Initially, one may have conjectured that an economy that starts with a high stock of physical capital relative to human capital will have a relatively high physical to human capital ratio for an extended period of time. However, Proposition 10.1 and in particular, equation (10.24) show that this is not the case. The reason for this is that we have not imposed any non-negativity constraints on the investment levels. If the economy starts with a high level of physical capital and low level of human capital, at the first instant it will experience a very high level of $i_h(0)$, compensated with a very negative $i_k(0)$, so that at the next instant the physical to human capital ratio will have been brought back to balance. After this, the dynamics of the economy will be identical to those of the baseline neoclassical growth model. Therefore, issues of imbalance will not arise in this version of the neoclassical growth model. This result is an artifact of the fact that there are no non-negativity constraints on physical and human capital investments. The situation is somewhat different when there are such non-negativity or "irreversibility" constraints, that is, if we assume that $i_k(t) \geq 0$ and $i_h(t) \geq 0$ for all t. In this case, initial imbalances will persist for a while. In particular, it can be shown that starting with a ratio of physical to human capital stock $(k(0)/h(0))$ that does not satisfy (10.24), the optimal path will involve investment only in one of the two stocks until balance is reached (see Exercise 10.14). Therefore, with irreversibility constraints, some amount of imbalance can arise, but the economy quickly moves towards correcting this imbalance.

Another potential application of the neoclassical growth model with physical and human capital is in the analysis of the impact of policy distortions. Recall the discussion in Section 8.9 in Chapter 8, and suppose that the resource constraint of the economy is modified to

$$c(t) + (1+\tau)(i_k(t) + i_h(t)) \leq f(k(t), h(t)),$$

where $\tau \geq 0$ is a tax affecting both types of investments. Using an analysis parallel to that in Section 8.9, we can characterize the steady-state income ratio of two countries with different policies represented by τ and τ'. In particular, let us suppose that the aggregate production function takes the Cobb-Douglas form

$$\begin{aligned} Y &= F(K, H, L) \\ &= K^{\alpha_k} H^{\alpha_h} L^{1-\alpha_k-\alpha_h}. \end{aligned}$$

In this case, the ratio of income in the two economies with taxes/distortions of τ and τ' is given by (see Exercise 10.15):

$$\frac{Y(\tau)}{Y(\tau')} = \left(\frac{1+\tau'}{1+\tau}\right)^{\frac{\alpha_k+\alpha_h}{1-\alpha_k-\alpha_h}}. \tag{10.25}$$

If we again take α_k to be approximately 1/3, then the ability of this modified model to account for income differences using tax distortions increases because of the responsiveness of human capital accumulation to these distortions. For example, with $\alpha_k = \alpha_h = 1/3$ and eightfold distortion differences, we would have

$$\frac{Y(\tau)}{Y(\tau')} \approx 8^2 \approx 64,$$

which is a huge difference in economic performance across countries.

Therefore, incorporating human capital into the neoclassical growth model provides one potential way of generating larger income per capita differences. Nevertheless, this result has to be interpreted with caution. First, the large impact of distortions on income per capita here is driven by a very elastic response of human capital accumulation. It is not clear whether human capital investments will indeed respond so much to tax distortions. For instance, if these distortions correspond to differences in corporate taxes or corruption, we would expect them to affect corporations rather than individual human capital decisions. This is of course not to deny that in societies where policies discourage capital accumulation, there are also barriers to schooling and other types of human capital investments. Nevertheless, the impact of these on physical and human capital investments may be quite different. Second, and more important, the large implied elasticity of output to distortions when both physical and human capital are endogenous has an obvious similarity to the Mankiw-Romer-Weil's approach to explaining cross-country differences in terms of physical and human capital stocks. As discussed in Chapter 3, while this is a logical possibility, existing evidence does not support the notion that human capital differences across countries can have such a large impact on income differences. This conclusion equally sheds doubt on the importance of the large contribution of human capital differences induced by policy differences in the current model. Nevertheless, the conclusions in Chapter 3 were subject to two caveats, which could potentially increase the role of human capital; large human capital externalities and significant differences in the quality of schooling across countries. These issues will be discussed below.

10.5. Capital-Skill Complementarity in an Overlapping Generations Model

Our analysis in the previous section suggests that the neoclassical growth model with physical and human capital does not generate significant imbalances between these two different types of capital (unless we impose irreversibilities, in which case it can do so along the transition path). We next investigate possibility of capital-skill imbalances in a simple overlapping generations model with impure altruism, similar to the models introduced in Section 9.6 of the previous chapter. We will see that this class of models also generates only limited capital-skill imbalances. Nevertheless, it provides a simple framework in which labor market frictions can be introduced, and capital-skill imbalances become much more important in the presence of such frictions. We will also use the model in this section to go back to the more natural production function, which features capital and effective units of labor (with human capital-augmenting the effective units of labor), as opposed to the production function used in the previous section with human capital as a third separate factor of production.

The economy is in discrete time and consists of a continuum 1 of dynasties. Each individual lives for two periods, childhood and adulthood. Individual i of generation t works during their adulthood at time t, earns labor income equal to $w(t)\,h_i(t)$, where $w(t)$ is the wage rate per unit of human capital and $h_i(t)$ is the individual's human capital. The individual also earns capital income equal to $R(t)\,b_i(t-1)$, where $R(t)$ is the gross rate of return on capital and $b_i(t-1)$ is his asset holdings, inherited as bequest from his parent. The human capital of the individual is determined at the beginning of his adulthood by an effort decision. Labor is supply to the market after this effort decision. At the end of adulthood, after labor and capital incomes are received, the individual decides his consumption and the level of bequest to his offspring.

Preferences of individual i (or of dynasty i) of generation t are given by

$$\eta^{-\eta}(1-\eta)^{-(1-\eta)}\,c_i(t)^{\eta}\,b_i(t)^{1-\eta}-\gamma(e_i(t)),$$

where $\eta\in(0,1)$, $c_i(t)$ is own consumption, $b_i(t)$ is the bequest to the offspring, $e_i(t)$ is effort expended for human capital acquisition, and $\gamma(\cdot)$ is a strictly increasing, continuously differentiable and strictly convex cost of effort function. The term $\eta^{-\eta}(1-\eta)^{-(1-\eta)}$ is included as a normalizing factor to simplify the algebra.

The human capital of individual i is given by

$$h_i(t)=ae_i(t), \tag{10.26}$$

where a corresponds to "ability" and increases the effectiveness of effort in generating human capital for the individual. Substituting for $e_i(t)$ in the above expression, the preferences of individual i of generation t can be written as

$$\eta^{-\eta}(1-\eta)^{-(1-\eta)}c_i(t)^{\eta}b_i(t)^{1-\eta}-\gamma\left(\frac{h_i(t)}{a}\right). \tag{10.27}$$

The budget constraint of the individual is

$$c_i(t)+b_i(t)\leq m_i(t)=w(t)h_i(t)+R(t)b_i(t-1), \tag{10.28}$$

which defines $m_i(t)$ as the current income of individual i at time t consisting of labor earnings, $w(t)h_i(t)$, and asset income, $R(t)b_i(t-1)$ (we use m rather than y, since y will have a different meaning below).

The production side of the economy is given by an aggregate production function

$$Y(t)=F(K(t),H(t)),$$

that satisfies Assumptions 1 and 2, where $H(t)$ is "effective units of labor" or alternatively the total stock of human capital given by,

$$H(t)=\int_0^1 h_i(t)\,di,$$

while $K(t)$, the stock of physical capital, is given by

$$K(t)=\int_0^1 b_i(t-1)\,di.$$

Note also that this specification ensures that capital and skill (K and H) are complements. This is because a production function with two factors and constant returns to scale necessarily implies that the two factors are complements (see Exercise 10.7), that is,

$$\frac{\partial^2 F(K,H)}{\partial K\partial H}\geq 0. \tag{10.29}$$

Furthermore, we again simplify the notation by assuming capital depreciates fully after use, that is, $\delta=1$ (see Exercise 10.8).

Since the amount of human capital per worker is an endogenous variable in this economy, it is more useful to define a normalized production function expressing output per unit of human capital rather than the usual per capita production function. In particular, let $\kappa\equiv K/H$ be the capital to human capital ratio (or the "effective capital-labor ratio"), and

$$\begin{aligned}y(t) &\equiv \frac{Y(t)}{H(t)}\\ &= F\left(\frac{K(t)}{H(t)},1\right)\\ &= f(\kappa(t)),\end{aligned}$$

where the second line uses the linear homogeneity of $F(\cdot,\cdot)$, while the last line uses the definition of κ. Here we use κ instead of the more usual k, in order to preserve the notation

k for capital per worker in the next section. From the definition of κ, the law of motion of effective capital-labor ratios can be written as

$$\kappa(t) \equiv \frac{K(t)}{H(t)} = \frac{\int_0^1 b_i(t-1)\,di}{\int_0^1 h_i(t)\,di}. \tag{10.30}$$

Factor prices are then given by the usual competitive pricing formulae:

$$R(t) = f'(\kappa(t)) \text{ and } w(t) = f(\kappa(t)) - \kappa(t) f'(\kappa(t)), \tag{10.31}$$

with the only noteworthy feature that $w(t)$ is now wage per unit of human capital, in a way consistent with (10.28).

An equilibrium in this overlapping generations economy is a sequence of bequest and consumption levels for each individual, $\left\{[h_i(t)]_{i\in[0,1]}, [c_i(t)]_{i\in[0,1]}, [b_i(t)]_{i\in[0,1]}\right\}_{t=0}^{\infty}$, that solve (10.27) subject to (10.28) a sequence of effective capital-labor ratios, $\{\kappa(t)\}_{t=0}^{\infty}$, given by (10.30) with some initial distribution of bequests $[b_i(0)]_{i\in[0,1]}$, and sequences of factor prices, $\{w(t), R(t)\}_{t=0}^{\infty}$, that satisfy (10.31).

The characterization of an equilibrium is simplified by the fact that the solution to the maximization problem of (10.27) subject to (10.28) involves

$$c_i(t) = \eta m_i(t) \text{ and } b_i(t) = (1-\eta)\, m_i(t), \tag{10.32}$$

and substituting these into (10.27), we obtain the indirect utility function (see Exercise 10.16):

$$m_i(t) - \gamma\left(\frac{h_i(t)}{a}\right), \tag{10.33}$$

which the individual maximizes by choosing $h_i(t)$ and recognizing that $m_i(t) = w(t)\, h_i(t) + R(t)\, b_i(t-1)$. The first-order condition of this maximization gives the human capital investment of individual i at time t as:

$$a w(t) = \gamma'\left(\frac{h_i(t)}{a}\right), \tag{10.34}$$

or inverting this relationship, defining $\gamma'^{-1}(\cdot)$ as the inverse function of $\gamma'(\cdot)$ (which is strictly increasing) and using (10.31), we obtain

$$h_i(t) = h(t) \equiv a\gamma'^{-1}\left[a\left(f(\kappa(t)) - \kappa(t) f'(\kappa(t))\right)\right]. \tag{10.35}$$

An important implication of this equation is that the human capital investment of each individual is identical, and only depends on the effective of capital-labor ratio in the economy. This is a consequence of the specific utility function in (10.27), which ensures that there are no income effects in human capital decisions so that all agents choose the same "income-maximizing" level of human capital (as in Theorem 10.1).

Next, note that since bequest decisions are linear as shown (10.32), we have

$$\begin{aligned} K(t+1) &= \int_0^1 b_i(t)\,di \\ &= (1-\eta)\int_0^1 m_i(t)\,di \\ &= (1-\eta) f(\kappa(t)) h(t), \end{aligned}$$

where the last line uses the fact that, since all individuals choose the same human capital level given by (10.35), $H(t) = h(t)$, and thus $Y(t) = f(\kappa(t))h(t)$.

Now combining this with (10.30), we obtain

$$\kappa(t+1) = \frac{(1-\eta) f(\kappa(t)) h(t)}{h(t+1)}.$$

Using (10.35), this becomes

$$\begin{aligned} & \kappa(t+1)\gamma'^{-1}\left[a\left(f(\kappa(t+1)) - \kappa(t+1) f'(\kappa(t+1))\right)\right] \\ = \;& (1-\eta) f(\kappa(t))\gamma'^{-1}\left[af(\kappa(t)) - \kappa(t) f'(\kappa(t))\right]. \end{aligned} \tag{10.36}$$

A steady state, as usual, involves a constant effective capital-labor ratio, i.e., $\kappa(t) = \kappa^*$ for all t. Substituting this into (10.36) yields

$$\kappa^* = (1-\eta) f(\kappa^*), \tag{10.37}$$

which defines the unique positive steady-state effective capital-labor ratio, κ^* (since $f(\cdot)$ is strictly concave).

Proposition 10.2. *In the overlapping generations economy with physical and human capital described above, there exists a unique steady state with positive activity, and the physical to human capital ratio is κ^* as given by (10.37).*

This steady-state equilibrium is also typically stable, but some additional conditions need to be imposed on the $f(\cdot)$ and $\gamma(\cdot)$ to ensure this (see Exercise 10.17).

An interesting implication of this equilibrium is that, the capital-skill (k-h) complementarity in the production function $F(\cdot,\cdot)$ implies that a certain target level of physical to human capital ratio, κ^*, has to be reached in equilibrium. In other words, physical capital should not be too abundant relative to human capital, and neither should human capital be excessive relative to physical capital. Consequently, this model does not allow equilibrium "imbalances" between physical and human capital either. A possible and arguably attractive way of introducing such imbalances is to depart from perfectly competitive labor markets. This also turns out to be useful to illustrate how the role of human capital can be quite different in models with imperfect labor markets.

10.6. Physical and Human Capital with Imperfect Labor Markets

In this section, we analyze the implications of labor market frictions that lead to factor prices different from the ones we have used so far (in particular, in terms of the model of the last section, deviating from the competitive pricing formulea (10.31)). The literature on labor market imperfections is vast and our purpose here is not to provide an overview. For this reason, we will adopt the simplest representation. In particular, imagine that the economy is identical to that described in the previous section, except that there is a measure 1 of firms as well as a measure 1 of individuals at any point in time, and each firm can only hire one worker. The production function of each firm is still given by

$$y_j(t) = F(k_j(t), h_i(t)),$$

where $y_j(t)$ refers to the output of firm j, $k_j(t)$ is its capital stock (equivalently capital per worker, since the firm is hiring only one worker), and $h_i(t)$ is the human capital of worker i that the firm has matched with. This production function again satisfies Assumptions 1 and 2. The main departure from the models analyzed so far is that we now assume the following structure for the labor market:

(1) Firms choose their physical capital level irreversibly (incurring the cost $R(t)k_j(t)$, where $R(t)$ is the market rate of return on capital), and simultaneously workers choose their human capital level irreversibly.
(2) After workers complete their human capital investments, they are randomly matched with firms. Random matching here implies that high human capital workers are *not* more likely to be matched with high physical capital firms.
(3) After matching, each worker-firm pair bargains over the division of output between themselves. We assume that they simply divide the output according to some pre-specified rule, and the worker receives total earnings of

$$W_j(k_j(t), h_i(t)) = \lambda F(k_j(t), h_i(t)),$$

for some $\lambda \in (0, 1)$.

This is admittedly a very simple and reduced-form specification. Nevertheless, it will be sufficient to emphasize the main economic issues. A more detailed game-theoretic justification for a closely related environment is provided in Acemoglu (1996).

Let us next introduce heterogeneity in the cost of human capital acquisition by modifying (10.26) to

$$h_i(t) = a_i e_i(t),$$

where a_i differs across dynasties (individuals). A high-value of a_i naturally corresponds to an individual who can more effectively accumulate human capital.

An equilibrium is defined similarly except that factor prices are no longer determined by (10.31). Let us start the analysis with the physical capital choices of firms. At the time each firm chooses its physical capital it is unsure about the human capital of the worker he will be facing. Therefore, the expected return of firm j can be written as

$$(1-\lambda)\int_0^1 F\left(k_j\left(t\right),h_i\left(t\right)\right)di-R\left(t\right)k_j\left(t\right). \tag{10.38}$$

This expression takes into account that the firm will receive a fraction $1-\lambda$ of the output produced jointly by itself and of worker that it is matched with. The integration takes care of the fact that the firm does not know which worker it will be matched with and thus we are taking the expectation of $F\left(k_j\left(t\right),h_i\left(t\right)\right)$ over all possible human capital levels that are possible (given by $\left[h_i\left(t\right)\right]_{i\in[0,1]}$). The last term is the cost of making irreversible capital investment at the market price $R\left(t\right)$. This investment is made before the firm knows which worker it will be matched with. The important observation is that the objective function in (10.38) is strict concave in $k_j\left(t\right)$ given the strict concavity of $F\left(\cdot,\cdot\right)$ from Assumption 1. Therefore, each firm will choose the same level of physical capital, $\hat{k}\left(t\right)$, such that

$$(1-\lambda)\int_0^1 \frac{\partial F\left(\hat{k}\left(t\right),h_i\left(t\right)\right)}{\partial k\left(t\right)}di=R\left(t\right).$$

Now given this (expected) capital investment by firms, and following (10.33) from the previous section, each worker's objective function can be written as:

$$\lambda F\left(\hat{k}\left(t\right),h_i\left(t\right)\right)+R\left(t\right)b_i\left(t-1\right)-\gamma\left(\frac{h_i\left(t\right)}{a_i}\right),$$

where we have substituted for the income $m_i\left(t\right)$ of the worker in terms of his wage earnings and capital income, and introduced the heterogeneity in human capital decisions. This implies the following choice of human capital investment by a worker i:

$$\lambda a_i\frac{\partial F\left(\hat{k}\left(t\right),h_i\left(t\right)\right)}{\partial h_i(t)}=\gamma'\left(\frac{h_i\left(t\right)}{a_i}\right).$$

This equation yields a unique equilibrium human capital investment $\hat{h}_i\left(\hat{k}\left(t\right)\right)$ for each i. This human capital investment directly depends on the capital choices of all the firms, $\hat{k}\left(t\right)$ (since this affects the marginal product of human capital) and also depends implicitly on a_i. Moreover, given (10.29), $\hat{h}_i\left(\hat{k}\left(t\right)\right)$ is strictly increasing in $\hat{k}\left(t\right)$. Also, since $\gamma\left(\cdot\right)$ is strictly convex, $\hat{h}_i\left(\hat{k}\left(t\right)\right)$ is a strictly concave function of $\hat{k}\left(t\right)$. Substituting this into the first-order condition of firms, we obtain

$$(1-\lambda)\int_0^1 \frac{\partial F\left(\hat{k}\left(t\right),\hat{h}_i\left(\hat{k}\left(t\right)\right)\right)}{\partial k\left(t\right)}di=R\left(t\right).$$

Finally, to satisfy market clearing in the capital market, the rate of return to capital, $R(t)$, has to adjust, such that

$$\hat{k}(t) = \int_0^1 b_i(t-1)\,di,$$

which follows from the facts that all firms choose the same level of capital investment and that the measure of firms is normalized to 1. This equation implies that in the closed economy version of the current model, capital per firm is fixed by bequest decisions from the previous period. The main economic forces we would like to emphasize here are seen more clearly when physical capital is not predetermined. For this reason, let us imagine that the economy in question is small and open, so that $R(t) = R^*$ is pinned down by international financial markets (the closed economy version is further discussed in Exercise 10.18). Under this assumption, the equilibrium level of capital per firm is determined by

$$(1-\lambda)\int_0^1 \frac{\partial F\left(\hat{k}, \hat{h}_i\left(\hat{k}\right)\right)}{\partial k}di = R^*. \tag{10.39}$$

Proposition 10.3. *In the open economy version of the model described here, there exists a unique positive level of capital per worker $\hat{k}$ given by (10.39) such that the equilibrium capital per worker is always equal to $\hat{k}$. Given $\hat{k}$, the human capital investment of worker i is uniquely determined by $\hat{h}_i\left(\hat{k}\right)$ such that*

$$\lambda a_i \frac{\partial F\left(\hat{k}, \hat{h}_i\left(\hat{k}\right)\right)}{\partial h} = \gamma'\left(\frac{\hat{h}_i\left(\hat{k}\right)}{a_i}\right). \tag{10.40}$$

We have that $\hat{h}_i\left(\hat{k}\right)$ is increasing in $\hat{k}$, and a decline in R^* increases $\hat{k}$ and $\hat{h}_i$ for all $i \in [0,1]$.

In addition to this equilibrium, there also exists a no-activity equilibrium in which $\hat{k} = 0$ and $\hat{h}_i = 0$ for all $i \in [0,1]$.

Proof. Since $F(k,h)$ exhibits constant returns to scale and $\hat{h}_i\left(\hat{k}\right)$ is a concave function of $\hat{k}$ for each i, $\int_0^1 \left(\partial F\left(\hat{k}, \hat{h}_i\left(\hat{k}\right)\right)/\partial k\right) di$ is decreasing in $\hat{k}$ for a distribution of $[a_i]_{i\in[0,1]}$. Thus $\hat{k}$ is uniquely determined. Given $\hat{k}$, (10.40) determines $\hat{h}_i\left(\hat{k}\right)$ uniquely. Applying the Implicit Function Theorem to (10.40) implies that $\hat{h}_i\left(\hat{k}\right)$ is increasing in $\hat{k}$. Finally, (10.39) implies that a lower R^* increases $\hat{k}$, and from the previous observation $\hat{h}_i$ for all $i \in [0,1]$ increase as well.

The no-activity equilibrium follows, since when all firms choose $\hat{k} = 0$, output is equal to zero and it is best response for workers to choose $\hat{h}_i = 0$, and when $\hat{h}_i = 0$ for all $i \in [0,1]$, $\hat{k} = 0$ is the best response for all firms. □

We have therefore obtained a simple characterization of the equilibrium in this economy with labor market frictions and physical and human capital investments. It is straightforward to observe that there is underinvestment both in human capital and physical capital (this refers to the positive activity equilibrium; clearly, there is even a more severe underinvestment in the no-activity equilibrium). Consider a social planner wishing to maximize output (or one who could transfer resources across individuals in a lump-sum fashion). Suppose that the social planner is restricted by the same random matching technology, so that she cannot allocate workers to firms as she wishes. A similar analysis to above implies that the social planner would also like each firm to choose an identical level of capital per firm, say $\bar{k}$. However, this level of capital per firm will be different than in the competitive equilibrium and she will also choose a different relationship between human capital and physical capital investments. In particular, given $\bar{k}$, she would make human capital decisions to satisfy

$$a_i \frac{\partial F\left(\bar{k}, \bar{h}_i\left(\bar{k}\right)\right)}{\partial h} = \gamma' \left(\frac{\bar{h}_i\left(\bar{k}\right)}{a_i} \right),$$

which is similar to (10.40), except that λ is absent from the left-hand side. This is because each worker considered only his share of output, λ, when undertaking his human capital investment decisions, while the social planner considers the entire output. Consequently, as long as $\lambda < 1$,

$$\bar{h}_i\left(k\right) > \hat{h}_i\left(k\right) \text{ for all } k > 0.$$

Similarly, the social planner would also choose a higher level of capital investment for each firm, in particular, to satisfy the equation

$$\int_0^1 \frac{\partial F\left(\bar{k}, \bar{h}_i\left(\bar{k}\right)\right)}{\partial k} di = R^*,$$

which differs from (10.39) both because now the term $1-\lambda$ is not present on the left-hand side and also because the planner takes into account the differential human capital investment behavior of workers given by $\bar{h}_i\left(\bar{k}\right)$. This discussion establishes the following result:

Proposition 10.4. *In the equilibrium described in Proposition 10.3, there is underinvestment both in physical and human capital.*

More interesting than the underinvestment result is the imbalance in the physical to human capital ratio of the economy, which did not feature in the previous two environments we discussed. The following proposition summarizes this imbalance result in a sharp way:

Proposition 10.5. *Consider the positive activity equilibrium described in Proposition 10.3. Output is equal to 0 if either $\lambda = 0$ or $\lambda = 1$. Moreover, there exists $\lambda^* \in (0, 1)$ that maximizes output.*

Proof. See Exercise 10.19. □

Intuitively, different levels of λ create different types of "imbalances" between physical and human capital. A high level of λ implies that workers have a strong bargaining position, and this encourages their human capital investments. But symmetrically, it discourages the physical capital investments of firms, since they will only receive a small fraction of the output. Therefore, high level of λ (as long as we have $\lambda < 1$) creates an imbalance with too high a level of human capital relative to physical capital. This imbalance effect becomes more extreme as $\lambda \to 1$. In this limit, workers' investment behavior is converging to the first-order condition of the social planner (i.e., $\hat{h}_i(k) \to \bar{h}_i(k)$ for all $k > 0$). However, simultaneously, the physical capital investment of each firm, $\hat{k}$, is converging to zero, and this implies that $\hat{h}_i(k) \to 0$, and production collapses. The same happens, in reverse, when λ is too low. Now there is too high a level of physical capital relative to human capital. An intermediate value of λ^* achieves a balance, though the equilibrium continues to be inefficient as shown in Proposition 10.5.

Physical-human capital imbalances can also increase the role of human capital in cross-country income differences. In the current model, the proportional impact of a change in human capital on aggregate output (or on labor productivity) is greater than the return to human capital, since the latter is determined not by the marginal product of human capital, but by the bargaining parameter λ. The deviation from competitive factor prices, therefore, decouples the contribution of human capital to productivity from market prices.

At the root of the inefficiencies and of the imbalance effect in this model are *pecuniary externalities*. Pecuniary externalities refer to external effects that work through prices (not through direct technological spillovers). By investing more, workers (and symmetrically firms) increase the return to capital (symmetrically wages), and there is underinvestment because they do not take these external effects into consideration. Pecuniary external effects are also present in competitive markets (since, for example, supply affects price), but these are typically "second order," because prices are such that they are equal to both the marginal benefit of buyers (marginal product of firms in the case of factors of production) and to the marginal cost of suppliers. The presence of labor market frictions causes a departure from this type of marginal pricing and is the reason why pecuniary externalities are not second order.

Perhaps even more interesting is the fact that pecuniary externalities in this model take the form of *human capital externalities*, meaning that greater human capital investments by a group of workers increase other workers' wages. Notice that in competitive markets (without externalities) this does not happen. For example, in the economy analyzed in the last section, if a group of workers increase their human capital investments, this would depress the physical to human capital ratio in the economy, reducing wages per unit of human capital and thus the earnings of the rest of the workers. We will now see that the opposite may happen in the presence of labor market imperfections. To illustrate this point, let us suppose that there are two types of workers, a fraction of workers χ with ability a_1 and $1 - \chi$ with ability $a_2 < a_1$.

Using this specific structure, the first-order condition of firms, (10.39), can be written as

$$(1-\lambda)\left[\chi\frac{\partial F\left(\hat{k},\hat{h}_1\left(\hat{k}\right)\right)}{\partial k}+(1-\chi)\frac{\partial F\left(\hat{k},\hat{h}_2\left(\hat{k}\right)\right)}{\partial k}\right]=R^*, \tag{10.41}$$

while the first-order conditions for human capital investments for the two types of workers take the form

$$\lambda a_j\frac{\partial F\left(\hat{k},\hat{h}_j\left(\hat{k}\right)\right)}{\partial h}=\gamma'\left(\frac{\hat{h}_j\left(\hat{k}\right)}{a_j}\right)\text{ for }j=1,2. \tag{10.42}$$

Clearly, $\hat{h}_1(k)>\hat{h}_2(k)$ since $a_1>a_2$. Now imagine an increase in χ, which corresponds to an increase in the fraction of high-ability workers in the population. Holding $\hat{h}_1\left(\hat{k}\right)$ and $\hat{h}_2\left(\hat{k}\right)$ constant, (10.41) implies that $\hat{k}$ should increase, since the left-hand side has increased (in view of the fact that $\hat{h}_1\left(\hat{k}\right)>\hat{h}_2\left(\hat{k}\right)$ and $\partial^2F(k,h)/\partial k\partial h>0$). Therefore, capital-skill complementarity combined with the pecuniary externalities implies that an improvement in the pool of workers that firms face leads to greater investments by firms. Intuitively, each firm expects the average worker that it will be matched with to have higher human capital and since physical and human capital are complements, this makes it more profitable for each firm to increase their physical capital investment. Greater investments by firms, in turn, raise $F\left(\hat{k},h\right)$ for each h, in particular for $\hat{h}_2\left(\hat{k}\right)$. Since the earnings of type 2 workers is equal to $\lambda F\left(\hat{k},\hat{h}_2\left(\hat{k}\right)\right)$, their earnings will also increase as a result of the response of firms to the change in the composition of the workforce. This is therefore an example of human capital externalities, since greater human capital investments by one group of workers have increased the earnings of the remaining workers. In fact, human capital externalities, in this economy, are even stronger, because the increase in $\hat{k}$ also raises $\partial F\left(\hat{k},\hat{h}_2\left(\hat{k}\right)\right)/\partial h$ and thus encourages further investments by type 2 workers. These feedback effects nonetheless do not lead to divergence or multiple equilibria, since we know from Proposition 10.3 that there exists a unique equilibrium with positive activity. We summarize this discussion with the following result:

Proposition 10.6. *The positive activity equilibrium described in Proposition 10.3 exhibits human capital externalities in the sense that an increase in the human capital investments of a group of workers raises the earnings of the remaining workers.*

10.7. Human Capital Externalities

The previous section illustrated how a natural form of human capital externalities can emerge in the presence of capital-skill complementarities combined with labor market imperfections. This is not the only channel through which human capital externalities may arise.

Many economists believe that the human capital stock of the workforce creates a direct non-pecuniary (technological) spillover on the productivity of each worker. In *The Economy of Cities,* Jane Jacobs, for example, argued for the importance of human capital externalities, and suggested that the concentration of economic activity in cities is partly a result of these externalities and also acts as an engine of economic growth because it facilitates the exchange of ideas among workers and entrepreneurs. In the growth literature, a number of well-known papers, including Robert Lucas' (1988) paper and Azariadis and Drazen (1990), suggest that such technological externalities are important and play a major role in the process of economic growth. Human capital externalities are interesting in their own right, since if such external effects are present, the competitive price system may be inefficient (since it will fail to internalize these externalities, particularly if they take place across firm boundaries). Human capital externalities are also important for our understanding of the sources of income differences across countries. Our discussion of the contribution of physical and human capital to cross-country income differences in Chapter 3 showed that differences in human capital are unlikely to account for a large fraction of cross-country income differences, unless external effects are important.

At this point, it is therefore useful to briefly review the empirical evidence on the extent of human capital externalities. Early work in the area, in particular, the paper by James Rauch (1993) tried to measure the extent of human capital externalities by estimating quasi-Mincerian wage regressions, with the major difference that average human capital of workers in the local labor market is also included on the right-hand side. More specifically, Rauch estimated models of the following form:

$$\ln W_{j,m} = \mathbf{X}_{j,m}'\boldsymbol{\beta} + \gamma_p S_{j,m} + \gamma_e S_m,$$

where $\mathbf{X}_{j,m}$ is a vector of controls, $S_{j,m}$ is the years of schooling of individual j living/working in labor market m, and S_m is the average years of schooling of workers in labor market m. Without this last term, this equation would be similar to the standard Mincerian wage regressions discussed above, and we would expect an estimate of the *private return* to schooling γ_p between 6 and 10%. When the average years of schooling, S_m, is also included in the regression, its coefficient γ_e measures the *external return* to schooling in the same units. For example, if γ_e is estimated to be of the same magnitude as γ_p, we would conclude that external returns to schooling are as important as private returns (which would correspond to very large externalities).

Rauch estimated significant external returns, with the magnitude of the external returns often exceeding the private returns. External returns of this magnitude would imply that human capital differences could play a much more important role as a proximate source of cross-country differences in income per capita than implied by the computations in Chapter

3. However, Rauch's regressions exploited differences in average schooling levels across cities, which could reflect many factors that also directly affect wages. For example, wages are much higher in New York City than Ames, Iowa, but this is not only the result of the higher average education of New Yorkers. A more convincing estimate of external returns necessitates a source of exogenous variation in average schooling.

Acemoglu and Angrist (2000) exploited differences in average schooling levels across states and cohorts resulting from changes in compulsory schooling and child labor laws. These laws appear to have had a large effect on schooling, especially at the high school margin. Exploiting changes in average schooling in state labor markets driven by these law changes, Acemoglu and Angrist estimate external returns to schooling that are typically around 1 or 2 percent and statistically insignificant (as compared to private returns of about 10%). These results suggest that there are relatively small human capital externalities in local labor markets. This result is confirmed by a study by Duflo (2004) using Indonesian data and by Ciccone and Perri (2006). Moretti (2002) also estimates human capital externalities, and he finds larger effects. This may be because he focuses on college graduation, but also partly reflects the fact that the source of variation that he exploits, changes in age composition and the presence of land-grant colleges, may have other effects on average earnings in area. Overall, the evidence appears to suggest that local human capital externalities are not very large, and calibration exercises as those in Chapter 3 that ignore these externalities are unlikely to lead to significant downward bias in the contribution of human capital to cross-country income differences.

The qualification "local" in the above discussion has to be emphasized, however. The estimates discussed above focus on local externalities originally emphasized by Jacobs. Nevertheless, if a few very talented scientists and engineers, or other very skilled workers, generate ideas that are then used in other parts of the country or even in the world economy, there may exist significant global human capital externalities. Such global external effects would not be captured by the currently available empirical strategies. Whether such global human capital externalities are important is an interesting area for future research.

10.8. Nelson-Phelps Model of Human Capital

The discussion in this chapter so far has focused on the productivity-enhancing role of human capital. This is arguably the most important role of human capital, emphasized by Becker and Mincer's seminal analyses. However, an alternative perspective on human capital is provided by Richard Nelson and Edmund Phelps in their short and influential paper, Nelson and Phelps (1966), and also by Ted Schultz (1965). According to this perspective, the major role of human capital is not to increase productivity in existing tasks, but to enable workers to cope with change, disruptions and especially new technologies. The Nelson-Phelps view of human capital has played an important role in a variety of different literatures and features in a number of growth models. Here we will provide a simple presentation of the main ideas along the lines of Nelson and Phelps' original model and a discussion of how this new dimension of human capital will change our views of its role in economic growth and development. This model will also act as a steppingstone towards our study of technology adoption later in the book.

Consider the following continuous time model to illustrate the basic ideas. Suppose that output in the economy in question is given by

$$Y(t) = A(t)L, \tag{10.43}$$

where L is the constant labor force, supplying its labor inelastically, and $A(t)$ is the technology level of the economy. There is no capital (and thus no capital accumulation decision) and also no labor supply margin. The only variable that changes over time is technology $A(t)$.

Suppose that the world technological frontier is given by $A_F(t)$. This could correspond to the technology in some other country or perhaps to the technological know-how of scientists that has not yet been applied to production processes. We assume that $A_F(t)$ evolves exogenously according to the differential equation

$$\frac{\dot{A}_F(t)}{A_F(t)} = g_F,$$

with initial condition $A_F(0) > 0$.

Let the human capital of the workforce be denoted by h. Notice that this human capital does not feature in the production function, (10.43). This is an extreme case in which human capital does not play any of the productivity enhancing role we have emphasized so far. Instead, the role of human capital in the current model will be to facilitate the implementation and use of frontier technology in the production process. In particular, the evolution of the technology in use, $A(t)$, is governed by the differential equation

$$\dot{A}(t) = gA(t) + \phi(h)A_F(t),$$

with initial condition $A(0) \in (0, A_F(0))$. The parameter g is strictly less than g_F and measures the growth rate of technology $A(t)$, resulting from learning by doing or other sources of productivity growth. But this is only one source of improvements in technology. The other one comes from the second term, and can be interpreted as improvements in technology because of implementation and adoption of frontier technologies. The extent of this second source of improvement is determined by the average human capital of the workforce, h. This captures the above-mentioned role of human capital, in facilitating coping with technological change. In particular, we assume that $\phi(\cdot)$ is increasing, with

$$\phi(0) = 0 \text{ and } \phi(h) = g_F - g > 0 \text{ for all } h \geq \bar{h},$$

where $\bar{h} > 0$. This specification implies that the human capital of the workforce regulates the ability of the economy to cope with new developments embedded in the frontier technologies; if the workforce has no human capital, there will be no adoption or implementation of frontier technologies and $A(t)$ will grow at the rate g. If, in contrast, $h \geq \bar{h}$, there will be very quick adaptation to the frontier technologies.

Since $A_F(t) = \exp(g_F t) A_F(0)$, the differential equation for $A(t)$ can be written as

$$\dot{A}(t) = gA(t) + \phi(h) A_F(0) \exp(g_F t).$$

Solving this differential equation, we obtain

$$A(t) = \left[\left(\frac{A(0)}{g} - \frac{\phi(h) A_F(0)}{g_F - g} \right) \exp(gt) + \frac{\phi(h) A_F(0)}{g_F - g} \exp(g_F t) \right],$$

which shows that the growth rate of $A(t)$ is faster when $\phi(h)$ is higher. Moreover, it can be verified that

$$A(t) \to \frac{\phi(h)}{g_F - g} A_F(t),$$

so that the ratio of the technology in use to the frontier technology is also determined by human capital.

The role of human capital emphasized by Nelson and Phelps is undoubtedly important in a number of situations. For example, a range of empirical evidence shows that more educated farmers are more likely to adopt new technologies and seeds (e.g., Foster and Rosenzweig, 1995). The Nelson and Phelps' conception of human capital has also been emphasized in the growth literature in connection with the empirical evidence already discussed in Chapter 1, which shows that there is a stronger correlation between economic growth and levels of human capital than between economic growth and changes in human capital. A number of authors, for example, Benhabib and Spiegel (1994), suggest that this may be precisely because the most important role of human capital is not to increase the productive capacity with existing tasks, but to facilitate technology adoption. One might then conjecture that if the role of human capital emphasized by Nelson and Phelps is important in practice, human capital could be playing a more major role in economic growth and development than the discussion so far has suggested. While this is an interesting hypothesis, it is not entirely convincing. If the role of human capital in facilitating technology adoption is taking place within the firm's boundaries, then this will be reflected in the marginal product of more skilled workers. Workers that contribute to faster and more effective technology adoption would be compensated in line with the increase in the net present value of the firm. Then the returns to schooling and human capital used in the calculations in Chapter 3 should have already taken into account the contribution of human capital to aggregate output (thus to economic growth). If, on the other hand, human capital facilitates technology adoption not at the level of the firm, but at the level of the labor market, this would be a form of local human capital externalities and it should have shown up in the estimates on local external effects of human capital. It therefore would appear that, unless this particular role of human capital is also external and these external effects work at a global level, the calibration-type exercises in Chapter 3 should not be seriously underestimating the contribution of human capital to cross-country differences in income per capita.

10.9. Taking Stock

Human capital differences are a major proximate cause of cross-country differences in economic performance. In addition, human capital accumulation may play an important role in the process of economic growth and economic development. These considerations justify a detailed analysis of human capital. This chapter has presented a number of models of human capital investments that have emphasized how human capital investments respond to future rewards and how they evolve over time (with schooling as well as on-the-job training).

Four sets of related but distinct issues arise in connection with the role of human capital in economic growth. First, if some part of the earnings of labor we observe are rewards to accumulated human capital, then the effect of policies (and perhaps technology) on income per capita could be larger, because these would affect not only physical capital accumulation but also human accumulation. The neoclassical economy with physical and human capital studied in Section 10.4 models and quantifies this effect. It also provides a tractable framework in which physical and human capital investments can be studied simultaneously. Nevertheless, any effect of human capital differences resulting from differences in distortions or policies across countries should have shown up in the measurements in Chapter 3. The findings there suggest that human capital differences, though important, can only explain a small fraction of cross-country income differences (unless there is a significant mismeasurement of the impact of human capital on productivity).

The second important issue related to the role of human capital relates to the measurement of the contribution of education and skills to productivity. A possible source of mismeasurement of these effects is the presence of human capital externalities. There are many compelling reasons why there might exist significant pecuniary or technological human capital externalities. Section 10.6 illustrated how capital-skill complementarities in imperfect labor markets can lead to pecuniary externalities. Nevertheless, existing evidence suggests that the extent of human capital externalities is rather limited—with the important caveat that there might be global externalities that remain unmeasured. A particular channel through which global externalities may arise is R&D and technological progress, which are the topics of the next part of the book. An alternative source of mismeasurement of the contribution of human capital is differences in human capital quality. There are significant differences in school and teacher quality even within a narrow geographical area, so we may expect much larger differences across countries. In addition, most available empirical approaches measure human capital differences across countries by using differences in formal schooling. However, the Ben-Porath model, analyzed in Section 10.3, suggests that human capital continues to be accumulated even after individuals complete their formal schooling. When human capital is highly rewarded, we expect both higher levels of formal schooling and greater levels of on-the-job investments. Consequently, the Ben-Porath model suggests that there might be higher quality of human capital (or greater amount of unmeasured human capital) in economies where the levels of formal schooling are higher. If this is the case, the empirical measurements reported in Chapter 3 may understate the contribution of human capital to productivity. Whether or not this is so is an interesting area for future research.

The third set of novel issues raised by the modeling of human capital is the possibility of an imbalance between physical and human capital. Empirical evidence suggests that physical and human capital are complementary. This implies that productivity will be high when the correct balance is achieved between physical and human capital. Could equilibrium incentives lead to an imbalance, whereby too much or too little physical capital is accumulated relative to human capital? We saw that such imbalances are unlikely or rather short lived in models with competitive labor markets. However, our analysis in Section 10.6 shows that they become a distinct possibility when factor prices do not necessarily reflect marginal products, as in labor markets with frictions. The presence of such imbalances might increase the impact of human capital on aggregate productivity.

The final issue relates to the role of human capital. In Section 10.8, we discussed the Nelson-Phelps view of human capital, which emphasizes the role of skills in facilitating the adoption and implementation of new technologies. While this perspective is likely to be important in a range of situations, it seems that, in the absence of significant external effects, this particular role of human capital should not lead to a major mismeasurement of the contribution of human capital to aggregate productivity either, especially, in the types of exercises reported in Chapter 3.

This chapter has also contributed to our quest towards understanding the sources of economic growth and cross-country income differences. We now have arrived to a relatively simple and useful framework for understanding both physical and human capital accumulation decisions. Our next task is to develop models for the other major proximate source of economic growth and income differences; technology. Before doing this, however, we will have our first look at models of sustained long-run growth.

10.10. References and Literature

The concept of human capital is due to Ted Shultz (1965), Gary Becker (1965), and Jacob Mincer (1974). The standard models of human capital, used extensively in labor economics and in other areas economics, have been developed by Becker (1965), Mincer (1974) and Yoram Ben-Porath (1967). These models have been the basis of the first three sections of this chapter. Recently there has been a renewed interest in the Ben-Porath model among macroeconomists. Two recent contributions include Manuelli and Seshadri (2005) and Guvenen and Kuruscu (2006). These models make parametric assumptions (Cobb-Douglas functional forms) and try to gauge the quantitative implications of the Ben-Porath model for cross-country income differences and for the evolution on wage inequality, respectively. Manuelli and Seshadri (2005) also emphasize how differences in on-the-job training investments will create systematic differences in unmeasured human capital across countries and argue that once these "quality" differences are taken into account, human capital differences could explain a very large fraction of cross-country income differences. Caselli (2006), on the other hand, argues that quality differences are unlikely to increase the contribution of human capital to aggregate productivity.

There is a large literature on returns to schooling. As noted in the text and also in Chapter 3, this literature typically finds that one more year of schooling increases earnings by about 6 to 10% (see, for example, the survey in Card, 1999).

There is also a large literature on capital-skill complementarity. The idea was first put forward and empirically supported in Griliches (1969). Katz and Autor (1999) summarize more recent evidence on as capital-skill complementarities.

Technological human capital externalities are emphasized in Jacobs (1965), Lucas (1988), Azariadis and Drazen (1990), while pecuniary human capital externalities were first discussed by Marshall (1961), who argued that increasing the geographic concentration of specialized inputs increases productivity since the matching between factor inputs and industries is improved. Models of pecuniary human capital externalities are constructed in Acemoglu (1996, 1997a). The model with capital-skill complementarity and labor market imperfections is based on Acemoglu (1996), who provides a more detailed and microfounded model leading to similar results to those presented in Section 10.6 and derives the results on pecuniary externalities and human capital externalities.

The empirical literature on human capital externalities includes Rauch (1993), Acemoglu and Angrist (2000), Duflo (2004), Moretti (2002) and Ciccone and Perri (2006).

The role of human capital in adapting to change and implementing new technologies was first suggested by Schultz (1965) in the context of agricultural technologies (he emphasized the role of ability rather than human capital and stressed the importance of "disequilibrium" situations). Nelson and Phelps (1966) formulated the same ideas and presented a simple model, essentially identical to that presented in Section 10.8 above. Foster and Rosenzweig (1995) provide evidence consistent with this role of human capital. Benhabib and Spiegel (1994) and Aghion and Howitt (1999) also include extensive discussions of the Nelson-Phelps view of human capital. Recent macroeconomic models that feature this role of human capital include Galor and Tsiddon (1997), Greenwood and Yorukoglu (1997), Caselli (1999), Galor and Moav (2001), and Aghion, Howitt and Violante (2004).

10.11. Exercises

Exercise 10.1. Formulate, state and prove the Separation Theorem, Theorem 10.1, in an economy in discrete time.

Exercise 10.2.
(1) Consider the environment discussed in Section 10.1. Write the flow budget constraint of the individual as

$$\dot{a}(t) = ra(t) - c(t) + W(t),$$

and suppose that there are credit market imperfections so that $a(t) \geq 0$. Construct an example in which Theorem 10.1 does not apply. Can you generalize this to the case in which the individual can save at the rate r, but can only borrow at the rate $r' > r$?

(2) Now modify the environment so that the instantaneous utility function of the individual is

$$u(c(t), 1 - l(t)),$$

where $l(t)$ denotes total hours of work, labor supply at the market is equal to $l(t) - s(t)$, so that the individual has a non-trivial leisure choice. Construct an example in which Theorem 10.1 does not apply.

Exercise 10.3. Derive equation (10.9) from (10.8).

Exercise 10.4. Consider the model presented in Section 10.2 and suppose that the discount rate r varies across individuals (for example, because of credit market imperfections). Show that individuals facing a higher r would choose lower levels of schooling. What would happen if you estimate the wage regression similar to (10.12) in a world in which the source of difference in schooling is differences in discount rates across individuals?

Exercise 10.5. Consider the following variant of the Ben-Porath model, where the human capital accumulation equation is given by

$$\dot{h}(t) = s(t)\phi(h(t)) - \delta_h h(t),$$

where ϕ is strictly increasing, continuously differentiable and strictly concave, with $s(t) \in [0, 1]$. Assume that individuals are potentially infinitely lived and face a Poisson death rate of $\nu > 0$. Show that the optimal path of human capital investments involves $s(t) = 1$ for some interval $[0, T]$ and then $s(t) = s^*$ for $t \geq T$.

Exercise 10.6. Modify the Ben-Porath model studied in Section 10.3 as follows. First, assume that the horizon is finite. Second, suppose that $\phi'(0) < \infty$. Finally, suppose that $\lim_{x \to h(0)} \phi'(x) > 0$. Show that under these conditions the optimal path of human capital accumulation will involve an interval of full-time schooling with $s(t) = 1$, followed by another interval of on-the-job investment $s(t) \in (0, 1)$, and finally an interval of no human capital investment, $s(t) = 0$. How do the earnings of the individual evolve over the life cycle?

Exercise 10.7. Prove that as long as $Y(t) = F(K(t), H(t))$ satisfies Assumptions 1 and 2, the inequality in (10.29) holds.

Exercise 10.8. Show that equilibrium dynamics in Section 10.5 remain unchanged if $\delta < 1$.

Exercise 10.9. Prove that the current-value Hamiltonian in (10.23) is jointly concave in $(k(t), h(t), i_k(t), i_h(t))$.

Exercise 10.10. Prove that (10.24) implies the existence of a relationship between physical and human capital of the form $h = \xi(k)$, where $\xi(\cdot)$ is uniquely defined, strictly increasing and continuously differentiable.

Exercise 10.11. Prove 10.1. Show that the differential equation for consumption growth could have alternatively been written as

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))} \left[f_h(k(t), \xi(k(t))) - \delta_h - \rho \right].$$

Exercise 10.12. Consider the neoclassical growth model with physical and human capital discussed in Section 10.4.

(1) Specify the consumer maximization problem in this economy.
(2) Define a competitive equilibrium (specifying firm optimization and market clearing conditions).
(3) Characterize the competitive equilibrium and show that it coincides with the solution to the optimal growth problem.

Exercise 10.13. Introduce labor-augmenting technological progress at the rate g into the neoclassical growth model with physical and human capital discussed in Section 10.4.

(1) Define a competitive equilibrium.
(2) Determine transformed variables that will remain constant in a steady state allocation.
(3) Characterize the steady state equilibrium and the transitional dynamics.
(4) Why does faster technological progress lead to more rapid accumulation of human capital?

Exercise 10.14. * Characterize the optimal growth path of the economy in Section 10.4 subject to the additional constraints that $i_k(t) \geq 0$ and $i_h(t) \geq 0$.

Exercise 10.15. Derive equation (10.25).

Exercise 10.16. Derive equations (10.32) and (10.33).

Exercise 10.17. Provide conditions on $f(\cdot)$ and $\gamma(\cdot)$ such that the unique steady-state equilibrium in the model of Section 10.5 is locally stable.

Exercise 10.18. Analyze the economy in Section 10.6 under the closed economy assumption. Show that an increase in a_1 for group 1 will now create a dynamic externality, in the sense that current output will increase and this will lead to greater physical and human capital investments next periods.

Exercise 10.19. Prove Proposition 10.5.

CHAPTER 11

First-Generation Models of Endogenous Growth

The models presented so far focused on physical and human capital accumulation. Economic growth is generated by exogenous technological progress. While such models are useful in thinking about sources of income differences among countries that have (free) access to the same set of technologies, they do not generate sustained long-run growth (of the country or of the world economy) and have relatively little to say about sources of technology differences. A full analysis of both cross-country income differences and the process of world economic growth requires models in which technology choices and technological progress are endogenized. This will be the topic of the next part of the book. While models in which technology evolves as a result of firms' and workers' decisions are most attractive in this regard, sustained economic growth is possible in the neoclassical model as well. We end this part of the book by investigating sustained endogenous economic growth in neoclassical or quasi-neoclassical models.

We have already encountered the AK model in Chapter 2. This model relaxed one of the key assumptions on the aggregate production function of the economy (Assumption 2) and prevented diminishing returns to capital. Consequently, continuous capital accumulation could act as the engine of sustained economic growth. In this chapter we start with a neoclassical version of the AK model, which not only shows the possibility of endogenous growth in the neoclassical growth model, but also provides us with a very tractable model that find applications in many areas. This model is not without shortcomings, however. The most major one is that capital is the only (or essentially the only) factor of production, and asymptotically, the share of national income accruing to capital tends to 1. This, however, is not an essential feature of neoclassical endogenous growth models. We present two different two-sector endogenous growth models, which behave very similarly to the baseline AK model, but avoid this counterfactual prediction. The first of these incorporates physical and human capital accumulation, and is thus a close cousin of the neoclassical growth model with physical and human capital studied in Section 10.4 in Chapter 10. The second, which builds on the work by Rebelo (1991), is a substantially richer model and is also interesting since it allows investment and consumption goods sectors to have different capital intensities.

We conclude this section with a presentation of Paul Romer's (1986) path breaking article. In many ways, Romer's paper started the endogenous growth literature and rejuvenated the interest in economic growth among economists. While Romer's objective was to model "technological change," he achieved this by introducing technological spillovers—similar to those we encountered in Chapter 10. Consequently, while the competitive equilibrium of Romer's model is not Pareto optimal and the engine of economic growth can be interpreted as a form "knowledge accumulation," in many ways the model is still neoclassical in nature. In particular, we will see that in reduced-form it is very similar to the baseline AK model (except its welfare implications).

11.1. The AK Model Revisited

Let us start with the simplest neoclassical model of sustained growth, which we already encountered in the context of the Solow growth model, in particular, Proposition 2.10 in subsection 2.5.1. This is the so-called AK model, where the production technology is linear in capital. We will also see that in fact what matters is that the accumulation technology is linear, not necessarily the production technology. But for now it makes sense to start with the simpler case of the AK economy.

11.1.1. Demographics, Preferences and Technology. Our focus in this chapter and the next part of the book is on economic growth, and as a first pass, we will focus on balanced economic growth, defined as a growth path consistent with the Kaldor facts (recall Chapter 2). As demonstrated in Chapter 8, balanced growth forces us to adopt the standard CRRA preferences as in the canonical neoclassical growth model (to ensure a constant intertemporal elasticity of substitution).

Throughout this chapter, we assume that the economy admits an infinitely-lived representative household, with household size growing at the exponential rate n. The preferences of the representative household at time $t = 0$ are given by

$$U = \int_0^\infty \exp\left(-\left(\rho - n\right)t\right)\left[\frac{c\left(t\right)^{1-\theta} - 1}{1-\theta}\right]dt. \tag{11.1}$$

Labor is supplied inelastically. The flow budget constraint facing the household can be written as

$$\dot{a}\left(t\right) = \left(r\left(t\right) - n\right)a\left(t\right) + w\left(t\right) - c\left(t\right), \tag{11.2}$$

where $a\left(t\right)$ denotes assets per capita at time t, $r\left(t\right)$ is the interest rate, $w\left(t\right)$ is the wage rate per capita, and n is the growth rate of population. As usual, we also need to impose the no-Ponzi game constraint:

$$\lim_{t\to\infty}\left\{a(t)\exp\left[-\int_0^t\left[r(s) - n\right]ds\right]\right\} \geq 0. \tag{11.3}$$

The Euler equation for the representative household is the same as before and implies the following rate of consumption growth per capita:

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta}(r(t) - \rho). \tag{11.4}$$

The other necessary condition for optimality of the consumer's plans is the transversality condition,

$$\lim_{t\to\infty} \left\{ a(t) \exp\left[-\int_0^t [r(s) - n]\, ds \right] \right\} = 0. \tag{11.5}$$

As before, the problem of the consumer is concave, thus any solution to these necessary conditions is in fact an optimal plan.

The final good sector is similar to before, except that Assumptions 1 and 2 are *not* satisfied. More specifically, we adopt the following aggregate production function:

$$Y(t) = AK(t),$$

with $A > 0$. Notice that this production function does not depend on labor, thus wage earnings, $w(t)$, in (11.2) will be equal to zero. This is one of the unattractive features of the baseline AK model, but will be relaxed below (and it is also relaxed in Exercises 11.3 and 11.4). Dividing both sides of this equation by $L(t)$, and as usual, defining $k(t) \equiv K(t)/L(t)$ as the capital-labor ratio, we obtain per capita output as

$$\begin{aligned} y(t) &\equiv \frac{Y(t)}{L(t)} \\ &= Ak(t). \end{aligned} \tag{11.6}$$

Equation (11.6) has a number of notable differences from our standard production function satisfying Assumptions 1 and 2. First, output is only a function of capital, and there are no diminishing returns (i.e., it is no longer the case that $f''(\cdot) < 0$). We will see that this feature is only for simplicity and introducing diminishing returns to capital does not affect the main results in this section (see Exercise 11.4). The more important assumption is that the Inada conditions embedded in Assumption 2 are no longer satisfied. In particular,

$$\lim_{k\to\infty} f'(k) = A > 0.$$

This feature is essential for sustained growth.

The conditions for profit-maximization are similar to before, and require that the marginal product of capital be equal to the rental price of capital, $R(t) = r(t) + \delta$. Since, as is obvious from equation (11.6), the marginal product of capital is constant and equal to A, thus $R(t) = A$ for all t, which implies that the net rate of return on the savings is constant and equal to:

$$r(t) = r = A - \delta, \text{ for all } t. \tag{11.7}$$

Since the marginal product of labor is zero, the wage rate, $w(t)$, is zero as noted above.

11.1.2. Equilibrium. A competitive equilibrium of this economy consists of paths of per capita consumption, capital-labor ratio, wage rates and rental rates of capital, $[c(t), k(t), w(t), R(t)]_{t=0}^{\infty}$, such that the representative household maximizes (11.1) subject to (11.2) and (11.3) given initial capital-labor ratio $k(0)$ and factor prices $[w(t), r(t)]_{t=0}^{\infty}$ such that $w(t) = 0$ for all t, and $r(t)$ is given by (11.7).

To characterize the equilibrium, we again note that $a(t) = k(t)$. Next using the fact that $r = A - \delta$ and $w = 0$, equations (11.2), (11.4), and (11.5) imply

$$\dot{k}(t) = (A - \delta - n)k(t) - c(t) \tag{11.8}$$

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta}(A - \delta - \rho), \tag{11.9}$$

$$\lim_{t\to\infty} k(t)\exp(-(A - \delta - n)t) = 0. \tag{11.10}$$

The important result immediately follows from equation (11.9). Since the right-hand side of this equation is constant, there must be a constant rate of consumption growth (as long as $A - \delta - \rho > 0$). The rate of growth of consumption is therefore independent of the level of capital stock per person, $k(t)$. This will also imply that there are no transitional dynamics in this model. Starting from any $k(0)$, consumption per capita (and as we will see, the capital-labor ratio) will immediately start growing at a constant rate. To develop this point, let us integrate equation (11.9) starting from some initial level of consumption $c(0)$, which as usual is still to be determined later (from the lifetime budget constraint). This gives

$$c(t) = c(0)\exp\left(\frac{1}{\theta}(A - \delta - \rho)t\right). \tag{11.11}$$

Since there is growth in this economy, we have to ensure that the transversality condition is satisfied (i.e., that lifetime utility is bounded away from infinity), and also we want to ensure positive growth (the condition $A - \delta - \rho > 0$ mentioned above). We therefore impose:

$$A > \rho + \delta > (1 - \theta)(A - \delta) + \theta n + \delta. \tag{11.12}$$

The first part of this condition ensures that there will be positive consumption growth, while the second part is the analog to the condition that $\rho + \theta g > g + n$ in the neoclassical growth model with technological progress, which was imposed to ensure bounded utility (and thus was used in proving that the transversality condition was satisfied).

11.1.3. Equilibrium Characterization. We first establish that there are no transitional dynamics in this economy. In particular, we will show that not only the growth rate of consumption, but the growth rates of capital and output are also constant at all points in time, and equal the growth rate of consumption given in equation (11.9).

To do this, let us substitute for $c(t)$ from equation (11.11) into equation (11.8), which yields

$$\dot{k}(t) = (A - \delta - n)k(t) - c(0)\exp\left(\frac{1}{\theta}(A - \delta - \rho)t\right), \tag{11.13}$$

which is a first-order, non-autonomous linear differential equation in $k(t)$. This type of equation can be solved easily. In particular recall that if

$$\dot{z}(t) = az(t) + b(t),$$

then, the solution is

$$z(t) = z_0 \exp(at) + \exp(at)\int_0^t \exp(-as)\, b(s)ds,$$

for some constant z_0 chosen to satisfy the boundary conditions. Therefore, equation (11.13) solves for:

(11.14)

$$k(t) = \left\{\kappa \exp((A - \delta - n)\,t) + \left[(A - \delta)(\theta - 1)\theta^{-1} + \rho\theta^{-1} - n\right]^{-1}\left[c(0)\exp\left(\theta^{-1}(A - \delta - \rho)t\right)\right]\right\},$$

where κ is a constant to be determined. Assumption (11.12) ensures that

$$(A - \delta)(\theta - 1)\theta^{-1} + \rho\theta^{-1} - n > 0.$$

From (11.14), it may look like capital is not growing at a constant rate, since it is the sum of two components growing at different rates. However, this is where the transversality condition becomes useful. Let us substitute from (11.14) into the transversality condition, (11.10), which yields

$$\lim_{t\to\infty}[\kappa + \left[(A - \delta)(\theta - 1)\theta^{-1} + \rho\theta^{-1} - n\right]^{-1} c(0)\exp\left(-\left(A - \delta)(\theta - 1)\theta^{-1} + \rho\theta^{-1} - n\right)t\right)] = 0.$$

Since $(A - \delta)(\theta - 1)\theta^{-1} + \rho\theta^{-1} - n > 0$, the second term in this expression converges to zero as $t \to \infty$. But the first term is a constant. Thus the transversality condition can only be satisfied if $\kappa = 0$. Therefore we have from (11.14) that:

$$\begin{aligned} k(t) &= \left[(A - \delta)(\theta - 1)\theta^{-1} + \rho\theta^{-1} - n\right]^{-1}\left[c(0)\exp\left(\theta^{-1}(A - \delta - \rho)t\right)\right] \\ &= k(0)\exp\left(\theta^{-1}(A - \delta - \rho)t\right), \end{aligned} \tag{11.15}$$

where the second line immediately follows from the fact that the boundary condition has to hold for capital at $t = 0$. This equation naturally implies that capital and output grow at the same rate as consumption.

It also pins down the initial level of consumption as

$$c(0) = \left[(A - \delta)(\theta - 1)\theta^{-1} + \rho\theta^{-1} - n\right]k(0). \tag{11.16}$$

Note also that in this simple AK model, growth is not only sustained, but it is also endogenous in the sense of being affected by underlying parameters. For example, consider an increase in the rate of discount, ρ. Recall that in the Ramsey model, this only influenced the level of income per capita—it could have no effect on the growth rate, which was determined by the exogenous labor-augmenting rate of technological progress. Here, is straightforward to verify that an increase in the discount rate, ρ, will reduce the growth rate, because it will make consumers less patient and will therefore reduce the rate of capital accumulation. Since capital accumulation is the engine of growth, the equilibrium rate of growth will decline. Similarly, changes in A and θ affect the levels and growth rates of consumption, capital and output.

Finally, we can calculate the saving rate in this economy. It is defined as total investment (which is equal to increase in capital plus replacement investment) divided by output. Consequently, the saving rate is constant and given by

$$\begin{aligned} s &= \frac{\dot{K}(t) + \delta K(t)}{Y(t)} \\ &= \frac{\dot{k}(t)/k(t) + n + \delta}{A} \\ &= \frac{A - \rho + \theta n + (\theta - 1)\delta}{\theta A}, \end{aligned} \tag{11.17}$$

where the last equality exploited the fact that $\dot{k}(t)/k(t) = (A - \delta - \rho)/\theta$. This equation implies that the saving rate, which was taken as constant and exogenous in the basic Solow model, is again constant, but is now a function of parameters, and more specifically of exactly the same parameters that determine the equilibrium growth rate of the economy.

Summarizing, we have:

Proposition 11.1. *Consider the above-described AK economy, with a representative household with preferences given by (11.1), and the production technology given by (11.6). Suppose that condition (11.12) holds. Then, there exists a unique equilibrium path in which consumption, capital and output all grow at the same rate* $g^* \equiv (A - \delta - \rho)/\theta > 0$ *starting from any initial positive capital stock per worker* $k(0)$*, and the saving rate is endogenously determined by (11.17).*

One important implication of the AK model is that since all markets are competitive, there is a representative household, and there are no externalities, the competitive equilibrium will be Pareto optimal. This can be proved either using First Welfare Theorem type reasoning, or by directly constructing the optimal growth solution.

Proposition 11.2. *Consider the above-described AK economy, with a representative household with preferences given by (11.1), and the production technology given by (11.6). Suppose that condition (11.12) holds. Then, the unique competitive equilibrium is Pareto optimal.*

Proof. See Exercise 11.2 □

11.1.4. The Role of Policy. It is straightforward to incorporate policy differences in to this framework and investigate their implications on the equilibrium growth rate. The simplest and arguably one of the most relevant classes of policies are, as also discussed above, those affecting the rate of return to accumulation. In particular, suppose that there is an effective tax rate of τ on the rate of return from capital income, so that the flow budget constraint of the representative household becomes:

$$\dot{a}(t) = ((1-\tau)\,r(t) - n)a(t) + w(t) - c(t)\,. \tag{11.18}$$

Repeating the analysis above immediately implies that this will adversely affect the growth rate of the economy, which will now become (see Exercise 11.5):

$$g = \frac{(1-\tau)(A-\delta)-\rho}{\theta}. \tag{11.19}$$

Moreover, it can be calculated that the saving rate will now be

$$s = \frac{(1-\tau)\,A - \rho + \theta n - (1-\tau-\theta)\,\delta}{\theta A}, \tag{11.20}$$

which is a decreasing function of τ if $A - \delta > 0$. Therefore, in this model, the equilibrium saving rate is constant as in the basic Solow model, but in contrast to that model, it responds endogenously to policy. In addition, the fact that the saving rate is constant implies that differences in policies will lead to permanent differences in the rate of capital accumulation. This observation has a very important implication. While in the baseline neoclassical growth model, even reasonably large differences in distortions (for example, eightfold differences in τ) could only have limited effects on differences in income per capita, here even small differences in τ can have very large effects. In particular, consider two economies, with respective (constant) tax rates on capital income τ and $\tau' > \tau$, and exactly the same technology and preferences otherwise. It is straightforward to verify that for any $\tau' > \tau$,

$$\lim_{t\to\infty} \frac{Y(\tau',t)}{Y(\tau,t)} = 0,$$

where $Y(\tau,t)$ denotes aggregate output in the economy with tax τ at time t. Therefore, even small policy differences can have very large effects in the long run. So why does the literature focus on the inability of the standard neoclassical growth model to generate large differences rather than the possibility that the AK model can generate arbitrarily large differences? The reason is twofold: first, for the reasons already discussed, the AK model, with no diminishing returns and the share of capital in national income asymptoting to 1, is not viewed as a good approximation to reality. Second, and related to our discussion in Chapter 1, most economists believe that the relative stability of the world income distribution in the post-war era makes it more attractive to focus on models in which there is a stationary world income distribution, rather than models in which small policy differences can lead to permanent growth differences. Whether this last belief is justified is, in part, an empirical question.

11.2. The AK Model with Physical and Human Capital

As pointed out in the previous section, a major shortcoming of the baseline AK model is that the share of capital accruing to national income is equal to 1 (or limits to 1 as in the variant of the AK model studied in Exercises 11.3 and 11.4). One way of enriching the AK model and avoiding these problems is to include both physical and human capital. We now briefly discuss this extension. Suppose the economy admits a representative household with preferences given by (11.1). The production side of the economies represented by the aggregate production function

$$Y(t) = F(K(t), H(t)), \tag{11.21}$$

where $H(t)$ denotes efficiency units of labor (or human capital), which will be accumulated in the same way as physical capital. We assume that the production function $F(\cdot, \cdot)$ now satisfies our standard assumptions, Assumptions 1 and 2.

Suppose that the budget constraint of the representative household is given by

$$\dot{a}(t) = (r(t) - n)a(t) + w(t)h(t) - c(t) - i_h(t), \tag{11.22}$$

where $h(t)$ denotes the effective units of labor (human capital) on the representative household, $w(t)$ is wage rate per unit of human capital, and $i_h(t)$ is investment in human capital. The human capital of the representative household evolves according to the differential equation:

$$\dot{h}(t) = i_h(t) - \delta_h h(t), \tag{11.23}$$

where δ_h is the depreciation rate of human capital. The evolution of the capital stock is again given from the observation that $k(t) = a(t)$, and we now denote the depreciation rate of physical capital by δ_k to avoid confusion with δ_h. In this model, the representative household maximizes its utility by choosing the paths of consumption, human capital investments and asset holdings. Competitive factor markets imply that

$$R(t) = f'(k(t)) \text{ and } w(t) = f(k(t)) - k(t)f'(k(t)), \tag{11.24}$$

where, now, the effective capital-labor ratio is given by dividing the capital stock by the stock of human capital in the economy,

$$k(t) \equiv \frac{K(t)}{H(t)}.$$

A competitive equilibrium of this economy consists of paths of per capita consumption, capital-labor ratio, wage rates and rental rates of capital, $[c(t), k(t), w(t), R(t)]_{t=0}^{\infty}$, such that the representative household maximizes (11.1) subject to (11.3), (11.22) and (11.23) given initial effective capital-labor ratio $k(0)$ and factor prices $[w(t), R(t)]_{t=0}^{\infty}$ that satisfy (11.24).

To characterize the competitive equilibrium, let us first set up at the current-value Hamiltonian for the representative household with costate variables μ_a and μ_h:

$$\begin{aligned}\mathcal{H}(a,h,c,i_h,\mu_a,\mu_k) &= \frac{c(t)^{1-\theta}-1}{1-\theta}+\mu_a(t)\left[(r(t)-n)a(t)+w(t)h(t)-c(t)-i_h(t)\right] \\ &\quad +\mu_h(t)\left[i_h(t)-\delta_h h(t)\right].\end{aligned}$$

Now the necessary conditions of this optimization problem imply the following (see Exercise 11.8):

$$\begin{aligned}\mu_a(t) &= \mu_h(t)=\mu(t) \text{ for all } t \\ w(t)-\delta_h &= r(t)-n \text{ for all } t \\ \frac{\dot{c}(t)}{c(t)} &= \frac{1}{\theta}(r(t)-\rho) \text{ for all } t.\end{aligned} \tag{11.25}$$

Combining these with (11.24), we obtain that

$$f'(k(t))-\delta_k-n=f(k(t))-k(t)f'(k(t))-\delta_h \text{ for all } t.$$

Since the left-hand side is decreasing in $k(t)$, while the right-hand side is increasing, this implies that the effective capital-labor ratio must satisfy

$$k(t)=k^* \text{ for all } t.$$

We can then prove the following proposition:

Proposition 11.3. *Consider the above-described AK economy with physical and human capital, with a representative household with preferences given by (11.1), and the production technology given by (11.21). Let k^* be given by*

$$f'(k^*)-\delta_k-n=f(k^*)-k^*f'(k^*)-\delta_h. \tag{11.26}$$

Suppose that $f'(k^*)>\rho+\delta_k>(1-\theta)(f'(k^*)-\delta)+n\theta+\delta_k$. Then, in this economy there exists a unique equilibrium path in which consumption, capital and output all grow at the same rate $g^*\equiv(f'(k^*)-\delta_k-\rho)/\theta>0$ starting from any initial conditions, where k^* is given by (11.26).The share of capital in national income is constant at all times.

Proof. See Exercise 11.9 □

The advantage of the economy studied here, especially as compared to the baseline *AK* model is that, it generates a stable factor distribution of income, with a significant fraction of national income accruing to labor as rewards to human capital. Consequently, the current model cannot be criticized on the basis of generating counter-factual results on the capital share of GDP. A similar analysis to that in the previous section also shows that the current model generates long-run growth rate differences from small policy differences. Therefore, it can account for arbitrarily large differences in income per capita across countries. Nevertheless, it would do so partly by generating large human capital differences across countries. As such, the empirical mechanism through which these large cross-country income differences are generated may again not fit with the empirical patterns discussed in Chapter 3. Moreover, given substantial differences in policies across economies in the postwar period, like the baseline AK economy, the current model would suggest significant changes in the world income distribution, whereas the evidence in Chapter 1 points to a relatively stable postwar world income distribution.

11.3. The Two-Sector AK Model

The models studied in the previous two sections are attractive in many respects; they generate sustained growth, and the equilibrium growth rate responds to policy, to underlying preferences and to technology. Moreover, these are very close cousins of the neoclassical model. In fact, as argued there, the endogenous growth equilibrium is Pareto optimal.

One unattractive feature of the baseline AK model is that all of national income accrues to capital. Essentially, it is a one-sector model with only capital as the factor of production. This makes it difficult to apply this model to real world situations. The model in the previous section avoids this problem, but at some level it does so by creating another factor of production that accumulates linearly, so that the equilibrium structure is again equivalent to the one-sector AK economy. Therefore, in some deep sense, the economies of both sections are one-sector models. More important than this one-sector property, these models potentially blur key underlying characteristic driving growth in these environments. What is important is not that the production technology is AK, but the related feature that the *accumulation technology* is linear. In this section, we will discuss a richer two-sector model of neoclassical endogenous growth, based on Rebelo's (1991) work. This model will generate constant factor shares in national income without introducing human capital accumulation. Perhaps more importantly, it will illustrate the role of differences in the capital intensity of the production functions of consumption and investment.

The preference and demographics are the same as in the model of the previous section, in particular, equations (11.1)-(11.5) apply as before (but with a slightly different interpretation for the interest rate in (11.4) as will be discussed below). Moreover, to simplify the analysis, suppose that there is no population growth, i.e., $n = 0$, and that the total amount of labor in the economy, L, is supplied inelastically.

The main difference is in the production technology. Rather than a single good used for consumption and investment, we now envisage an economy with two sectors. Sector 1 produces consumption goods with the following technology

$$C(t) = B(K_C(t))^{\alpha} L_C(t)^{1-\alpha}, \tag{11.27}$$

where the subscript "C" denotes that these are capital and labor used in the consumption sector, which has a Cobb-Douglas technology. In fact, the Cobb-Douglas assumption here is quite important in ensuring that the share of capital in national income is constant (see Exercise 11.12). The capital accumulation equation is given by:

$$\dot{K}(t) = I(t) - \delta K(t),$$

where $I(t)$ denotes investment. Investment goods are produced with a different technology than (11.27), however. In particular, we have

$$I(t) = AK_I(t). \tag{11.28}$$

The distinctive feature of the technology for the investment goods sector, (11.28), is that it is linear in the capital stock and does not feature labor. This is an extreme version of an assumption often made in two-sector models, that the investment-good sector is more capital-intensive than the consumption-good sector. In the data, there seems to be some support for this, though the capital intensities of many sectors have been changing over time as the nature of consumption and investment goods has changed.

Market clearing implies:

$$K_C(t) + K_I(t) \leq K(t),$$

for capital, and

$$L_C(t) \leq L,$$

for labor (since labor is only used in the consumption sector).

An equilibrium in this economy is defined similarly to that in the neoclassical economy, but also features an allocation decision of capital between the two sectors. Moreover, since the two sectors are producing two different goods, consumption and investment goods, there will be a relative price between the two sectors which will adjust endogenously.

Since both market clearing conditions will hold as equalities (the marginal product of both factors is always positive), we can simplify notation by letting $\kappa(t)$ denote the share of capital used in the investment sector

$$K_C(t) = (1 - \kappa(t)) K(t) \text{ and } K_I(t) = \kappa(t) K(t).$$

From profit maximization, the rate of return to capital has to be the same when it is employed in the two sectors. Let the price of the investment good be denoted by $p_I(t)$ and that of the consumption good by $p_C(t)$, then we have

$$p_I(t) A = p_C(t) \alpha B \left(\frac{L}{(1 - \kappa(t)) K(t)} \right)^{1-\alpha}. \tag{11.29}$$

Define a steady-state (a balanced growth path) as an equilibrium path in which $\kappa(t)$ is constant and equal to some $\kappa \in [0, 1]$. Moreover, let us choose the consumption good as the numeraire, so that $p_C(t) = 1$ for all t. Then differentiating (11.29) implies that at the steady state:

$$\frac{\dot{p}_I(t)}{p_I(t)} = -(1-\alpha) g_K, \tag{11.30}$$

where g_K is the steady-state (BGP) growth rate of capital.

As noted above, the Euler equation for consumers, (11.4), still holds, but the relevant interest rate has to be for *consumption-denominated loans*, denoted by $r_C(t)$. In other words, it is the interest rate that measures how many units of consumption good an individual will receive tomorrow by giving up one unit of consumption today. Since the relative price of consumption goods and investment goods is changing over time, the proper calculation goes as follows. By giving up one unit of consumption, the individual will buy $1/p_I(t)$ units of capital goods. This will have an instantaneous return of $r_I(t)$. In addition, the individual will get back the one unit of capital, which has now experienced a change in its price of $\dot{p}_I(t)/p_I(t)$, and finally, he will have to buy consumption goods, whose prices changed by $\dot{p}_C(t)/p_C(t)$. Therefore, the general formula of the rate of return denominated in consumption goods in terms of the rate of return denominated in investment goods is

$$r_C(t) = \frac{r_I(t)}{p_I(t)} + \frac{\dot{p}_I(t)}{p_I(t)} - \frac{\dot{p}_C(t)}{p_C(t)}. \tag{11.31}$$

In our setting, given our choice of numeraire, we have $\dot{p}_C(t)/p_C(t) = 0$. Moreover, $\dot{p}_I(t)/p_I(t)$ is given by (11.30). Finally,

$$\frac{r_I(t)}{p_I(t)} = A - \delta$$

given the linear technology in (11.28). Therefore, we have

$$r_C(t) = A - \delta + \frac{\dot{p}_I(t)}{p_I(t)}.$$

and in steady state, from (11.30), the steady-state consumption-denominated rate of return is:

$$r_C = A - \delta - (1-\alpha) g_K.$$

From (11.4), this implies a consumption growth rate of

$$g_C \equiv \frac{\dot{C}(t)}{C(t)} = \frac{1}{\theta}\left(A - \delta - (1-\alpha) g_K - \rho\right). \tag{11.32}$$

Finally, differentiate (11.27) and use the fact that labor is always constant to obtain

$$\frac{\dot{C}(t)}{C(t)} = \alpha \frac{\dot{K}_C(t)}{K_C(t)},$$

which, from the constancy of $\kappa(t)$ in steady state, implies the following steady-state relationship:

$$g_C = \alpha g_K.$$

Substituting this into (11.32), we have

$$g_K^* = \frac{A - \delta - \rho}{1 - \alpha (1 - \theta)} \tag{11.33}$$

and

$$g_C^* = \alpha \frac{A - \delta - \rho}{1 - \alpha (1 - \theta)}. \tag{11.34}$$

What about wages? Because labor is being used in the consumption good sector, there will be positive wages. Since labor markets are competitive, the wage rate at time t is given by

$$w(t) = (1 - \alpha) p_C(t) B \left(\frac{(1 - \kappa(t)) K(t)}{L} \right)^{\alpha}.$$

Therefore, in the balanced growth path, we obtain

$$\begin{aligned} \frac{\dot{w}(t)}{w(t)} &= \frac{\dot{p}_C(t)}{p_C(t)} + \alpha \frac{\dot{K}(t)}{K(t)} \\ &= \alpha g_K^*, \end{aligned}$$

which implies that wages also grow at the same rate as consumption.

Moreover, with exactly the same arguments as in the previous section, it can be established that there are no transitional dynamics in this economy. This establishes the following result:

Proposition 11.4. *In the above-described two-sector neoclassical economy, starting from any $K(0) > 0$, consumption and labor income grow at the constant rate given by (11.34), while the capital stock grows at the constant rate (11.33).*

It is straightforward to conduct policy analysis in this model, and as in the basic AK model, taxes on investment income will depress growth. Similarly, a lower discount rate will increase the equilibrium growth rate of the economy

One important implication of this model, different from the neoclassical growth model, is that there is continuous *capital deepening.* Capital grows at a faster rate than consumption and output. Whether this is a realistic feature is debatable. The Kaldor facts, discussed above, include constant capital-output ratio as one of the requirements of balanced growth. Here we have steady state and "balanced growth" without this feature. For much of the 20th century, capital-output ratio has been constant, but it has been increasing steadily over the past 30 years. Part of the reason why it has been increasing recently but not before is because of relative price adjustments. New capital goods are of higher quality, and this needs to be incorporated in calculating the capital-output ratio. These calculations have only been performed in the recent past, which may explain why capital-output ratio has been constant in the earlier part of the century, but not recently.

11.4. Growth with Externalities

The model that started much of endogenous growth theory and revived economists' interest in economic growth was Paul Romer's (1986) paper. Romer's objective was to model the process of "knowledge accumulation". He realized that this would be difficult in the context of a competitive economy. His initial solution (later updated and improved in his and others' work during the 1990s) was to consider knowledge accumulation to be a *byproduct* of capital accumulation. In other words, Romer introduced technological spillovers, similar to those discussed in the context of human capital in Chapter 10. While arguably crude, this captures an important dimension of knowledge, that knowledge is a largely *non-rival* good—once a particular technology has been discovered, many firms can make use of this technology without preventing others using the same knowledge. Non-rivalry does not imply knowledge is also non-excludable (which would have made it a pure public good). A firm that discovers a new technology may use patents or trade secrecy to prevent others from using it, for example, in order to gain a competitive advantage. These issues will be discussed in the next part of the book. For now, it suffices to note that some of the important characteristics of "knowledge" and its role in the production process can be captured in a reduced-form way by introducing technological spillovers. We next discuss a version of the model in Romer's (1986) paper, which introduces such technological spillovers as an engine of economic growth. While the type of technological spillovers used in this model are unlikely to be important in practice, this model is a good starting point for our analysis of endogenous technological progress, since its similarity to the baseline AK economy makes it a very tractable model of knowledge accumulation.

11.4.1. Preferences and Technology. Consider an economy without any population growth (we will see why this is important) and a production function with labor-augmenting knowledge (technology) that satisfies the standard assumptions, Assumptions 1 and 2. For reasons that will become clear, instead of working with the aggregate production function, let us assume that the production side of the economy consists of a set $[0, 1]$ of firms. The production function facing each firm $i \in [0, 1]$ is

$$Y_i(t) = F(K_i(t), A(t) L_i(t)), \tag{11.35}$$

where $K_i(t)$ and $L_i(t)$ are capital and labor rented by a firm i. Notice that $A(t)$ is not indexed by i, since it is technology common to all firms. Let us normalize the measure of final good producers to 1, so that we have the following market clearing conditions:

$$\int_0^1 K_i(t)\, di = K(t)$$

and

$$\int_0^1 L_i(t)\,di = L,$$

where L is the constant level of labor (supplied inelastically) in this economy. Firms are competitive in all markets, which implies that they will all hire the same capital to effective labor ratio, and moreover, factor prices will be given by their marginal products, thus

$$\begin{aligned} w(t) &= \frac{\partial F(K(t), A(t)L)}{\partial L} \\ R(t) &= \frac{\partial F(K(t), A(t)L)}{\partial K(t)}. \end{aligned}$$

The key assumption of Romer (1986) is that although firms take $A(t)$ as given, this stock of technology (knowledge) advances endogenously for the economy as a whole. In particular, Romer assumes that this takes place because of spillovers across firms, and attributes spillovers to physical capital. Lucas (1988) develops a similar model in which the structure is identical, but spillovers work through human capital (i.e., while Romer has physical capital externalities, Lucas has human capital externalities).

The idea of externalities is not uncommon to economists, but both Romer and Lucas make an extreme assumption of sufficiently strong externalities such that $A(t)$ can grow continuously at the economy level. In particular, Romer assumes

$$A(t) = BK(t), \tag{11.36}$$

i.e., the knowledge stock of the economy is proportional to the capital stock of the economy. This can be motivated by "learning-by-doing" whereby, greater investments in certain sectors increases the experience (of firms, workers, managers) in the production process, making the production process itself more productive. Alternatively, the knowledge stock of the economy could be a function of the cumulative output that the economy has produced up to now, thus giving it more of a flavor of "learning-by-doing".

In any case, substituting for (11.36) into (11.35) and using the fact that all firms are functioning at the same capital-effective labor ratio, we obtain the production function of the representative firm as

$$Y(t) = F(K(t), BK(t)L).$$

Using the fact that $F(\cdot,\cdot)$ is homogeneous of degree 1, we have

$$\begin{aligned} \frac{Y(t)}{K(t)} &= F(1, BL) \\ &= \tilde{f}(L). \end{aligned}$$

Output per capita can therefore be written as:

$$\begin{aligned} y(t) &\equiv \frac{Y(t)}{L} \\ &= \frac{Y(t)}{K(t)} \frac{K(t)}{L} \\ &= k(t) \tilde{f}(L), \end{aligned}$$

where again $k(t) \equiv K(t)/L$ is the capital-labor ratio in the economy.

As in the standard growth model, marginal products and factor prices can be expressed in terms of the normalized production function, now $\tilde{f}(L)$. In particular, we have

$$w(t) = K(t) \tilde{f}'(L) \quad (11.37)$$

and

$$R(t) = R = \tilde{f}(L) - L\tilde{f}'(L), \quad (11.38)$$

which is constant.

11.4.2. Equilibrium. An equilibrium is defined similarly to the neoclassical growth model, as a path of consumption and capital stock for the economy, $[C(t), K(t)]_{t=0}^{\infty}$ that maximize the utility of the representative household and wage and rental rates $[w(t), R(t)]_{t=0}^{\infty}$ that clear markets. The important feature is that because the knowledge spillovers, as specified in (11.36), are external to the firm, factor prices are given by (11.37) and (11.38)—that is, they do not price the role of the capital stock in increasing future productivity.

Since the market rate of return is $r(t) = R(t) - \delta$, it is also constant. The usual consumer Euler equation (e.g., (11.4) above) then implies that consumption must grow at the constant rate,

$$g_C^* = \frac{1}{\theta}\left(\tilde{f}(L) - L\tilde{f}'(L) - \delta - \rho\right). \quad (11.39)$$

It is also clear that capital grows exactly at the same rate as consumption, so the rate of capital, output and consumption growth are all given by g_C^* as given by (11.39)—see Exercise 11.15.

Let us assume that

$$\tilde{f}(L) - L\tilde{f}'(L) - \delta - \rho > 0, \quad (11.40)$$

so that there is positive growth, but also that growth is not fast enough to violate the transversality condition, in particular,

$$(1-\theta)\left(\tilde{f}(L) - L\tilde{f}'(L) - \delta\right) < \rho. \quad (11.41)$$

Proposition 11.5. *Consider the above-described Romer model with physical capital externalities. Suppose that conditions (11.40) and (11.41) are satisfied. Then, there exists* *a unique equilibrium path where starting with any level of capital stock* $K(0) > 0$*, capital, output and consumption grow at the constant rate (11.39).*

Proof. Much of this proposition is proved in the preceding discussion. You are asked to verify the transversality conditions and show that there are no transitional dynamics in Exercise 11.16. □

Population must be constant in this model because of the *scale effect*. Since $\tilde{f}(L) - L\tilde{f}'(L)$ is always increasing in L (by Assumption 1), a higher population (labor force) L leads to a higher growth rate. The scale effect refers to this relationship between population and the equilibrium rate of economic growth. Now if population is growing, the economy will not admit a steady state and the growth rate of the economy will increase over time (output reaching infinity in finite time and violating the transversality condition). The implications of positive population growth are discussed further in Exercise 11.17. Scale effects and how they can be removed will be discussed in detail in Chapter 13.

11.4.3. Pareto Optimal Allocations. Given the presence of externalities, it is not surprising that the decentralized equilibrium characterized in Proposition 11.5 is not Pareto optimal. To characterize the allocation that maximizes the utility of the representative household, let us again set up on the current-value Hamiltonian. The per capita accumulation equation for this economy can be written as

$$\dot{k}(t) = \tilde{f}(L)\,k(t) - c(t) - \delta k(t).$$

The current-value Hamiltonian is

$$\hat{H}(k, c, \mu) = \frac{c(t)^{1-\theta} - 1}{1-\theta} + \mu\left[\tilde{f}(L)\,k(t) - c(t) - \delta k(t)\right],$$

and has the necessary conditions:

$$\begin{aligned} \hat{H}_c(k, c, \mu) &= c(t)^{-\theta} - \mu(t) = 0 \\ \hat{H}_k(k, c, \mu) &= \mu(t)\left[\tilde{f}(L) - \delta\right] = -\dot{\mu}(t) + \rho\mu(t), \\ \lim_{t\to\infty}\left[\exp(-\rho t)\,\mu(t)\,k(t)\right] &= 0. \end{aligned}$$

These equations imply that the social planner's allocation will also have a constant growth rate for consumption (and output) given by

$$g_C^S = \frac{1}{\theta}\left(\tilde{f}(L) - \delta - \rho\right),$$

which is always greater than g_C^* as given by (11.39)—since $\tilde{f}(L) > \tilde{f}(L) - L\tilde{f}'(L)$. Essentially, the social planner takes into account that by accumulating more capital, she is improving productivity in the future. Since this effect is external to the firms, the decentralized economy fails to internalize this externality. Therefore we have:

Proposition 11.6. *In the above-described Romer model with physical capital externalities, the decentralized equilibrium is Pareto suboptimal and grows at a slower rate than the allocation that would maximize the utility of the representative household.*

Exercise 11.18 asks you to characterize various different types of policies that can close the gap between the equilibrium and Pareto optimal allocations.

11.5. Taking Stock

This chapter ends our investigation of neoclassical growth models. It also opens the way for the analysis of endogenous technological progress in the next part of the book. The models presented in this chapter are, in many ways, more tractable and easier than those we have seen in earlier chapters. This is a feature of the linearity of the models (most clearly visible in the AK model). This type of linearity removes transitional dynamics and leads to a more tractable mathematical structure. Linearity, of course, is an essential feature of any model that will exhibit sustained economic growth. If strong concavity sets in (especially concavity consistent with the Inada conditions as in Assumption 2), sustained growth will not be possible. Therefore, (asymptotic) linearity is an essential ingredient of any model that will lead to sustain growth. The baseline AK model and its cousins make this linear structure quite explicit. While this type of linearity will be not as apparent (and often will be derived rather than assumed), it will also be a feature of the endogenous technology models studied in the next part of the book. Consequently, many of these endogenous technology models will be relatively tractable as well. Nevertheless, we will see that the linearity will often result from much more interesting economic interactions than being imposed in the aggregate production function of the economy. There is another sense in which the material in this chapter does not do justice to issues of sustained growth. As the discussion in Chapter 3 showed, modern economic growth is largely the result of technological progress. Except for the Romer model of Section 11.4, in the models studied in this chapter do not feature technological progress. This does not imply that they are necessarily inconsistent with the data. As our discussion in Chapter 3 indicated there is a lively debate about whether the observed total factor productivity growth is partly a result of mismeasurement of inputs. If this is the case, it could be that much of what we measure as technological progress is in fact capital deepening, which is the bread-and-butter of economic growth in the AK model and its variants. Consequently, the debate about the measurement of total factor productivity has important implications for what types of models we should use for thinking about world economic growth and cross-country income differences.

The discussion in this chapter has also revealed another important tension. Chapters 3 and 8 demonstrated that the neoclassical growth model (or the simpler Solow growth model) have difficulty in generating the very large income differences across countries that we observe in the data. Even if we choose quite large differences in cross-country distortions (for example, eightfold differences in effective tax rates), the implied steady-state differences in income per capita are relatively modest. We have seen that this has generated a large literature that seeks reasonable extensions of the neoclassical growth model in order to derive more elastic responses to policy distortions or other differences across countries. The models presented in this chapter, like those that we will encounter in the next part of the book, suffer from the opposite problem. They imply that even small differences in policies, technological opportunities or other characteristics of societies will lead to permanent differences in long-run growth rates. Consequently, these models can explain very large differences in living standards from small policy, institutional or technological differences. But this is both a blessing and a curse. Though capable of explaining large cross-country differences, these models also predict an ever expanding world distribution, since countries with different characteristics should grow at permanently different rates. The relative stability of the world income distribution in the postwar era is then a challenge to the baseline endogenous growth models. However, as we have seen, the world income distribution is not exactly stationary. While economists more sympathetic to the exogenous growth version of the neoclassical model emphasize the relative stability of the world income distribution, others see stratification and increased inequality. This debate can, in principle, be resolved by carefully mapping various types of endogenous growth theories to postwar data.

Nevertheless, there is more to understanding the nature of the growth process and the role of technological progress than simply looking at the postwar data. First, as illustrated in Chapter 1, the era of divergence is not the past 60 years, but the 19th century. Therefore, it is equally important to confront these models with historical data. Second, a major assumption of most endogenous growth models is that each country can be treated in isolation. This "each country as an island" approach is unlikely to be a good approximation to reality in most circumstances, and much less so when we endogenize technology. Most economies do not generate their own technology by R&D or other processes, but partly import or adopt these technologies from more advanced nations (or from the world technology frontier). Consequently, a successful mapping of the theories to data requires us to enrich these theories and abandon the "each country as an island" assumption. We will do this later in the book both in the context of technology flows across countries and of international trade linkages. But the next part will follow the established literature and develop the models of endogenous technological progress without paying much attention to cross-country knowledge flows.

11.6. References and Literature

The AK model is a special case of Rebelo's (1991), which was discussed in greater detail in Section 11.3 of this chapter. Solow's (1965) book also discussed the AK model (naturally with exogenous savings), but dismissed it as uninteresting. A more complete treatment of sustained neoclassical economic growth is provided in Jones and Manuelli (1990), who show that even convex models (with production function is that satisfy Assumption 1, but naturally not Assumption 2) are consistent with sustained long-run growth. Exercise 11.4 is a version of the convex neoclassical endogenous growth model of Jones and Manuelli.

Barro and Sala-i-Martin (2004) discuss a variety of two-sector endogenous growth models with physical and human capital, similar to the model presented in Section 11.2, though the model presented here is much simpler than similar ones analyzed in the literature.

Romer (1986) is the seminal paper of the endogenous growth literature and the model presented in Section 11.4 is based on this paper. Frankel (1962) analyzed a similar growth economy, but with exogenous constant saving rate. The importance of Romer's paper stems not only from the model itself, but from two other features. The first is its emphasis on potential non-competitive elements in order to generate long-run economic growth (in this case knowledge spillovers). The second is its emphasis on the non-rival nature of knowledge and ideas. These issues will be discussed in greater detail in the next part of the book.

Another paper that has played a major role in the new growth literature is Lucas (1988), which constructs an endogenous growth model similar to that of Romer (1986), but with human capital accumulation and human capital externalities. Lucas' model is also similar to the earlier contribution by Uzawa (1964). Lucas's paper has played two major roles in the literature. First, it emphasized the empirical importance of sustained economic growth and thus was instrumental in generating interest in the newly emerging endogenous growth models. Second, it emphasized the importance of human capital and especially of human capital externalities. Since the role of human capital was discussed extensively in Chapter 10, which also showed that the evidence for human capital externalities is rather limited, we focused on the Romer model rather than the Lucas model. It turns out that Lucas model also generates transitional dynamics, which are slightly more difficult to characterize than the standard neoclassical transitional dynamics. A version of the Lucas model is discussed in Exercise 11.20.

11.7. Exercises

Exercise 11.1. Derive equation (11.14).

Exercise 11.2. Prove Proposition 11.2.

Exercise 11.3. Consider the following continuous time neoclassical growth model:

$$U(0) = \int_0^\infty \exp(-\rho t) \frac{(c(t))^{1-\theta} - 1}{1-\theta},$$

with aggregate production function

$$Y(t) = AK(t) + BL(t),$$

where $A, B > 0$.

(1) Define a competitive equilibrium for this economy.
(2) Set up the current-value Hamiltonian for an individual and characterize the necessary conditions for consumer maximization. Combine these with equilibrium factor market prices and derive the equilibrium path. Show that the equilibrium path displays non-trivial transitional dynamics.
(3) Determine the evolution of the labor share of national income over time.
(4) Analyze the impact of an unanticipated increase in B on the equilibrium path.
(5) Prove that the equilibrium is Pareto optimal.

Exercise 11.4. Consider the following continuous time neoclassical growth model:

$$U(0) = \int_0^\infty \exp(-\rho t) \frac{(c(t))^{1-\theta} - 1}{1-\theta},$$

with production function

$$Y(t) = A\left[L(t)^{\frac{\sigma-1}{\sigma}} + K(t)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}.$$

(1) Define a competitive equilibrium for this economy.
(2) Set up the current-value Hamiltonian for an individual and characterize the necessary conditions for consumer maximization. Combine these with equilibrium factor market prices and derive the equilibrium path.
(3) Prove that the equilibrium is Pareto optimal in this case.
(4) Show that if $\sigma \leq 1$, sustained growth is not possible.
(5) Show that if A and σ are sufficiently high, this model generates asymptotically sustained growth due to capital accumulation. Interpret this result.
(6) Characterize the transitional dynamics of the equilibrium path.
(7) What is happening to the share of capital in national income? Is this plausible? How would you modify the model to make sure that the share of capital in national income remains constant?
(8) Now assume that returns from capital are taxed at the rate τ. Determine the asymptotic growth rate of consumption and output.

Exercise 11.5. Derive equations (11.19) and (11.20).

Exercise 11.6. Consider the neoclassical growth model with Cobb-Douglas technology $y(t) = Ak(t)^\alpha$ (expressed in per capita terms) and log preferences. Characterize the equilibrium path of this economy and show that as $\alpha \to 1$, equilibrium path approaches that of the baseline AK economy. Interpret this result.

Exercise 11.7. Consider the baseline AK model of Section 11.1 and suppose that two otherwise-identical countries have different taxes on the rate of return on capital. Consider the following calibration of the model where $A = 0.15$, $\delta = 0.05$, $\rho = 0.02$, and $\theta = 3$.

Suppose that the first country has a capital income tax rate of $\tau = 0.2$, while the second country has a tax rate of $\tau' = 0.4$. Suppose that the two countries start with the same level of income in 1900 and experience no change in technology or policies for the next 100 years. What will be the relative income gap between the two countries in the year 2000? Discuss this result and explain why you do (or do not) find the implications plausible.

Exercise 11.8. Prove that the necessary conditions for consumer optimization in Section 11.2 lead to the conditions enumerated in (11.25).

Exercise 11.9. Prove Proposition 11.3.

Exercise 11.10. Prove that the competitive equilibrium of the economy in Section 11.2, characterized in Proposition 11.3, is Pareto optimal and coincides with the solution to the optimal growth problem.

Exercise 11.11. Show that the rate of population growth has no effect on the equilibrium growth rate of the economies studied in Sections 11.1 and 11.2. Explain why this is. Do you find this to be a plausible prediction?

Exercise 11.12. * Show that in the model of Section 11.3, if the Cobb-Douglas assumption is relaxed, there will not exist a balanced growth path with a constant share of capital income in GDP.

Exercise 11.13. Consider the effect of an increase in α on the competitive equilibrium of the model in Section 11.3. Why does it increase the rate of capital accumulation in the economy?

Exercise 11.14. Consider a variant of the model studied in Section 11.3, where the technology in the consumption-good sector is still given by (11.27), while the technology in the investment-good sector is modified to

$$I(t) = A (K_I(t))^{\beta} (L_I(t))^{1-\beta},$$

where $\beta \in (\alpha, 1)$. The labor market clearing condition requires $L_C(t) + L_I(t) \leq L(t)$. The rest of the environment is unchanged.

(1) Define a competitive equilibrium.
(2) Characterize the steady-state equilibrium and show that it does not involve sustained growth.
(3) Explain why the long-run growth implications of this model differ from those of Section 11.3.
(4) Analyze the steady-state income differences between two economies taxing capital at the rates τ and τ'. What are the roles of the parameters α and β in determining these relative differences? Why do the implied magnitudes differ from those in the one-sector neoclassical growth model?

Exercise 11.15. In the Romer model presented in Section 11.4, let g_C^* be the growth rate of consumption and g^* the growth rate of aggregate output. Show that $g_C^* > g^*$ is not feasible, while $g_C^* < g^*$ would violate the transversality condition.

Exercise 11.16. Consider the Romer model presented in Section 11.4. Prove that the allocation in Proposition 11.5 satisfies the transversality condition. Prove also that there are no transitional dynamics in this equilibrium.

Exercise 11.17. Consider the Romer model presented in Section 11.4 and suppose that population grows at the exponential rate n. Characterize the labor market clearing conditions. Formulate the dynamic optimization problem of a representative household and show that any interior solution to this problem violates the transversality condition. Interpret this result.

Exercise 11.18. Consider the Romer model presented in Section 11.4. Provide two different types of tax/subsidy policies that would make the equilibrium allocation identical to the Pareto optimal allocation.

Exercise 11.19. Consider the following infinite-horizon economy in discrete time that admits a representative household with preferences at time $t = 0$ as

$$U(0) = \sum_{t=0}^{\infty} \beta^t \left[\frac{C(t)^{1-\theta} - 1}{1-\theta} \right],$$

where $C(t)$ is consumption, and $\beta \in (0,1)$. Total population is equal to L and there is no population growth and labor is supplied inelastically. The production side of the economy consists of a continuum 1 of firms, each with production function

$$Y_i(t) = F(K_i(t), A(t) L_i(t)),$$

where $L_i(t)$ is employment of firm i at time t, $K_i(t)$ is capital used by firm i at time t, and $A(t)$ is a common technology term. Market clearing implies that $\int_0^1 K_i(t)\, di = K(t)$, where $K(t)$ is the total capital stock at time t,and $\int_0^1 L_i(t)\, di = L(t)$. Assume that capital fully depreciates, so that the resource constraint of the economy is

$$K(t+1) = \int_0^1 Y_i(t)\, di - C(t).$$

Assume also that labor-augmenting productivity at time t, $A(t)$, is given by

$$A(t) = K(t). \tag{11.42}$$

(1) Explain (11.42) and why it implies a (non-pecuniary) externality.
(2) Define a competitive equilibrium (where all agents are price takers—but naturally not all markets are complete).
(3) Show that there exists a unique balanced growth path competitive equilibrium, where the economy grows (or shrinks) at a constant rate every period. Provide a

condition on F, β and θ such that this growth rate is positive, but the transversality condition is still satisfied.

(4) Argue (without providing the math) why any equilibrium must be along the balanced growth path characterized in part 3 at all points.

(5) Is this a good model of endogenous growth? If yes, explain why. If not, contrast it with what you consider to be better models.

EXERCISE 11.20. * Consider the following endogenous growth model due to Uzawa and Lucas. The economy admits a representative household and preferences are given by

$$\int_0^{\infty} \exp(-\rho t) \frac{C(t)^{1-\theta} - 1}{1-\theta} dt,$$

where $C(t)$ is consumption of the final good, which is produced as

$$Y(t) = AK(t)^{\alpha} H_P^{1-\alpha}(t)$$

where $K(t)$ is capital and $H(t)$ is human capital, and $H_P(t)$ denotes human capital used in production. The accumulation equations are as follows:

$$\dot{K}(t) = I(t) - \delta K(t)$$

for capital and

$$\dot{H}(t) = BH_E(t) - \delta H(t)$$

where $H_E(t)$ is human capital devoted to education (further human capital accumulation), and the depreciation of human capital is assumed to be at the same rate as physical capital for simplicity (δ). The resource constraints of the economy are

$$I(t) + C(t) \leq Y(t)$$

and

$$H_E(t) + H_P(t) \leq H(t).$$

(1) Interpret the second resource constraint.

(2) Denote the fraction of human capital allocated to production by $\phi(t)$, and calculate the growth rate of final output as a function of $\phi(t)$ and the growth rates of accumulable factors.

(3) Assume that $\phi(t)$ is constant, and characterize the balanced growth path of the economy (with constant interest rate and constant rate of growth for capital and output). Show that in this balanced growth path, we have $r^* \equiv B - \delta$ and the growth rate of consumption, capital, human capital and output are given by $g^* \equiv (B - \delta - \rho)/\theta$. Show also that there exists a unique value of $k^* \equiv K/H$ consistent with balanced growth path.

(4) Determine the parameter restrictions to make sure that the transversality condition is satisfied.

(5) Now analyze the transitional dynamics of the economy starting with K/H different from k^* [Hint: look at dynamics in three variables, $k \equiv K/H$, $\chi \equiv C/K$ and ϕ, and consider the cases $\alpha < \theta$ and $\alpha \geq \theta$ separately].

Part 4

Endogenous Technological Change

This part of the book focuses on models of endogenous technological change. Chapter 12 discusses various different approaches to technological change and provides a brief overview of some models of technological progress from the industrial organization literature. Chapters 13 and 14 present the baseline endogenous technological progress models developed by Romer, Grossman and Helpman and Aghion and Howitt. Chapter 15 considers a richer class of models in which the direction of technological change, for example, which factors technological change will augment or complement, is also endogenous. The models presented in this part of the book are useful for two related purposes. First, they enable us to endogenize technology, thus allowing a more in-depth study of cross-country and over-time differences in technologies. This is useful since, as discussed in Chapter 3, differences in technology and in the efficiency with which factors of production are used are a major proximate cause of growth over time and cross-country differences in economic performance. Second, endogenous technology models provide a tractable approach to modeling sustained growth, which we will later combine with other factors to investigate the determinants of country and world growth rates.

CHAPTER 12

Modeling Technological Change

We have so far investigated models of economic growth of exogenous or endogenous variety. But economic growth has not resulted from technological change. Either it has been exogenous, or it has been sustained because of a linear neoclassical technology, or it has taken place as a byproduct of knowledge spillovers. Since our purpose is to understand the process of economic growth, models in which growth results from technological progress and technological change itself is a consequence of purposeful investments by firms and individuals are much more attractive. These models not only endogenize technological progress, but they also relate the process of technological change to the underlying market structure, anti-trust and competition policy, and intellectual property rights policy. They will also enable us to discuss issues of directed technical change. In this chapter, we begin with a brief discussion of different conceptions of technological change and provide some foundations for the models that will come later.

12.1. Different Conceptions of Technology

12.1.1. Types of Technological Change. The literature on technological change often distinguishes between different types of innovations. A first common distinction is between *process* and *product* innovation. While the latter refers to the introduction of a new product (for example, the introduction of the first DVD player), the former is concerned with innovations that reduce the costs of production of existing products (for example, the introduction of new machines to produce existing goods). Although the distinction between process and product innovations is important in the literature and in practice, we will shortly see that many models with process innovation, in particular, those in which new innovations introduce new types of inputs, are mathematically identical to models of product innovation. Nevertheless, the distinction is still useful in mapping some of these theories to data.

A somewhat different type of process innovation is perhaps most important in practice and involves the introduction of a cost-reducing technological improvement or a higher-quality version of an existing good. Both the introduction of a better DVD player, when there are already DVD players in the market, and the innovation to manufacture exactly the same DVD player at lower costs would be examples of this type of innovation. The most important implication is that these types of innovations will typically lead to the replacement of older vintages of the same good or machine and also to potential competition between existing producers and the innovator. In addition, in the context of this type of innovation, one might want to distinguish between the introduction of a higher-quality DVD player and the production of a cheaper DVD player because heterogeneous consumers may have differential willingness to pay for quality than for quantity. Issues of differential willingness to pay for quality are important in the theory of industrial organization and for constructing accurate quality-adjusted price indices. However, most growth models represent the consumer side by a representative household and also implicitly assume perfect substitution between quality and quantity. These features create a close connection between innovations that increase the quality of existing products and process innovations. The following example illustrates why, in the context of the typical growth models, quality improvements and cost reductions are essentially equivalent.

Example 12.1. Consider an economy admitting a representative household with preferences $U\left(c\left(q\right), y \mid q\right)$, where y stands for a generic good (perhaps representing all other goods), c is a particular consumption good available in different qualities. Here $c\left(q\right)$ denotes the amount consumed of the "vintage" of this good of quality q. The utility function is also conditioned on q. This specification implies that quality and quantity are perfect substitutes, so that higher-quality increases the "effective units" of consumption. This is a typical assumption in growth models, though it is clearly restrictive; the consumption (use) of five Pentium I computers would not give the same services as the use of a single Pentium III computer.

Let the budget constraint of the representative consumer be

$$p\left(q\right)c\left(q\right) + y \leq m,$$

where $p\left(q\right)$ is the price of a good of quality q, the price of the generic good is normalized to 1, and m denotes the resources available to the consumer. The problem of the consumer can then be equivalently written as

$$\begin{aligned}
&\max_{x(q),y} U\left(x\left(q\right), y \mid q\right) \\
&\text{subject to} \\
&\frac{p\left(q\right)}{q}x\left(q\right) + y \leq m,
\end{aligned}$$

where $x\left(q\right) \equiv qc\left(q\right)$ corresponds to the effective units of consumption of good c. It is straightforward to see from this problem formulation that an $s\%$ increase in quality q and an $s\%$ decline in the price $p\left(q\right)$ have exactly the same effect on the effective units of consumption and on welfare. This justifies the claim above that in many models, process innovations reducing costs of production and quality improvements have identical effects.

Another important distinction in the technological change literature is between "macro" and "micro" innovations (see Mokyr, 1990). The first refers to radical innovations, perhaps the introduction of general-purpose technologies, such as electricity or the computer, which potentially changed the organization of production in many different product lines. In contrast, micro innovations refer to the more common innovations that introduce newer models of existing products, improve the quality of a certain product line, or simply reduce costs. Most of the innovations we will be discussing can be viewed as "micro innovations". Moreover, empirically, it appears that micro innovations are responsible for most of the productivity growth in practice (see the evidence and discussion in Freeman (1982), Myers and Marquis (1969) and Abernathy (1980)). We will discuss the implications of macro or general-purpose innovations below.

12.1.2. A Production Function for Technology. A potentially confusing issue in the study of technological progress is how to conceptualize the menu of technologies available to firms or individuals. Since our purpose is to develop models of endogenous technology, firms and/or individuals must have a choice over different types of technologies, with greater effort, research spending and investment leading to better technologies. At some level, this implies that there must exist a *meta production function* (a production function over production functions), which determines how new technologies are generated as a function of inputs. We will sometimes refer to the meta production function as the *innovation possibilities frontier* or as the R&D production function.

While a meta production function may appear natural to some, there are various economists and social scientists who do not find this a compelling approach. Their argument against the production function approach to technology is that, by its nature, innovation includes the discovery of the "unknown"; thus how could we put that in the context of a production function where inputs go in and outputs come out in a deterministic fashion?

Although this question has some descriptive merit (in the sense that describing the discovery of new technologies with a production function obscures some important details of the innovation process), the concern is largely irrelevant. There is no reason to assume that the meta production function for technology is deterministic. For example, we can assume that when a researcher puts l hours and x units of the final good into a research project, then there will be some probability $p(l, x)$ that any innovation will be made. Conditional on an innovation, the quality of the good will have a distribution $F(q \mid l, x)$. In this particular formulation, both the success of the research project and the quality of the research output conditional on success are uncertain. Nevertheless, all this can be formulated as part of the meta production function with stochastic output. Therefore, the production function approach to technology is not particularly restrictive, as long as uncertain outcomes are allowed and we are willing to assume that individuals can make calculations about the effect of their actions on the probability of success and quality of the research project. Naturally, some may argue that such calculations are not possible. But, without such calculations we would have little hope of modeling the process of technological change (or technology adoption). Since our objective is to model purposeful innovations, to assume that individuals and firms can make such calculations is entirely natural, and the existence of individuals and firms making such calculations is equivalent to assuming the existence of a meta production function for technologies.

12.1.3. Non-Rivalry of Ideas. Another important aspect of technology is emphasized in Paul Romer's work. As we already discussed in the previous chapter, Romer's first model of endogenous growth, Romer (1986), introduced increasing returns to scale to physical capital accumulation. The justification for this was that the accumulation of knowledge could be considered a byproduct of the economic activities of firms. Later work by Romer, which we will study in the next chapter, took a very different approach to modeling the process of economic growth, but the same key idea is present in both his early and later work: the *non-rivalry* of ideas matters.

By non-rivalry, Romer means that the use of an idea by one producer to increase efficiency does not preclude its use by others. While the same unit of labor or capital cannot be used by multiple producers, the same idea can be used by many, potentially increasing everybody's productivity. Let us consider a production function of the form

$$F(K, L, A),$$

with A denoting "technology". Romer argues that an important part of this technology is the ideas or blueprints concerning how to produce new goods, how to increase quality, or how to reduce costs. We are generally comfortable assuming that the production function $F(K, L, A)$ exhibits constant returns to scale in capital and labor (K and L), and we adopted this assumption throughout the first three parts of the book. For example, replication arguments could be used to justify this type of constant returns to scale; when capital and labor double, the society can always open a replica of the same production facility, and in the absence of externalities, this will (at least) double output.

Romer, then, argues that when we endogenize A, this will naturally lead to increasing returns to scale to all three inputs, K, L and A. To understand why "non-rivalry" is important here, imagine that A is just like any other input. Then the replication argument would require the new production facility to replicate A as well, thus we should expect constant returns to scale when we vary all three inputs, K, L and A. But, instead, assume that ideas are non-rival. The new production facility does not need to re-create or to replicate A, because it is already out there available for all firms to use. In that case, $F(K, L, A)$ will exhibits constant returns in K and L, and *increasing returns to scale* in K, L and A.

Thus the non-rivalry of ideas and increasing returns to scale to all factors of production, including technology, are intimately linked. This has motivated Romer to develop different types of endogenous growth models, exhibiting different sources of increasing returns to scale, but the non-rivalry of ideas has been a central element in all of them.

Another important implication of the non-rivalry of ideas is the *market size* effect, which we will frequently encounter below. If, once discovered, an idea can be used as many times as one wishes, then the size of its potential market will be a crucial determinant of whether or not it is profitable to implement it and whether to research it in the first place. This is well captured by a famous quote from Matthew Boulton, James Watt's business partner, who wrote to Watt:

> "It is not worth my while to manufacture your engine for three countries only, but I find it very well worth my while to make it for all the world." (quoted in Scherer, 1984, p. 13).

To see why non-rivalry is related to the market size effect, imagine another standard (rival) input that is also essential for production. A greater market size will not typically induce firms to use this other input more intensively, since a greater market size and thus greater sales means that more of this input has to be used. It is the fact that, once invented, non-rival ideas can be embedded in as many units desired without further costs that makes the market size effect particularly important. In the next section, we will discuss some empirical evidence on the importance of the market size effect.

Nevertheless, the non-rivalry of ideas does not make ideas or innovations *pure public goods*. Recall that pure public goods are both non-rival and non-excludable. While some discoveries may be, by their nature, non-excludable (for example, the "discovery" that providing excessively high-powered incentives to CEOs in the form of stock options will lead to counterproductive incentives and cheating), most discoveries can be made partly excludable by *patenting*. An important aspect of the models of technological progress will be whether and how discoveries are protected from rivals. For this reason, intellectual property rights protection and patent policy often play an important role in models of technological progress.

12.2. Science and Profits

Another major question for the economic analysis of technological change is whether innovation is mainly determined by scientific constraints and stimulated by scientific breakthroughs in particular fields, or whether it is driven by profit motives. Historians and economists typically give different answers to this question. Many historical accounts of technological change come down on the side of the "science-driven" view, emphasizing the autonomous progress of science, and how important breakthroughs—perhaps macro innovations discussed above—have taken place as scientists build on each other's work, with little emphasis on profit opportunities. For example, in his *History of Modern Computing*, Ceruzzi emphasizes the importance of a number of notable scientific discoveries and the role played by certain talented individuals, such as John von Neumann, J. Presper Eckert, John Maucly, John Backus, Kenneth H. Olsen, Harlan Anderson and those taking part in the Project Whirlwind at MIT, rather than profit motives and the potential market for computers. He points out, for example, how important developments took place despite the belief of many important figures in the development of the computer, such as Howard Aiken, that there would not be more than a handful of personal computers in the United States (2000, p. 13). Many economic historians, including Rosenberg (1974) and Sherer (1984) similarly argue that a key determinant of innovation in a particular field is the largely-exogenous growth of scientific and engineering knowledge in that field.

In contrast, most economists believe that profit opportunities play a much more important role, and the demand for innovation is key to understanding the process of technological change. John Stuart Mill provides an early and clear statement of this view in his *Principles of Political Economy*, when he writes:

> "The labor of Watt in contriving the steam-engine was as essential a part of production as that of the mechanics who build or the engineers who work the instrument; and was undergone, no less than theirs, in the prospect of a renumeration from the producers." (1890, p. 68, also quoted in Schmookler, 1966, p. 210).

In fact, profits were very much in the minds of James Watt and his business partner, Matthew Bolton as the previous quote illustrates. James Watt also praised the patent system for the same reasons, arguing that: "...an engineer's life without patent was not worthwhile" (quoted in Mokyr, 1990, p. 248). The view that profit opportunities are the primary determinant of innovation and invention is articulated by Griliches and Schmookler (1963), and most forcefully by Schmookler's seminal study, *Invention and Economic Growth*. Schmookler writes:

> "...invention is largely an economic activity which, like other economic activities, is pursued for gain." (1966, p. 206)

Moreover, Schmookler argues against the importance of major breakthroughs in science on economic innovation. He concludes his analysis of innovations in petroleum refining, papermaking, railroading, and farming by stating that there is no evidence that past breakthroughs have been the major factor in new innovations. In particular, he argues: "Instead, in hundreds of cases the stimulus was the recognition of a costly problem to be solved or a potentially profitable opportunity to be seized..." (1966, p. 199). Other studies of innovation in particular industries also reach similar conclusions, see, for example, Myers and Marquis (1969) or Langrish et al. (1974).

If potential profits are a main driver of technological change, then *the market size* that will be commanded by new technologies or products will be a key determinant of innovations. A greater market size increases profits and makes innovation and invention more desirable. To emphasize this point, Schmookler called two of his chapters "*The amount of invention is governed by the extent of the market.*" Schmokler's argument is most clearly illustrated by the example of the horseshoe. He documented that there was a very high rate of innovation throughout the late nineteenth and early twentieth centuries in the ancient technology of horseshoe making, and no tendency for inventors to run out of additional improvements. On the contrary, inventions and patents increased because demand for horseshoes was high. Innovations came to an end only when "the steam traction engine and, later, internal combustion engine began to displace the horse..." (1966, p. 93). The classic study by Griliches (1957) on the spread of hybrid seed corn in the U.S. agriculture also provides support for the view that technological change and technology adoption are closely linked to profitability and market size.

A variety of more recent papers also reach similar conclusions. An interesting paper by Newell, Jaffee and Stavins (1999) shows that between 1960 and 1980, the typical air-conditioner sold at Sears became significantly cheaper, but not much more energy-efficient. On the other hand, between 1980 and 1990, there was little change in costs, but air-conditioners became much more energy-efficient, which, they argue, was a response to higher energy prices. This seems to be a clear example of the pace and the type of innovation responding to profit incentives. In a related study, Popp (2002) finds a strong positive correlation between patents for energy-saving technologies and energy prices and thus confirms the overall picture resulting from the Newell, Jaffee and Stavins study.

Evidence from the pharmaceutical industry also illustrates the importance of profit incentives and especially of the market size on the rate of innovation. Finkelstein (2003) exploits three different policy changes affecting the profitability of developing new vaccines against 6 infectious diseases: the 1991 Center for Disease Control recommendation that all infants be vaccinated against hepatitis B, the 1993 decision of Medicare to cover the costs of influenza vaccinations, and the 1986 introduction of funds to insure vaccine manufactures against product liability lawsuits for certain kinds of vaccines. She finds that increases in vaccine profitability resulting from these policy changes are associated with a significant increase in the number of clinical trials to develop new vaccines against the relevant diseases. Acemoglu and Linn (2004) look at demographic-driven exogenous changes in the market size for drugs of different types and find a significant response in the rate of innovation to these changes in market sizes.

Overall, the evidence suggests that the market size is a major determinant of innovation incentives and the amount and type of technological change. This evidence motivates the types of models we will study, where technological change will be an economic activity and will respond to profit incentives rather than simply being driven by exogenous scientific processes.

12.3. The Value of Innovation in Partial Equilibrium

Let us now turn to the analysis of the value of innovation and R&D to a firm. The equilibrium value of innovation and the difference between this private value and the social value (i.e., the value to a social planner internalizing externalities) will play a central role in our analysis below. All of the growth models we have studied so far, as well as most of those we will study next, are dynamic general equilibrium models. In fact, as emphasized at the beginning, economic growth is a process we can only understand in the context of general equilibrium analysis. Nevertheless, it is useful to start our investigation of the value of innovation in partial equilibrium, where much of the industrial organization literature starts.

Throughout this section, we consider a single industry. Firms in this industry have access to an existing technology that enables firms to produce one unit of the product at the marginal cost $\psi > 0$. The demand side of the industry is modeled with a demand curve

$$Q = D(p),$$

where p is the price of the product and Q is the demand at this price. Throughout we assume that $D(p)$ is strictly decreasing, continuously differentiable and satisfies the following conditions:

$$D(\psi) > 0 \text{ and } \varepsilon_D(p) \equiv -\frac{pD'(p)}{D(p)} \in (1, \infty).$$

The first ensures that there is positive demand when price is equal to marginal cost, and the second ensures that the elasticity of demand, $\varepsilon_D(p)$, is always greater than 1, so that when we consider monopoly pricing, there will exist a well-defined profit-maximizing price. Moreover, this elasticity is less than infinity, so that the monopoly price will be above marginal cost.

Throughout this chapter and whenever we analyze economies with monopolistic competition, oligopolies or potential monopolies, equilibrium refers to Nash equilibrium or subgame perfect Nash equilibrium (when the game in question is dynamic).

12.3.1. No Innovation with Pure Competition. Suppose first that there is a large number of firms, say N firms, with access to the existing technology. Now imagine that one of these firms, say firm 1, also has access to a research technology for generating a process innovation. In particular, let us simplify the discussion and suppose that there is no uncertainty in research, and if the firm incurs a cost $\mu > 0$, it can innovate and reduce the marginal cost of production to ψ/λ, where $\lambda > 1$. Let us suppose that this innovation is non-rival and is also non-excludable, either because it is not patentable or because the patent system does not exist.

Let us now analyze the incentives of this firm in undertaking this innovation. We first look at the equilibrium without the innovation. Clearly, the presence of a large number of N firms, all with the same technology with marginal cost ψ, implies that the equilibrium price will be $p^N = \psi$, where the superscript N denotes "no innovation". The total quantity demanded will be $D(\psi) > 0$ and can be distributed among the N firms in any arbitrary fashion. Since price is equal to marginal cost, the profits of firm 1 in this equilibrium will be

$$\begin{aligned}\pi_1^N &= \left(p^N - \psi\right) q_1^N \\ &= 0,\end{aligned}$$

where q_1^N denotes the amount supplied by this firm.

Now imagine that firm 1 innovates, but because of non-excludability, the innovation can be used by all the other firms in the industry. The same reasoning implies that the equilibrium price will be $p^I = \lambda^{-1}\psi$, and total quantity supplied by all the firms will equal $D\left(\lambda^{-1}\psi\right) > D(\psi)$. Then, the net profits of firm 1 following innovation will be

$$\begin{aligned}\pi_1^I &= \left(p^I - \lambda^{-1}\psi\right) q_1^I - \mu \\ &= -\mu < 0.\end{aligned}$$

Therefore, if it undertakes the innovation, firm 1 will lose money. The reason for this is simple. The firm incurs the cost of innovation, μ, but because the knowledge generated by the innovation is non-excludable, it is unable to *appropriate* any of the gains from innovation. This simple example underlies a claim dating back to Schumpeter that pure competition will not generate innovation.

Clearly, this outcome is potentially very inefficient. For example, μ could be arbitrarily small (but still positive), while λ, the gain from innovation, can be arbitrarily large, but the equilibrium would still involve no innovation. For future reference, let us calculate the social value of innovation, which is the additional gain resulting from innovation. A natural measure of social value is in the sum of the consumer and producer surpluses generated from the innovation. Presuming that after innovation, the good will be priced at marginal cost, this social value is

$$\begin{aligned}\mathcal{S}^I &= \int_{\lambda^{-1}\psi}^{\psi} D(p)\,dp - \mu \\ &= \int_{\lambda^{-1}\psi}^{\psi} \left[D(p) - D(\psi)\right] dp + D(\psi)\,\lambda^{-1}(\lambda - 1)\,\psi - \mu.\end{aligned} \tag{12.1}$$

The first term in the second line is the increase in consumer surplus because of the expansion of output as the price falls from ψ to $\lambda^{-1}\psi$ (recall that price is equal to marginal cost in this social planner's allocation). The second term is the savings in costs for already produced units; in particular, there is a saving of $\lambda^{-1}(\lambda-1)\psi$ on $D(\psi)$ units. Finally, the last term is the cost of innovation. Depending on the shape of the function $D(p)$, the values of λ and μ, this social value of innovation can be quite large.

12.3.2. Some Caveats. The above example illustrates the problem of innovation under pure competition in a very sharp way. The main problem is the inability of the innovator to exclude others from using this innovation. One way of ensuring such excludability is via the protection of intellectual property rights or a patent system, which will create ex post monopoly power for the innovator. This type of intellectual property right protection is present in most countries and will play an important role in many of the models we study below.

Before embarking on an analysis of the implications of ex post monopoly power of innovators, there are a number of caveats we should emphasize. First, even without patents, "trade secrecy" may be sufficient to provide some incentives for innovation. Second, firms may engage in innovations that are only appropriate for their own firm, making their innovations de facto excludable. For example, imagine that at the same cost, the firm can develop a new technology that reduces the marginal cost of production by only $\lambda' < \lambda$. But this technology is *specific* to the needs and competencies of the current firm and cannot be used by any other (or alternatively, λ/λ' is the "proportional" cost of making the innovation "excludable"). We will show that the adoption of this technology may be profitable for the firm, since the specificity of the innovation firm acts exactly like patent protection (see next subsection and also Exercise 12.5). Therefore, some types of innovations, in particular those protected by trade secrecy, can be undertaken under pure competition.

Finally, a number of authors have recently argued that innovations in competitive markets are possible, when firms are able to replicate the technology and sell it to competitors during a certain interval of time before being imitated by others (e.g., Hellwig and Irmen, 2001, Walde, 2002, and Boldrin and Levine, 2003).

12.3.3. Innovation and Ex Post Monopoly. Let us now return to the same environment as above, and suppose that if firm 1 undertakes a successful innovation it can obtain a fully-enforced patent. Once this happens, firm 1 will have better technology than the rest of the firms, and will possess *ex post* monopoly power. This monopoly power will enable the firm to earn profits from the innovation, potentially encouraging its research activity in the first place. This is the basis of the claim by Schumpeter, Arrow, Romer and others that there is an intimate link between ex post monopoly power and innovation.

Let us now analyze this situation in a little more detail. It is useful to separate two cases:

(1) *Drastic innovation*: a drastic innovation corresponds to a sufficiently high value of λ such that firm 1 becomes an effective monopolist after the innovation. To determine which values of λ will lead to a situation of this sort, let us first suppose that firm 1 does indeed act like a monopolist. This implies that it will choose its price to maximize

$$\pi_1^I = D(p)\left(p - \lambda^{-1}\psi\right).$$

The first-order condition of this maximization is

$$D'(p)\left(p - \lambda^{-1}\psi\right) + D(p) = 0.$$

Clearly the solution to this equation gives the standard monopoly pricing formula (see Exercise 12.1):

$$p^M \equiv \frac{\lambda^{-1}\psi}{1 - \varepsilon_D(p)^{-1}}. \tag{12.2}$$

We say that the innovation is *drastic* if $p^M \leq \psi$. It is clear that this will be the case when

$$\lambda \geq \lambda^* \equiv \frac{1}{1 - \varepsilon_D(p)^{-1}}.$$

When the innovation is drastic, firm 1 can set its unconstrained monopoly price, p^M, and capture the entire market.

(2) *Limit pricing*: when the innovation is not drastic, so that $p^M > \psi$ or alternatively, when $\lambda < \lambda^*$, the equilibrium will involve limit pricing, where firm 1 sets the price

$$p_1 = \psi,$$

so as to make sure that it still captures the entire market (since in this case if it were to set $p_1 = p^M$, other firms can profitably undercut firm 1). This type of limit pricing arises in many situations. In the case we have just discussed, limit pricing results from process innovations by some firms that now have access to a better technology than their rivals. Alternatively, it can also arise when *a fringe* of potential entrants can imitate the technology of a firm (either at some cost or with lower efficiency) and the firm may be forced to set a limit price in order to prevent the fringe from stealing its customers.

We summarize this discussion in the next proposition. Recall that the difference between the drastic innovation and limit pricing cases corresponds to different values of λ—i.e., it depends on whether or not λ is greater than λ^*.

Proposition 12.1. *Consider the above-described industry. Suppose that firm 1 undertakes an innovation reducing marginal cost of production from ψ to $\lambda^{-1}\psi$. If $p^M \leq \psi$ (or if*

$\lambda \geq \lambda^*$), then firm 1 sets the unconstrained monopoly price $p_1 = p^M$ and makes profits

$$\hat{\pi}_1^I = D\left(p^M\right)\left(p^M - \lambda^{-1}\psi\right) - \mu. \tag{12.3}$$

If $p^M > \psi$ (if $\lambda < \lambda^*$), then firm 1 sets the limit price $p_1 = \psi$ and makes profits

$$\pi_1^I = D\left(\psi\right)\lambda^{-1}\left(\lambda - 1\right)\psi - \mu < \hat{\pi}_1^I. \tag{12.4}$$

Proof. The proof of this proposition involves solving for the equilibrium of an asymmetric cost Bertrand competition game. While this is standard, it is useful to repeat it, especially to see why in equilibrium, all demand must be met by the low cost firm. Exercise 12.2 asks you to work through the steps of the proof. □

The fact that $\hat{\pi}_1^I > \pi_1^I$ is intuitive, since the former refers to the case where λ is greater than λ^*, whereas in the latter, firm 1 has a sufficiently low λ that it is forced to charge a price lower than the profit-maximizing monopoly price because of the competition by the remaining firms (still producing at marginal cost ψ).

It can also be easily verified that both $\hat{\pi}_1^I$ and π_1^I can be strictly positive, so that with ex post monopoly innovation becomes possible. This corresponds to a situation in which we start with pure competition, but one of the firms undertakes an innovation in order to *escape competition* and gains ex post monopoly power. The fact that the ex post monopoly power is important for providing incentives to undertake innovations is consistent with Schumpeter's emphasis on the role of monopoly in generating innovations.

Now returning to the discussion in the previous subsection, we can also see that trade secrecy or innovations that are specific only for the needs of the firm in question will act in the same way as ex post patent protection in encouraging innovation (see Exercise 12.5).

Note also that the expressions for $\hat{\pi}_1^I$ and π_1^I in this proposition also give the value of innovation to firm 1, since without innovation, it would make zero profits. Given this observation, we now contrast the value of innovation for firm 1 in these two regimes with the social value of innovation, which is still given by (12.1). Moreover, we can also compare social values in the equilibrium in which innovation is undertaken by firm 1 (who will charge the profit-maximizing price) to the full social value of innovation in (12.1), which applies when the product is priced at marginal cost. The equilibrium social surplus in the regimes with monopoly and limit pricing (again corresponding to the cases in which λ is greater than or less than λ^*) can be computed as

$$\begin{aligned} \hat{\mathcal{S}}_1^I &= D\left(p^M\right)\left(p^M - \lambda^{-1}\psi\right) + \int_{p^M}^{\psi} D\left(p\right)dp - \mu, \text{ and} \\ \mathcal{S}_1^I &= D\left(\psi\right)\lambda^{-1}\left(\lambda - 1\right)\psi - \mu. \end{aligned} \tag{12.5}$$

We then have the following result:

Proposition 12.2. *We have that*

$$\begin{aligned}\pi_1^I &< \hat{\pi}_1^I < \mathcal{S}^I.\\ \mathcal{S}_1^I &< \widehat{\mathcal{S}}_1^I < \mathcal{S}^I.\end{aligned}$$

Proof. See Exercise 12.3. □

This proposition states that the social value of innovation is always greater than the private value in two senses. First, the first line states that a social planner interested in maximizing consumer and producer surplus will always be more willing to adopt an innovation, because of *an appropriability effect*; the firm, even if it has ex post monopoly rights, will be able to appropriate only a portion of the gain in consumer surplus created by the better technology. Second, the second line implies that even conditional on innovation, the gain in social surplus is always less in the equilibrium supported by ex post monopoly than the gain that the social planner could have achieved (by also controlling prices). Therefore, even though ex post monopoly power (for example, generated by patents) can induce innovation, the incentives for innovation and the equilibrium allocations following an innovation are still inefficient. Note also that $\widehat{\mathcal{S}}_1^I$ can be negative, so that a potentially productivity-enhancing process innovation can reduce social surplus because of the cost of innovation, μ. However, it can be shown that if $\hat{\pi}_1^I > 0$, then $\widehat{\mathcal{S}}_1^I > 0$, which implies that excessive innovation is not possible in this competitive environment (see Exercise 12.4). This will contrast with the results in the next subsection.

12.3.4. The Value of Innovation to a Monopolist: The Replacement Effect. Let us now analyze the same industry as in the previous subsection, but assuming that firm 1 is already an unconstrained monopolist with the existing technology. Then with the existing technology, this firm would set the monopoly price of

$$\hat{p}^M \equiv \frac{\psi}{1-\varepsilon_D(p)^{-1}}$$

and make profits equal to

$$\hat{\pi}_1^N = D\left(\hat{p}^M\right)\left(\hat{p}^M-\psi\right). \tag{12.6}$$

If it undertakes the innovation, it will reduce its marginal cost to $\lambda^{-1}\psi$ and still remain an unconstrained monopolist. Therefore, its profits will be given by $\hat{\pi}_1^I$ as in (12.3), with the monopoly price p^M given by (12.2). Now the value of innovation to the monopolist is

$$\begin{aligned}\Delta\hat{\pi}_1^I &= \hat{\pi}_1^I-\hat{\pi}_1^N\\ &= D\left(p^M\right)\left(p^M-\lambda^{-1}\psi\right)-D\left(\hat{p}^M\right)\left(\hat{p}^M-\psi\right)-\mu,\end{aligned}$$

where $\hat{\pi}_1^I$ is given by (12.3) and $\hat{\pi}_1^N$ by (12.6).

Proposition 12.3. *We have that*

$$\Delta\hat{\pi}_1^I < \pi_1^I < \hat{\pi}_1^I,$$

so that a monopolist always has lower incentives to undertake innovation than a competitive firm.

Proof. See Exercise 12.6. □

This result, which was first pointed out in Arrow's (1962) seminal paper, is referred to as the *replacement effect*. The terminology reflects the intuition for the result; the monopolist has lower incentives to undertake innovation than the firm in a competitive industry because with its innovation will replace its own already existing profits. In contrast, a competitive firm would be making zero profits and thus had no profits to replace.

An immediate and perhaps more useful corollary of this proposition is the following:

Corollary 12.1. *An entrant will have stronger incentives to undertake an innovation than an incumbent monopolist.*

The potential entrant is making zero profits without the innovation. If it undertakes the innovation it will become the ex post monopolist and make profits equal to π_1^I or $\hat{\pi}_1^I$. Both of these are greater than the additional profits that the incumbent would make by innovating, $\Delta\hat{\pi}_1^I$. This is a direct consequence of the replacement effect; while the incumbent would be replacing its own profit-making technology, the entrant would be replacing the incumbent. The replacement effect and this corollary imply that in many models entrants have stronger incentives to invest in R&D than incumbents.

The observation that entrants will often be the engine of process innovations takes us to the realm of Schumpeterian models. Joseph Schumpeter characterized the process of economic growth as one of *creative destruction*, meaning a process in which economic progress goes hand-in-hand with the destruction of some existing productive units. Put differently, innovation is driven by the prospect of monopoly profits. Because of the replacement effect, it will be entrants, not incumbents, that undertake greater R&D towards inventing and implementing process innovations. Consequently, innovations will displace incumbents and destroy their rents. According to Schumpeter, this process of creative destruction is the essence of the capitalist economic system. We will see, especially in Chapter 14, that the process of creative destruction can be the essence of economic growth as well.

In addition to providing an interesting description of the process of economic growth and highlighting the importance of the market structure, the process of creative destruction is important because it also brings political economy interactions to the fore of the question of economic growth. If economic growth will take place via creative destruction, it will create losers, in particular, the incumbents who are currently enjoying profits and rents. Since we expect incumbents to be politically powerful, this implies that many economic systems will create powerful barriers against the process of economic growth. Political economy of growth is partly about understanding the opposition of certain firms and individuals to technological progress and studying whether this opposition will be successful.

There is another, perhaps more surprising, implication of the analysis in this subsection. This relates to *the business stealing effect*, which is closely related to the replacement effect. The entrant, by replacing the incumbent, is also stealing the business of the incumbent. The above discussion suggests that this business stealing effect helps closing the gap between the private and the social values of innovation. It is also possible, however, for the business stealing effect to lead to *excessive innovation* by the entrant. To see the possibility of excessive innovation, let us first look at the total surplus gain from an innovation starting with the monopolist. Suppose to simplify the discussion that the innovation in question is drastic, so that if the entrant undertakes this innovation, it can set the unconstrained monopoly price p^M as given by (12.2) above. Therefore, the social value of innovation is $\widehat{\mathcal{S}}_1^I$ as given by (12.5).

Proposition 12.4. *It is possible that*

$$\widehat{\mathcal{S}}_1^I < \hat{\pi}_1^I,$$

so that the entrant has excessive incentives to innovate.

Proof. See Exercise 12.8. □

Intuitively, the social planner values the profits made by the monopolist, since these are part of the "producer surplus". In contrast, the entrant only values the profits that it will make if it undertakes the innovation. This is the essence of the business stealing effect and creates the possibility of excessive innovations. This result is important because it points out that, in general, it is not clear whether the equilibrium will involve too little or too much innovation. Whether or not it does so depends on how strong the business stealing effect is relative to the appropriability effect discussed above.

12.4. The Dixit-Stiglitz Model and "Aggregate Demand Externalities"

The analysis in the previous section focused on the private and the social values of innovations in a partial equilibrium setting. In growth theory, most of our interest will be in general equilibrium models of innovation. This requires a tractable model of industry equilibrium, which can then be embedded in a general equilibrium framework. The most widely-used model of industry equilibrium is the model developed by Dixit and Stiglitz (1977) and Spence (1976), which captures many of the key features of Chamberlin's (1933) discussion of monopolistic competition. Chamberlin (1933) suggested that a good approximation to the market structure of many industries is one in which each firm faces a downward sloping demand curve (thus has some degree of monopoly power), but there is also free entry into the industry, so that each firm (or at the very least, the marginal firm) makes zero profits.

The distinguishing feature of the Dixit-Stiglitz-Spence model (or Dixit-Stiglitz model for short) is that it allows us to specify a structure of preferences that leads to constant monopoly markups. This turns out to be a very convenient feature in many growth models, though it also implies that this model may not be particularly well suited to situations in which market structure and competition affect monopoly markups. In this section, we present a number of variants of the Dixit Stiglitz model and emphasize its advantages and shortcomings.

12.4.1. The Dixit-Stiglitz Model with a Finite Number of Products. Consider a static economy that admits a representative household with preferences given by

$$U(c_1, ..., c_N, y) = u(C, y), \tag{12.7}$$

where

$$C \equiv \left(\sum_{i=1}^{N} c_i^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}} \tag{12.8}$$

is a consumption index of N differentiated "varieties," $c_1, ..., c_N$, of a particular good and y stands for a generic good, representing all other consumption. The function $u(\cdot, \cdot)$ is strictly increasing and continuously differentiable in both of its arguments and is jointly strictly concave. The parameter ε in (12.8) represents the *elasticity of substitution* between the differentiated varieties and we assume that $\varepsilon > 1$. The key feature of (12.8) is that it features *love-for-variety*, meaning that the greater is the number of differentiated varieties that the individual consumes, the higher is his utility. The aggregator over the different consumption varieties in (12.8) will appear in many different models of technological change and economic growth in the remainder of the book. We will refer to it as a *Dixit-Stiglitz aggregator* or more often, as a *CES aggregator* (where CES stands for constant elasticity of substitution).

More specifically, consider the case in which

$$c_1 = ... = c_N = \frac{\bar{C}}{N},$$

so that the individual purchases a total of $\bar{C}$ units of differentiated varieties, distributed equally across all N varieties. Substituting this into (12.7) and (12.8), we obtain

$$U\left(\frac{\bar{C}}{N}, ..., \frac{\bar{C}}{N}, y\right) = u\left(N^{\frac{1}{\varepsilon-1}}\bar{C}, y\right),$$

which is strictly increasing in N (since $\varepsilon > 1$) and implies that for a fixed total $\bar{C}$ units of differentiated commodities, the larger is the number of varieties over which this total number of units are distributed, the higher is the utility of the individual. This is the essence of the love-for-variety utility function. What makes this utility function convenient is not only this feature, but also the fact that individual demands take a very simple iso-elastic form. To derive the demand for individual varieties, let us normalize the price of the y good to 1 and denote the price of variety i by p_i and the total money income of the individual by m. Then the budget constraint of the individual takes the form

$$\sum_{i=1}^{N} p_i c_i + y \leq m. \tag{12.9}$$

The maximization of (12.7) subject to (12.9) implies the following first-order condition between varieties:

$$\left(\frac{c_i}{c_{i'}}\right)^{-\frac{1}{\varepsilon}} = \frac{p_i}{p_{i'}} \text{ for any } i, i'.$$

To write this first-order condition in a more convenient form, let P denote the price index corresponding to the consumption index C. Then combining the first-order conditions, we obtain

$$\left(\frac{c_i}{C}\right)^{-\frac{1}{\varepsilon}} = \frac{p_i}{P} \text{ for } i = 1, ..., N. \tag{12.10}$$

This first-order condition for the consumption index immediately implies that (see Exercise 12.10):

$$P \equiv \left(\sum_{i=1}^{N} p_i^{1-\varepsilon}\right)^{\frac{1}{1-\varepsilon}}. \tag{12.11}$$

Since P is the price index corresponding to the consumption index C, it is typically referred to as the *ideal price index*. In many circumstances, it will be convenient to choose this ideal price index as the numeraire. Note, however, that we cannot set this as the price index in this particular instance, since we have already written the budget constraint in terms of money income, m, and normalized the price of good y to 1.

The choice between C and y is straightforward in this case and boils down to the maximization of the utility function $u(C, y)$ subject to the budget constraint

$$PC + y \leq m,$$

where we combined (12.10) and (12.11) with the budget constraint, (12.9), in order to obtain a budget constraint expressed in terms of C and y. Now this maximization yields the following intuitive first-order condition:

$$\frac{\partial u(C, y)/\partial y}{\partial u(C, y)/\partial C} = \frac{1}{P},$$

which assumes that the solution is interior, an assumption we maintain throughout this section to simplify the discussion. The strict joined concavity of u, combined with the budget constraint, implies that this first-order condition can be expressed as

$$\begin{aligned} y &= g(P, m) \\ C &= \frac{m - g(P, m)}{P}, \end{aligned} \tag{12.12}$$

for some function $g(\cdot, \cdot)$.

Next, let us consider the production of the varieties. Suppose that each variety can only be produced by a single firm, who is thus an effective monopolist for this particular commodity. Also assume that all monopolists are owned by the representative household and maximize profits.

Recall that the marginal cost of producing each of these varieties is constant and equal to ψ. Let us first write down the profit maximization problem of one of these monopolists:

$$\max_{p_i} \left(\left(\frac{p_i}{P} \right)^{-\varepsilon} C \right) (p_i - \psi), \tag{12.13}$$

where the term in the first parentheses is c_i (recall (12.10)) and the second is the difference between price and marginal cost. The complication in this problem comes from the fact that P and C are potentially functions of p_i. However, for N sufficiently large, the effect of p_i on these can be ignored and the solution to this maximization problem becomes very simple (see Exercise 12.11). This enables us to derive the optimal price in the form of a constant markup over marginal cost:

$$p_i = p = \frac{\varepsilon}{\varepsilon - 1} \psi \text{ for each } i = 1, ..., N. \tag{12.14}$$

This result follows because when the effect of firm i's price choice on P and C are ignored, the demand function facing the firm, (12.10), is iso-elastic with an elasticity $\varepsilon > 1$. Since each firm charges the same price, the ideal price index P can be computed as

$$P = N^{-\frac{1}{\varepsilon - 1}} \frac{\varepsilon}{\varepsilon - 1} \psi. \tag{12.15}$$

Using this expression the profits for each firm are obtained as

$$\pi_i = \pi = N^{-\frac{\varepsilon}{\varepsilon - 1}} C \frac{1}{\varepsilon - 1} \psi \text{ for each } i = 1, ..., N.$$

Profits are decreasing in the price elasticity for the usual reasons. In addition, profits are increasing in C because this is the total amount of expenditure on these differentiated goods, and they are decreasing in N, since for given C a larger number of varieties means less spending on each variety.

However, the total impact of N on profits can be positive. To see this, let us substitute for P from (12.15) to obtain

$$C = N^{\frac{1}{\varepsilon - 1}} \frac{\varepsilon - 1}{\varepsilon \psi} \left(m - g \left(N^{-\frac{1}{\varepsilon - 1}} \frac{\varepsilon}{\varepsilon - 1} \psi, m \right) \right)$$

and

$$\pi = \frac{1}{\varepsilon N}\left(m - g\left(N^{-\frac{1}{\varepsilon-1}}\frac{\varepsilon}{\varepsilon-1}\psi, m\right)\right).$$

It can be verified that depending on the form of the $g(\cdot)$ function, which in turn depends on the shape of the utility function u in (12.7), profits can be increasing in the number of varieties (see Exercise 12.12). This may at first appear somewhat surprising: typically, we expect a greater number of competitors to reduce profits. But the love-for-variety effect embedded in the Dixit-Stiglitz preferences creates a countervailing effect, which is often referred to as an *aggregate demand externality* in the macroeconomics literature. The basic idea is that a higher N raises the utility from consuming each of the varieties because of the love-for-variety effect. The impact of the entry of a particular variety (or the impact of the increase in the production of a particular variety) on the demand for other varieties is a pecuniary externality. This pecuniary externality will play an important role in many of the models of endogenous technological change and we will encounter it again in models of poverty traps in Chapter 21.

12.4.2. The Dixit-Stiglitz Model with a Continuum of Products. As discussed in the last subsection and analyzed further in Exercise 12.12, when N is finite, the equilibrium in which each firm charges the price given by (12.14) may be viewed as an approximation (where each firm only has a small effect on the ideal price index and thus ignores this effect). An alternative modeling assumption would be to assume that there is a continuum of varieties. When there is a continuum of varieties, (12.14) is no longer an approximation. Moreover, such a model will be more tractable because the number of firms, N, need not be an integer. For this reason, the version of the Dixit-Stiglitz model with a continuum of products is often used in the literature and will also be used in the rest of this book.

This version of the model is very similar to the one discussed in the previous subsection, except that the utility function of the representative household now takes the form

$$U\left([c_i]_{i=0}^{N}, y\right) = u(C, y),$$

where now

$$C \equiv \left(\int_0^N c_i^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}}$$

and N denotes the measure of varieties. The budget constraint facing the representative household is

$$\int_0^N p_i c_i di + y \leq m.$$

An identical analysis leads to utility maximizing decisions given by (12.10) and to the ideal price index

$$P = \left(\int_0^N p_i^{1-\varepsilon} di\right)^{\frac{1}{1-\varepsilon}}.$$

Using the definition of the ideal price index and (12.10), we obtain the budget constraint as

$$PC + y \leq m.$$

Equation (12.12) then determines y and C. Since the supplier of each variety is infinitesimal, their prices have no effect on P and C. Consequently, the profit-maximizing pricing decision in (12.14) obtains exactly, and each firm has profits given by

$$\pi = \frac{1}{\varepsilon N}\psi\left(m - g\left(N^{-\frac{1}{\varepsilon-1}}\frac{\varepsilon}{\varepsilon-1}\psi, m\right)\right),$$

where $g(\cdot)$ is defined as (12.12) in the previous subsection.

Now using this expression, we can endogenize the entry margin. Imagine, for example, that there is an infinite number of potential different varieties, and a particular firm can adopt one of these varieties at some fixed cost $\mu > 0$ and enter the market. Consequently, as in the Chamberlin's (1933) model of monopolistic competition, in equilibrium all varieties will make zero profits because of free entry. This implies that the following zero-profit condition has to hold for all entrants and thus for all varieties:

$$\frac{1}{\varepsilon N}\psi\left(m - g\left(N^{-\frac{1}{\varepsilon-1}}\frac{\varepsilon}{\varepsilon-1}\psi, m\right)\right) = \mu. \tag{12.16}$$

As we will see in the next chapter, there is an intimate link between entry by new products (firms) and technological change. Leaving a detailed discussion of this connection to the next chapter, here we can ask a simpler question: do the aggregate demand externalities imply that there is too little entry in a model of this sort? The answer is not necessarily. While the aggregate demand externalities imply that firms do not take into account the positive benefits their entry creates on other firms, the business stealing effect identified above is still present and implies that entry may also reduce the demand for existing products. Thus, in general, whether there is too little or too much entry in models of product differentiation depends on the details of the model and the values of the parameters (see Exercise 12.13).

12.4.3. Objectives of Monopolistic Firms. It is also useful to briefly discuss the objectives of monopolistically competitive firms in the Dixit-Stiglitz model (and related models). Throughout, I follow the industrial organization and the growth literatures and assume that all firms maximize profits, even when they are owned by a "representative household". One may object to this assumption, noting that the representative household would be better-off if firms pursued a non-profit maximizing strategy. However, profit maximization is still the right objective function for firms. This is because an allocation in which firms do not maximize profits (instead act in the way that a social planner would like them to act) cannot be an equilibrium. To see this, note that the representative household is itself a price taker—for example, it represents a large number of identical price-taking households. If some firms did not maximize profits, then the households would refuse to hold the stocks of these firms in their portfolios and there would be entry by other profit-maximizing firms instead. Thus, as long as the representative household or the set of households on the consumer side act as price-takers (as we have assumed to be the case throughout), profit maximization is the only consistent strategy for the monopolistically competitive firms.

The only caveat to this arises from a different type of deviation on the production side. In particular, a single firm may buy all of the monopolistically competitive firms and act as the single producer in the economy. This firm might then ensure an allocation that makes consumers better-off relative to the equilibrium allocation considered here. Nevertheless, I ignore this type of deviation for two reasons. First, as usual we are taking the market structure as given, and the market structure here is monopolistic competition not pure monopoly. A single firm owning all production units would correspond to an entirely different market structure, with much less realism and relevance to the issues studied here. Second, in a related model Acemoglu and Zilibotti (1997) show that a single firm owning all production units would not be an equilibrium either because the possibility of free entry would encourage the entry of profit-maximizing firms at the margin, disrupting the equilibrium with the single producer. Given these considerations, throughout the book I assume that firms are profit maximizing.

12.4.4. Limit Prices in the Dixit-Stiglitz Model. We have already encountered how limit prices can arise in the previous section, when process innovations are non-drastic relative to the existing technology. Another reason why limit prices can arise is because of the presence of a "competitive" fringe of firms that can imitate the technology of monopolists. This type of competitive pressure from the fringe of firms is straightforward to incorporate into the Dixit-Stiglitz model and will be useful in later chapters as a way of parameterizing competitive pressures.

Let us assume that there is a large number of fringe firms that can imitate the technology of the incumbent monopolists. Let us assume that this imitation is equivalent to the production of a similar good and is not protected by patents. It may be reasonable to assume that the imitating firms will be less efficient than those who have invented the variety and produced it for a while. A simple way of capturing this would be to assume that while the monopolist creates a new variety by paying the fixed cost μ and then having access to a technology with the marginal cost of production of ψ, the fringe of firms do not pay any fixed costs, but can only produce with a marginal cost of $\gamma\psi$, where $\gamma > 1$.

Similar to the analysis in the previous section, if $\gamma \geq \varepsilon/(\varepsilon - 1)$, then the fringe is sufficiently unproductive that they cannot profitably produce even when the monopolists charge the unconstrained monopoly price given in (12.14). Instead, when $\gamma < \varepsilon/(\varepsilon - 1)$, the monopolists will be forced to charge a limit price. The same arguments as in the previous section establish that this limit price must take the form

$$p = \gamma\psi < \frac{\varepsilon}{\varepsilon - 1}\psi.$$

It is then straightforward to see that the entry condition that determines the number of varieties in the market will change to

$$N^{-\frac{\varepsilon}{\varepsilon-1}}\gamma\psi\left(m - g\left(N^{-\frac{1}{\varepsilon-1}}\frac{\varepsilon}{\varepsilon-1}\psi\right)\right) = \mu.$$

12.4.5. Limitations. The most important limitation of the Dixit-Stiglitz model is the feature that makes it tractable: the constancy of markups as in equation (12.14). In particular, the model implies that the markup of each firm is independent of the number of varieties in the market. But this is a very special feature. Most industrial organization models imply that markups over marginal cost are declining in the number of competing products (see, for example, Exercise 12.14). While plausible, this makes endogenous growth models less tractable, because in many classes of models, endogenous technological change will correspond to a steady increase in the number of products N. If markups decline towards zero as N increases, this would ultimately stop the process of innovation and thus prevent sustained economic growth. The alternative would be to have a model in which some other variable, perhaps capital, simultaneously increases the potential markups that firms can charge. While such models can be developed, they are more difficult than the standard Dixit-Stiglitz setup. For this reason, the literature typically focuses on Dixit-Stiglitz specifications.

12.5. Individual R&D Uncertainty and the Stock Market

The final issue we will discuss in this chapter involves uncertainty in the research process. As discussed at the beginning of the chapter, it is reasonable to presume that the output of research will be uncertain. This implies that individual firms undertaking research will face a stochastic revenue stream. When individuals are risk averse, this may imply that there should be a risk premium associated with such stochastic streams of income. This is not necessarily the case, however, when the following three conditions are satisfied:

(1) there are many firms involved in research;
(2) the realization of the uncertainty across firms is independent;
(3) consumers and firms have access to a "stock market," where each consumer can hold a *balanced portfolio* of various research firms.

In many of the models we study in the next two chapters, firms will face uncertainty (for example, regarding whether their R&D will be successful or how long their monopoly position will last), but the three conditions outlined here will be satisfied. When this is the case, even though each firm's revenue is risky, the balanced portfolio held by the representative the consumer will have deterministic returns. Here we illustrate this with a simple example.

EXAMPLE 12.2. Suppose that the representative household has a utility function over consumption given by $u(c)$, where $u(\cdot)$ is strictly increasing, continuously differentiable and strictly concave, so that individual is risk averse. Moreover, let us assume that $\lim_{c\to 0} u'(c) = \infty$, so that the marginal utility of consumption at zero is very high. The individual starts with an endowment equal to $y > 0$. This endowment can be consumed or it can be invested in a risky R&D project. Imagine that the R&D project is successful with probability p and will have a return equal to $1+R > 1/p$ per unit of investment. It is unsuccessful with probability $1-p$, in which case it will have a zero return. When this is the only project available, the individual would be facing consumption risk when it invests in this project. In particular, the maximization problem that determines how much he should invest will be a solution to the following expected utility maximization

$$\max_x (1-p)\, u(y-x) + pu(y+Rx).$$

The first-order condition of this problem implies that the optimal amount of investment in the risky research activity will be given by:

$$\frac{u'(y-x)}{u'(y+Rx)} = \frac{pR}{1-p}.$$

The assumption $\lim_{c\to 0} u'(c) = \infty$ implies that $x < y$, thus less than the full endowment of the individual will be invested in the research activity, even though this is a positive expected return project. Intuitively, the individual requires a risk premium to bear the consumption risk associated with the risky investment.

Next imagine a situation in which many different firms can independently invest in similar risky research ventures. Suppose that the success or failure of each project is independent of the others. Imagine that the individual invests an amount x/N in each of N projects. The Strong Law of Large Numbers implies that as $N \to \infty$, a fraction p of these projects will be successful and the remaining fraction $1-p$ will be unsuccessful. Therefore, the individual will receive (almost surely) a utility of

$$u(y + (p(1+R) - 1)\, x).$$

Since $1+R > 1/p$, this is strictly increasing in x, and implies that the individual would prefer to invest all of its endowment in the risky projects, i.e., $x = y$. Therefore, the ability to hold a balanced portfolio of projects with independently disputed returns allows the individual to diversify the risks and act in a risk-neutral manner. A similar logic will apply in many of the models we will study in the next three chapters; even though individual firms will have stochastic returns, the representative household will hold a balanced portfolio of all the firms in the economy and thus will have risk-neutral preferences in the aggregate. This will imply that the objective of each firm will be to maximize expected profits (without a risk premium).

12.6. Taking Stock

This chapter has reviewed a number of conceptual and modeling issues related to the economics of research and development. We have introduced the distinction between process and product innovations, macro and micro innovations, and also discussed the concept of the innovation possibilities set and the importance of the non–rivalry of ideas.

We have also seen why ex post monopoly power is important to create incentives for research spending, how incentives to undertake innovations differ between competitive firms and monopolies, and how these compare to the social value of innovation. In this context, we have emphasized the importance of the appropriability effect, which implies that the private value of innovation often falls short of the social value of innovation, because even with ex post monopoly power an innovating firm will not be able to appropriate the entire consumer surplus created by a better product or a cheaper process. We have also encountered the Arrow's replacement effect, which implies that unless they have a cost advantage, incumbent monopolists will have weaker incentives for innovation than the entrants. Despite the appropriability and the replacement effects, the amount of innovation in equilibrium can be excessive, because of another, countervailing force, the business stealing effect, which encourages firms to undertake innovations in order to become the new monopolist and take over ("steal") the monopoly rents. Therefore, whether there is too little or too much innovation in equilibrium depends on the market structure and the parameters of the model.

This chapter has also introduced the Dixit-Stiglitz-Spence (or for short the Dixit-Stiglitz) model, which will play an important role in the analysis of the next few chapters. This model enables a very tractable approach to Chamberlin type of monopolistic competition, where each firm has some monopoly power, but free entry ensures that all firms (or the marginal entrants) make zero profits. The Dixit-Stiglitz model is particularly tractable because the markup charged by monopolists is independent of the number of competing firms. This makes it an ideal model to study endogenous growth, because it will enable innovation to remain profitable even when the number of products or the number of machines increase continuously.

12.7. References and Literature

The literature on R&D in industrial organization is vast, and our purpose in this chapter has not been to review this literature, but to highlight the salient features that will be used in the remainder of the book. The reader who is interested in this area can start with Tirole (1990, Chapter 10), which contains an excellent discussion of the contrast between private and the social values of innovation. It also provides an excellent introduction to patent races, which we will encounter in Section 14.4 in Chapter 14. A more up-to-date reference that surveys the recent developments in the economics of innovation is Scotchmer (2005).

The classic reference on the private and social values of innovation is Arrow (1962). Schumpeter (1943) was the first to emphasize the role of monopoly in R&D and innovation. The importance of monopoly power for innovation and the indications of the non-rival nature of ideas are discussed in Romer (1990, 1993) and Jones (2006). Most of the industrial organization literature also emphasizes the importance of ex post monopoly power and patent systems in providing incentives for innovation. See, for example, Scotchmer (2005). This perspective has recently been criticized by Boldrin and Levine (2003).

The idea of creative destruction was also originally developed by Schumpeter. Models of creative destruction in the industrial organization literature include Reinganum (1983, 1985). Similar models in the growth literature are developed in Aghion and Howitt (1992, 1998).

Chamberlin (1933) is the classic reference on monopolistic competition. The Dixit-Stiglitz model is developed in Dixit and Stiglitz (1977) and is also closely related to Spence (1976). This model was first used for an analysis of R&D in Dasgupta and Stiglitz (1979). An excellent exposition of the Dixit-Stiglitz model is provided in Matsuyama (1995). Tirole (1990, Chapter 7) also discusses the Dixit-Stiglitz-Spence model as well as other models of product innovation, including the Salop model, due to Salop (1979), which is presented in Exercise 12.14.

An excellent general discussion of issues of innovation and the importance of market size and profit incentives is provided in Schmookler (1966). Recent evidence on the effect of market size and profit incentives on innovation is discussed in Popp (2002), Finkelstein (2003) and Acemoglu and Linn (2004).

Mokyr (1990) contains an excellent history of innovation. Freeman (1982) also provides a survey of the qualitative literature on innovation and discusses the different types of innovations.

In this chapter and the rest of this part of the book, we will deal with monopolistic environments, where the appropriate equilibrium concept is not the competitive equilibrium, but one that incorporates game-theoretic interactions. Throughout the games we will study in this book will have complete information, thus the appropriate notion of equilibrium is the standard Nash equilibrium concept or when the game is multi-stage or dynamic, it is the subgame perfect Nash equilibrium. In these situations, equilibrium always refers to a Nash equilibrium or a subgame perfect Nash equilibrium, and we typically do not add the additional "Nash" qualification. We presume that the reader is familiar with these concepts. A quick introduction to the necessary game theory is provided in the Appendix of Tirole (1990), and a more detailed treatment can be found in Fudenberg and Tirole (1994), Myerson (1995) and Osborne on Rubinstein (1994).

12.8. Exercises

Exercise 12.1. Derive equation (12.2).

Exercise 12.2. Prove Proposition 12.1. In particular:

(1) Show that even if $p^M = \psi$, the unique (Nash) equilibrium involves $q_1 = D\left(p^M\right)$ and $q_j = 0$ for all $j > 1$. Why is this?
(2) Show that when $p^M > \psi$, any price $p_1 > \psi$ or $p_1 < \psi$ cannot be profit-maximizing. Show that there cannot be an equilibrium in which $p_1 = \psi$ and $q_j > 0$ for some $j > 1$ [Hint: find a profitable deviation for firm 1].
(3) Prove that $\hat{\pi}_1^I > \pi_1^I$.

Exercise 12.3. Derive equation (12.5). Using these relationships, prove Proposition 12.2.

Exercise 12.4. Prove that if $\hat{\pi}_1^I > 0$, then $\widehat{\mathcal{S}}_1^I > 0$ (where these terms are defined in Proposition 12.2).

Exercise 12.5. Consider the model in Section 12.3, and suppose that there is no patent protection for the innovating firm. The firm can undertake two different types of innovations at the same cost η. The first is a general technological improvement, which can be copied by all firms. It reduces the marginal cost of production to $\lambda^{-1}\psi$. The second is *specific* to the needs of the current firm and cannot be copied by others. It reduces the marginal cost of production by $\lambda' < \lambda$. Show that the firm would never adopt the λ technology, but may adopt λ' technology. Calculate the difference in the social values generated by these two technologies.

Exercise 12.6. Prove Proposition 12.3. In particular, verify that the conclusion is true even with limit pricing, i.e., $\Delta\hat{\pi}_1^I < \pi_1^I$.

Exercise 12.7. Consider the model in Section 12.3 with an incumbent monopolist and an entrant. Suppose that the cost of innovation for the incumbent is μ, while for the entrant it is $\chi\mu$, where $\chi \geq 1$.

(1) Explain why we may have $\chi > 1$.
(2) Show that there exists $\bar{\chi} > 1$ such that if $\chi < \bar{\chi}$, the entrant has greater incentives to undertake innovation, and if $\chi > \bar{\chi}$, the incumbent has greater incentives to undertake innovation.
(3) What is the effect of the elasticity of demand on the relative incentives of the incumbent and the entrant to undertake innovation.

Exercise 12.8.
(1) Prove Proposition 12.4 by providing an example in which there is excessive innovation incentives.
(2) What factors make excessive innovation more likely?

Exercise 12.9. The discussion in the text presumed a particular form of patent policy, which provided ex post monopoly power to the innovator. An alternative intellectual property right policy is licensing, where firms that have made an innovation can license the rights to use this innovation to others. This exercise asks you to work through the implications of this type of licensing. Throughout, we think of the licensing stage as follows: the innovator can make a take-it-or-leave-it-offer to one or many firms so that they can buy the rights to use the innovation (and produce as many units of the output as they like) in return for some licensing fee ν.

(1) Consider the competitive environment we started with and show that if firm 1 is allowed to license its innovation to others, this can never raise its profits and it can never increase its incentives to undertake the innovation. Provide an intuition for this result.
(2) Now modify the model, so that each firm has a strictly convex and increasing cost of producing, $\psi_1(q)$, and also has to pay a fixed cost of $\psi_0 > 0$ to be active (so that the average costs take the familiar inverse U shape). Show that licensing can be beneficial for firm 1 in this case and therefore increase incentives to undertake the innovation. Explain why the results differ between the two cases.

Exercise 12.10. Derive the expression for the ideal price index, (12.11), from (12.10) and the definition of the consumption index C.

Exercise 12.11. Consider the maximization problem in (12.13) and write down the first-order conditions taking into account the impact of p_i on P and C. Show that as $N \to \infty$, the solution to this problem converges to (12.14).

Exercise 12.12. In the Dixit-Stiglitz model, determine the conditions on the function $v(\cdot)$ such that an increase in N raises the profits of a monopolist.

Exercise 12.13. Suppose that $U(C, y) = Cv + (y)$, where $v(y) = y^{1-\alpha}/(1-\alpha)$ with $\alpha \in (0,1)$. Suppose also that new varieties can be introduced at the fixed cost μ.

(1) Consider the allocation determined by a social planner also controlling prices. Characterize the number of varieties that a social planner would choose in order to maximize the utility of the representative household in this case.
(2) Suppose that prices are given by (12.14). Characterize the number of varieties that the social planner would choose in order to maximize utility of the representative household in this case.
(3) Characterize the equilibrium number of varieties (at which all monopolistically competitive variety producers makes zero profits) and compare this with the answers to the previous two parts. Explain the sources of differences between the equilibrium and the social planner's solution in each case.

Exercise 12.14. This exercise asks you to work through the Salop (1979) model of product differentiation, which differs from the Dixit-Stiglitz model in that equilibrium markups are declining in the number of firms. Imagine that consumers are located uniformly around a circle with perimeter equal to 1. The circle indexes both the preferences of heterogeneous consumers and the types of goods. The point where the consumer is located along the circle corresponds to the type of product that he most prefers. When a consumer at point x around the circle consumes a good of type z, his utility is

$$R - t\,|z - x| - p,$$

while if he chooses not to consume, his utility is 0. Here R can be thought of as the reservation utility of the individual, while t parameterizes the "transport" costs that the individual has to pay in order to consume a good that is away from his ideal point along the circle. Suppose that each firm has a marginal cost of ψ per unit of production

(1) Imagine a consumer at point x, with the two neighboring firms at points $z_1 > x > z_2$. As long as the prices of these firms are not much higher than those further a far, the consumer will buy from one of these two firms. Denote the prices of these two firms by p_1 and p_2. Show that the price difference that would make the consumer indifferent between purchasing from the two firms satisfies

$$p_1 - p_2 = (2x - z_1 - z_2)\,t$$

with

$$t\,(z_1 - x) + p_1 \leq R.$$

(2) Suppose that p_1 and p_2 satisfy the above inequality. Then show that all $x' \in [z_2, x)$ strictly prefer to buy from firm 2 and all $x' \in (x, z_1]$ strictly prefer to buy from firm 1.

(3) Now assume that there are three firms along the circle at locations $z_1 > z_2 > z_3$. Show that firm 2's profits are given by

$$\pi_2\,(p_1, p_2, p_3 \mid z_1, z_2, z_3) = (p_2 - \psi)\left(\frac{p_1 - p_2}{2t} + \frac{z_1 - z_2}{2} + \frac{p_3 - p_2}{2t} + \frac{z_2 - z_3}{2}\right)$$

and calculate its profit maximizing price.

(4) Now look at the location choice of firm 2. Suppose that $p_1 = p_3$. Show that it would like to locate half way between z_1 and z_3.

(5) Prove that in a symmetric equilibrium with N firms, the distance between any two firms will be $1/N$.

(6) Show that the symmetric equilibrium price with N equity-distant firms is

$$p = \psi + \frac{t}{N}.$$

(7) Explain why the markup here is a decreasing function of the number of firms, whereas it was independent of the number of firms in the Dixit-Stiglitz model.

CHAPTER 13

Expanding Variety Models

As emphasized in the previous chapter, the key to understanding endogenous technological progress is that R&D is a purposeful activity, undertaken for profits, and the knowledge (machines, blueprints, or new technologies) that it generates increases the productivity of existing factors of production. The first endogenous technological change models were formulated by Romer (1987 and 1990). Different versions have been analyzed by Segerstrom, Anant and Dinopoulos (1990), Grossman and Helpman (1991a,b), Aghion and Howitt (1992). Some of those will be discussed in the next chapter.

The simplest models of endogenous technological change are those in which R&D expands the variety of inputs or machines used in production. In this chapter, we focus on models with expanding input varieties; research will lead to the creation of new varieties of inputs (machines) and a greater variety of inputs will increase the "division of labor," raising the productivity of final good firms. This can therefore be viewed as a form of *process innovation*. An alternative, formulated and studied by Grossman and Helpman (1991a,b), focuses on *product innovation*. In this model, research leads to the invention of new goods, and because individuals have love-for-variety, they derive greater utility when they consume a greater variety of products. Consequently "real" income increases as a result of these product innovations. The comparison of Grossman-Helpman's model of product innovation with our baseline model of process innovation will show that the two models are mathematically very similar (though the model of product innovation is slightly more involved and is thus treated at the end of this chapter).

In all of these models, and also in the models of quality competition we will see below, we will use the Dixit-Stiglitz constant elasticity structure introduced in the previous chapter.

13.1. The Lab Equipment Model of Growth with Input Varieties

We start with a particular version of the endogenous growth model with expanding varieties of inputs and an R&D technology that only uses output for creating new inputs. This is sometimes referred to as the *lab equipment* model, since all that is required for research is investment in equipment or in laboratories—rather than the employment of skilled or unskilled workers or scientists.

13.1.1. Demographics, Preferences and Technology. Imagine an infinite-horizon economy in continuous time admitting a representative household with preferences

$$\int_0^\infty \exp(-\rho t) \frac{C(t)^{1-\theta} - 1}{1-\theta} dt. \tag{13.1}$$

There is no population growth, and the total population of workers, L, supplies labor inelastically throughout. We also assume, as discussed in the previous chapter, that the representative household owns a balanced portfolio of all the firms in the economy. Alternatively, we can think of the economy as consisting of many households with the same preferences as the representative household and each household holding a balanced portfolio of all the firms.

The unique consumption good of the economy is produced with the following aggregate production function:

$$Y(t) = \frac{1}{1-\beta}\left[\int_0^{N(t)} x(\nu,t)^{1-\beta} d\nu\right] L^\beta, \tag{13.2}$$

where L is the aggregate labor input, $N(t)$ denotes the different number of varieties of inputs (machines) available to be used in the production process at time t, and $x(\nu,t)$ is the total amount of input (machine) type ν used at time t. We assume that x's depreciate fully after use, thus they can be interpreted as generic inputs, as intermediate goods, as machines, or even as capital as long as we are comfortable with the assumption that there is immediate depreciation. The assumption that the inputs or machines are "used up" in production or depreciate immediately after being used makes sure that the amounts of inputs used in the past do not become additional state variables, and simplifies the exposition of the model (though the results are identical without this assumption, see Exercise 13.22). Nevertheless, we refer to the inputs as "machines," which makes the economic interpretation of the problem easier.

The term $(1-\beta)$ in the denominator is included for notational simplicity. Notice that for given $N(t)$, which final good producers take as given, equation (13.2) exhibits constant returns to scale. Therefore, final good producers are competitive and are subject to constant returns to scale, justifying our use of the aggregate production function to represent their production possibilities set.

One can also write (13.2) in the following form

$$Y(t) = \frac{1}{1-\beta}\tilde{\mathbf{X}}(t)^{1-\beta} L^\beta,$$

where

$$\tilde{\mathbf{X}}(t) \equiv \left[\int_0^{N(t)} x(\nu,t)^{\frac{\varepsilon_\beta - 1}{\varepsilon_\beta}} d\nu\right]^{\frac{\varepsilon_\beta}{\varepsilon_\beta - 1}},$$

with $\varepsilon_\beta \equiv 1/\beta$ as the elasticity of substitution between inputs. This form emphasizes both the constant returns to scale properties of the production function and the continuity between the model here and the Dixit-Stiglitz model of the previous chapter.

The resource constraint of the economy at time t is

$$C(t) + X(t) + Z(t) \leq Y(t), \tag{13.3}$$

where $X(t)$ is investment or spending on inputs at time t and $Z(t)$ is expenditure on R&D at time t, which comes out of the total supply of the final good.

We next need to specify how quantities of machines are created and how new machines are invented. Let us first assume that once the blueprint of a particular input is invented, the research firm can create one unit of that machine at marginal cost equal to $\psi > 0$ units of the final good. We also assume the following form for *innovation possibilities frontier*, where new machines are created as follows:

$$\dot{N}(t) = \eta Z(t), \tag{13.4}$$

where $\eta > 0$, and the economy starts with some initial technology stock $N(0) > 0$. This implies that greater spending on R&D leads to the invention of new machines. Throughout, we assume that there is free entry into research, which means that any individual or firm can spend one unit of the final good at time t in order to generate a flow rate η of the blueprints of new machines. The firm that discovers these blueprints receives a fully-enforced perpetual patent on this machine.

There is no aggregate uncertainty in the innovation process. Naturally, there will be uncertainty at the level of the individual firm, but with many different research labs undertaking such expenditure, at the aggregate level, equation (13.4) holds deterministically.

Given the patent structure specified above, a firm that invents a new machine variety is the sole supplier of that type of machine, say machine of type ν, and sets a price of $p^x(\nu, t)$ at time t to maximize profits. Since machines depreciate after use, $p^x(\nu, t)$ can also be interpreted as a "rental price" or the use and cost of this machine.

The demand for machine of type ν is obtained by maximizing net aggregate profits of the final good sector as given by (13.2) minus the cost of inputs. Since machines depreciate after use and labor is hired on the spot market for its flow services, the maximization problem on the final good sector can be considered for each point in time separately, and simply requires the maximization of the instantaneous profits of a representative final good producer. These instantaneous profits can be obtained by subtracting the total inputs costs, the user costs of renting machines and labor costs, from the value of our production. Therefore, the maximization problem at time t is:

$$\max_{[x(\nu,t)]_{l\nu\in[0,N(t)]},L} \frac{1}{1-\beta}\left[\int_0^{N(t)} x(\nu,t)^{1-\beta} d\nu\right] L^\beta - \int_0^{N(t)} p^x(\nu,t)\, x(\nu,t) d\nu - w(t) L. \tag{13.5}$$

The first-order condition of this maximization problem with respect to $x(\nu, t)$ for any $\nu \in [0, N(t)]$ yields the demand for machines from the final good sector. These demands take the convenient isoelastic form:

$$x(\nu,t) = p^x(\nu,t)^{-1/\beta} L, \tag{13.6}$$

which is intuitive in view of the fact that elasticity of demand for different machine varieties is $\varepsilon_\beta = 1/\beta$ (so that $x(\nu,t) = p^x(\nu,t)^{-\varepsilon_\beta} L$). This equation implies that the demand for machines only depends on the user cost of the machine and on equilibrium labor supply but not on the interest rate, $r(t)$, the wage rate, $w(t)$, or the total measure of available machines, $N(t)$. This feature makes the model very tractable.

Now consider the maximization problem of a monopolist owning the blueprint of a machine of type ν invented at time t. Since the representative household holds a balanced portfolio of all the firms in the economy and there is a continuum of firms, there will be no aggregate uncertainty, so each monopolist's objective is to maximize profits. Consequently, this monopolist chooses an investment plan and a sequence of capital stocks so as to maximize the present discounted value of profits starting from time t. Recalling that the interest rate at time t is $r(t)$ and the marginal cost of producing machines (in terms of the final good) is ψ, the net present discounted value can be written as:

$$V(\nu,t) = \int_t^\infty \exp\left[-\int_t^s r\left(s'\right) ds'\right] \pi(\nu,s)\, ds \tag{13.7}$$

where $\pi(\nu,t) \equiv p^x(\nu,t)x(\nu,t) - \psi x(\nu,t)$ denotes profits of the monopolist producing intermediate ν at time t, $x(\nu,t)$ and $p^x(\nu,t)$ are the profit-maximizing choices for the monopolist, and $r(t)$ is the market interest rate at time t. Alternatively, assuming that the value function is differentiable in time, this could be written in the form of Hamilton-Jacobi-Bellman equations as in Theorem 7.10 in Chapter 7:

$$r(t)\, V(\nu,t) - \dot{V}(\nu,t) = \pi(\nu,t). \tag{13.8}$$

Exercise 13.1 asks you to provide a different derivation of this equation than in Theorem 7.10.

13.1.2. Characterization of Equilibrium. An allocation in this economy is defined by the following objects: time paths of consumption levels, aggregate spending on machines, and aggregate R&D expenditure $[C(t), X(t), Z(t)]_{t=0}^\infty$, time paths of available machine types, $[N(t)]_{t=0}^\infty$, time paths of prices and quantities of each machine and the net present discounted value of profits from that machine, $[p^x(\nu,t), x(\nu,t), V(\nu,t)]_{\nu\in N(t),t=0}^\infty$, and time paths of interest rates and wage rates, $[r(t), w(t)]_{t=0}^\infty$.

An equilibrium is an allocation in which all existing research firms choose $[p^x(\nu,t), x(\nu,t)]_{\nu\in[0,N(t)],t=0}^\infty$ to maximize profits, the evolution of $[N(t)]_{t=0}^\infty$ is determined by free entry, the time paths of interest rates and wage rates, $[r(t), w(t)]_{t=0}^\infty$, are consistent with market clearing, and the time paths of $[C(t), X(t), Z(t)]_{t=0}^{\infty}$ are consistent with consumer optimization. We now characterize the unique equilibrium of this economy.

Let us start with the firm side. Since (13.6) defines isoelastic demands, the solution to the maximization problem of any monopolist $\nu \in [0, N(t)]$ involves setting the same price in every period (see Exercise 13.2):

$$p^x(\nu, t) = \frac{\psi}{1-\beta} \text{ for all } \nu \text{ and } t. \tag{13.9}$$

That is, all monopolists charge a constant rental rate, equal to a mark-up over the marginal cost. Without loss of generality, let us normalize the marginal cost of machine production to $\psi \equiv (1-\beta)$, so that

$$p^x(\nu, t) = p^x = 1 \text{ for all } \nu \text{ and } t.$$

Profit-maximization also implies that each monopolist rents out the same quantity of machines in every period, equal to

$$x(\nu, t) = L \text{ for all } \nu \text{ and } t. \tag{13.10}$$

This gives monopoly profits as:

$$\pi(\nu, t) = \beta L \text{ for all } \nu \text{ and } t. \tag{13.11}$$

The important implication of this equation is that each monopolist sells exactly the same amount of machines, charges the same price and makes the same amount of profits at all time points. This particular feature simplifies the analysis of endogenous technological change models with expanding variety.

Substituting (13.6) and the machine prices into (13.2) yields a final good production function of the form

$$Y(t) = \frac{1}{1-\beta} N(t) L. \tag{13.12}$$

This is one of the main equations of the expanding product or input variety models. It shows that even though the aggregate production function exhibits constant returns to scale from the viewpoint of final good firms (which take $N(t)$ as given), there are *increasing returns to scale* for the entire economy; (13.12) makes it clear that an increase in the variety of machines, $N(t)$, raises the productivity of labor and that when $N(t)$ increases at a constant rate so will output per capita.

The labor demand of the final good sector follows from the first-order condition of maximizing (13.5) with respect to L and implies the equilibrium condition

$$w(t) = \frac{\beta}{1-\beta} N(t). \tag{13.13}$$

Finally, free entry into research implies that at all points in time we must have

$$\eta V(\nu, t) \leq 1, \; Z(\nu, t) \geq 0 \text{ and } (\eta V(\nu, t) - 1) Z(\nu, t) = 0, \text{ for all } \nu \text{ and } t, \tag{13.14}$$

where $V(\nu,t)$ is given by (13.7). To understand (13.14), recall that one unit of final good spent on R&D leads to the invention of η units of new inputs, each with a net present discounted value of profits given by (13.7). This free entry condition is written in the complementary slackness form, since research may be very unprofitable and there may be zero R&D effort, in which case $\eta V(\nu,t)$ could be strictly less than 1. Nevertheless, for the relevant parameter values there will be positive entry and economic growth (and technological *progress*), so we often simplify the exposition by writing the free-entry condition as

$$\eta V(\nu,t) = 1.$$

Note also that since each monopolist $\nu \in [0, N(t)]$ produces machines given by (13.10), and there are a total of $N(t)$ monopolists, the total expenditure on machines is

$$X(t) = (1-\beta)\, N(t)\, L. \tag{13.15}$$

Finally, the representative household's problem is standard and implies the usual Euler equation:

$$\frac{\dot{C}(t)}{C(t)} = \frac{1}{\theta}(r(t) - \rho) \tag{13.16}$$

and the transversality condition

$$\lim_{t\to\infty}\left[\exp\left(-\int_0^t r(s)\,ds\right) N(t)\, V(t)\right] = 0, \tag{13.17}$$

which is written in the "market value" form and requires the value of the total wealth of the representative household, which is equal to the value of corporate assets $N(t)\,V(t)$, not to grow faster than the discount rate (see Exercise 13.3).

In light of the previous equations, we can now define an equilibrium more formally as time paths of consumption, expenditures, R&D decisions and total number of varieties, $[C(t), X(t), Z(t), N(t)]_{t=0}^{\infty}$, such that (13.3), (13.15), (13.16), (13.17) and (13.14) are satisfied; time paths of prices and quantities of each machine and the net present discounted value of profits from that machine, $[p^x(\nu,t), x(\nu,t)]_{\nu\in N(t),t=0}^{\infty}$ that satisfy (13.9) and (13.10), time paths of interest rate and wages such that $[r(t), w(t)]_{t=0}^{\infty}$ (13.13) and (13.16), hold.

We define a *balanced growth path* as an equilibrium path where $C(t), X(t), Z(t)$ and $N(t)$ grow at a constant rate. Such an equilibrium can alternatively be referred to as a "steady state", since it is a steady state in transformed variables (even though the original variables grow at a constant rate). This is a feature of all the growth models and we will throughout use the terms steady state and balanced growth path interchangeably when referring to endogenous growth models.

13.1.3. Balanced Growth Path. A balanced growth path (BGP) requires that consumption grows at a constant rate, say g_C. This is only possible from (13.16) if the interest rate is constant. Let us therefore look for an equilibrium allocation in which

$$r(t) = r^* \text{ for all } t,$$

where "*" refers to BGP values. Since profits at each date are given by (13.11) and since the interest rate is constant, (13.8) implies that $\dot{V}(t) = 0$. Substituting this in either (13.7) or (13.8), we obtain

$$V^* = \frac{\beta L}{r^*}. \tag{13.18}$$

This equation is intuitive: a monopolist makes a flow profit of βL, and along the BGP, this is discounted at the constant interest rate r^*.

Let us next suppose that the (free entry) condition (13.14) holds as an equality, in which case we also have

$$\frac{\eta \beta L}{r^*} = 1$$

This equation pins down the steady-state interest rate, r^*, as:

$$r^* = \eta \beta L$$

The consumer Euler equation, (13.16), then implies that the rate of growth of consumption must be given by

$$g_C^* = \frac{\dot{C}(t)}{C(t)} = \frac{1}{\theta}(r^* - \rho). \tag{13.19}$$

Moreover, it can be verified that the current-value Hamiltonian for the consumer's maximization problem is concave, thus this condition, together with the transversality condition, characterizes the optimal consumption plans of the consumer.

In a balanced growth path, consumption cannot grow at a different rate than total output (see Exercise 13.5), thus we must also have the growth rate of output in the economy is

$$g^* = g_C^*.$$

Therefore, given the BGP interest rate we can simply determine the long-run growth rate of the economy as:

$$g^* = \frac{1}{\theta}(\eta \beta L - \rho) \tag{13.20}$$

We next assume that

$$\eta \beta L > \rho \text{ and } (1-\theta)\eta \beta L < \rho, \tag{13.21}$$

which will ensure that $g^* > 0$ and that the transversality condition is satisfied.

We then obtain:

Proposition 13.1. *Suppose that condition (13.21) holds. Then, in the above-described lab equipment expanding input variety model, there exists a unique balanced growth path in which technology, output and consumption all grow at the same rate, g^*, given by (13.20).*

Proof. The preceding discussion establishes all the claims in the proposition except that the transversality condition holds. You are asked to check this in Exercise 13.7. □

An important feature of this class of endogenous technological progress models is the presence of the *scale effect*, which we encountered in Section 11.4 in Chapter 11: the larger is L, the greater is the growth rate. The scale effect comes from a very strong form of the *market size effect* discussed in the previous chapter. As illustrated there, the increasing returns to scale nature of the technology (for example, as highlighted in equation (13.12)) is responsible for this strong form of the market size effect and thus for the scale effect. We will see in Section 15.5 in Chapter 15 that it is possible to have variants of the current model that feature the market size effect, but not the scale effect.

13.1.4. Transitional Dynamics. It is also straightforward to see that there are no transitional dynamics in this model. To derive this result, let us go back to the value function for each monopolist. Substituting for profits, this gives

$$r(t)\, V(\nu,t) - \dot{V}(\nu,t) = \beta L.$$

The key observation is that positive growth at any point implies that $\eta V(\nu,t) = 1$ for all t. In other words, if $\eta V(\nu,t') = 1$ for some t', then $\eta V(\nu,t) = 1$ for all t (see Exercise 13.4). Now differentiating $\eta V(\nu,t) = 1$ with respect to time yields $\dot{V}(\nu,t) = 0$, which is only consistent with $r(t) = r^*$ for all t, thus

$$r(t) = \eta\beta L \text{ for all } t.$$

This establishes:

Proposition 13.2. *Suppose that condition (13.21) holds. In the above-described lab equipment expanding input-variety model, with initial technology stock* $N(0) > 0$*, there is a unique equilibrium path in which technology, output and consumption always grow at the rate* g^* *as in (13.20).*

At some level, this result is not too surprising. While the microfoundations and the economics of the expanding varieties model studied here are very different from the neoclassical AK economy, the mathematical structure of the model is very similar to the AK model (as most clearly illustrated by the derived equation for output, (13.12)). Consequently, as in the AK model, the economy always grows at a constant rate.

Even though the mathematical structure of the model is similar to the neoclassical AK economy, it is important to emphasize that the economics here is very different. The equilibrium in Proposition 13.2 exhibits *endogenous technological progress*. In particular, research firms spend resources in order to invent new inputs. They do so because, given their patents, they can profitably sell these inputs to final good producers. It is therefore profit incentives that drive R&D, and R&D drives economic growth. We have therefore arrived to our first model in which market-shaped incentives determine the rate at which the technology of the economy evolves over time.

13.1.5. Pareto Optimal Allocations. The presence of monopolistic competition implies that the competitive equilibrium is not necessarily Pareto optimal. In particular, the current model exhibits a version of the *aggregate demand externalities* discussed in the previous chapter and features two sources of potential inefficiencies. First, there is a markup over the marginal cost of production of inputs. Second, the number of inputs produced at any point in time may not be optimal. The first source of inefficiency is familiar from models of static monopoly, while the second emerges from the fact that in this economy the set of traded (Arrow-Debreu) commodities is endogenously determined. This latter source of potential inefficiency relates to the issue of endogenously incomplete markets (there is no way to purchase an input that is not supplied in equilibrium) and will be discussed in greater detail in Section 17.6 in Chapter 17.

To contrast the equilibrium allocations with the Pareto optimal allocations, we set up the problem of the social planner and derive the optimal growth rate. Notice that the social planner will also use the same quantity of all types of machines in production, but because of the absence of a markup, this quantity will be different than in the equilibrium allocation. The social planner will also take into account the effect of an increase in the variety of inputs on the overall productivity in the economy, which monopolists did not because they did not appropriate the full surplus from inventions.

More explicitly, given $N(t)$, the social planner will choose

$$\max_{[x(\nu,t)]_{v\in[0,N(t)]},L} \frac{1}{1-\beta}\left[\int_0^{N(t)} x(\nu,t)^{1-\beta}d\nu\right]L^{\beta}-\int_0^{N(t)}\psi x(\nu,t)d\nu,$$

which only differs from the equilibrium profit maximization problem, (13.5), because the marginal cost of machine creation, ψ, is used as the cost of machines rather than the monopoly price, and the cost of labor is not subtracted. Recalling that $\psi\equiv 1-\beta$, the solution to this program involves

$$x^S(\nu,t)=(1-\beta)^{-1/\beta}L,$$

so that the social planner's output level will be

$$\begin{aligned} Y^S(t) &= \frac{(1-\beta)^{-(1-\beta)/\beta}}{1-\beta}N^S(t)L \\ &= (1-\beta)^{-1/\beta}N^S(t)L, \end{aligned}$$

where superscripts "S" are used to emphasize that the level of technology and thus the level of output will differ between the social planner's allocation and the equilibrium allocation.

The aggregate resource constraint is still given by (13.3). Let us define net output, which subtracts the cost of machines from total output, as

$$\tilde{Y}^S(t) \equiv Y^S(t) - X^S(t).$$

This is relevant, since it is net output that will be distributed between R&D expenditure and consumption. We obtain

$$\begin{aligned}\tilde{Y}^S(t) &= (1-\beta)^{-1/\beta} N^S(t) L - \int_0^{N^S(t)} \psi x^S(\nu,t)\, d\nu \\ &= (1-\beta)^{-1/\beta} N^S(t) L - (1-\beta)^{-(1-\beta)/\beta} N^S(t) L \\ &= (1-\beta)^{-1/\beta} \beta N^S(t) L.\end{aligned}$$

Given this and (13.4), the maximization problem of the social planner can be written as

$$\max \int_0^\infty \frac{C(t)^{1-\theta}-1}{1-\theta} \exp(-\rho t)\, dt$$

subject to

$$\dot{N}(t) = \eta (1-\beta)^{-1/\beta} \beta N(t) L - \eta C(t).$$

In this problem, $N(t)$ is the state variable, and $C(t)$ is the control variable. Let us set up the current-value Hamiltonian

$$\hat{H}(N,C,\mu) = \frac{C(t)^{1-\theta}-1}{1-\theta} + \mu(t)\left[\eta(1-\beta)^{-1/\beta} \beta N(t) L - \eta C(t)\right].$$

The necessary conditions are

$$\begin{aligned}\hat{H}_C(N,C,\mu) &= C(t)^{-\theta} - \eta\mu(t) = 0 \\ \hat{H}_N(N,C,\mu) &= \mu(t)\eta(1-\beta)^{-1/\beta}\beta L = \rho\mu(t) - \dot{\mu}(t) \\ \lim_{t\to\infty}\left[\exp(-\rho t)\mu(t) N(t)\right] &= 0.\end{aligned}$$

It can be verified easily that the current-value Hamiltonian of the social planner is concave, thus the necessary conditions are also sufficient for an optimal solution.

Combining these necessary conditions, we obtain the following growth rate for consumption in the social planner's allocation (see Exercise 13.9):

$$\frac{\dot{C}^S(t)}{C^S(t)} = \frac{1}{\theta}\left(\eta(1-\beta)^{-1/\beta}\beta L - \rho\right). \tag{13.22}$$

Like the equilibrium, the social planner also chooses and allocation with a constant rate of consumption growth (thus no transitional dynamics). The growth rate of consumption chosen by the social planner, (13.22), can be directly compared to the growth rate in the decentralized equilibrium, (13.20). The comparison boils down to that of

$$(1-\beta)^{-1/\beta}\beta \text{ to } \beta,$$

and it is straightforward to see that the former is always greater since $(1-\beta)^{-1/\beta} > 1$ by virtue of the fact that $\beta \in (0,1)$. This implies that the socially-planned economy will always grow faster than the decentralized economy

Proposition 13.3. *In the above-described expanding input variety model, the decentralized equilibrium is always Pareto suboptimal. Starting with any $N(0) > 0$, the Pareto optimal allocation involves a constant growth rate*

$$g^S = \frac{1}{\theta}\left(\eta\,(1-\beta)^{-1/\beta}\,\beta L - \rho\right),$$

which is strictly greater than the equilibrium growth rate g^* given in (13.20).

Proof. Most of the proof is provided in the preceding discussion. Exercise 13.11 asks you to show that the Pareto optimal allocation always involves a constant growth rate and no transitional dynamics. □

Intuitively, the Pareto optimal growth rate is greater than the equilibrium growth rate because the social planner values innovation more. The greater social value of innovations stems from the fact that the social planner is able to use the machines more intensively after innovation, since the monopoly markup reducing the demand for machines is absent in the social planner's allocation. This is related to the *pecuniary externality* resulting from the monopoly markups (and is thus related to the aggregate demand externalities discussed in the previous chapter) and also indirectly affects the set of traded commodities (thus the rate of growth of inputs and technology). Other models of endogenous technological progress we will study in this chapter incorporate technological spillovers and thus generate inefficiencies both because of the pecuniary externality isolated here and because of the standard technological spillovers.

13.1.6. Policy in Models of Endogenous Technological Progress. The divergence between the decentralized equilibrium and the socially planned allocation introduces the possibility of Pareto-improving policy interventions. There are two natural alternatives to consider:

(1) *Subsidies to Research:* by subsidizing research, the government can increase the growth rate of the economy, and this can be turned into a Pareto improvement if taxation is not distortionary and there can be appropriate redistribution of resources so that all parties benefit.
(2) *Subsidies to Capital Inputs:* inefficiencies also arise from the fact that the decentralized economy is not using as many units of the machines/capital inputs (because of the monopoly markup); so subsidies to capital inputs given to final good producers would also be useful in increasing the growth rate.

Moreover, it is noteworthy that as in the first-generation endogenous growth models, a variety of different policy interventions, including taxes on investment income and subsidies of various forms, will have growth effects not just level effects in this framework (see, for example, Exercise 13.13).

Naturally, once we start thinking of policy in order to close the gap between the decentralized equilibrium and the Pareto optimal allocation, we also have to think of the objectives of policymakers and this brings us to political economy issues, which are the subject matter of Part 8. For that reason, we will not go into a detailed discussion of optimal policy (leaving some of this to you in Exercises 13.12-13.14). Nevertheless, it is useful to briefly discuss the role of competition policy in models of endogenous technological progress.

Recall that the optimal price that the monopolist charges for machines is

$$p^x = \frac{\psi}{1-\beta}.$$

Imagine, instead, that a fringe of competitive firms can copy the innovation of any monopolist, but they will not be able to produce at the same level of costs (because the inventor has more know-how). In particular, as in the previous chapter, suppose that instead of a marginal cost ψ, they will have marginal cost of $\gamma\psi$ with $\gamma > 1$. If $\gamma > 1/(1-\beta)$, this fringe is not a threat to the monopolist, since the monopolist could set its ideal, profit maximizing, markup and the fringe would not be able to enter without making losses. However, if $\gamma < 1/(1-\beta)$, the fringe would prevent the monopolist from setting its ideal monopoly price. In particular, in this case the monopolist would be forced to set a "limit price", exactly equal to

$$p^x = \gamma\psi. \tag{13.23}$$

This price formula follows immediately by noting that, if the price of the monopolist were higher than $\gamma\psi$, the fringe could undercut the price of the monopolist, take over to market and make positive profits. If it were below this, the monopolist could increase its price towards the unconstrained monopoly price and make more profits. Thus, there is a unique equilibrium price given by (13.23).

When the monopolist charges this limit price, its profits per unit would be

$$\text{profits per unit } = (\gamma - 1)\,\psi = (\gamma - 1)\,(1-\beta)\,,$$

which is less than β, the profits per unit that the monopolist made in the absence of the competitive fringe.

What is the implication of this on the rate of economic growth? It is straightforward to work out that in this case the economy would grow at a slower rate. For example, in the baseline model with the lab-equipment technology, this growth rate would be (see Exercise 13.15):

$$\hat{g} = \frac{1}{\theta}\left(\eta\gamma^{-1/\beta}\,(\gamma-1)\,(1-\beta)^{-(1-\beta)/\beta}\,L - \rho\right),$$

which is less than g^* given in (13.20). Therefore, in this model, greater competition, which reduces markups (and thus static distortions), also reduces long-run growth. This might at first appear counter-intuitive, since the monopoly markup may be thought to be the key source of inefficiency and greater competition (lower γ) reduces this markup. Nevertheless, as mentioned above, inefficiency results both because of monopoly markups and because the set of available inputs may not be appropriately chosen. As γ declines, monopoly markups decline, but the problem of underprovision of inputs becomes more severe. This is because profits are important in this model to encourage innovation by new research firms. If these profits are cut, incentives for research are also reduced. Since γ can also be interpreted as a parameter of anti-trust (competition) policy, this result implies that in the baseline endogenous technological change models more strict anti-trust policy reduces economic growth.

Welfare is not the same as growth, however, and some degree of competition reducing prices below the unconstrained monopolistic level might be useful for welfare depending on the discount rate of the representative household. Essentially, with a lower markup, households are happier in the present, but suffer slower consumption growth. The tradeoff between these two opposing effects depends on the discount rate of the representative household (see Exercise 13.15 for the details).

Similar results apply when we consider patent policy. In practice, patents are for limited durations. In the baseline model, we assumed that patents are perpetual; once a firm invents a new good, it has a fully-enforced patent forever and it becomes the monopolist for that good forever. If patents are enforced strictly, then this might rule out the competitive fringe from competing, restoring the growth rate of the economy to (13.20). Also, even in the absence of the competitive fringe, we can imagine that once the patent runs out, the firm will cease making profits on its innovation. In this case, it can easily be shown that growth is maximized by having as long patents as possible. Again there is a tradeoff here between the equilibrium growth rate of the economy and the static level of welfare.

Perhaps, more important than these tradeoffs between growth and level is the fact that the models discussed in this chapter do not feature an interesting type of competition among firms. The quality competition (Schumpeterian) models introduced in the next chapter will allow a richer analysis of the effect of competition on innovation and economic growth.

13.2. Growth with Knowledge Spillovers

In the model of the previous section, growth resulted from the use of final output for R&D. This is similar, in some way, to the endogenous growth model of Rebelo (1991) we studied in Chapter 11, since the accumulation equation is linear in accumulable factors. As a result, we saw that, in equilibrium, output took a linear form in the stock of knowledge (new machines), thus a AN form instead of Rebelo's AK form.

An alternative is to have "scarce factors" used in R&D. In other words, instead of the lab equipment specification, we now have scientists as the key creators of R&D. The lab equipment model generated sustained economic growth by investing more and more resources in the R&D sector. This is impossible with scarce factors, since, by definition, a sustained increase in the use of these factors in the R&D sector is not possible. Consequently, with this alternative specification, there cannot be endogenous growth unless there are knowledge spillovers from past R&D, making the scarce factors used in R&D more and more productive over time. In other words, we now need current researchers to *"stand on the shoulder of past giants"*. In fact, the original formulation of the endogenous technological change model by Romer (1990) relied on this type of knowledge spillovers. While these types of knowledge spillovers might be important in practice, the lab equipment model studied in the previous section was a better starting point for us, since it clearly delineated the role of technology accumulation and showed that growth need not be generated by technological externalities or spillovers.

Nevertheless, knowledge spillovers play a very important role in many models of economic growth and it is useful to see how the baseline endogenous technological progress model works in the presence of such spillovers. We now present the simplest version of the endogenous technological change model with knowledge spillovers. The environment is identical to that of the previous section, with the exception of the innovation possibilities frontier, which now takes the form

$$\dot{N}(t) = \eta N(t) L_R(t) \tag{13.24}$$

where $L_R(t)$ is labor allocated to R&D at time t. The term $N(t)$ on the right-hand side captures spillovers from the stock of existing ideas. The greater is $N(t)$, the more productive is an R&D worker. Notice that (13.24) imposes that these spillovers are proportional or linear. This linearity will be the source of endogenous growth in the current model. In the next section, we will see that a different kind of endogenous growth model can be formulated with less than proportional spillovers.

In (13.24), $L_R(t)$ is research employment, which comes out of the regular labor force. An alternative, which was originally used by Romer (1990), would be to suppose that only skilled workers or scientists can work in the knowledge-production (R&D) sector. Here we use the assumption that a homogeneous workforce is employed both in the R&D sector and in the final good sector. The advantage of this formulation is that competition between the production and the R&D sectors for workers ensures that the cost of workers to the research sector is given by the wage rate in final good sector. The only other change we need to make to the underlying environment is that now the total labor input *employed* in the final good sector, represented by the production function (13.2), is $L_E(t)$ rather than L, since some of the workers are working in the R&D sector. Labor market clearing then requires that

$$L_R(t) + L_E(t) \leq L.$$

The fact that not all workers are in the final good sector implies that the aggregate output of the economy (by an argument similar to before) is given by

$$Y(t) = \frac{1}{1-\beta} N(t) L_E(t), \tag{13.25}$$

and profits of monopolists from selling their machines is

$$\pi(t) = \beta L_E(t). \tag{13.26}$$

The net present discounted value of a monopolist (for a blueprint ν) is still given by $V(\nu, t)$ as in (13.7) or (13.8), with the flow profits given by (13.26). However, the free entry condition is no longer the same as that which followed from equation (13.4). Instead, (13.24) implies the following free entry condition (when there is positive research):

$$\eta N(t) V(\nu, t) = w(t), \tag{13.27}$$

where $N(t)$ is on the left-hand side because it parameterizes the productivity of an R&D worker, while the flow cost of undertaking research is hiring workers for R&D, thus is equal to the wage rate $w(t)$.

The equilibrium wage rate must be the same as in the lab equipment model of the previous section, in particular, as in equation (13.13), since the final good sector is unchanged. Thus, we still have $w(t) = \beta N(t) / (1-\beta)$. Moreover, balanced growth again requires that the interest rate must be constant at some level r^*. Using these observations together with the free entry condition, we obtain:

$$\eta N(t) \frac{\beta L_E(t)}{r^*} = \frac{\beta}{1-\beta} N(t). \tag{13.28}$$

Hence the BGP equilibrium interest rate must be

$$r^* = (1-\beta) \eta L_E^*,$$

where L_E^* is the number of workers employed in production in BGP (given by $L_E^* = L - L_R^*$). The fact that the number of workers in production must be constant in BGP follows from (13.28). Now using the Euler equation of the representative household, (13.16), we have that for all t,

$$\begin{aligned} \frac{\dot{C}(t)}{C(t)} &= \frac{1}{\theta}\left((1-\beta)\eta L_E^* - \rho\right) \\ &\equiv g^*. \end{aligned} \tag{13.29}$$

To complete the characterization of the BGP equilibrium, we need to determine L_E^*. In BGP, (13.24) implies that the rate of technological progress satisfies $\dot{N}(t)/N(t) = \eta L_R^* = \eta(L - L_E^*)$. Moreover, by definition, we have the BGP growth rate of consumption equal to the rate of technological progress, thus $g^* = \dot{N}(t)/N(t)$. This implies that the BGP level of employment is uniquely pinned down as

$$L_E^* = \frac{\theta\eta L + \rho}{(1-\beta)\eta + \theta\eta}. \tag{13.30}$$

The rest of the analysis is unchanged. It can also be verified that there are no transitional dynamics in the decentralized equilibrium (see Exercise 13.17). It is also useful to note that there is again a scale effect here—greater L increases the interest rate and the growth rate in the economy.

Proposition 13.4. *Consider the above-described expanding input-variety model with knowledge spillovers and suppose that*

$$(1-\theta)(1-\beta)\eta L_E^* < \rho < (1-\beta)\eta L_E^*, \tag{13.31}$$

where L_E^* is the number of workers employed in production in BGP, given by (13.30). Then there exists a unique balanced growth path in which technology, output and consumption grow at the same rate, $g^* > 0$, given by (13.29) starting from any initial level of technology stock $N(0) > 0$.

Proof. Most of the proof is given by the preceding discussion. Exercise 13.16 asks you to verify that the transversality condition is satisfied and that there are no transitional dynamics. □

Also, as in the lab equipment model, the equilibrium allocation is Pareto suboptimal, and the Pareto optimal allocation involves a higher rate of output and consumption growth. Intuitively, while firms disregard the future increases in the productivity of R&D resulting from their own R&D spending, the social planner internalizes this effect (see Exercise 13.17).

13.3. Growth without Scale Effects

As we have seen, the models used so far feature a scale effect in the sense that a larger population, L, translates into a higher interest rate and a higher growth rate. This is problematic for three reasons as argued in a series of papers by Charles Jones and others:

(1) Larger countries do not necessarily grow faster (though the larger market of the United States or European economies may have been an advantage during the early phases of the industrialization process. We will return to this issue in Chapter 21).
(2) The population of most nations has not been constant. If we have population growth as in the standard neoclassical growth model, e.g., $L(t) = \exp(nt)L(0)$, these models would not feature balanced growth, rather, the growth rate of the economy would be increasing over time.

(3) In the data, the total amount of resources devoted to R&D appears to increase steadily, but there is no associated increase in the aggregate growth rate.

Each one of these arguments against scale effects can be debated (for example, by arguing that countries do not provide the right level of analysis because of international trade linkages or that the growth rate of the world economy has indeed increased when we look at the past 2000 years rather than the past 100 years). Nevertheless together they do suggest that the strong form of scale effects embedded in the baseline endogenous technological change models may not provide a good approximation to reality. These observations have motivated Jones (1995) to suggest a modified version of the baseline endogenous technological progress model. While the type of modification to remove scale effect can be formulated in the lab equipment model (see Exercise 13.21), it is conceptually simpler to do so in the context of the model with knowledge spillovers discussed in the previous section. In particular, in that model the scale effect can be removed by reducing the impact of knowledge spillovers.

More specifically, consider the model of the previous section with only two differences. First, there is population growth at the constant exponential rate n, so that $\dot{L}(t) = nL(t)$. The economy admits a representative household, which is also growing at the rate n, so that its preferences can be represented by the standard CRRA form:

$$\int_0^\infty \exp\left(-\left(\rho - n\right)t\right)\frac{C(t)^{1-\theta} - 1}{1-\theta}dt, \tag{13.32}$$

where $C(t)$ is consumption of the final good of the economy at time t, which is produced as before (with the production function (13.2)).

Second, in contrast to the knowledge-spillovers model studied in the previous section, the R&D sector only admits limited knowledge spillovers and (13.24) is replaced by

$$\dot{N}(t) = \eta N(t)^{\phi} L_R(t) \tag{13.33}$$

where $\phi < 1$ and $L_R(t)$ is labor allocated to R&D activities at time t. Labor market clearing requires

$$L_E(t) + L_R(t) = L(t), \tag{13.34}$$

where $L_E(t)$ is the level of employment in the final good sector, and the labor market clearing condition takes into account that population is changing over time.

The key assumption for the model is that $\phi < 1$. The case where $\phi = 1$ is the one analyzed in the previous section, and as commented above, with population growth this would lead to an exploding path, leading to infinite utility. However, the model is well behaved when $\phi < 1$.

Aggregate output and profits are given by (13.25) and (13.26) as in the previous section. An equilibrium is also defined similarly. Let us focus on the BGP, where a constant fraction of workers are allocated to R&D, and the interest rate and the growth rate are constant.

Suppose that this BGP involves positive growth, so that the free entry condition holds as equality. Then, the BGP free entry condition can be written as (see Exercise 13.18)

$$\eta N(t)^{\phi} \frac{\beta L_E(t)}{r^* - n} = w(t). \tag{13.35}$$

As before, the equilibrium wage is determined by the production side, (13.13), as $w(t) = \beta N(t) / (1-\beta)$. Combining this with the previous equation gives the following free entry condition

$$\eta N(t)^{\phi-1} \frac{(1-\beta) L_E(t)}{r^*} = 1.$$

Now differentiating this condition with respect to time, we obtain

$$(\phi - 1) \frac{\dot{N}(t)}{N(t)} + \frac{\dot{L}_E(t)}{L_E(t)} = 0.$$

Since in BGP, the fraction of workers allocated to research is constant, we must have $\dot{L}_E(t) / L_E(t) = n$. This implies that the BGP growth rate of technology is given by

$$g_N^* \equiv \frac{\dot{N}(t)}{N(t)} = \frac{n}{1-\phi}. \tag{13.36}$$

From equation (13.12), this implies that total output grows at the rate $g_N^* + n$. But now there is population growth, so consumption per capita grows only at the rate

$$\begin{aligned} g_C^* &= g_N^* \\ &= \frac{n}{1-\phi}. \end{aligned} \tag{13.37}$$

We can then use the consumer Euler equation, equivalent of (11.4) incorporating the fact that the discount factor is $\rho - n$ instead of ρ, to determine BGP interest rate as

$$\begin{aligned} r^* &= \theta g_N^* + \rho - n \\ &= \frac{\phi - (1-\theta)}{1-\phi} n + \rho > 0. \end{aligned}$$

The most noteworthy feature is that this model generates sustained and exponential growth in income per capita in the presence of population growth. More interestingly, in order to achieve this growth rate, it allocates more and more of the labor force to R&D. The reason for this is that the technology for creating new ideas, (13.33), only features limited spillovers, thus to maintain sustained growth, more resources need to be allocated to R&D.

Proposition 13.5. *In the above-described expanding input-variety model with limited knowledge spillovers as given by (13.33), starting from any initial level of technology stock $N(0) > 0$, there exists a unique balanced growth path in which, technology and consumption per capita grow at the rate g_N^* as given by (13.36), and output grows at rate $g_N^* + n$.*

This analysis therefore shows that sustained equilibrium growth of per capita income is possible in an economy with growing population. Intuitively, instead of the linear (proportional) spillovers in the baseline Romer model, the current model allows only a limited amount of spillovers. Without population growth, these spillovers would affect the level of output, but would not be sufficient to sustain long-run growth. Continuous population growth, on the other hand, steadily increases the market size for new technologies and generates growth from these limited spillovers. While this pattern is referred to as "growth without scale effects," it is useful to note that there are two senses in which there are limited scale effects in these models. First, a faster rate of population growth translates into a higher equilibrium growth rate. Second, a larger population size leads to higher output per capita (see Exercise 13.20). It is not clear whether the data support these types of scale effects either. Put differently, some of the evidence suggested against the scale effects in the baseline endogenous technological change models may be inconsistent with this class of models as well. For example, there does not seem to be any evidence in the postwar data or from the historical data of the past 200 years that faster population growth leads to a higher equilibrium growth rate.

It is also worth noting that these models are sometimes referred to as "semi-endogenous growth" models, because while they exhibit sustained growth, the per capita growth rate of the economy, (13.37), is determined only by population growth and technology, and does not respond to taxes or other policies. Some papers in the literature have developed models of endogenous growth without scale effects, with equilibrium growth responding to policies, though this normally requires a combination of restrictive assumptions.

13.4. Growth with Expanding Product Varieties

Finally, we will briefly discuss the equivalent model in which growth is driven by *product innovations*, that is, by expanding product varieties rather than expanding varieties of inputs. The economy is in continuous time and has constant population L. It admits a representative household with preferences given by

$$\int_0^\infty \exp(-\rho t) \log C(t)\, dt, \tag{13.38}$$

where

$$C(t) \equiv \left[\int_0^{N(t)} c(\nu, t)^{\frac{\varepsilon-1}{\varepsilon}} d\nu\right]^{\frac{\varepsilon}{\varepsilon-1}} \tag{13.39}$$

is the consumption index, which is a CES aggregate of the consumption of different varieties. Here $c(\nu, t)$ denotes the consumption of product ν at time t, while $N(t)$ is the total measure of products. We assume throughout that $\varepsilon > 1$. Therefore, we have replaced expanding input varieties with expanding product varieties. The log specification in this utility function is for simplicity, and can be replaced by a CRRA utility function.

The patent to produce each product $\nu \in [0, N(t)]$ belongs to a monopolist, and the monopolist who invents the blueprints for a new product receives a fully enforced perpetual patent on this product. Each product can be produced with the technology

$$y(\nu,t) = l(\nu,t), \tag{13.40}$$

where $l(\nu,t)$ is labor allocated to the production of this variety. Since the economy is closed, $y(\nu,t) = c(\nu,t)$.

As in model with knowledge spillovers of Section 13.2, we assume that new products can be produced with the production function

$$\dot{N}(t) = \eta N(t) L_R(t). \tag{13.41}$$

The reader will notice that there is a very close connection between the model here and the models of expanding input variety studied so far, especially the model with knowledge spillovers in Section 13.2. For instance, if $y(\nu,t)$s were interpreted as intermediate goods or inputs instead of products and if $C(t)$ in (13.39) were interpreted as the production function for the final good rather than part of the utility function of the representative consumer, the two models would be essentially identical. The only difference would be that, with this interpretation, labor would now be used in the production of the inputs, while in Section 13.2 it is used in the final good sector. This similarity emphasizes that the distinction between process and product innovations is fairly minor in theory, though this distinction might still be useful in mapping these models to reality.

An equilibrium and a balanced growth path are defined similarly to before. The representative household now determines both the allocation of its expenditure on different varieties and the time path of consumption expenditures. We assume that the economy is closed and there is no capital, thus all output must be consumed. Nevertheless, the consumer Euler equation will apply to determine the equilibrium interest rate. Labor market clearing requires that

$$\int_0^{N(t)} l(\nu,t)\,d\nu + L_R(t) \leq L. \tag{13.42}$$

Let us start with expenditure decisions. Since the representative household has Dixit-Stiglitz preferences, the following consumer demands can be derived (see Exercise 13.24):

$$c(\nu,t) = \frac{p^x(\nu,t)^{-\varepsilon}}{\left(\int_0^{N(t)} p^x(\nu,t)^{1-\varepsilon}\,d\nu\right)^{\frac{-\varepsilon}{1-\varepsilon}}} C(t), \tag{13.43}$$

where $p^x(\nu,t)$ is the price of product variety ν at time t, and $C(t)$ is defined in (13.39). The term in the denominator is the ideal price index raised to the power $-\varepsilon$. As before, it is most convenience to set this ideal price index as the numeraire, so that the price of output at every instant is normalized to 1. Thus we impose

$$\left(\int_0^{N(t)} p^x(\nu,t)^{1-\varepsilon}\,d\nu\right)^{\frac{1}{1-\varepsilon}} = 1 \text{ for all } t. \tag{13.44}$$

With this choice of numeraire, we obtain the consumer Euler equation as (see Exercise 13.25):

$$\frac{\dot{C}(t)}{C(t)} = r(t) - \rho. \tag{13.45}$$

With similar arguments to before, the net present discounted value of the monopolist owning the patent for product ν can be written as

$$V(\nu,t) = \int_t^\infty \exp\left[-\int_t^s r\left(s'\right) ds'\right] \left[p^x(\nu,s)c(\nu,s) - w(s)\, c(\nu,s)\right] ds,$$

where $w(t)\, c(\nu,t)$ is the total expenditure of the firm to produce a total quantity of $c(\nu,t)$ (given the production function (13.40) and the wage rate at time t equal to $w(t)$), while $p^x(\nu,t)c(\nu,t)$ is its revenue, consistent with the demand function (13.43). The maximization of the net present discounted value again requires profit maximization at every instant. Since each monopolist faces the iso-elastic demand curve given in (13.43), the profit-maximizing monopoly price is

$$p^x(\nu,t) = \frac{\varepsilon}{\varepsilon-1} w(t) \text{ for all } \nu \text{ and } t.$$

Since all firms charge the same price, they will all produce the same amount and employ the same amount of labor. At time t, there are $N(t)$ products, so the labor market clearing condition (13.42) implies that

$$c(\nu,t) = l(\nu,t) = \frac{L - L_R(t)}{N(t)} \text{ for all } \nu \text{ and } t. \tag{13.46}$$

Consequently, the instantaneous profits of each monopolist at time t can be written as

$$\begin{aligned} \pi(\nu,t) &= p^x(\nu,t)c(\nu,t) - w(t)\, c(\nu,t) \\ &= \frac{1}{\varepsilon-1}\frac{L-L_R(t)}{N(t)} w(t) \text{ for all } \nu \text{ and } t. \end{aligned} \tag{13.47}$$

Since prices, sales and profits are equal for all monopolists, we can simplify notation by letting

$$V(t) = V(\nu,t) \text{ for all } \nu \text{ and } t.$$

In addition, since $c(\nu,t) = c(t)$ for all ν,

$$\begin{aligned} C(t) &= N(t)^{\frac{\varepsilon}{\varepsilon-1}} c(t). \\ &= (L - L_R(t))\, N(t)^{\frac{1}{\varepsilon-1}}, \end{aligned} \tag{13.48}$$

where the second equality uses (13.46).

Labor demand comes from the research sector as well as from the final good producers. Labor demand from research can again be determined using the free entry condition. Assuming that there is positive research, so that the free entry condition holds as an equality, this takes the form

$$\eta N(t)\, V(t) = w(t). \tag{13.49}$$

Combining this equation with (13.47), we see that

$$\pi(t) = \frac{1}{\varepsilon - 1}(L - L_R(t))\,\eta V(t),$$

where we use $\pi(t)$ to denote the profits of all monopolists at time t, which are equal. In BGP, where the fraction of the workforce working in research is constant, this implies that profits and the net present discounted value of monopolists are also constant. Moreover, in this case we must have

$$V(t) = \frac{\pi(t)}{r^*},$$

where r^* denotes the BGP interest rate. The previous two equations then imply

$$r^* = \frac{\eta}{\varepsilon - 1}(L - L_R^*),$$

with L_R^* denoting the BGP size of the research sector. The R&D employment level of L_R^* combined with the R&D sector production function, (13.41) then implies

$$\frac{\dot{N}(t)}{N(t)} = \eta L_R^*.$$

However, we also know from the consumer Euler equation, (13.45) combined with (13.48)

$$\begin{aligned}\frac{\dot{C}(t)}{C(t)} &= r(t) - \rho \\ &= \frac{1}{\varepsilon - 1}\frac{\dot{N}(t)}{N(t)},\end{aligned}$$

which implies

$$\frac{\eta}{\varepsilon - 1}(L - L_R^*) - \rho = \frac{1}{\varepsilon - 1}\eta L_R^*,$$

or

$$L_R^* = \frac{L}{2} - \frac{\varepsilon - 1}{2\eta}\rho.$$

Consequently, the growth rate of consumption expenditure (and utility) is

$$g^* = \frac{1}{2}\left(\frac{\eta}{\varepsilon - 1}L - \rho\right). \tag{13.50}$$

This establishes:

Proposition 13.6. *In the above-described expanding product variety model, there exists a unique BGP, in which aggregate consumption expenditure, $C(t)$, grows at the rate g^* given by (13.50).*

A couple of features are worth noting about this equilibrium. First, in this equilibrium, there is growth of "real income," even though the production function of each good remains unchanged. This is because, while there is no process innovation reducing costs or improving quality, the number of products available to consumers expands because of product innovations. Since the utility function of the representative household, (13.38), exhibits *love-for-variety*, the expanding variety of products increases utility. What happens to income depends on what we choose as the numeraire. The natural numeraire is the one setting the ideal price index, (13.44), equal to 1, which amounts to measuring incomes in similar units at different dates. With this choice of numeraire, real incomes grow at the same rate as $C(t)$, at the rate g^*. Second, even though the equilibrium was characterized in a somewhat different manner than our baseline expanding input variety model, there is a close parallel between expanding product varieties and expanding input varieties. This can be seen, for example, in Exercise 13.23, which looks at an economy with expanding input varieties produced by labor. It can be verified that the structure of the equilibrium is very similar to the one studied here. Third, Exercise 13.26 will show that as in the other models of endogenous technological progress we have seen in this chapter, there are no transitional dynamics and the equilibrium is again Pareto suboptimal. Moreover, log preferences now ensure that the transversality condition is always satisfied. Finally, it can be verified that there is again a scale effect here. This discussion then reveals that whether one wishes to use the expanding input variety or the expanding product model is mostly a matter of taste, and perhaps one of context. Both models lead to a similar structure of equilibria, to similar equilibrium growth rates, and to similar welfare properties.

13.5. Taking Stock

In this chapter, we had our first look at models of endogenous technological progress. The distinguishing feature of these models is the fact that profit incentives shape R&D spending and investments, which in turn determines the rate at which the technology of the economy evolves over time. At some level, there are many parallels between the models studied here and the Romer (1986) model of growth with externalities studied in Section 11.4 in Chapter 11; both have a mathematical structure similar to the neoclassical AK models (constant long-run growth rate, no transitional dynamics) and both generate externalities causing an equilibrium growth rate less than the Pareto optimal growth rate (because of physical capital externalities in the Romer (1986) model, because of *aggregate demand externalities* in the lab equipment model of Section 13.1 here, and because of a mixture of these in the other models studied in this chapter). The difference between the Romer (1986) model and the endogenous technological change model should not be understated, however. While one may interpret the Romer (1986) model as involving "knowledge accumulation," the accumulation of knowledge and technology is *not* an economic activity—it is a byproduct of other decisions (in this particular instance, individual physical capital accumulation decisions). Hence, while such a model may "endogenize" technology, it does so without explicitly specifying the costs and benefits of investing in new technologies. Since, as discussed in Chapter 3, technology differences across countries are likely to be important in accounting for their income differences, understanding the sources of technology differences is a major part of our effort to understand the mechanics of economic growth. In this respect, the models presented in this chapter constitute a major improvement over those we have encountered so far.

The models studied in this chapter, like those of the previous chapter, emphasize the importance of profits in shaping technology choices. We have also seen the role of monopoly power and patent length on the equilibrium growth rate. In addition, the same factors that influenced the equilibrium growth rate in the neoclassical AK model also affect equilibrium economic growth here. These include the discount rate, ρ, as well as taxes on capital income or corporate profits. Nevertheless, the effect of the market structure on equilibrium growth and innovation rates is somewhat limited in the current models because the Dixit-Stiglitz structure and expanding product or input varieties limit the extent to which firms can compete with each other. The models of quality competition in the next chapter will feature a richer interaction between market structure and equilibrium growth.

While the models in this chapter highlight certain major determinants of the rate of technological progress, another shortcoming of these models should be noted. The technology stock of a society is determined only by its own R&D. Thus technological differences will result simply from R&D differences. In the world of relatively free knowledge-flows, many countries will not only generate technological know-how by their own R&D but will also benefit from the advances in the world technology frontier. Consequently, in practice, technology adoption decisions and the patterns of technology diffusion may be equally important as, or more important than, R&D rates towards the invention of new technologies (see Chapter 18 below). Therefore, the major contribution of the models studied in this chapter to our knowledge may be not in pinpointing the exact source of technology differences across countries, but in their emphasis on the endogenous nature of technology and the set of factors that affect technological investments.

In addition, models of endogenous technological change are essential for understanding world economic growth, since presumably the world technology frontier does largely advance because of R&D. Therefore, for our purpose of understanding that world economic growth, the perspectives we have gained on the determinants of technological progress are important. Nevertheless, the AK structure of these models implies that they may have relatively little to say about why R&D investments and rapid technological progress has been a feature of the past 200 years, and the stock of knowledge and output per capita did not exhibit steady growth before the 19th century. Some of these questions will be addressed later in the book.

13.6. References and Literature

Models of endogenous technological progress were introduced in Romer (1987 and 1990), and then subsequently analyzed by, among others, Segerstrom, Anant and Dinopoulos (1990), Grossman and Helpman (1991a,b), Aghion and Howitt (1992). The lab equipment model presented in Section 13.1 appears in Rivera-Batiz and Romer (1991). The model in Romer (1990) is similar to that presented in Section 13.2, but with skilled workers working in R&D. Gancia and Zilibotti (2005) provide an excellent survey of many of the models discussed in this chapter. Matsuyama (1995) gives a very lucid and informative discussion of the sources of inefficiency in Dixit-Stiglitz type models, which is related to the sources of inefficiency in the lab equipment model presented in Section 13.1.

The critique of endogenous growth models because of scale effect is contained in Backus, Kehoe and Kehoe (1992) and in Jones (1995). The first of these papers pointed out that countries with larger sizes (either without adjustment or adjusted for international trade) do not grow faster in the postwar era. Jones (1995), on the other hand, focused on time-series patterns and pointed out the substantial increase in R&D inputs, for example, the total number of workers involved in research, with no corresponding increase in the equilibrium growth rate. Others argued that looking at the 20th century data may not be sufficient to reach a conclusion on whether there is a scale effect or not. Kremer (1993) argues, on the basis of estimates of world population, that there must have been an increase in economic growth over the past one million years. Laincz and Perreto (1996) argue that R&D resources allocated to specific product lines have not increased.

The model in Section 13.3 is similar to that presented in Jones (1995) and Jones (1999). As pointed out there, these models generate sustained growth of per capita income, but the growth rate of the economy does not respond to policies or preferences (given the rate of population growth). A number of authors have developed models of endogenous growth without scale effect, where policy might have an effect on the equilibrium growth rate. See, among others, Dinopoulos and Thompson (1998), Segerstrom (1998), Howitt (1999) and Young (1998). Aghion and Howitt (1998) and Ha and Howitt (2005) argue that semi-endogenous growth models along these lines also faced difficulties when confronted with the time-series evidence.

The model of expanding product variety was first suggested by Judd (1985), but in the context of a model of exogenous growth. The endogenous growth models with expanding product variety is presented in Grossman and Helpman (1991a,b). The treatment here is somewhat different from that in Grossman and Helpman, especially because we used the ideal price index rather than Grossman and Helpman's choice of total expenditure as the numeraire.

13.7. Exercises

Exercise 13.1. This exercise asks you to derive (13.8) from (13.7)

(1) Rewrite (13.7) at time t as:

$$\begin{aligned} V(\nu,t) &= \int_t^{t+\Delta t} \exp\left[-\int_t^s r(\tau)\,d\tau\right](p^x(\nu,s)-\psi)\,x(\nu,s)ds \\ &\quad + \int_{t+\Delta t}^{\infty} \exp\left[-\int_{t+\Delta t}^s r(\tau)\,d\tau\right][p^x(\nu,s)x(\nu,s)-\psi x(\nu,s)]\,ds \end{aligned}$$

which is just an identity for any Δt. Interpret this equation and relate this to the *Principle of Optimality.*

(2) Show that for small Δt, this can be written as

$$V(\nu,t) = \Delta t\,(p^x(\nu,t)-\psi)\,x(\nu,t) + \exp(r(t)\,\Delta t)\,V(\nu,t+\Delta t) + o(\Delta t),$$

and thus derive the equation

$$\Delta t\,(p^x(\nu,t)-\psi)\,x(\nu,t) + \exp(r(t)\,\Delta t)\,V(\nu,t+\Delta t) - \exp(r(t)\times 0)\,V(\nu,t) + o(\Delta t) = 0,$$

where, recall that, $\exp(r(t)\times 0) = 1$. Interpret this equation and the significance of the term $o(\Delta t)$.

(3) Now divide both sides by Δt and take the limit $\Delta t \to 0$, to obtain

$$(p^x(\nu,t)-\psi)\,x(\nu,t) + \lim_{\Delta t\to 0}\frac{\exp(r(t)\,\Delta t)\,V(\nu,t+\Delta t) - \exp(r(t)\times 0)\,V(\nu,t)}{\Delta t} = 0.$$

(4) When the value function is differentiable in its time argument, the previous equations is equivalent to

$$(p^x(\nu,t)-\psi)\,x(\nu,t) + \left.\frac{\partial\,(\exp(r(t)\,\Delta t)\,V(\nu,t+\Delta t))}{\partial t}\right|_{\Delta t=0} = 0.$$

Now derive (13.8).

(5) Provide an economic intuition for the equation (13.8).

EXERCISE 13.2. Derive (13.9) and (13.10) from the profit maximization problem of a monopolist.

EXERCISE 13.3. Formulate the consumer optimization problem in terms of the current-value Hamiltonian and derive the necessary conditions. Show that these are equivalent to (13.16) and (13.17).

EXERCISE 13.4. Prove that in the model of Section 13.1, if $\eta V(\nu,t') = 1$ and there is entry at some t', then $\eta V(\nu,t) = 1$ for all t and conversely that if $\eta V(\nu,t') < 1$ for some t', then $\eta V(\nu,t) \leq 1$ for all t and there is no entry at any t.

EXERCISE 13.5. Consider the expanding variety model of Section 13.1 and denote the BGP growth rates of consumption and total output by g_C^* and g^*.

(1) Show that $g_C^* > g^*$ is not feasible.
(2) Show that $g_C^* < g^*$ violates the transversality condition.

Exercise 13.6. This exercise asks you to construct and analyze the equivalent of the lab-equipment expanding variety model of Section 13.1 in discrete time. Suppose that the economy admits a representative household with preferences at time 0 given by

$$\sum_{t=0}^{\infty} \beta^t \frac{C(t)^{1-\theta} - 1}{1-\theta},$$

with $\beta \in (0,1)$ and $\theta \geq 0$. Production technology is the same as in the text and the innovation possibilities frontier of the economy is given by

$$N(t+1) - N(t) = \eta Z(t).$$

(1) Define an equilibrium.
(2) Characterize the balanced growth path and compare the structure of the equilibrium to that in Section 13.1.
(3) Show that there are no transitional dynamics, so that starting with any $N(0) > 0$, the economy grows at a constant rate.

Exercise 13.7. Complete the proof of Proposition 13.1, in particular, showing that condition (13.21) is sufficient for the transversality condition to be satisfied.

Exercise 13.8. Consider a world economy consisting of $j = 1, ..., M$ economies. Suppose that each of those are closed and have access to the same production and R&D technology as described in Section 13.1. The only differences across countries are in the size of labor force, L_j, productivity of R&D, η_j, and discount rate ρ_j. You may also want to assume that one unit of R&D expenditure costs ζ_j units of final good in country j, and this varies across countries. There are no technological exchange across countries.

(1) Define the "world equilibrium" in which each country is in equilibrium in analogy with the equilibrium path of the one country economy in Section 13.1.
(2) Characterize the world equilibrium. Show that in the world equilibrium, each country will grow at a constant rate starting at $t = 0$. Provide explicit solutionse for these growth rates.
(3) Show that except in "knife-edge" cases, output in each country will grow at a different long-run rate.
(4) Now return to the discussion in Chapters 3 and 8 regarding the effect of policy and taxes on long-run income per capita differences. Show that, in the model discussed in this exercise, arbitrarily small differences in policy or discount factors across countries will lead to "infinitely large" differences in long-run income per capita. Does this resolve the empirical challenges discussed in those chapters? Does the environment in this exercise provide a satisfactory model for the study of long-run income per capita differences across countries? If yes, please elaborate how such a

model can be mapped to reality. If not, explain which features of the model appear unsatisfactory to you and how you would want to change them.

Exercise 13.9. Derive the consumption growth rate in the socially-planned economy, (13.22).

Exercise 13.10. Consider the expanding input variety model of Section 13.1. Show that it is possible for the equilibrium allocation to satisfy the transversality condition, while the social planner's solution may violate it. Interpret this result. Does it imply that the social planner's allocation is less compelling?

Exercise 13.11. Complete the proof of Proposition 13.3, in particular showing that the Pareto optimal allocation always involves a constant growth rate and no transitional dynamics.

Exercise 13.12. Consider the expanding input variety model of Section 13.1.

(1) Suppose that a benevolent government has access only to research subsidies, which can be financed by lump-sum taxes. Can these subsidies be chosen so as to ensure that the equilibrium growth rate is the same as the Pareto optimal growth rate? Can they be used to replicate the Pareto optimal equilibrium path? Would it be desirable for the government to use subsidies so as to achieve the Pareto optimal growth rate (from the viewpoint of maximizing social welfare at time $t = 0$)?
(2) Suppose that the government now has only access to subsidies to machines, which can again be financed by lump-sum taxes. Can these be chosen to induce the Pareto optimal growth rate? Can they be used to replicate the Pareto optimal equilibrium path?
(3) Will the combination of subsidies to machines and subsidies to research be better than either of these two policies by themselves?

Exercise 13.13. Consider the expanding input variety model of Section 13.1 and assume that corporate profits are taxed at the rate τ.

(1) Characterize the equilibrium allocation.
(2) Consider two economies with identical technologies and identical initial conditions, but with different corporate tax rates, τ and τ'. Determine the relative income of these two economies (possibly as a function of time).

Exercise 13.14. * Consider the expanding input variety model of Section 13.1, with one difference. A firm that invents a new machine receives a patent, which expires at the Poisson rate ι. Once the patent expires, that machine is produced competitively and is supplied to final good producers at marginal cost.

(1) Characterize the equilibrium in this case and show how the equilibrium growth rate depends on ι. [Hint: notice that there will be two different types of machines, supplied at different prices].
(2) What is the value of ι that maximizes the equilibrium rate of economic growth?
(3) Show that a policy of $\iota = 0$ does not necessarily maximize social welfare at time $t = 0$.

Exercise 13.15. Consider the formulation of competition policy in subsection 13.1.6.

(1) Characterize the equilibrium fully.
(2) Write down the welfare of the representative household at time $t = 0$ in this equilibrium.
(3) Maximize this welfare function by choosing a value of γ.
(4) Why is the optimal value of γ not equal to some $\gamma^* \geq 1/(1-\beta)$? Provide an interpretation in terms of the tradeoff between level and growth effects.
(5) What is the relationship between the optimal value of γ and ρ. Interpret.

Exercise 13.16. Complete the proof of Proposition 13.4. In particular, show that the equilibrium path involves no transitional dynamics and that under (13.31), the transversality condition is satisfied.

Exercise 13.17. Characterize the Pareto optimal allocation in the economy of Section 13.2. Show that it involves a constant growth rate greater than the equilibrium growth rate in Proposition 13.4 and no transitional dynamics.

Exercise 13.18. Derive equation (13.35) and explain why the denominator is equal to $r^* - n$.

Exercise 13.19. Consider the model of endogenous technological progress with limited knowledge spillover as discussed in Section 13.3.

(1) Characterize the transitional dynamics of the economy starting from an arbitrary $N(0) > 0$.
(2) Characterize the Pareto optimal allocation and compare it to the equilibrium allocation in Proposition 13.5.
(3) Analyze the effect of the following two policies: first, a subsidy to research; second, the patent policy, where each patent expires at the rate $\iota > 0$. Explain why the effects of these policies on economic growth are different than their effects in the baseline endogenous growth model.

Exercise 13.20. Consider the model in Section 13.3. Suppose that there are two economies with identical preferences, technology and initial conditions, except country 1 starts with population $L_1(0)$ and country 2 starts with $L_2(0) > L_1(0)$. Show that income per capita is always higher in country 2 than in country 1.

Exercise 13.21. Consider the lab equipment model of Section 13.1, but modify the innovation possibilities frontier to

$$\dot{N}(t) = \eta N(t)^{-\phi} Z(t),$$

where $\phi > 0$.

(1) Define an equilibrium.
(2) Characterize the market clearing factor prices and determine the free entry condition.
(3) Show that without population growth, there will be no sustained growth in this economy.
(4) Now consider population growth at the exponential rate n, and show that this model generates sustained equilibrium growth as in the model analyzed in Section 13.3.

Exercise 13.22. Consider the baseline endogenous technological change model with expanding machine varieties in Section 13.1. Suppose that x's now denote machines that do not immediately depreciate. In contrast, once produced these machines depreciate as an exponential rate δ. References and the rest of the production structure remain unchanged.

(1) Define an equilibrium in this economy.
(2) Formulate the maximization problem of producers of machines. [Hint: it is easier to formulate the problem in terms of machine rentals rather than machine sales].
(3) Characterize the equilibrium in this economy and show that all the results are identical to those in Section 13.1.

Exercise 13.23. Consider the following model. Population at time t is $L(t)$ and grows at the constant rate n (i.e., $\dot{L}(t) = nL(t)$). All agents have preferences given by

$$\int_0^\infty \exp(-\rho t) \frac{C(t)^{1-\theta} - 1}{1-\theta} dt,$$

where C is consumption defined over the final good of the economy. This good is produced as

$$Y(t) = \left[\int_0^N y(\nu, t)^\beta d\nu\right]^{1/\beta},$$

where $y(\nu, t)$ is the amount of intermediate good ν used in production at time t. The production function of each intermediate is

$$y(\nu, t) = l(\nu, t)$$

where $l(\nu, t)$ is labor allocated to this good at time t.

New goods are produced by allocating workers to the R&D process, with the production function

$$\dot{N}(t) = \eta N^\phi(t) L_R(t)$$

where $\phi \leq 1$ and $L_R(t)$ is labor allocated to R&D at time t. So labor market clearing requires $\int_0^{N(t)} l(\nu, t) dv + L_R(t) = L(t)$. Risk-neutral firms hire workers for R&D. A firm who discovers a new good becomes the monopoly supplier, with a perfectly and indefinitely enforced patent.

(1) Characterize the balanced growth path in the case where $\phi = 1$ and $n = 0$, and show that there are no transitional dynamics. Why is this? Why does the long-run growth rate depend on θ? Why does the growth rate depend on L? Do you find this plausible?
(2) Now suppose that $\phi = 1$ and $n > 0$. What happens? Interpret.
(3) Now characterize the balanced growth path when $\phi < 1$ and $n > 0$. Does the growth rate depend on L? Does it depend on n? Why? Do you think that the configuration $\phi < 1$ and $n > 0$ is more plausible than the one with $\phi = 1$ and $n = 0$?

Exercise 13.24. Derive equation (13.43). [Hint: use the first-order condition between two products ν and ν', and then substitute into the budget constraint of the representative household with total expenditure denoted by $C(t)$].

Exercise 13.25. Using (13.43) and the choice of numeraire in (13.44), set up the consumer optimization problem in the form of the current-value Hamiltonian. Derive the consumer Euler equation (13.45).

Exercise 13.26. Consider the model analyzed in Section 13.4.

(1) Show that the allocation described in Proposition 13.6 always satisfies the transversality condition.
(2) Show that in this model there are no transitional dynamics.
(3) Characterize the Pareto optimal allocation and show that the equilibrium growth rate in Proposition 13.6 is less than the growth rate in the Pareto optimal allocation.

CHAPTER 14

Models of Schumpeterian Growth

The previous chapter presented the basic endogenous technological change models based on expanding input or product varieties. The advantage of these models is their relative tractability. While the expansion of the variety of machines used in production captures certain aspects of process innovation, most process innovations we observe in practice either increase the quality of an existing product or reduce the costs of production. Therefore, typical process innovations have a number of distinct features compared to the "horizontal innovations" of the previous chapter. For example, in the expanding machine variety model a newly-invented computer is used alongside all previous vintages of computers, though, in practice, a newly-invented superior computer often *replaces* existing vintages. Thus in some fundamental sense, models of expanding machine variety do not provide a good description of innovation dynamics in practice because they do not capture the *competitive* aspect of innovations. These competitive aspects bring us to the realm of Schumpeterian *creative destruction* in which economic growth is driven, at least in part, by new firms replacing incumbents. For this reason, the models discussed in this chapter are often referred to as *Schumpeterian growth models*. My purpose in this chapter is to develop tractable models of Schumpeterian growth or of growth with "competitive innovations".

As Chapter 12 discussed, innovations that involve quality improvements or cost reductions will feature the replacement effect, which implies that entrants should be more active in the research process than incumbents. Schumpeterian growth therefore raises a number of novel and important issues. First, in contrast to the models of expanding varieties, there may be direct price competition between different producers with different vintages of quality or different costs of producing the same product. This will affect both the description of the growth process and a number of its central implications. For example, market structure and anti-trust policy can play potentially richer roles in models exhibiting this type of price competition. Second, competition between incumbents and entrants brings the business stealing effect we encountered in Chapter 12 to the fore and raises the possibility of excessive innovations.

This description suggests that a number of new and perhaps richer issues arise when we model Schumpeterian growth. One may then expect models of Schumpeterian models to be significantly more complicated than expanding varieties models. This is not necessarily the case, however. In this chapter, we will present the basic models of competitive innovations, first proposed by Aghion and Howitt (1992) and then further developed by Grossman and Helpman (1991a,b) and Aghion and Howitt (1998). The literature on models of Schumpeterian economic growth is now large and an excellent survey is presented in Aghion and Howitt (1998). Our purpose here is not to provide a detailed survey, but to emphasize the most important implications of these models for understanding cross-country income differences and the process of economic growth. We will also present these models in a way that parallels the mathematical structure of the expanding varieties models, both to emphasize the similarities and clarify the differences. A number of distinct applications of these models are also discussed later in the chapter and in the exercises.

14.1. A Baseline Model of Schumpeterian Growth

14.1.1. Preferences and Technology. In this section, we present a tractable model of Schumpeterian growth. We choose the economy to be as similar to the baseline (lab equipment) expanding machine variety model as possible, both to emphasize the parallels in the mathematical structures between these models and to highlight the basic economic differences that come from the presence of competition between new innovations and existing inputs (or products). The economy is in continuous time and admits a representative household with the standard CRRA preferences, (13.1), as in the previous chapter. Population is constant at L and labor is supplied inelastically. The resource constraint at time t again takes the form

$$C(t) + X(t) + Z(t) \leq Y(t), \tag{14.1}$$

where $C(t)$ is consumption, $X(t)$ is aggregate spending on machines, and $Z(t)$ is total expenditure on R&D at time t.

We again assume that there is a continuum of machines used in the production of a unique final good. Since there will be no expansion of inputs/machine variety, we normalize the measure of inputs to 1, and denote each machine line by $\nu \in [0,1]$. The engine of economic growth here will be process innovations that lead to *quality improvement*. Let us first specify how the qualities of different machine lines change over time. Let $q(\nu,t)$ be the quality of machine line ν at time t. We assume the following "quality ladder" for each machine type:

$$q(\nu,t) = \lambda^{n(\nu,t)} q(\nu,0) \text{ for all } \nu \text{ and } t, \tag{14.2}$$

where $\lambda > 1$ and $n(\nu,t)$ denotes the number of innovations on this machine line between time 0 and time t. This specification implies that there is a ladder of quality for each machine type, and each innovation takes the machine quality up by one rung in this ladder. These rungs are proportionally equi-distant, so that each improvement leads to a proportional increase in quality by an amount $\lambda > 1$. Growth will be the result of these quality improvements.

The production function of the final good is similar to that in the previous chapter, except that now the quality of the machines matters for productivity. We write the aggregate production function of the economy as follows:

$$Y(t)=\frac{1}{1-\beta}\left[\int_0^1 q(\nu,t)x(\nu,t\mid q)^{1-\beta}d\nu\right]L^{\beta}, \tag{14.3}$$

where $x(\nu,t\mid q)$ is the quantity of the machine of type ν of quality q used in the production process. As in the previous chapter, this production function can be written as

$$Y(t)=\frac{1}{1-\beta}\tilde{\mathbf{X}}(t)^{1-\beta}L^{\beta},$$

where

$$\tilde{\mathbf{X}}(t)\equiv\left[\int_0^1 q(\nu,t)x(\nu,t\mid q)^{\frac{\varepsilon_\beta-1}{\varepsilon_\beta}}d\nu\right]^{\frac{\varepsilon_\beta}{\varepsilon_\beta-1}},$$

with $\varepsilon_\beta\equiv 1/\beta$, which again emphasizes the continuity with the Dixit-Stiglitz model of Chapter 12.

An implicit assumption in (14.3) is that at any point in time only one quality of any machine is used. This is without loss of any generality, since in equilibrium only the highest-quality machines of each type will be used. This production function already indicates the source of *creative destruction* in this class on models: when a higher-quality machine is invented it will replace ("destroy") the previous vintage of machines.

We next specify the technology for producing machines of different qualities and the innovation possibilities frontier of this economy. First, new machine vintages are invented by R&D. The R&D process is cumulative, in the sense that new R&D builds on an existing machine type. For example, consider the machine line ν that has quality $q(\nu,t)$ at time t. R&D on this machine line will attempt to improve over this quality. If a firm spends $Z(\nu,t)$ units of the final good for research on this machine line, then it generates a flow rate $\eta Z(\nu,t)/q(\nu,t)$ of innovation. The innovation advances the knowhow on the production of this machine to the new rung of the quality ladder, thus creates a machine of type ν with quality $\lambda q(\nu,t)$. Note that one unit of R&D spending is proportionately less effective when applied to a more advanced machine. This is intuitive, since we expect research on more advanced machines to be more difficult. It is also convenient from a mathematical point of view, since the benefit of research is also increasing with the quality of the machine (in particular, the quality improvements are *proportional*, with an innovation increasing quality from $q(\nu,t)$ to $\lambda q(\nu,t)$). Note that the costs of R&D are identical for the current incumbent and new firms (see Exercise 14.5 for alternative formulations). We assume that there is free entry into research, thus any firm or individual can undertake this type of research on any of the machine lines.

As in the expanding varieties models of the previous chapter, the firm that makes an innovation has a perpetual patent on the new machine has invented. However, note that the patent system does not preclude other firms undertaking research based on the product invented by this firm. We will discuss below how different patenting arrangements might affect incentives in this model.

Once a particular machine of quality $q(\nu, t)$ has been invented, any quantity of this machine can be produced at the marginal cost $\psi q(\nu, t)$. Once again, the fact that the marginal cost is proportional to the quality of the machine is natural, since producing higher-quality machines should be more expensive.

One noteworthy issue here concerns the identity of the firm that will undertake R&D and innovation. In the expanding varieties model, this was irrelevant, since machines could not be improved upon, so there was only R&D for new machines, and who undertook the R&D was not important. Here, in contrast, existing machines can be (and are) improved, and this is the source of economic growth. We have already seen in Chapter 12 that if the cost of R&D are identical for incumbents and new firms, Arrow's replacement effect will imply that it will be the new entrants that undertake the R&D. The same applies in this model. The incumbent has weaker incentives to innovate, since it would be replacing its own machine, and thus destroying the profits that it is already making. In contrast, a new entrant does not have this replacement calculation in mind. As a result, with the same technology of innovation, it will always be the entrants that undertake the R&D investments in this model (see Exercise 14.1). This is an attractive implication, since it creates a real sense of creative destruction or churning. Of course in practice we observe established and leading firms undertaking innovations. This might be because the technology of innovation differs between incumbents and new potential entrants, or there is only a limited number of new entrants as in the model studied in Section 14.4 below (though in the current model this will not be sufficient, see Exercise 14.1).

14.1.2. Equilibrium. An allocation in this economy is similar to that in the previous chapter. It consists of time paths of consumption levels, aggregate spending on machines, and aggregate R&D expenditure $[C(t), X(t), Z(t)]_{t=0}^{\infty}$, time paths of machine qualities denoted by, $[q(\nu, t)]_{\nu\in[0,1],t=0}^{\infty}$, time paths of prices and quantities of each machine and the net present discounted value of profits from that machine, $[p^x(\nu, t \mid q), x(\nu, t), V(\nu, t \mid q)]_{\nu\in[0,1],t=0}^{\infty}$, and time paths of interest rates and wage rates, $[r(t), w(t)]_{t=0}^{\infty}$. We will now characterize the equilibrium in this economy.

Let us start with the aggregate production function for the final good producers. A similar analysis to that in the previous chapter implies that the demand for machines is given by

$$x(\nu, t \mid q) = \left(\frac{q(\nu, t)}{p^x(\nu, t \mid q)}\right)^{1/\beta} L \quad \text{for all } \nu \in [0,1] \text{ and all } t, \tag{14.4}$$

where $p^x(\nu, t \mid q)$ refers to the price of machine type ν of quality $q(\nu, t)$ at time t. This expression stands for $p^x(\nu, t \mid q(\nu, t))$, but there should be no confusion in this notation since it is clear that q here refers to $q(\nu, t)$, and we will use this notation for other variables as well. The price $p^x(\nu, t \mid q)$ will be determined by the profit-maximization of the monopolist holding the patent for machine of type ν of quality $q(\nu, t)$. Note that the demand from the final good sector for machines in (14.4) is iso-elastic as in the previous chapter, so the unconstrained monopoly price is again a constant markup over marginal cost. However, contrary to the situation in the previous chapter, there is now competition between firms that have access to different vintages of the machine. This implies that, as in our discussion in Chapter 12, we need to consider two regimes, one in which the innovation is "drastic" so that each firm can charge the unconstrained monopoly price, and the other one in which limit prices have to be used. Which regime we are in does not make any difference to the mathematical structure or to the substantive implications of the model. Nevertheless, we have to choose one of these two alternatives for consistency. Here we assume that the quality gap between a new machine and the machine that it replaces, λ, is sufficiently large, in particular, satisfies

$$\lambda \geq \left(\frac{1}{1-\beta}\right)^{\frac{1-\beta}{\beta}}, \tag{14.5}$$

so that we are in the drastic innovations regime (see Exercise 14.7 for the derivation of this condition and Exercise 14.8 for the structure of the equilibrium under the alternative assumption). Let us also normalize $\psi \equiv 1-\beta$ as in the previous chapter, which implies that the profit-maximizing monopoly price is

$$p^x(\nu, t \mid q) = q(\nu, t). \tag{14.6}$$

Combining this with (14.4) implies that

$$x(\nu, t \mid q) = L. \tag{14.7}$$

Consequently, the flow profits of a firm with the monopoly rights on the machine of quality $q(v, t)$ can be computed as:

$$\pi(\nu, t \mid q) = \beta q(\nu, t) L.$$

This only differs from the flow profits in the previous chapter because of the presence of the quality term, $q(\nu, t)$. Next, substituting (14.7) into (14.3), we obtain that total output is given by

$$Y(t) = \frac{1}{1-\beta} Q(t) L, \tag{14.8}$$

where

$$Q(t) = \int_0^1 q(\nu, t) d\nu \tag{14.9}$$

is the average total quality of machines. This expression closely parallels the derived production function (13.12) in the previous chapter, except that instead of the number of machine varieties, $N(t)$, labor productivity is determined by the average quality of the machines, $Q(t)$. This expression also clarifies the reasoning for the particular functional form assumptions above. In particular, the reader can verify that it is the linearity of the aggregate production function of the final good, (14.3) in the quality of machines that makes labor productivity depend on average qualities. With alternative assumptions, a similar expression to (14.8) would still obtain, but with a more complicated aggregator of machine qualities than the simple average (see, for example, Section 14.4). As a byproduct, we also obtain that aggregate spending on machines is

$$X(t) = (1-\beta) Q(t) L. \tag{14.10}$$

Similar to the previous chapter, labor, which is only used in the final good sector, receives an equilibrium wage rate of

$$w(t) = \frac{\beta}{1-\beta} Q(t). \tag{14.11}$$

We next specify the value function for the monopolist of variety ν of quality $q(\nu, t)$ at time t. As in the previous two chapters, despite the fact that each firm generates a stochastic stream of revenues, the presence of many firms with independent risks implies that each should maximize expected profits. The net present value of expected profits can be written in the Hamilton-Jacobi-Bellman form as follows

$$r(t) V(\nu, t \mid q) - \dot{V}(\nu, t \mid q) = \pi(\nu, t \mid q) - z(\nu, t \mid q) V(\nu, t \mid q), \tag{14.12}$$

where $z(\nu, t \mid q)$ is the rate at which new innovations occur in sector ν at time t, while $\pi(\nu, t \mid q)$ is the flow of profits. This value function is somewhat different from the ones in the previous chapter (e.g., (13.8)), because of the last term on the right-hand side, which captures the essence of Schumpeterian growth. When a new innovation occurs, the existing monopolist loses its monopoly position and is replaced by the producer of the higher-quality machine. From then on, it receives zero profits, and thus has zero value. In writing this equation, we have made use of the fact that because of Arrow's replacement effect, it is an entrant that is undertaking the innovation, thus $z(\nu, t \mid q)$ corresponds to the flow rate at which the incumbent will be replaced by a new entrant.

Free entry again implies that we must have

$$\eta V(\nu, t \mid q) \leq \lambda^{-1} q(\nu, t) \text{ and } \eta V(\nu, t \mid q) = \lambda^{-1} q(\nu, t) \text{ if } Z(\nu, t \mid q) > 0. \tag{14.13}$$

In other words, the value of spending one unit of the final good should not be strictly positive. Recall that one unit of the final good spent on R&D for a machine of quality $\lambda^{-1} q$ has a flow rate of success equal to $\eta / \left(\lambda^{-1} q\right)$, and in this case, it generates a new machine of quality

q, which will have a net present value gain of $V(\nu, t \mid q)$. If there is positive R&D, i.e., $Z(\nu, t \mid q) > 0$, then the free entry condition must hold as equality.

Note also that even though the quality of individual machines, the $q(\nu, t)$'s, are stochastic (and depend on success in R&D), as long as R&D expenditures, the $Z(\nu, t \mid q)$'s, are nonstochastic, average quality $Q(t)$, and thus total output, $Y(t)$, and total spending on machines, $X(t)$, will be nonstochastic. This feature will significantly simplify notation and the analysis of this economy.

Consumer maximization again implies the familiar Euler equation,

$$\frac{\dot{C}(t)}{C(t)} = \frac{1}{\theta}(r(t) - \rho), \tag{14.14}$$

and the transversality condition takes the form

$$\lim_{t \to \infty} \left[\exp\left(- \int_0^t r(s)\, ds \right) \int_0^1 V(\nu, t \mid q)\, d\nu \right] = 0 \tag{14.15}$$

for all q. This transversality condition follows because now the total value of corporate assets is $\int_0^1 V(\nu, t \mid q)\, d\nu$. Even though the evolution of the quality of each machine line is stochastic, the value of a machine of type ν of quality q at time t, $V(\nu, t \mid q)$, is nonstochastic. Either q is not the highest quality in this machine line, in which case $V(\nu, t \mid q)$ is equal to 0, or alternatively, it is given by (14.12).

These equations complete the description of the environment. An equilibrium can then be represented as time paths of consumption, aggregate spending on machines, and aggregate R&D, $[C(t), X(t), Z(t)]_{t=0}^{\infty}$ that satisfy (14.1), (14.10), (14.15), time paths of aggregate machine quality $[Q(t)]_{t=0}^{\infty}$ and value functions $[V(\nu, t \mid q)]_{\nu \in [0,1], t=0}^{\infty}$ consistent with (14.9), (14.12) and (14.13), time paths of prices and quantities of machines that have highest quality in their lines at that point in time and the net present discounted value of profits from those machines, $[p^x(\nu, t \mid q), x(\nu, t)]_{\nu \in [0,1], t=0}^{\infty}$ given by (14.6) and (14.7), and time paths of interest rates and wage rates, $[r(t), w(t)]_{t=0}^{\infty}$ that are consistent with (14.11) and (14.14).

We will first focus on an equilibrium path with balance growth, i.e., on the the balanced growth path (BGP), where output and consumption grow at constant rates.

14.1.3. Balanced Growth Path. In the balanced growth path (BGP), consumption grows at the constant rate g_C^*. With familiar arguments, this must be the same rate as output growth, g^*. Moreover, from (14.14), the interest rate must be constant, i.e., $r(t) = r^*$ for all t.

If there is positive growth in this BGP equilibrium, then there must be research at least in some sectors. Since both profits and R&D costs are proportional to quality, whenever the free entry condition (14.13) holds as equality for one machine type, it will hold as equality for all of them. This, in turn, implies that

$$V(\nu, t \mid q) = \frac{q(\nu, t)}{\lambda \eta}. \tag{14.16}$$

Moreover, if it holds between t and $t + \Delta t$, $\dot{V}(\nu, t \mid q) = 0$, because the right-hand side of equation (14.16) is constant over time—$q(\nu, t)$ refers to the quality of the machine supplied by the incumbent, which does not change. Since R&D for each machine type has the same productivity, this implies that $z(\nu, t)$ must also be the same for all machine types, thus equal to some $z(t)$. Moreover, in BGP, this rate will be constant and we will denote it by z^*. Then (14.12) implies

$$V(\nu, t \mid q) = \frac{\beta q(\nu, t) L}{r^* + z^*}. \tag{14.17}$$

Notice the difference between this value function and those in the previous chapter: instead of the discount rate r^*, the effective discount rate is $r^* + z^*$, since incumbent monopolists understand that competitive innovations will replace them.

Combining this equation with (14.16), we obtain

$$r^* + z^* = \lambda \eta \beta L. \tag{14.18}$$

Moreover, from the fact that $g_C^* = g^*$ and (14.14), we have that $g^* = (r^* - \rho)/\theta$, or

$$r^* = \theta g^* + \rho. \tag{14.19}$$

To solve for the BGP equilibrium, we need a final equation relating the BGP growth rate of the economy, g^*, to z^*. From (14.8)

$$\frac{\dot{Y}(t)}{Y(t)} = \frac{\dot{Q}(t)}{Q(t)}.$$

Next, note that, by definition, in an interval of time Δt, there will be $z(t)\Delta t$ sectors that experience one innovation, and this will increase their productivity by λ. The measure of sectors experiencing more than one innovation within this time interval is $o(\Delta t)$—i.e., it is second-order in Δt, so that as $\Delta t \to 0$, $o(\Delta t)/\Delta t \to 0$. Therefore, we have

$$Q(t + \Delta t) = \lambda Q(t) z(t) \Delta t + (1 - z(t)\Delta t) Q(t) + o(\Delta t).$$

Now subtracting $Q(t)$ from both sides, dividing by Δt and taking the limit as $\Delta t \to 0$, we obtain

$$\dot{Q}(t) = (\lambda - 1) z(t) Q(t).$$

Therefore,

$$g^* = (\lambda - 1) z^*. \tag{14.20}$$

Now combining (14.18)-(14.20), we obtain the BGP growth rate of output and consumption as:

$$g^* = \frac{\lambda \eta \beta L - \rho}{\theta + (\lambda - 1)^{-1}}. \tag{14.21}$$

This establishes the following proposition

Proposition 14.1. *Consider the model of Schumpeterian growth described above. Suppose that*

$$\lambda \eta \beta L > \rho > (1 - \theta) \lambda \eta \beta L \ . \tag{14.22}$$

Then, there exists a unique balanced growth path in which average quality of machines, output and consumption grow at rate g^* given by (14.21). The rate of innovation is $g^* / (\lambda - 1)$.

Proof. Most of the proof is given in the preceding analysis. In Exercise 14.3 you are asked to check that the BGP equilibrium is unique and satisfies the transversality condition. □

The above analysis illustrates that the mathematical structure of the model is quite similar to those analyzed in the previous chapter. Nevertheless, the feature of creative destruction, the process of incumbent monopolists being replaced by new entrants, is new and provides a very different interpretation of the growth process. We will return to some of the applications of creative destruction below.

Before doing this, we can also analyze transitional dynamics in this economy. Similar arguments to those used in the previous chapter establish the following result:

Proposition 14.2. *In the model of Schumpeterian growth described above, starting with any average quality of machines $Q(0) > 0$, there are no transitional dynamics and the equilibrium path always involves constant growth at the rate g^* given by (14.21).*

Proof. See Exercise 14.4. □

A notable feature of the model, which is again related to the functional form of the aggregate production function (14.3), is that only the average quality of machines, $Q(t)$, matters for the allocation of resources. Moreover, the incentives to undertake research are identical for two machine types ν and ν', with different quality levels $q(\nu, t)$ and $q(\nu', t)$, thus there is no incentive to undertake different R&D investments for more and less advanced machines. This is again a feature of the functional forms chosen here, and Exercise 14.13 shows that in different circumstances this result may not apply. Nevertheless, the specification chosen in this section is appealing, since it implies that research will be directed at a broad range of machines rather than a specific subset of the available types of machines.

14.1.4. Pareto Optimality. This equilibrium, like that of the endogenous technology model with expanding varieties, is typically Pareto suboptimal. The first reason for this is the appropriability effect, which results because monopolists are not able to capture the entire social gain created by an innovation. However, Schumpeterian growth also introduces the business stealing effect discussed in Chapter 12. Consequently, the equilibrium rate of innovation and growth can now be too high or too low. We now investigate this question.

We proceed as in the previous chapter, first deriving quantities of machines that will be used in the final good sector by the social planner. In the social planner's allocation there is no markup on machines, thus we have

$$\begin{aligned} x^S(\nu, t \mid q) &= \psi^{-1/\beta} L \\ &= (1-\beta)^{-1/\beta} L. \end{aligned}$$

Substituting this into (14.3), we obtain

$$Y^S(t) = (1-\beta)^{-1/\beta} Q^S(t) L,$$

where again the superscript S refers to the social planner's allocation. The net output that can be distributed between consumption and research expenditure is

$$\begin{aligned} \tilde{Y}^S(t) &\equiv Y^S(t) - X^S(t) \\ &= (1-\beta)^{-1/\beta} Q^S(t) L - \int_0^1 \psi q(\nu, t) x^S(\nu, t \mid q) d\nu \\ &= (1-\beta)^{-1/\beta} \beta Q^S(t) L. \end{aligned} \tag{14.23}$$

Moreover, given the specification of the innovation possibilities frontier above, the social planner can improve the aggregate technology as follows:

$$\dot{Q}^S(t) = \eta(\lambda - 1) Z^S(t),$$

since an R&D spending of $Z^S(t)$ will lead to discoveries of better vintages at the flow rate of η and each of these vintages increases average quality of machines by a proportional amount $\lambda - 1$.

Now, given this equation, the maximization problem of the social planner can be written as

$$\max \int_0^\infty \exp(-\rho t) \frac{C^S(t)^{1-\theta} - 1}{1-\theta} dt$$

subject to

$$\dot{Q}^S(t) = \eta(\lambda - 1)(1-\beta)^{-1/\beta} \beta Q^S(t) L - \eta(\lambda - 1) C^S(t),$$

where the constraint equation uses net output, (14.23), and the resource constraint, (14.1). In this problem, $Q^S(t)$ is the state variable, and $C^S(t)$ is the control variable. It can be verified that this problem satisfies all the assumptions of Theorems 7.9 and 7.12, so a solution that satisfies the necessary conditions in Theorem 7.9 will give the unique optimal growth path. To characterize this solution, let us set up the current-value Hamiltonian as

$$\hat{H}\left(Q^S, C^S, \mu^S\right) = \frac{C^S(t)^{1-\theta} - 1}{1-\theta} + \mu^S(t)\left[\eta(\lambda-1)(1-\beta)^{-1/\beta}\beta Q^S(t)L - \eta(\lambda-1)C^S(t)\right].$$

The necessary conditions for a maximum are

$$\begin{aligned}
\hat{H}_C\left(Q^S, C^S, \mu^S\right) &= C^S(t)^{-\theta} - \mu^S(t)\eta(\lambda-1) = 0 \\
\hat{H}_Q\left(Q^S, C^S, \mu^S\right) &= \mu^S(t)\eta(\lambda-1)(1-\beta)^{-1/\beta}\beta L = \rho\mu^S(t) - \dot{\mu}^S(t) \\
\lim_{t\to\infty}\left[\exp(-\rho t)\mu^S(t)Q^S(t)\right] &= 0.
\end{aligned}$$

Moreover, it is straightforward to verify that the current-value Hamiltonian is concave in C and Q, so any solution to these necessary conditions is an optimal plan. Combining these conditions, we obtain the following growth rate for consumption in the social planner's allocation (see Exercise 14.6):

$$\frac{\dot{C}^S(t)}{C^S(t)} = g^S \equiv \frac{1}{\theta}\left(\eta(\lambda-1)(1-\beta)^{-1/\beta}\beta L - \rho\right). \tag{14.24}$$

Clearly, total output and average quality will also grow at the rate g^S in this allocation.

Comparing g^S to g^* in (14.21), we can see that either could be greater. In particular, when λ is very large, $g^S > g^*$, and there is insufficient growth in the equilibrium. We can see this as follows: as $\lambda \to \infty$, $g^S/g^* \to (1-\beta)^{-1/\beta} > 1$. In contrast, to obtain an example in which there is excessive growth in the equilibrium, suppose that $\theta = 1$, $\beta = 0.9$, $\lambda = 1.3$, $\eta = 1$, $L = 1$ and $\rho = 0.38$. In this case, it can be verified that $g^S \approx 0$, while $g^* \approx 0.18 > g^S$.[1]

This illustrates the counteracting influences of the appropriability and business stealing effects discussed above. The following proposition summarizes this result:

Proposition 14.3. *In the model of Schumpeterian growth described above, the decentralized equilibrium is generally Pareto suboptimal, and may have a higher or lower rate of innovation and growth than the Pareto optimal allocation.*

It is also straightforward to verify that as in the models of the previous section, there is a scale effect, and thus population growth would lead to an exploding growth path. Exercise 14.10 asks you to construct an endogenous growth model of Schumpeterian growth without scale effects.

14.1.5. Policy in the Model of Schumpeterian Growth. We now use the Schumpeterian growth model to analyze the effects of policy on economic growth. As in the model

[1] Notice that the combination of $\beta = 0.9$ and $\lambda = 1.3$ are consistent with (14.5), which was used in deriving the equilibrium growth rate g^*.

of the previous few chapters, anti-trust policy, patent policy and taxation will affect the equilibrium growth rate. For example, two economies that tax corporate incomes at different rates, say τ and τ', will grow at different rates.

There is a sense in which the current model is much more appropriate for conducting policy analysis than the expanding varieties models, however. In those models, there was no reason for any agent in the economy to support distortionary taxes (which reduce the growth rate).[2] In contrast, the fact that growth takes place through creative destruction here implies that there is a natural conflict of interest, and certain types of policies may have a constituency. To illustrate this point, which will be discussed in greater detail in Part 8 of the book, suppose that there is a tax τ imposed on R&D spending. This has no effect on the profits of existing monopolists, and only influences their net present discounted value via replacement. Since taxes on R&D will discourage R&D, there will be replacement at a slower rate, i.e., z^* will fall. This increases the steady-state value of all monopolists given by (14.17). In particular, denoting the value of a monopolist with a machine of quality q by $V(q)$, we have

$$V(q) = \frac{\beta q L}{r^*(\tau) + z^*(\tau)},$$

where the equilibrium interest rate and the replacement rate have been written as functions of τ. With the tax rate on R&D, the free entry condition, (14.13) becomes

$$V(q) = \frac{(1+\tau)}{\lambda\eta} q.$$

This equation shows that $V(q)$ is clearly increasing in the tax rate on R&D, τ. Combining the previous two equations, we see that in response to a positive rate of taxation, $r^*(\tau) + z^*(\tau)$ must adjust downward, so that the value of current monopolists increases (consistent with the previous equation). Intuitively, when the costs of R&D are raised because of tax policy, the value of a successful innovation, $V(q)$, must increase to satisfy the free entry condition. This can only happen through a decline the effective discount rate $r^*(\tau) + z^*(\tau)$. A lower effective discount rate, in turn, is achieved by a decline in the equilibrium growth rate of the economy, which now takes the form

$$g^*(\tau) = \frac{(1+\tau)^{-1}\lambda\eta\beta L - \rho}{\theta + (\lambda-1)^{-1}}.$$

It is straightforward to verify that this growth rate is strictly decreasing in τ. Nevertheless, as the previous expression shows, incumbent monopolists would be in favor of increasing τ in order to shield themselves from the competition of new entrants. Essentially, in this

[2]Naturally, one can enrich these models so that tax revenues are distributed unequally across agents, for example, with taxes on capital distributed to workers. In this case, even in the basic neoclassical growth model, some groups could prefer distortionary taxes. Such models will be discussed in Part 8 of the book.

model, slowing down the process of creative destruction is beneficial for incumbents, creating a rationale for growth-retarding policies to emerge in equilibrium.

Therefore, an important advantage of models of Schumpeterian growth is that they start providing us clues about why some societies may adopt policies that reduce the growth rate. Since taxing R&D by new entrants benefits incumbent monopolists, when incumbents are sufficiently powerful politically, such distortionary taxes can emerge in the political economy equilibrium, even though they are not in the interest of the society at large.

14.2. A One-Sector Schumpeterian Growth Model

The model of Schumpeterian growth presented in the previous section was designed to maximize the parallels between this class of models and those based on expanding varieties. In this section, we discuss a model more closely related to the original Aghion and Howitt (1992) setup, which is simpler in some ways and more complicated in others. Relative to the model presented in the previous section, it has two major differences. First, there is only one sector experiencing quality improvements rather than a continuum of machine types. Second, the innovation possibilities frontier uses a scarce factor, labor, as in the model of knowledge spillovers in Section 13.2 of the previous chapter. Since there are many parallels between this model and those we have studied so far, we will provide only a brief exposition of this model.

14.2.1. The Basic Aghion-Howitt Model. The consumer side is the same as before, with the only difference that we now assume consumers are risk neutral, so that the interest rate is determined as

$$r^* = \rho$$

at all points in time. Population is again constant at a level L and all individuals supply labor inelastically. The aggregate production function of the unique final good is now given by

$$Y(t) = \frac{1}{1-\beta} x(t \mid q)^{1-\beta} (q(t) L_E(t))^{\beta}, \tag{14.25}$$

where $q(t)$ is the quality of the unique machine used in production and is written in the labor-augmenting form for simplicity; $x(t \mid q)$ is the quantity of this machine used at time t; and $L_E(t)$ denotes the amount labor used in production at time t, which is less than L, since $L_R(t)$ workers will be employed in the R&D sector. Market clearing requires that

$$L_E(t) + L_R(t) \leq L.$$

Once invented, a machine of quality $q(t)$ can be produced at the constant marginal cost ψ in terms of final goods. We again normalize $\psi \equiv 1-\beta$. The innovation possibilities frontier now involves labor being used for R&D. In particular, each worker employed in the R&D sector generates a flow rate η of a new machine. When the current machine used in production has quality $q(t)$, the new machine has quality $\lambda q(t)$.

Let us once again assume that (14.5) above is satisfied, so that the monopolist can charge the unconstrained monopoly price. Then, an analysis similar to that in the previous section immediately implies that the demand for the leading-edge machine of quality q is given by

$$x(t \mid q) = p^x(t)^{-1/\beta} q(t) L_E(t),$$

where again $p^x(\nu, t)$ denotes the price of the machine of quality q. The monopoly price for the highest quality machine is:[3]

$$p^x(t \mid q) = \frac{\psi}{1-\beta} = 1,$$

for all q and t. Consequently, the demand for the machine of quality q at time t is given by

$$x(t \mid q) = q(t) L_E(t),$$

and monopoly profits are

$$\pi(t \mid q) = \beta q(t) L_E(t).$$

We can also write aggregate output as

$$Y(t \mid q) = \frac{1}{1-\beta} q(t) L_E(t),$$

where we again condition on the quality of the machine available at the time, q. This also implies that the equilibrium wage, determined from the final good sector, is given by

$$w(t \mid q) = \frac{\beta}{1-\beta} q(t).$$

When there is no need to emphasize time dependence, we will write this wage rate as a function of machine quality, i.e., as $w(q)$.

Let us now focus on a "steady-state equilibrium" in which the flow rate of innovation is constant and equal to z^*. Steady state here is an quotation marks since, even though the flow rate of innovation is constant, consumption and output growth will not be constant because of the stochastic nature of innovation (and this is the reason why we do not use the term "balanced growth path" in this context). This implies that a constant number (and thus a constant fraction) of workers, L_R^*, must be working in research. Since the interest rate is equal to $r^* = \rho$, this implies that the steady-state value of a monopolist with a machine of quality q is given by

$$V(q) = \frac{\beta q (L - L_R^*)}{\rho + z^*},$$

[3]This expression follows the original Aghion and Howitt (1992) paper and assumes that innovators ignore their effect on equilibrium wages. In the baseline model presented above, since there was a continuum of monopolistically competitive firms, each had no effect on equilibrium wages. Here, instead, the single monopolist could recognize that its price will also affect equilibrium wages and thus the cost of R&D for its competitors. In this case, it may want to set a lower price than $\psi/(1-\beta)$. Following Aghion and Howitt (1992), I ignore this issue.

where we have used the fact that in steady state total employment in the final good sector is equal to $L_E^* = L - L_R^*$. To simplify the notation, we also wrote V as a function of q only rather than a function of both q and time. Free entry requires that when the current machine quality is q, the wage paid to one more R&D worker, $w(q)$, must be equal to the flow benefits, $\eta V(\lambda q)$, thus

$$w(q) = \eta V(\lambda q).$$

Flow benefits from R&D are equal to $\eta V(\lambda q)$, since, when current machine quality is q, one more worker in R&D leads to the discovery of a new machine of quality λq at the flow rate η. In addition, given the R&D technology, we must have $z^* = \eta L_R^*$. Combining the last four equations we obtain

$$\frac{\lambda(1-\beta)\eta(L - L_R^*)}{\rho + \eta L_R^*} = 1,$$

which uniquely determines the steady-state number of workers in research as

$$L_R^* = \frac{\lambda(1-\beta)\eta L - \rho}{\eta + \lambda(1-\beta)\eta}, \tag{14.26}$$

as long as this expression is positive.

Contrary to the model in the previous section, however, this does not imply that output grows at a constant rate. Since there is only one sector undergoing technological change and this sector experiences growth only at finite intervals, the growth rate of the economy will have an *uneven* nature; in particular, it can be verified that the economy will have constant output for an interval of time (of average length $1/\eta L_R^*$; see Exercise 14.15) and then will have a burst of growth when a new machine is invented. This pattern of uneven growth is a consequence of having only one sector rather than the continuum of sectors in the model of the previous section. Whether it provides a better approximation to reality is open to debate. While modern capitalist economies do not grow at constant rates, they also do not have as jagged a growth performance as that implied by this model.

The results of this analysis are summarized in the next proposition.

PROPOSITION 14.4. *Consider the one-sector Schumpeterian growth model presented in this section and suppose that*

$$0 < \lambda(1-\beta)\eta L - \rho < \frac{1 + \lambda(1-\beta)\rho}{\ln \lambda}. \tag{14.27}$$

Then there exists a unique steady-state equilibrium in which L_R^* workers work in the research sector, where L_R^* is given in equation (14.26). The economy has an average growth rate of $g^* = \eta L_R^* \ln \lambda$. Equilibrium growth is "uneven," in the sense that the economy has constant output for a while and then grows by a discrete amount when an innovation takes place.

Proof. Much of the proof is provided by the preceding analysis. Exercise 14.16 asks you to verify that the average growth is given by $g^* = \eta L_R^* \ln \lambda$ and that (14.27) is necessary for the above described equilibrium to exist and to satisfy the transversality condition. □

Therefore, this analysis shows that the basic insights of the one-sector Schumpeterian model, as originally developed by Aghion and Howitt (1992), are very similar to the baseline model of Schumpeterian growth presented in the previous section. The main difference is that growth has an uneven flavor in the one-sector model, because it is driven by infrequent bursts of innovation, preceded and followed by periods of no growth.

14.2.2. Uneven Growth and Endogenous Cycles*. The analysis in the previous subsection showed how the basic one-sector Schumpeterian growth leads to an uneven pattern of economic growth. This is driven by the discrete nature of innovations in continuous time. There is another source of uneven growth in this basic model, which is more closely related to the process of creative destruction. The nature of Schumpeterian growth implies that future growth reduces the value of current innovations, because it causes more rapid replacement of existing technologies. This effect did not play a role in our analysis so far, because in the model with a continuum of sectors, growth takes a smooth form and as Proposition 14.2 showed, there is a unique equilibrium path with no transitional dynamics. The one-sector growth model analyzed in this section allows these effects to manifest themselves. To show the potential for these creative destruction effects, we now construct a variant of the model which exhibits endogenous growth cycles. Throughout, we focus on an equilibrium path with such a cycle.

The only difference is that we now assume that the technology of R&D implies that L_R workers in research leads to innovation at the rate

$$\eta\left(L_R\right) L_R,$$

where $\eta(\cdot)$ is a strictly decreasing function, representing an externality in the research process. When more firms try to discover the next generation of technology, there will be more crowding-out in the research process, making it less likely for each of them to innovate. Each firm ignores its effect on the aggregate rate of innovation, thus takes $\eta(L_R)$ as given (this assumption is not important as shown by Exercise 14.21). Consequently, when the current machine quality is q, the free entry condition takes the form

$$\eta\left(L_R\left(q\right)\right) V\left(\lambda q\right) = w\left(q\right),$$

where $L_R(q)$ is the number of workers employed in research when the current machine quality is q.

Let us now look for an equilibrium with the following cyclical property: the rate of innovation differs when the innovation in question is an odd-numbered innovation versus an even-numbered innovation (say with the number of innovations counted starting from some arbitrary date $t = 0$). This type of equilibrium is possible when all agents in the economy expect there to be such an equilibrium (i.e., it is a "self-fulfilling" equilibrium). Denote the number of workers in R&D for odd and even-numbered innovations by L_R^1 and L_R^2. Then, following the analysis in the previous subsection, in any equilibrium with a cyclical pattern the values of odd and even-numbered innovations (with a machine of quality q) can be written as (see Exercise 14.19):

$$V^2(\lambda q) = \frac{\beta q \left(L - L_R^2\right)}{\rho + \eta\left(L_R^2\right) L_R^2} \text{ and } V^1(\lambda q) = \frac{\beta q \left(L - L_R^1\right)}{\rho + \eta\left(L_R^1\right) L_R^1}, \tag{14.28}$$

and the free entry conditions take the form

$$\eta\left(L_R^1\right) V^2(\lambda q) = w(q) \text{ and } \eta\left(L_R^2\right) V^1(\lambda q) = w(q), \tag{14.29}$$

where $w(q)$ is the equilibrium wage with technology of quality q. The reason why $\eta\left(L_R^1\right)$ multiplies the value for an even-numbered innovation is because L_R^1 researchers are employed for innovation today, when the current technology is odd-numbered, but the innovation that this research will produce will be even-numbered and thus will have value $V^2(\lambda q)$. Therefore, we have the following two equilibrium conditions:

$$\eta\left(L_R^1\right) \frac{\lambda(1-\beta) q \left(L - L_R^2\right)}{\rho + \eta\left(L_R^2\right) L_R^2} = 1 \text{ and } \eta\left(L_R^2\right) \frac{\lambda(1-\beta) q \left(L - L_R^1\right)}{\rho + \eta\left(L_R^1\right) L_R^1} = 1. \tag{14.30}$$

It can easily be verified that these two equations can have solutions L_R^1 and $L_R^2 \neq L_R^1$, which would correspond to the possibility of a two-period endogenous cycle (see Exercise 14.20).

14.2.3. Labor Market Implications of Creative Destruction. Another important implication of creative destruction is related to the fact that growth destroys existing productive units. So far this only led to the destruction of the monopoly rents of incumbent producers, without any loss of employment. In more realistic economies, creative destruction may dislocate previously employed workers and these workers may experience some unemployment before finding a new job. How creative destruction may lead to unemployment is discussed in Exercise 14.18.

A final implication of creative destruction that is worth noting relates to the destruction of firm-specific skills. It may be efficient for workers to accumulate human capital that is specific to their employers. Creative destruction implies that productive units may have shorter horizons in an economy with rapid economic growth. An important consequence of this might be that in rapidly growing economies, workers (and sometimes firms) may be less willing to make a range of specific human capital and other investments.

14.3. Innovation by Incumbents and Entrants and Sources of Productivity Growth

A key aspect of the growth process is the interplay between innovations and productivity improvements by existing firms on the one hand and entry by more productive, new firms on the other. The evidence from industry studies, which will be discussed in greater detail in Section 18.1, suggests that a large part of productivity growth at the industry level (and thus in the aggregate) comes from productivity improvements by continuing plants, though entry by new plants also makes a nontrivial contribution to industry productivity growth. The Schumpeterian models presented in this section have emphasized entry by new firms as the engine of growth. Interpreted literally, these models predict that all growth should be driven by entry, which is at odds with the facts. The expanding variety models presented in the previous chapter also do not provide a framework for the analysis of the interplay between existing firms and new entrants.[4] In this and the next section, I will discuss models that feature productivity growth by continuing plants (firms). The model in this section will feature productivity growth both by incumbents and entrants. The model in the next section will be richer in many respects, but will not allow entry. The two models together provide a first glimpse of the type of models that might be useful for studying the industrial organization of innovation and productivity growth. Both models will also be useful in linking the size distribution of firms to innovation and productivity growth.

14.3.1. Model. The economy is again in continuous time and admits a representative household with the standard CRRA preferences, as in (13.1) in the previous chapter. Population is constant at L and labor is supplied inelastically. The resource constraint at time t takes the usual form

$$C(t) + X(t) + Z(t) \leq Y(t), \tag{14.31}$$

where $C(t)$ is consumption, $X(t)$ is aggregate spending on machines, and $Z(t)$ is total expenditure on R&D at time t.

The production function of the unique final good is given by:

$$Y(t) = \frac{1}{1-\beta}\left[\int_0^1 q(\nu,t)^{\beta} x(\nu,t \mid q)^{1-\beta} d\nu\right] L^{\beta}, \tag{14.32}$$

where $x(\nu,t \mid q)$ is the quantity of the machine of type ν of quality q used in the production process and the measure of machines is again normalized to 1. This production is very similar to (14.3) used above, except that the quality of machines comes in with an exponent β. This has no effects on any of the results concerning growth, but will imply that firms

[4] In the expanding variety models, the identity of the firms that are undertaking the innovation does not matter, so one could assume that it is the existing producers that are inventing new varieties, though this will be essentially determining the distribution of productivity improvements across firms by assumption.

with different productivity levels will have different levels of sales (see Exercise 14.26). It will therefore enable us to make predictions about the size distribution of firms as well.

The engine of economic growth is again quality improvements, but these will be driven by two types of innovations:

(1) Innovation by incumbents
(2) Creative destruction by entrants.

Let $q(\nu, t)$ be the quality of machine line ν at time t. We assume the following "quality ladder" for each machine type:

$$q(\nu, t) = \lambda^{n(\nu,t)} q(\nu, s) \text{ for all } \nu \text{ and } t,$$

where $\lambda > 1$ and $n(\nu, t)$ denotes the number of *incremental* innovations on this machine line between time $s \leq t$ and time t, where time s is the date at which this particular type of technology was first invented and $q(\nu, s)$ refers to its quality at that point. The incumbent has a fully enforced patent on the machines that it has developed (though this patent does not prevent entrants leapfrogging the incumbent's machine quality). We assume that at time $t = 0$, each machine line starts with some quality $q(\nu, 0) > 0$ owned by an incumbent with fully enforce patent on this initial machine quality. Incremental innovations can only be performed by the incumbent producer. So we can think of those as "tinkering" innovations that improve the quality of the machine. The assumption that incumbents have access to a technology to create incremental innovations is consistent with case study evidence on industry level innovation (e.g., Freemen, 1982, or Sherer, 1984).

More specifically, if the current incumbent spends an amount $z(\nu, t) q(\nu, t)$ of the final good for this type of innovation activity on a machine of current quality $q(\nu, t)$, it has a flow rate of innovation equal to $\phi z(\nu, t)$ for $\phi > 0$ (more formally, this implies that for any interval $\Delta t > 0$, the probability of one incremental innovation is $\phi z(\nu, t) \Delta t$ and the probability of more than one incremental innovation is $o(\Delta t)$ (with $o(\Delta t)/\Delta t \to 0$ as $\Delta t \to 0$). Recall that such an innovation results in an improvement in quality and the resulting new machine will be of quality $\lambda q(\nu, t)$.

Alternatively, a new firm (entrant) can undertake R&D to innovate over the existing machines in machine line ν at time t. If the current quality of machine is $q(\nu, t)$, then by spending one unit of the final good, this new firm has a flow rate of innovation equal to $\eta(\hat{z}(\nu, t))/q(\nu, t)$, where $\eta(\cdot)$ is a strictly decreasing, continuously differentiable function and $\hat{z}(\nu, t)$ is the total amount of R&D by new entrants towards machine line ν at time t. Incumbents can also be allowed to have access to the same technology for radical innovation as the entrants. However, the Arrow replacement effect then immediately implies that incumbents would never use this technology (since entrants will be making zero profits from this technology, the profits of incumbents would be negative; see Exercise 14.22). Incumbents will still find it profitable to use the technology for incremental innovations, which is not available to entrants.

The presence of the strictly decreasing function η, which was also used in Section 14.2, captures the fact that when many firms are undertaking R&D to replace the same machine line, they are likely to try similar ideas, thus there will be some amount of "external" diminishing returns (new entrants will be "fishing out of the same pond"). Since each entrant attempting R&D on this line is potentially small, they will all take $\eta(\hat{z}(\nu,t))$ as given. Throughout I assume that $z\eta(z)$ is strictly increasing in z so that greater aggregate R&D towards a particular machine line increases the overall probability of discovering a superior machine. I also suppose that $\eta(z)$ satisfies the following Inada-type assumptions:

$$\lim_{z\to\infty}\eta(z)=0 \text{ and } \lim_{z\to 0}\eta(z)=\infty. \tag{14.33}$$

An innovation by an entrant leads to a new machine of quality $\kappa q(\nu,t)$, where $\kappa>\lambda$. Therefore, innovation by entrants are more "radical" than those of incumbents. Existing empirical evidence from studies of innovation support the notion that innovations by new entrants are more significant or radical than those of incumbents (though it may take a while for the successful entrants to realize the full productivity gains from these innovations; I am abstracting from this aspect). Importantly, whether the entrant was a previous incumbent on this specific machine line or whether it its currency an incumbent in some other machine line do not matter for its technology of innovation.

Once a particular machine of quality $q(\nu,t)$ has been invented, any quantity of this machine can be produced at the marginal cost ψ. I again normalize $\psi\equiv 1-\beta$. This implies that the total amount of expenditure on the production of intermediate goods at time t is

$$X(t)=\int_0^1 \psi x(\nu,t)\,d\nu, \tag{14.34}$$

where $x(\nu,t)$ is the quantity of this machine used in final good production. Similarly, the total expenditure on R&D is

$$Z(t)=\int_0^1 [z(\nu,t)+\hat{z}(\nu,t)]\,q(\nu,t)\,d\nu, \tag{14.35}$$

where $q(\nu,t)$ refers to the highest quality of the machine of type ν at time t. Notice also that total R&D is the sum of R&D by incumbents and entrants ($z(\nu,t)$ and $\hat{z}(\nu,t)$ respectively).

An allocation in this economy consists of time paths of consumption levels, aggregate spending on machines, and aggregate R&D expenditure $[C(t),X(t),Z(t)]_{t=0}^{\infty}$, time paths for R&D expenditure by incumbent and entrants $[z(\nu,t),\hat{z}(\nu,t)]_{\nu\in[0,1],t=0}^{\infty}$, time paths of prices and quantities of each machine and the net present discounted value of profits from that machine, $[p^x(\nu,t\mid q),x(\nu,t),V(\nu,t\mid q)]_{\nu\in[0,1],t=0}^{\infty}$, and time paths of interest rates and wage rates, $[r(t),w(t)]_{t=0}^{\infty}$. An equilibrium is given by an allocation in which R&D decisions by entrants maximize their net present discounted value, pricing, quantity and R&D decisions by incumbents maximize their net present discounted value, consumers choose the path of consumption optimally, and the labor market clears.

Let us start with the aggregate production function for the final good producers. Profit-maximization by the final good sector implies the demand for machines of highest-quality is given by a slight variant of equation (14.4) in Section 14.1:

$$x(\nu,t \mid q)=p^{x}\left(\nu,t \mid q\right)^{-1/\beta} q\left(\nu,t\right) L \quad \text{for all } \nu\in[0,1] \text{ and all } t, \tag{14.36}$$

where $p^{x}\left(\nu,t \mid q\right)$ refers to the price of machine type ν of quality $q\left(\nu,t\right)$ at time t. Since the demand from the final good sector for machines in (14.36) is iso-elastic, the unconstrained monopoly price will given by the usual formula as a constant markup over marginal cost. In this context, I introduced the analogue of condition (14.5) above,

$$\kappa>\left(\frac{1}{1-\beta}\right)^{\frac{1-\beta}{\beta}}, \tag{14.37}$$

which ensures that new entrants can charge the unconstrained monopoly price. By implication, incumbents that make further innovations can also charge the unconstrained monopoly price.

14.3.2. Equilibrium. Since the demand for machines in (14.36) is iso-elastic and $\psi = 1-\beta$, the profit-maximizing monopoly price is

$$p^{x}\left(\nu,t \mid q\right)=1. \tag{14.38}$$

Combining this with (14.36) implies that

$$x\left(\nu,t \mid q\right)=qL. \tag{14.39}$$

Consequently, the flow profits of a firm with the monopoly rights on the machine of quality q can be computed as:

$$\pi\left(\nu,t \mid q\right)=\beta qL. \tag{14.40}$$

Next, substituting (14.39) into (14.32), we obtain that total output is given by an expression identical to (14.8) above

$$Y\left(t\right)=\frac{1}{1-\beta}Q\left(t\right)L, \tag{14.41}$$

with average quality of machines $Q\left(t\right)$ given as in (14.9) in Section 14.1.

As a byproduct, we also obtain that aggregate spending on machines is

$$X\left(t\right)=\left(1-\beta\right)Q\left(t\right)L. \tag{14.42}$$

Moreover, since the labor market is competitive, the wage rate at any point in time is given by (14.11) as before.

To characterize the full equilibrium, we need to determine R&D effort levels by incumbents and entrants. To do this, let us write the net present value of a monopolist with the highest quality of machine q at time t in machine line ν. This value satisfies the standard Hamilton-Jacobi-Bellman equation:

$$(14.43)\quad r(t)V(\nu,t\mid q)-\dot{V}(\nu,t\mid q)=\max_{z(\nu,t|q)\geq o}\{\pi(\nu,t\mid q)-z(\nu,t\mid q)\,q(\nu,t)$$
$$+\phi z(\nu,t\mid q)\left(V(\nu,t\mid\lambda q)-V(\nu,t\mid q)\right)-\hat{z}(\nu,t\mid q)\eta\left(\hat{z}(\nu,t\mid q)\right)V(\nu,t\mid q)\},$$

where $\hat{z}(\nu,t\mid q)\eta(\hat{z}(\nu,t\mid q))$ is the rate at which radical innovations by entrants occur in sector ν at time t and $z(\nu,q\mid q)$ is the rate at which the incumbent improves its technology. The first term in (14.43), $\pi(\nu,t\mid q)$, is the flow of profits given by (14.40), while the second term is the expenditure of the incumbent for improving the quality of its product. The second line includes changes in the value of the incumbent due to innovation either by itself (at the rate $\phi z(\nu,t\mid q)$, the quality of its product increases from q to λq) or by an entrant (at the rate $\hat{z}(\nu,t\mid q)\eta(\hat{z}(\nu,t\mid q))$, the incumbent is replaced and receives zero value from then on).[5] The value function is written with a maximum on the right hand side, since $z(\nu,t\mid q)$ is a choice variable for the incumbent.

Free entry by entrants implies that we must have a free entry condition similar to (14.13) in Section 14.1:

$$(14.44)\quad \begin{aligned}\eta\left(\hat{z}(\nu,t\mid q)\right)V(\nu,t\mid\kappa q) &\leq q(\nu,t),\text{ and}\\ \eta\left(\hat{z}(\nu,t\mid q)\right)V(\nu,t\mid\kappa q) &= q(\nu,t)\text{ if }\hat{z}(\nu,t\mid q)>0,\end{aligned}$$

which takes into account that by spending an amount $q(\nu,t)$, the entrant generates a flow rate of innovation of $\eta(\hat{z})$, and if this innovation is successful, the entrant will end up with a product of quality κq, thus earning the value $\eta(\hat{z}(\nu,t\mid q))V(\nu,t\mid\kappa q)$.

In addition, the incumbent's choice of R&D effort implies a similar complementary slackness condition

$$(14.45)\quad \begin{aligned}\phi\left(V(\nu,t\mid\lambda q)-V(\nu,t\mid q)\right) &\leq q(\nu,t)\text{ and}\\ \phi\left(V(\nu,t\mid\lambda q)-V(\nu,t\mid q)\right) &= q(\nu,t)\text{ if }z(\nu,t\mid q)>0.\end{aligned}$$

Finally, consumer maximization implies the familiar Euler equation and the transversality condition given by (14.14) and (14.15) as before.

In light of this analysis, an equilibrium can be more succinctly defined as time path of $[C(t),X(t),Z(t)]_{t=0}^{\infty}$ that satisfy (14.31), (14.35), (14.42) and (14.15); time paths for R&D expenditure by incumbent and entrants $[z(\nu,t),\hat{z}(\nu,t)]_{\nu\in[0,1],t=0}^{\infty}$ that satisfy (14.44) and (14.45); time paths of prices and quantities of each machine and the net present discounted value of profits from that machine, $[p^x(\nu,t\mid q),x(\nu,t),V(\nu,t\mid q)]_{\nu\in[0,1],t=0}^{\infty}$ given by (14.38), (14.7) and (14.43); and time paths of wage and interest rates, $[w(t),r(t)]_{t=0}^{\infty}$ that satisfy (14.11) and (14.14).

[5] The fact that the incumbent receives a zero value from then on follows from the assumption that a previous incumbent has no advantage relative to other entrants in competing for another round of innovations.

As usual, I define a *BGP* (balanced growth path) as an equilibrium path in which innovation, output and consumption growth a constant rate. Notice that in BGP, aggregates grow at the constant rate, but there will be firm deaths and births, and the firm size distribution may also change.

The requirement that consumption grows at a constant rate in the BGP implies that $r(t) = r^*$ (from (14.14)). Moreover, in BGP, we must also have $z(\nu, t \mid q) = z(q)$ and $\hat{z}(\nu, t \mid q) = \hat{z}(q)$. These together imply that in BGP $\dot{V}(\nu, t \mid q) = 0$ and $V(\nu, t \mid q) = V(q)$. Finally, since profits and costs are both proportional to quality q, we can also see that $\hat{z}(q) = \hat{z}$ and $V(q) = vq$ (see the proof of Proposition 14.6, which in fact shows that , $\hat{z}(\nu, t \mid q) = \hat{z}(t)$ and $V(\nu, t \mid q) = v(t)\, q$ in any equilibrium, even outside the BGP). These results enable a straightforward characterization of the BGP and the dynamic equilibrium. While $\hat{z}(q) = \hat{z}$ for all q, it is not necessarily true that $z(q) = z$ for all q. In fact, as we will see the equilibrium only pins down the average R&D intensity of incumbents.

Let us first look for an "interior" BGP equilibrium (we will verify below that such an interior BGP exists and is the unique equilibrium). This implies that incumbents undertake research, thus

$$\phi\left(V(\nu, t \mid \lambda q) - V(\nu, t \mid q)\right) = q(\nu, t). \tag{14.46}$$

Given the linearity of V in q, this implies the following convenient equation for the value of a firm with a machine of quality q:

$$V(q) = \frac{q}{\phi(\lambda - 1)}. \tag{14.47}$$

Moreover, from the free entry condition (again holding as equality since the BGP is interior) we have that

$$V(q) = \frac{\beta L q}{r^* + \hat{z}\eta(\hat{z})}. \tag{14.48}$$

Moreover, from the free entry condition (again holding as equality from the fact that the equilibrium is interior):

$$\eta(\hat{z})\, V(\kappa q) = q \text{ or } V(q) = \frac{q}{\kappa \eta(\hat{z})}.$$

Combining this expression with (14.46) and (14.47), we obtain

$$\frac{\phi(\lambda - 1)}{\kappa \eta(\hat{z})} = 1.$$

This implies that the BGP R&D level by entrants $\hat{z}^*$ is implicitly defined by

$$\hat{z}(q) = \hat{z}^* \equiv \eta^{-1}\left(\frac{\phi(\lambda - 1)}{\kappa}\right) \text{ for all } q > 0. \tag{14.49}$$

Next, combining this with (14.48), we obtain the BGP interest rate as

$$r^* = \phi(\lambda - 1)\,\beta L - \hat{z}^*\eta(\hat{z}^*). \tag{14.50}$$

From (14.14), the growth rate of consumption and output is therefore given by

$$g^* = \frac{1}{\theta}\left(\phi\left(\lambda - 1\right)\beta L - \hat{z}^* \eta\left(\hat{z}^*\right) - \rho\right). \tag{14.51}$$

Equation (14.51) already has some interesting implications. In particular, it determines the relationship between the rate of innovation by entrants $\hat{z}^*$ and the BGP growth rate g^*. In standard Schumpeterian models, this relationship is positive. In contrast, here we have the following immediate result:

Proposition 14.5. *There is a negative relationship between $\hat{z}^*$ and g^*.*

Equation (14.51), together with (14.49), determines the BGP growth rate of the economy, but does not specify how much of productivity growth is driven by creative destruction (innovation by entrants) and how much of it by productivity improvements by incumbents. To determine this, we repeat the same analysis as in Section 14.1. Recall, at this point, that $z\left(\nu, t \mid q\right)$ is not a function of ν, but could still depend on q. Consequently, we can obtain the law of motion of average quality, $Q\left(t\right)$, as

$$\begin{aligned} Q\left(t + \Delta t\right) &= \left(\lambda\phi z\left(t\right)\Delta t\right) Q\left(t\right) + \left(\kappa\hat{z}\left(t\right)\eta\left(\hat{z}\left(t\right)\right)\Delta t\right) Q\left(t\right) \\ &\quad + \left(1 - \phi z\left(t\right)\Delta t - \hat{z}\left(t\right)\eta\left(\hat{z}\left(t\right)\right)\Delta t\right) Q\left(t\right) + o\left(\Delta t\right), \end{aligned} \tag{14.52}$$

where

$$z\left(t\right) = \frac{\int_0^1 z\left(\nu, t \mid q\right) q\left(\nu, t\right) d\nu}{Q\left(t\right)} \tag{14.53}$$

is the average R&D effort of incumbents that time t. Now subtracting $Q\left(t\right)$ from both sides of (14.52), dividing by Δt and taking the limit as $\Delta t \to 0$, we obtain

$$\frac{\dot{Q}\left(t\right)}{Q\left(t\right)} = \left(\lambda - 1\right) z\left(t\right) + \left(\kappa - 1\right)\hat{z}\left(t\right)\eta\left(\hat{z}\left(t\right)\right). \tag{14.54}$$

Therefore, an alternative expression for the growth rate of the economy, which decomposes growth into the component coming from incumbent firms (the first term) and that coming from new entrants (the second term) is given as

$$g^* = \left(\lambda - 1\right)\phi z^* + \left(\kappa - 1\right)\hat{z}^*\eta\left(\hat{z}^*\right), \tag{14.55}$$

where z^* is the average BGP R&D effort of incumbents. The fact that this average R&D effort is constant in BGP follows from the fact from (14.54) together with the fact that in BGP the growth rate of average quality is g^* and the R&D effort by entrants on each machine line is $\hat{z}^*$. While (14.51) pins down the BGP growth rate of output and consumption, equation (14.55) determines how much of it is driven by innovation by incumbents and how much of it by innovation by entrants. Moreover, we can also verify that this economy does not have any transitional dynamics (see the proof of Proposition 14.6). Therefore, if an equilibrium with growth exists, it will involve growth at the rate g^*. To ensure that such an equilibrium exists, we need to verify that R&D is profitable both for entrants and incumbents. The condition that the BGP interest rate, r^*, given by (14.50), should be greater than the discount rate ρ is sufficient for there to be positive aggregate growth. In addition, this interest rate should not be so high that the transversality condition of the consumers is violated. Finally, we need to ensure that there is also innovation by incumbents. The following condition ensures all three of these requirements (see the proof of Proposition 14.6):

$$\phi(\lambda-1)\beta L - (\theta(\kappa-1)+1)\hat{z}^*\eta(\hat{z}^*) > \rho > (1-\theta)(\phi(\lambda-1)\beta L - \hat{z}^*\eta(\hat{z}^*)), \tag{14.56}$$

with $\hat{z}^*$ given by (14.49).

In addition, our main interest is with how much of productivity growth and innovation are driven by incumbents and how much by new entrants. This can now be easily obtained from (14.51) and (14.55) as

$$(\lambda-1)\phi z^* = g^* - (\kappa-1)\hat{z}^*\eta(\hat{z}^*), \tag{14.57}$$

where g^* is given by (14.51) and $\hat{z}^*$ by (14.49).

Another set of interesting implications of this model concern firm-size dynamics. The size of a firm can be measured by its sales, which is equal to

$$x(\nu, t \mid q) = qL \text{ for all } \nu \text{ and } t.$$

To determine the law of motion of firm sales and thus the firm-size dynamics, we need to know how incumbent R&D effort varies with q. To start with, let us suppose that in BGP this R&D effort is independent of q, so that $z(\nu, t \mid q) = z^*$ for all q. From the analysis so far, it is clear that such an equilibrium will exist (for a justification of the focus on this equilibrium, see Exercise 14.24; this issue will be discussed further below). Then the quality of each incumbent firm will increase at the flow rate ϕz^*, with z^* given by (14.57). At the same time, each incumbent is also replaced at the flow rate $\hat{z}^*\eta(\hat{z}^*)$. Therefore, for Δt sufficiently small, the stochastic process for the size of a particular firm is given by

$$x(\nu, t+\Delta t \mid q) = \begin{cases} \lambda x(\nu, t \mid q) & \text{with probability } \phi z^*\Delta t + o(\Delta t) \\ 0 & \text{with probability } \hat{z}^*\eta(\hat{z}^*)\Delta t + o(\Delta t) \\ x(\nu, t \mid q) & \text{with probability } (1 - \phi z^*\Delta t - \hat{z}^*\eta(\hat{z}^*)\Delta t) + o(\Delta t), \end{cases} \tag{14.58}$$

for all ν and t, where $\hat{z}^*$ is by (14.49) and z^* by (14.57). Therefore, firms have random growth, and surviving firms expand on average. However, firms also face a probability of bankruptcy (extinction). In particular, denoting the probability that a particular incumbent firm that started production in machine line ν at time s will be bankrupt by time $t \geq s$ by $P(t \mid s, \nu)$, we clearly have $\lim_{t\to\infty} P(t \mid s, \nu) = 1$, so that each firm will necessarily die eventually. The implications of equation (14.58) for the the stationary firm size distribution is discussed in subsection 14.3.5 below.

The following proposition summarizes the main results of this section.

Proposition 14.6. *Consider the above-described economy starting with an initial condition $Q(0) > 0$. Suppose that (14.33) and (14.56) are satisfied and focus on equilibrium in which all incumbents exert the same level of R&D effort. Then there exists a unique equilibrium. In this equilibrium growth is always balanced, and technology, $Q(t)$, aggregate output, $Y(t)$, and aggregate consumption, $C(t)$, grow at the rate g^* as in (14.51) with $\hat{z}^*$ given by (14.49). Equilibrium growth is driven both by innovation by incumbents and by creative destruction by entrants. Any given firm expands on average as long as it survives, but is eventually replaced by a new entrant with probability one.*

Proof. First, note that in an interior BGP where $\phi(V(\nu,t \mid \lambda q) - V(\nu,t \mid q)) = q$, V must be linear in q, thus $V(\nu,t \mid q) = vq$ as used in the text. Given this observation, the characterization of the BGP follows from the argument preceding the proposition. In particular, $\hat{z}^*$ is uniquely determined by (14.49) and (14.51) gives the unique BGP growth rate. To ensure that this is indeed an interior BGP, we need to check four conditions:

(1) *Positive $\hat{z}^*$:* this follows from (14.33), e.g., see observation [A] below.
(2) *Positive growth:* this requires $g^* = (\phi(\lambda-1)\beta L - \hat{z}^*\eta(\hat{z}^*) - \rho)/\theta > 0$. Since $(\theta(\kappa-1)+1) > 1$, the first inequality in (14.56), is sufficient for this.
(3) *Positive z^*:* combining (14.51) together with (14.55), we have that

$$\begin{aligned} z^* &= \frac{g^* - (\kappa-1)\hat{z}^*\eta(\hat{z}^*)}{(\lambda-1)\phi} \\ &= \frac{(\phi(\lambda-1)\beta L - \hat{z}^*\eta(\hat{z}^*) - \rho)/\theta - (\kappa-1)\hat{z}^*\eta(\hat{z}^*)}{(\lambda-1)\phi} > 0, \end{aligned}$$

in view of the first inequality (14.56).
(4) *The transversality condition:* condition (14.15) should hold so that the maximization problem of the representative household is well-defined. The condition $r^* > g^*$ is necessary and sufficient to ensure (14.15). The second inequality in (14.56) ensures that this inequality holds.

Therefore, the BGP is interior and is uniquely defined.

I next prove that the BGP also gives the unique dynamic equilibrium path. Let us start with two observations.

[A] Because of the Inada conditions, (14.33), the free entry condition (14.13) must hold as equality for all ν, t and q, so that

$$\eta(\hat{z}(\nu,t \mid q))\, V(\nu,t \mid \kappa q) = q \text{ for all } \nu,\, q \text{ and } t. \tag{14.59}$$

Since this equation holds for all t and the right-hand side is differentiable in q and t, so must be the left-hand side. Differentiating with respect to t, we obtain

$$\frac{\partial \hat{z}(\nu,t \mid q)/\partial t}{\hat{z}(\nu,t \mid q)} = \frac{1}{\varepsilon_\eta(\hat{z}(\nu,t \mid q))} \frac{\dot{V}(\nu,t \mid \kappa q)}{V(\nu,t \mid \kappa q)} \text{ for all } q \text{ and } t, \tag{14.60}$$

where

$$\varepsilon_\eta(\hat{z}) \equiv -\frac{\eta'(\hat{z})\hat{z}}{\eta(\hat{z})} > 0$$

is the elasticity of the η function.

[B] The value of a firm with a machine of quality q at time t can be written as

$$V(\nu, t \mid q) = \int_t^\infty \exp\left[-\int_t^s \left(r\left(s'\right) + \hat{z}(\nu, s' \mid q)\eta\left(\hat{z}(\nu, s' \mid q)\right)\right) ds'\right] \pi(\nu, s \mid q)\, ds. \tag{14.61}$$

For any finite q, this value is finite since, from observation [A], $(r(t) + \hat{z}(\nu, t \mid q)\eta(\hat{z}(\nu, t \mid q))) > 0$ for all ν, q and t, and because $\pi(\nu, s \mid q) = \beta q L \in (0, \infty)$. In addition, in view of (14.33), $V(\nu, t \mid q)$ is uniformly bounded away from 0 for any path of interest rates that does not limit to infinity. Moreover, $r(t) \to \infty$ is also impossible, since, from (14.14), it would imply $\dot{C}(t)/C(t) \to \infty$, violating the transversality condition, (14.15).

Now consider the following two cases.

Case 1: Suppose that (14.46), that is, $\phi(V(\nu, t \mid \lambda q) - V(\nu, t \mid q)) = q$, holds for all ν, q and t. Then, as in BGP, $V(\nu, t \mid q)$ is linear in q and can be written as $V(\nu, t \mid q) = v(t) q$ for some function $v(t)$. This implies that (14.46) can be written as $\phi(\lambda - 1) v(t) = 1$ for all t. Differentiating this equation with respect to time, we obtain $\dot{v}(t) = 0$ and $v(t) = v$ for all t. Moreover, from observation [A] above, the free entry condition (14.44) must hold as equality, so $\eta\left(\hat{z}(\nu, t \mid \kappa^{-1}q)\right) v(t) = 1$ for all t, which is only possible if $\hat{z}(\nu, t \mid q) = \hat{z}(t)$ for all q and t. Combining these conditions with (14.43) yields

$$r(t) v = \beta L - \hat{z}\eta(\hat{z}) v$$

for all t, which implies that $r(t)$ must be constant as well. Therefore, all variables must immediately take their BGP values, $r(t) = r^*$ and $\hat{z}(t) = \hat{z}^*$ for all t. Moreover, we are focusing on equilibria where $z(\nu, t \mid q)$ is independent of q, and thus we must have $z(\nu, t \mid q) = z^*$ for all t. This establishes that the economy must be in the BGP equilibrium starting at $t = 0$.

Case 2: Suppose that (14.46) does not hold for some $\nu \in \mathcal{N} \subset [0, 1]$, q and t. Since either $\phi(V(\nu, t \mid \lambda q) - V(\nu, t \mid q)) = q$ or $\phi(V(\nu, t \mid \lambda q) - V(\nu, t \mid q)) < q$ and $z(\nu, t \mid q) = 0$, and because from observation [A] above, $\eta\left(\hat{z}(\nu, t \mid \kappa^{-1}q)\right) V(\nu, t \mid q) = \kappa^{-1} q$, the value function (14.43) in this case can be written as

$$\frac{\dot{V}(\nu, t \mid q)}{V(\nu, t \mid q)} = r(t) + \hat{z}(\nu, t \mid q)\eta(\hat{z}(\nu, t \mid q)) - \kappa\beta L\eta\left(\hat{z}(\nu, t \mid \kappa^{-1}q)\right).$$

Combining this with (14.60), we obtain

$$\frac{\partial\hat{z}(\nu, t \mid \kappa^{-1}q)/\partial t}{\hat{z}(\nu, t \mid \kappa^{-1}q)} = \frac{1}{\varepsilon_\eta\left(\hat{z}(\nu, t \mid \kappa^{-1}q)\right)}\left[r(t) + \hat{z}(\nu, t \mid q)\eta(\hat{z}(\nu, t \mid q)) - \kappa\beta L\eta\left(\hat{z}(\nu, t \mid \kappa^{-1}q)\right)\right].$$

Similarly,

$$\frac{\partial \hat{z}(\nu,t \mid q)/\partial t}{\hat{z}(\nu,t \mid q)} = \frac{1}{\varepsilon_{\eta}\left(\hat{z}(\nu,t \mid q)\right)}\left[r\left(t\right)+\hat{z}\left(\nu,t \mid \kappa q\right)\eta\left(\hat{z}\left(\nu,t \mid \kappa q\right)\right)-\beta L\eta\left(\hat{z}(\nu,t \mid q)\right)\right],$$

and so on. It is then straightforward to see that the system of differential equations for $\hat{z}(\nu,t \mid q)$ for all q is unstable, in the sense that if $\partial \hat{z}(\nu,t \mid q)/\partial t > 0$, then it will grow continuously and if $\partial \hat{z}(\nu,t \mid q)/\partial t < 0$, then it will shrink continuously. This implies that we must have $\partial \hat{z}(\nu,t \mid q)/\partial t = 0$ for all $\nu \in [0,1]$. Suppose, to obtain a contradiction, that there exists (a positive measure) set $\mathcal{N} \subset [0,1]$ such that for all $\nu \in \mathcal{N}$, $\partial \hat{z}(\nu,t \mid q)/\partial t > 0$. Then $\hat{z}(\nu,t \mid q) \to \infty$ for all $\nu \in \mathcal{N}$, and thus $\eta\left(\hat{z}(\nu,t \mid q)\right) \to 0$. But to ensure the free entry condition, (14.44), which from observation [A] always holds as equality, we need $\eta\left(\hat{z}(\nu,t \mid q)\right) V\left(\nu,t \mid \kappa q\right) = q$ and thus $V\left(\nu,t \mid \kappa q\right) \to \infty$. But this is impossible in view of observation [B] above. Next, suppose that there exists $\mathcal{N} \subset [0,1]$ such that for all $\nu \in \mathcal{N}$, $\partial \hat{z}(\nu,t \mid q)/\partial t < 0$. Then $\hat{z}(\nu,t \mid q) \to 0$ and thus $\eta\left(\hat{z}(\nu,t \mid q)\right) \to \infty$, which from the free entry condition implies $V\left(\nu,t \mid \kappa q\right) \to 0$ and contradicts observation [B] that V is uniformly bounded away from zero. Therefore, $\partial \hat{z}(\nu,t \mid q)/\partial t = 0$ for all ν, q and t, which implies

$$r\left(t\right)+\hat{z}\left(\nu,t \mid \kappa q\right)\eta\left(\hat{z}\left(\nu,t \mid \kappa q\right)\right)-\beta L\eta\left(\hat{z}(\nu,t \mid q)\right)=0 \text{ for all } \nu,\ q \text{ and } t.$$

This is only possible if $r(t)$ is constant and thus equal to r^* and $\hat{z}(\nu,t \mid q) = \hat{z}^*$, establishing the result that also in this case there are no transitional dynamics. Moreover, since $\hat{z}(\nu,t \mid q) = \hat{z}^*$ for all ν, q and t, (14.61) implies that $V(\nu,t \mid q)$ is linear in q in this case as well.

Finally, the result that surviving firms expand on average and that all firms die with probability 1 follows from equation (14.58). □

Proposition 14.6 focuses on equilibria in which all incumbents exert the same R&D effort. Exercise 14.25 shows that the same conclusions hold when we do not focus on such equilibria.

14.3.3. Some Numbers. I will now try to flesh out the implications of this model on the decomposition of productivity growth between incumbents and entrants. Although some of the parameters of the current model are difficult to pin down with our current knowledge of the technology of R&D, some simple back-of-the-envelope calculations are still informative. Let us choose the following standard numbers:

$$g^* = 0.02,\ \rho = 0.01,\ r^* = 0.05, \text{ and } \theta = 2,$$

where the last number, the intertemporal elasticity of substitution, is pinned down by the choice of the other three numbers. The first three numbers refer to annual rates (implicitly defining $\Delta t = 1$ as one year). We have much greater uncertainty concerning the remaining parameters. We can normalize $\phi = L = 1$. For the rest, I will report a number of different possibilities. As a benchmark, As a benchmark, I take $\beta = 2/3$, which implies that two thirds of national income accrues to labor and one third to profits. The requirement in (14.37) then implies that $\kappa > 1.7$. I also take the benchmark value of $\kappa = 3$, so that entry by new firms is sufficiently "radical" as suggested by some of the qualitative accounts of the innovation process (e.g., Freeman, 1982, Scherer, 1984). Innovation by incumbents is taken to be correspondingly smaller $\lambda = 1.2$. This implies that productivity gains from a radical innovation is about ten times that of a standard "incremental" innovation by incumbents (i.e., $(\kappa - 1) / (\lambda - 1) = 10$). I will then show how the results change when the magnitudes of radical and incremental innovations are varied. For the function $\eta(z)$, I adopt the following frequently-used form:

$$\eta(z) = Bz^{-\alpha},$$

and choose the benchmark value of $\alpha = 0.5$. The remaining two parameters, ϕ and B will be chosen to ensure $g^* = 0.02$. I start with the benchmark value of $\phi = 0.4$, but this value will need to be modified in some of the variations in order to satisfy condition (14.56) above or to ensure $g^* = 0.02$. With these numbers, (14.49) implies

$$\hat{z}^* \eta(\hat{z}^*) = 0.0033,$$

and

$$(\kappa - 1) \hat{z}^* \eta(\hat{z}^*) = 0.0067.$$

The value for $\hat{z}^* \eta(\hat{z}^*)$ also implies that there is entry of a new firm (creative destruction) in each machine line on average once every 7.5 years (recall that $r^* = 0.05$ as the annual interest rate so that $r^* / \hat{z}^* \eta(\hat{z}^*) \approx 7.46$). Next, using (14.55), the contribution of productivity growth by continuing firms is

$$\begin{aligned} (\lambda - 1) \phi z^* &= g^* - (\kappa - 1) \hat{z}^* \eta(\hat{z}^*) \\ &= 0.0133. \end{aligned}$$

Therefore, in this benchmark parameterization, over two thirds of productivity growth comes from innovation by incumbents. Moreover, $\phi z^* = 0.0667$, so that there are on average 1.2 incremental innovations per year by an incumbent in a particular machine line ($r^* / \phi z^* \approx 1.2$). Using alternative values of the parameters κ, λ, β and α leads to broadly similar conclusions, though depending on the exact parameterization the contribution of entrants to productivity growth can be larger or smaller.

14.3.4. The Effects of Policy on Growth. Let us now use this model to analyze the effects of policies on equilibrium productivity growth and its decomposition between incumbents and entrants. Since the model has a Schumpeterian structure (with quality improvements as the engine of growth and creative destruction playing a major role), it may be conjectured that entry barriers (or taxes on potential entrance) will have negative effects on economic growth as in the baseline model we studied earlier in this chapter. To investigate whether this is the case, let us suppose that there is a tax τ_e on R&D expenditure by entrants and a tax τ_i on R&D expenditure by incumbents (naturally, these can be taken to be negative and interpreted as subsidies as well). Note also that the tax on entrants, τ_e, can be interpreted as a more strict patent policy than the one in the baseline model, where the entrant did not have to pay the incumbent for partially benefiting from its accumulated knowledge. Nevertheless, to keep the analysis brief, I only focus on the case in which tax revenues are collected by the government rather than rebated back to the incumbent as patent fees.

Repeating the analysis above, we obtain the following equilibrium conditions:

$$\eta\left(\hat{z}^*\right) V(\kappa q) = \left(1+\tau_e\right) q \text{ or } V(q) = \frac{q\left(1+\tau_e\right)}{\kappa\eta\left(\hat{z}^*\right)}. \tag{14.62}$$

The equation that determines the optimal R&D decisions of incumbents, (14.46), is also modified because of the tax rate τ_i and becomes

$$\phi\left(V(\lambda q) - V(q)\right) = \left(1+\tau_i\right) q. \tag{14.63}$$

Now combining (14.62) with (14.63), we obtain

$$\phi\left(\frac{(\lambda-1)\left(1+\tau_e\right)}{\kappa\eta\left(\hat{z}^*\right)\left(1+\tau_i\right)}\right) = 1.$$

Consequently, the BGP R&D level by entrants $\hat{z}^*$, when their R&D is taxed at the rate τ_e, is given by

$$\hat{z}^* \equiv \eta^{-1}\left(\frac{\phi(\lambda-1)\left(1+\tau_e\right)}{\kappa\left(1+\tau_i\right)}\right). \tag{14.64}$$

Equation (14.48) still applies, so that the the BGP interest rate is $r^* = \left(1+\tau_e\right)^{-1}\kappa\eta\left(\hat{z}^*\right)\beta L - \hat{z}^*\eta\left(\hat{z}^*\right)$, which, by substituting for (14.64), can be written as

$$r^* = \frac{\phi(\lambda-1)\beta L}{1+\tau_i} - \eta\left(\hat{z}^*\right)\hat{z}^*, \tag{14.65}$$

and the BGP growth rate is

$$g^* = \frac{1}{\theta}\left(\frac{\phi(\lambda-1)\beta L}{1+\tau_i} - \eta\left(\hat{z}^*\right)\hat{z}^* - \rho\right), \tag{14.66}$$

with $\hat{z}^*$ given by (14.64). The following is now immediate:

Proposition 14.7. *The growth rate of the economy is (strictly) decreasing in the tax rate on incumbents, i.e., $dg^*/d\tau_i < 0$, and is (strictly) increasing in the tax rate on entrants, i.e., $dg^*/d\tau_e > 0$.*

The result in this proposition is rather surprising and extreme. In Schumpeterian models, making entry more difficult, either with entry barriers or by taxing R&D by entrants, has negative effects on economic growth. Despite the Schumpeterian nature of the current model, here blocking entry increases equilibrium growth. Moreover, as Exercise 14.23 shows, in the decentralized equilibrium of this economy there tends to be too much entry, so a tax on entry also tends to improve welfare. The intuition for this result is related to the main departure of this model from the standard Schumpeterian models. In contrast to the baseline Schumpeterian models, the engine of growth is still quality improvements, but these are undertaken both by incumbents and entrants. Entry barriers, by protecting incumbents, increase their value and greater value by incumbents encourages more R&D investments and faster productivity growth. Taxing entrants makes incumbents more profitable and this encourages further innovation by the incumbents. Taxes on entrants or entry barriers also further increase the contribution of incumbents to productivity growth.

Nevertheless, the result in this proposition should be interpreted with caution. The model in this section is special in that the R&D technology of incumbents is a linear. This linearity is important for the results in Proposition 14.7. Exercise 14.25 shows that the equilibrium can be characterized even when $\phi(z)$ is a concave function of z , and in this case, the effect of taxes on entrants is ambiguous, because it encourages R&D by incumbents and discourages R&D by entrants. Therefore, Proposition 14.7 should be read as emphasizing a particular new channel in the stark as possible way, not as a realistic description of how innovation will respond to tax policies.

14.3.5. Equilibrium Firm Size Distribution. The model presented in this section generates a dynamic equilibrium in which the economy, and thus the size of average firm, as measured by sales $x(t)$, grows. To look at the firm size distribution, we therefore need to normalize firm sizes by the average size of firm, given by $X(t)$, in (14.10). In particular, let the *normalized firm size* be

$$\tilde{x}(t) \equiv \frac{x(t)}{X(t)}.$$

Let us continue to focus on equilibrium in which all incumbents exert the same R&D effort z^*. In this case, since the unique equilibrium involves $\dot{X}(t)/X(t) = g^* > 0$, the law of motion for the normalized size of the leading firm in each industry can be written as

(14.67)

$$\tilde{x}(t+\Delta t) = \begin{cases} \frac{\lambda}{1+g^*\Delta t}\tilde{x}(t) & \text{with probability } \phi z^* \Delta t \\ \frac{\kappa}{1+g^*\Delta t}\tilde{x}(t) & \text{with probability } \eta(\hat{z}^*)\hat{z}^*\Delta t + o(\Delta t) \\ \frac{1}{1+g^*\Delta t}\tilde{x}(t) & \text{with probability } (1-\phi z^*\Delta t - \hat{z}^*\eta(\hat{z}^*)\Delta t) + o(\Delta t). \end{cases}$$

Notice that this expression does not refer to the growth rate of a single firm, but to the leading firm in a representative industry, and in particular, when there is entry, this leads to an increase in size rather than extinction.

The following proposition gives the stationary distribution of (normalized) from sizes and shows that it takes the form of the Pareto distribution with an exponent of one, i.e., $\Pr[\tilde{x} \le y] = 1 - \Gamma/y$ (recall that the general form of the *Pareto distribution* is $\Pr[\tilde{x} \le y] =$

$1-\Gamma y^{-\chi}$). The Pareto distribution is attractive both because of its simplicity and tractability (see, for example, Section 15.8 in the next chapter), but also because the actual distribution of firm sizes in the US appears to fairly well approximated by a Pareto distribution with an exponent of one (e.g., Axtell, 2001). It is therefore a somewhat surprising and remarkable result that the simple model developed here, which was not designed to match the real-world firm size distributions, generates such a realistic distribution.

The following proposition shows that if a stationary distribution of (normalized) firm sizes exists, then it must take the form of the Pareto distribution with an exponent equal to 1. Recall that the Pareto distribution takes the form $\Pr[\tilde{x} \leq y] = 1 - \Gamma y^{-\chi}$ with $\Gamma > 0$.

Proposition 14.8. *Let us focus on the equilibrium in which all incumbents choose R&D effort z^*. Then, if a stationary distribution of (normalized) firm sizes exists, it is a Pareto distribution with exponent equal to 1, i.e.,* $\Pr[\tilde{x} \leq y] = 1 - \Gamma/y$ *with* $\Gamma > 0$.

Proof. To prove this claim, let us suppose that a stationary distribution exists and consider an arbitrary time interval of $\Delta t > 0$ and write

$$\begin{aligned}
\Pr[\tilde{x}(t+\Delta t) \leq y] &= \mathbb{E}\left[\mathbf{1}_{\{\tilde{x}(t+\Delta t) \leq y\}}\right] \\
&= \mathbb{E}\left[\mathbf{1}_{\{\tilde{x}(t) \leq y/(1+g^x(t+\Delta t))\}}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_{\{\tilde{x}(t) \leq y/(1+g^x(t+\Delta t))\}} \mid g^x(t+\Delta t)\right]\right],
\end{aligned}$$

where $\mathbf{1}_{\{\mathcal{P}\}}$ is the indicator function taking the value 1 if the statement $\mathcal{P}$ is correct and thus the first equation holds by definition. The second equation also holds by definition once $g^x(t+\Delta t)$ is designated as the (stochastic) growth rate of x between t and $t+\Delta t$. Finally, the third equation follows from the Law of Iterated Expectations. Next, denoting $\mathbb{G}_t(y) \equiv 1 - \Pr[\tilde{x}(t) \leq y]$ as the complement of the cumulative density function, the previous equation yields

$$\begin{aligned}
\Pr[\tilde{x}(t+\Delta t) \leq y] &= 1 - \mathbb{G}_{t+\Delta t}(y) \\
&= \mathbb{E}\left[1 - \mathbb{G}_t\left(\frac{y}{1+g^x(t+\Delta t)}\right)\right].
\end{aligned}$$

Therefore, we obtain the functional equation

$$\begin{aligned}
\mathbb{G}_{t+\Delta t}(y) &= \mathbb{E}\left[\mathbb{G}_t\left(\frac{y}{1+g^x(t+\Delta t)}\right)\right] \\
&= \phi z^* \Delta t \mathbb{G}_t\left(\frac{(1+g^*\Delta t)y}{\lambda}\right) + \hat{z}^*\eta(\hat{z}^*)\Delta t \mathbb{G}_t\left(\frac{(1+g^*\Delta t)y}{\kappa}\right) \\
&\quad + (1 - \phi z^*\Delta t - \hat{z}^*\eta(\hat{z}^*)\Delta t)\mathbb{G}_t((1+g^*\Delta t)y) + o(\Delta t).
\end{aligned} \tag{14.68}$$

A stationary equilibrium will correspond to a function $\mathbb{G}(y)$ such that $\mathbb{G}_{t+\Delta t}(y) = \mathbb{G}_t(y) = \mathbb{G}(y)$ for all t and Δt and (14.68) holds. Let us conjecture that $\mathbb{G}(y) = \Gamma y^{-\chi}$ with $\Gamma > 0$.

Substituting this conjecture into the previous expression, we obtain

$$\begin{aligned}\Gamma y^{-\chi} &= \phi z^{*}\Delta t\Gamma\left(\frac{(1+g^{*}\Delta t)\,y}{\lambda}\right)^{-\chi}+\hat{z}^{*}\eta\left(\hat{z}^{*}\right)\Delta t\Gamma\left(\frac{(1+g^{*}\Delta t)\,y}{\kappa}\right)^{-\chi}\\ &\quad+\left(1-\phi z^{*}\Delta t-\hat{z}^{*}\eta\left(\hat{z}^{*}\right)\Delta t\right)\Gamma\left((1+g^{*}\Delta t)\,y\right)^{-\chi}+o\left(\Delta t\right).\end{aligned}$$

Rewriting this,

$$\begin{aligned}&\phi z^{*}\Delta t\left(\frac{(1+g^{*}\Delta t)}{\lambda}\right)^{-\chi}+\hat{z}^{*}\eta\left(\hat{z}^{*}\right)\Delta t\left(\frac{(1+g^{*}\Delta t)}{\kappa}\right)^{-\chi}\\ +&\quad\left(1-\phi z^{*}\Delta t-\hat{z}^{*}\eta\left(\hat{z}^{*}\right)\Delta t\right)(1+g^{*}\Delta t)^{-\chi}+o\left(\Delta t\right)\Gamma^{-1}y^{-\chi}=1.\end{aligned}\tag{14.69}$$

Now subtracting 1 from both sides, dividing by Δt, and taking the limit as $\Delta t\rightarrow 0$, we obtain

$$\begin{aligned}\lim_{\Delta t\rightarrow 0}&\left\{\phi z^{*}\left(\frac{(1+g^{*}\Delta t)}{\lambda}\right)^{-\chi}+\hat{z}^{*}\eta\left(\hat{z}^{*}\right)\left(\frac{(1+g^{*}\Delta t)}{\kappa}\right)^{-\chi}\right.\\ &\left.+\frac{\left(1-\phi z^{*}\Delta t-\hat{z}^{*}\eta\left(\hat{z}^{*}\right)\Delta t\right)(1+g^{*}\Delta t)^{-\chi}-1}{\Delta t}+\frac{o\left(\Delta t\right)}{\Delta t}\Gamma^{-1}y^{\chi}\right\}=0.\end{aligned}$$

Taking the derivative of the penultimate term (which exists) and noting that $\lim_{\Delta t\rightarrow 0}o\left(\Delta t\right)/\Delta t=0$, we obtain the following equation for the exponent χ:

$$\phi\left(\lambda^{\chi}-1\right)z^{*}+\left(\kappa^{\chi}-1\right)\hat{z}^{*}\eta\left(\hat{z}^{*}\right)-\chi g^{*}=0.\tag{14.70}$$

It can be easily verified that (14.70) has two solutions $\chi=0$ and $\chi^{*}=1$, since, by definition, $g^{*}=\phi\left(\lambda-1\right)z^{*}+\left(\kappa-1\right)\hat{z}^{*}\eta\left(\hat{z}^{*}\right)$. To see that there are no other solutions, consider the derivative of this function, which is given by

$$g'\left(\chi\right)=\phi z^{*}\lambda^{\chi}\ln\lambda+\eta\left(\hat{z}^{*}\right)\hat{z}^{*}\kappa^{\chi}\ln\kappa-g^{*}.$$

Since $\ln a<a-1$ for any $a>1$, $g'\left(0\right)<0$. Moreover, $g''\left(0\right)>0$, so that the right-hand side of (14.70) is convex and as $\chi\rightarrow\infty$, it limits to infinity. Thus there is a unique nonzero solution, which as we saw above, is $\chi^{*}=1$. Finally, note that $\chi=0$ cannot be a solution, since it would imply $\mathbb{G}\left(y\right)=\Gamma$ and thus $\mathbb{G}\left(y\right)=0$ for $\Pr\left[\tilde{x}\left(t\right)\leq y\right]$ to be a well-defined distribution function. Yet this would imply that all firms have normalized size equal to zero, which violates the hypothesis that a stationary firm-size distribution exists. In contrast, $\mathbb{G}\left(y\right)=\Gamma/y$ (with $\Gamma>0$) yields $\Pr\left[\tilde{x}\left(t\right)\leq y\right]=1-\Gamma/y$, which is a well-defined distribution function.

It can also be verified that no other function than $\mathbb{G}\left(y\right)=\Gamma y^{-\chi}$ with $\Gamma>0$ can satisfy this functional equation, completing the proof of the proposition. □

In some ways, this result looks quite remarkable, since it generates a stationary firm-size distribution given by a Pareto distribution with an exponent of one, in a much simpler manner than any existing approaches, and does so despite the fact that the model was not designed to study firm size distributions. However, the proposition is proved under the hypothesis that a stationary firm-size distribution exists. Unfortunately, the next corollary shows that this will not be the case.

Corollary 14.1. *Let us focus on the equilibrium in which all incumbents choose R&D effort z^*. Then a stationary firm-size distribution does not exist.*

Proof. We know from Proposition 14.8 that if a stationary distribution exists, it must take the form $\Pr[\tilde{x} \leq y] = 1 - \Gamma/y$ with $\Gamma > 0$. But the Pareto distribution is defined for all $y \geq \Gamma$, thus Γ should be the minimum normalized firm size. However, the law of motion of $\tilde{x}(t)$, (14.67), shows that it is possible for the normalized size of the firm (or of the relevant firm in an industry), $\tilde{x}(t)$ to tend to zero. Therefore, Γ must be equal to 0, which implies that there does not exist a stationary firm-size distribution. □

The essence of Corollary 14.1 is that with the random growth process in (14.67), the distribution of firm sizes will continuously expand. The "limiting distribution" will involve all firms being arbitrarily small relative to the average $X(t)$ and a vanishingly small fraction of firms becomes arbitrarily large (so that average size $X(t)$ remains large and continues to grow).

The result in Corollary 14.1 is closely linked to our focus on the equilibrium in which all incumbents choose the same R&D effort level. This focus is justified on theoretical grounds in Exercise 14.24. On empirical grounds, this assumption ensures that firm growth is independent of firm size, a regularity commonly referred to as Gibrat's Law (e.g., Sutton, 1997). If $z(\nu, t \mid q)$ were a function of q, then firm growth would depend on firm size, violating Gibrat's Law. Gibrat's Law, despite its name, does not characterize the patterns of firm growth throughout the firm size distribution. In particular, the empirical evidence indicates that firm growth rates are indeed independent of firm size above a certain threshold, but are higher for small firms than for large firms (e.g., Hall, 1987). In light of this, consider the following candidate equilibrium. Let $\varepsilon > 0$ and suppose that

$$z^*(\tilde{q}) = \begin{cases} \bar{z}^* & \text{for all } \tilde{q} > \varepsilon \\ \bar{z}^+ & \text{if } \tilde{q} \leq \varepsilon, \end{cases} \tag{14.71}$$

where

$$\tilde{q}(\nu, t) \equiv \frac{q(\nu, t)}{Q(t)}$$

is relative quality in machine line ν at time t, and $\bar{z}^+ > \bar{z}^*$. This specification implies that there is a slight deviation from Gibrat's Law with firms below a certain relative size threshold growing faster than the rest (recall that average quality is proportional to average sales). Is straightforward to verify that there exists a BGP equilibrium in which incumbent R&D effort levels are given by (14.71) (see Exercise 14.25). Then we can prove the following result about the equilibrium firm-size distribution.

Proposition 14.9. *Consider the BGP equilibrium in which incumbent R&D effort levels are given by (14.71) for $\varepsilon > 0$ and $\bar{z}^+ > \bar{z}^* > 0$. Consider the limiting case where $\bar{z}^+ \to \infty$. Then there exists a unique stationary firm-size distribution given by the Pareto distribution with an exponent of one.*

Proof. As $\bar{z}^+ \to \infty$, no firm would have relative quality $\tilde{q} < \varepsilon$ (since there will be immediate innovation at $\tilde{q} = \varepsilon$). This implies that relative firm size $\tilde{x}$ is bounded below by ε. Moreover, since $z^*(\tilde{q}) \to \infty$ only at $\tilde{q} = \varepsilon$, the definition (14.53) implies that $\bar{z}^+ = z^*$, with z^* given by (14.57). In view of this, the functional equation (14.68) in the proof of Proposition 14.8 still applies and characterizes the stationary distribution (when it exists). The same arguments as in that proof then shows that this functional equation has a unique solution given by $\Pr[\tilde{x}(t) \leq y] = 1 - \Gamma/y$. Moreover, in this case a stationary distribution exists, since $\tilde{x} \geq \varepsilon$ and thus $\Pr[\tilde{x}(t) \leq y] = 1 - \Gamma/y$ with $\Gamma = \varepsilon$ is a well-defined probability distribution. □

This proposition therefore shows that, if we focus on equilibria in which firms below a certain relative threshold have much higher innovation intensity, the equilibrium size distribution (in terms of normalized sizes) will be Pareto with an exponent equal to one.. This result is interesting and encouraging, since the distribution of firm sizes in the United States is very well approximated by the Zipf's distribution (Axtell, 2001). Nevertheless, it should be noted that the approach adopted here is highly parsimonious (thus leaves out many relevant details) and the model takes a minimal departure from the baseline Schumpeterian model. Moreover, this model generates realistic firm-size distributions only when we select a particular distribution of R&D levels among incumbents. A more detailed study of the issues related to firm dynamics and firm-size distributions requires more complex approaches that may capture richer dynamics, including more realistic models of entry and exit behavior of firms.

14.4. Step-by-Step Innovations*

In the baseline Schumpeterian model and also in the extended Schumpeterian model of the previous section, new entrants could undertake innovation on any machine and did not need to have developed any knowhow on a particular line of business. This led to a simple structure, in many ways parallel to the models of expanding varieties studied in the previous chapter. However, quality improvements in practice may have a major cumulative aspect. For example, it may be that only firms that have already reached a certain level of knowledge in a particular product or machine line can engage in further innovations. This is in fact consistent with qualitative accounts of technological change and competition in specific industries. Abernathy (1980, p. 70), for instance, concludes his study of a number of diverse industries by stating that: "each of the major companies seems to have made more frequent contributions in a particular area," and argues that this is because previous innovations in a field facilitate future innovations. This aspect is entirely missing from the baseline model of Schumpeterian growth, where any firm can engage in research to develop the next higher-quality machine (and in addition Arrow's replacement effect implies that incumbents do not undertake R&D, though this aspect was relaxed and generalized in the previous section). A more realistic description of the research process may involve only a few firms engaging in continuous and cumulative innovation and competition in a particular product or machine line.

In this section, I will present a model of cumulative innovation of this type. Following Aghion, Harris, Howitt and Vickers (2001), we will refer to this as a model of *step-by-step innovation*. Such models are not only useful in providing a different conceptualization of the process of Schumpeterian growth, but they also enable us to endogenize the equilibrium market structure and allow a richer analysis of the effects of competition and intellectual property rights policy. Together with the model of innovation by incumbents and entrants presented in the previous section, this model enables us to have a framework in which existing firm (continuing establishments) contribute to productivity growth and build on their own past innovations (consistent with the empirical evidence as discussed in Section 18.1 in Chapter 18). In fact, the model in this section has a number of unique features relative to those presented so far in this and the previous chapter. For instance, these models predict that weaker patent protection and greater competition should reduce economic growth. Nevertheless, existing empirical evidence suggests that typically industries that are more competitive experience faster growth (or at the very least, there is a non-monotonic relationship between competition and economic growth, see, for example, Blundell (1999), Nickell (1999) and Aghion, Bloom, Blundell, Griffith and Howitt (2005)). Schumpeterian models with an endogenous market structure show that the effects of competition and intellectual property rights on economic growth are more complex, and greater competition (and weaker intellectual property rights protection) sometimes increases the growth rate of the economy.

14.4.1. Preferences and Technology. Consider the following continuous time economy with a unique final good. The economy is populated by a continuum of measure 1 of individuals, each with 1 unit of labor endowment, which they supply inelastically. To simplify the analysis, we assume that the instantaneous utility function takes a logarithmic form. Thus the representative household has preferences given by

$$\int_0^\infty \exp(-\rho t) \log C(t)\, dt, \tag{14.72}$$

where $\rho > 0$ is the discount rate and $C(t)$ is consumption at date t.

Let $Y(t)$ be the total production of the final good at time t. We assume that the economy is closed and the final good is used only for consumption (i.e., there is no investment or spending on machines), so that $C(t) = Y(t)$. The standard Euler equation from (14.72) then implies that

$$g(t) \equiv \frac{\dot{C}(t)}{C(t)} = \frac{\dot{Y}(t)}{Y(t)} = r(t) - \rho, \tag{14.73}$$

where this equation defines $g(t)$ as the growth rate of consumption and thus output, and $r(t)$ is the interest rate at date t.

The final good Y is produced using a continuum 1 of intermediate goods according to the Cobb-Douglas production function

$$\ln Y(t) = \int_0^1 \ln y(\nu, t)\, d\nu, \tag{14.74}$$

where $y(\nu, t)$ is the output of νth intermediate at time t. Throughout, we take the price of the final good (or the ideal price index of the intermediates) as the numeraire and denote the price of intermediate ν at time t by $p^y(\nu, t)$. We also assume that there is free entry into the final good sector. These assumptions, together with the Cobb-Douglas production function (14.74), imply that each final good producer will have the following demand for intermediates

$$y(\nu, t) = \frac{Y(t)}{p^y(\nu, t)}, \text{ for all } \nu \in [0, 1]. \tag{14.75}$$

Each intermediate $\nu \in [0, 1]$ comes in two different varieties, each produced by one of two infinitely-lived firms. We assume that these two varieties are perfect substitutes and compete a la Bertrand. No other firm is able to produce in this industry. Firm $i = 1$ or 2 in industry ν has the following technology

$$y(\nu, t) = q_i(\nu, t)\, l_i(\nu, t) \tag{14.76}$$

where $l_i(\nu, t)$ is the employment level of the firm and $q_i(\nu, t)$ is its level of technology at time t. The only difference between the two firms is their technology, which will be determined endogenously. As in the models studied so far, each consumer in the economy holds a balanced portfolio of the shares of all firms. Consequently, the objective function of each firm is to maximize expected profits.

The production function for intermediate goods, (14.76), implies that the marginal cost of producing intermediate ν for firm i at time t is

$$MC_i(\nu, t) = \frac{w(t)}{q_i(\nu, t)} \tag{14.77}$$

where $w(t)$ is the wage rate in the economy at time t.

Let us denote the *technological leader* in each industry by i and the *follower* by $-i$, so that we have:

$$q_i(\nu, t) \geq q_{-i}(\nu, t).$$

Bertrand competition between the two firms implies that all intermediates will be supplied by the leader at the limit price (see Exercise 14.27):

$$p_i^y(\nu, t) = \frac{w(t)}{q_{-i}(\nu, t)}. \tag{14.78}$$

Equation (14.75) then implies the following demand for intermediates:

$$y(\nu, t) = \frac{q_{-i}(\nu, t)}{w(t)} Y(t). \tag{14.79}$$

R&D by the leader or the follower stochastically leads to innovation. We assume that when the leader innovates, its technology improves by a factor $\lambda > 1$. The follower, on the other hand, can undertake R&D to catch up with the frontier technology. Let us assume that because this innovation is for the follower's variant of the product and results from its own R&D efforts, it does not constitute infringement of the patent of the leader, and the follower does not have to make any payments to the technological leader in the industry.

R&D investments by the leader and the follower may have different costs and success probabilities. Nevertheless, we simplify the analysis by assuming that they have the same costs and the same probability of success. In particular, in all cases, we assume that each firm (in every industry) has access to the following R&D technology (innovation possibilities frontier):

$$z_i(\nu, t) = \Phi(h_i(\nu, t)), \tag{14.80}$$

where $z_i(\nu, t)$ is the flow rate of innovation at time t and $h_i(\nu, t)$ is the number of workers hired by firm i in industry ν to work in the R&D process at t. Let us assume that Φ is twice continuously differentiable and satisfies $\Phi'(\cdot) > 0$, $\Phi''(\cdot) < 0$, $\Phi'(0) < \infty$ and that there exists $\bar{h} \in (0, \infty)$ such that $\Phi'(h) = 0$ for all $h \geq \bar{h}$. The assumption that $\Phi'(0) < \infty$ implies that there is no Inada condition when $h_i(\nu, t) = 0$. The last assumption, on the other hand, ensures that there is an upper bound on the flow rate of innovation (which is not essential but simplifies the proofs). Recalling that the wage rate for labor is $w(t)$, the cost for R&D is therefore $w(t) G(z_i(\nu, t))$ where

$$G(z_i(\nu, t)) \equiv \Phi^{-1}(z_i(\nu, t)), \tag{14.81}$$

and the assumptions on Φ immediately imply that G is twice continuously differentiable and satisfies $G'(\cdot) > 0$, $G''(\cdot) > 0$, $G'(0) > 0$ and $\lim_{z \to \bar{z}} G'(z) = \infty$, where $\bar{z} \equiv \Phi(\bar{h})$ is the maximal flow rate of innovation (with $\bar{h}$ defined above).

We next describe the evolution of technologies within each industry. Suppose that leader i in industry ν at time t has a technology level of

$$q_i(\nu, t) = \lambda^{n_i(\nu, t)}, \tag{14.82}$$

and that the follower $-i$'s technology at time t is

$$q_{-i}(\nu,t) = \lambda^{n_{-i}(\nu,t)}, \tag{14.83}$$

where, naturally, $n_i(\nu,t) \geq n_{-i}(\nu,t)$. Let us denote the technology gap in industry ν at time t by $n(\nu,t) \equiv n_i(\nu,t) - n_{-i}(\nu,t)$. If the leader undertakes an innovation within a time interval of Δt, then the technology gap rises to $n(\nu,t+\Delta t) = n(\nu,t)+1$ (the probability of two or more innovations within the interval Δt is again $o(\Delta t)$). If, on the other hand, the follower undertakes an innovation during the interval Δt, then $n(\nu,t+\Delta t) = 0$. In addition, let us assume that there is an intellectual property rights (IPR) policy of the following form: the patent held by the technological leader expires at the exponential rate $\kappa < \infty$, in which case, the follower can close the technology gap.

Given this specification, the law of motion of the technology gap in industry ν can be expressed as

$$n(\nu,t+\Delta t) = \begin{cases} n(\nu,t)+1 & \text{with probability} \quad z_i(\nu,t)\Delta t + o(\Delta t) \\ 0 & \text{with probability} \quad (z_{-i}(\nu,t)+\kappa)\Delta t + o(\Delta t) \\ n(\nu,t) & \text{with probability} \quad 1-(z_i(\nu,t)+z_{-i}(\nu,t)+\kappa)\Delta t - o(\Delta t)) \end{cases}. \tag{14.84}$$

Here $o(\Delta t)$ again represents second-order terms, in particular, the probabilities of more than one innovations within an interval of length Δt. The terms $z_i(\nu,t)$ and $z_{-i}(\nu,t)$ are the flow rates of innovation by the leader and the follower, while κ is the flow rate at which the follower is allowed to copy the technology of the leader. In the first line, when $n(\nu,t) = 0$, so that the two firms are neck and neck, $z_i(\nu,t)$ should be taken as equal to $2z(\nu,t)$, since the two firms will undertake the same amount of research effort given by $z(\nu,t)$ and the technology gap will increase to 1 if one of them is successful, which has probability $2z(\nu,t)$.

We next write the instantaneous "operating" profits for the leader (i.e., the profits exclusive of R&D expenditures and license fees). Profits of leader i in industry ν at time t are

$$\begin{aligned} \Pi_i(\nu,t) &= \left[p_i^y(\nu,t) - MC_i(\nu,t)\right] y_i(\nu,t) \\ &= \left(\frac{w(t)}{q_{-i}(\nu,t)} - \frac{w(t)}{q_i(\nu,t)}\right)\frac{Y(t)}{p_i^y(\nu,t)} \\ &= \left(1-\lambda^{-n(\nu,t)}\right) Y(t) \end{aligned} \tag{14.85}$$

where recall that $n(\nu,t)$ is the technology gap in industry j at time t. The first line simply uses the definition of operating profits as price minus marginal cost times quantity sold. The second line uses the fact that the equilibrium limit price of firm i is $p_i^y(\nu,t) = w(t)/q_{-i}(\nu,t)$ as given by (14.78), and the final equality uses the definitions of $q_i(\nu,t)$ and $q_{-i}(\nu,t)$ from (14.82) and (14.83). The expression in (14.85) also implies that there will be zero profits in an industry that is *neck-and-neck*, i.e., in industries with $n(j,t) = 0$. Followers also make zero profits, since they have no sales.

The Cobb-Douglas aggregate production function in (14.74) is responsible for the simple form of the profits (14.85), since it implies that profits only depend on the technology gap of the industry and aggregate output. This will simplify the analysis below by making the technology gap in each industry the only industry-specific payoff-relevant state variable.

The objective function of each firm is to maximize the net present discounted value of net profits (operating profits minus R&D expenditures and plus or minus patent fees). In doing this, each firm will take the sequence of interest rates, $[r(t)]_{t=0}^{\infty}$, the sequence of aggregate output levels, $[Y(t)]_{t=0}^{\infty}$, the sequence of wages, $[w(t)]_{t=0}^{\infty}$, the R&D decisions of all other firms and policies as given. Note that as in the baseline model of Schumpeterian growth in Section 14.1, even though technology and output in each sector are stochastic, total output, $Y(t)$, given by (14.74) is nonstochastic.

14.4.2. Equilibrium. Let $\boldsymbol{\mu}(t) \equiv \{\mu_n(t)\}_{n=0}^{\infty}$ denote the distribution of industries over different technology gaps, with $\sum_{n=0}^{\infty} \mu_n(t) = 1$. For example, $\mu_0(t)$ denotes the fraction of industries in which the firms are neck-and-neck at time t. Throughout, we focus on Markov Perfect Equilibria (MPE), where strategies are only functions of the payoff-relevant state variables. MPE is a natural equilibrium concept in this context, since it does not allow for implicit collusive agreements between the follower and the leader. While such collusive agreements may be likely when there are only two firms in the industry, in most industries there are many more firms and also many potential entrants, making collusion more difficult. Throughout, we assume that there are only two firms to keep the model tractable (see Appendix Chapter C for references and a further discussion of MPE). The focus on MPE allows us to drop the dependence on industry ν, thus we refer to R&D decisions by z_n for the technological leader that is n steps ahead and by z_{-n} for a follower that is n steps behind. Let us denote the list of decisions by the leader and follower with technology gap n at time t by $\xi_n(t) \equiv \langle z_n(t), p_i^y(\nu,t), y_i(\nu,t)\rangle$ and $\xi_{-n}(t) \equiv z_{-n}(t)$. Throughout, $\boldsymbol{\xi}$ will indicate the whole sequence of decisions at every state, $\boldsymbol{\xi}(t) \equiv \{\xi_n(t)\}_{n=-\infty}^{\infty}$.[6]

An allocation in this economy is then given by time paths of decisions for a leader that is $n = 0, 1, ..., \infty$ steps ahead, $[\xi_n(t)]_{t=0}^{\infty}$, time paths of R&D decisions for a follower that is $n = 1, ..., \infty$ steps behind, $\left[\xi_{-n}(t)\right]_{t=0}^{\infty}$, time path of wages and interest rates $[w(t), r(t)]_{t=0}^{\infty}$, and time paths of industry distributions over technology gaps $[\boldsymbol{\mu}(t)]_{t=0}^{\infty}$.

[6]There are two sources of abuse of notation here. First, pricing and output decisions, given by (14.78) and (14.79), depend on the aggregate level of output $Y(t)$ as well. However, profits, as given by (14.85), and other choices do not depend on $Y(t)$, and we suppress this dependence without any affect on the analysis. Second, the sequences $[p_i^{y*}(\nu,t)]_{t=0}^{\infty}$ and $[y_i^*(\nu,t)]_{t=0}^{\infty}$ are stochastic, while the rest of the objects specified above are not. Since the stochastic nature of these sequences has no effect on the analysis, we suppress this feature as well.

We can define an equilibrium as follows. A Markov Perfect Equilibrium is represented by time paths $[\boldsymbol{\xi}^*(t), w^*(t), r^*(t), Y^*(t)]_{t=0}^{\infty}$ such that (i) $\left[p_i^{y*}(\nu,t)\right]_{t=0}^{\infty}$ and $[y_i^*(\nu,t)]_{t=0}^{\infty}$ implied by $[\boldsymbol{\xi}^*(t)]_{t=0}^{\infty}$ satisfy (14.78) and (14.79); (ii) R&D policies $[\mathbf{z}^*(t)]_{t=0}^{\infty}$ are best responses to themselves, i.e., $[\mathbf{z}^*(t)]_{t=0}^{\infty}$ maximizes the expected profits of firms taking aggregate output $[Y^*(t)]_{t=0}^{\infty}$, factor prices $[w^*(t), r^*(t)]_{t=0}^{\infty}$, and the R&D policies of other firms $[\mathbf{z}^*(t)]_{t=0}^{\infty}$ as given; (iii) aggregate output $[Y^*(t)]_{t=0}^{\infty}$ is given by (14.74); and (iv) the labor and capital markets clear at all times given the factor prices $[w^*(t), r^*(t)]_{t=0}^{\infty}$.

We next characterize the equilibrium. Since only the technological leader produces, labor demand in industry ν with technology gap $n(\nu,t) = n$ can be expressed as

$$l_n(t) = \frac{\lambda^{-n} Y(t)}{w(t)} \text{ for } n \geq 0. \tag{14.86}$$

In addition, there is demand for labor coming from R&D of both followers and leaders in all industries. Using (14.80) and the definition of the G function, we can express industry demands for R&D labor as

$$h_n(t) = \begin{cases} G(z_n(t)) + G(z_{-n}(t)) & \text{if } n \geq 1 \\ 2G(z_0(t)) & \text{if } n = 0 \end{cases}, \tag{14.87}$$

where $z_{-n}(t)$ refers to the R&D effort of a follower that is n steps behind. Moreover, this expression takes into account that in an industry with neck-and-neck competition, i.e., with $n = 0$, there will be twice the demand for R&D coming from the two "symmetric" firms.

The labor market clearing condition can then be expressed as:

$$1 \geq \sum_{n=0}^{\infty} \mu_n(t) \left[\frac{1}{\omega(t)\lambda^n} + G(z_n(t)) + G(z_{-n}(t)) \right], \tag{14.88}$$

and $\omega(t) \geq 0$, with complementary slackness, where

$$\omega(t) \equiv \frac{w(t)}{Y(t)} \tag{14.89}$$

is the labor share at time t. The labor market clearing condition, (14.88), uses the fact that total supply is equal to 1, and the demand cannot exceed this amount. If demand falls short of 1, then the wage rate, $w(t)$, and thus the labor share, $\omega(t)$, have to be equal to zero (though this will never be the case in equilibrium). The right-hand side of (14.88) consists of the demand for production (the terms with ω in the denominator), the demand for R&D workers from the neck-and-neck industries ($2G(z_0(t))$ when $n = 0$) and the demand for R&D workers coming from leaders and followers in other industries ($G(z_n(t)) + G(z_{-n}(t))$ when $n > 0$).

The relevant index of aggregate quality in this economy is no longer the average, but reflects the Cobb-Douglas aggregator in the production function,

$$\ln Q(t) \equiv \int_0^1 \ln q(\nu,t)\, d\nu. \tag{14.90}$$

Given this, the equilibrium wage can be written as (see Exercise 14.28):

$$w(t) = Q(t)\,\lambda^{-\sum_{n=0}^{\infty} n\mu_n(t)}. \tag{14.91}$$

14.4.3. Steady-State Equilibrium. Let us now focus on steady-state (Markov Perfect) equilibria, where the distribution of industries $\boldsymbol{\mu}(t) \equiv \{\mu_n(t)\}_{n=0}^{\infty}$ is stationary, $\omega(t)$ defined in (14.89) and g^*, the growth rate of the economy, is constant over time (we refer to this as a steady-state Markov perfect equilibrium, since the potentially more accurate term "balanced growth path Markov perfect equilibrium" sounds awkward). We will establish the existence of such an equilibrium and characterize a number of its properties. If the economy is in steady state at time $t = 0$, then by definition, we have $Y(t) = Y_0 e^{g^* t}$ and $w(t) = w_0 \exp(g^* t)$. The two equations also imply that $\omega(t) = \omega^*$ for all $t \geq 0$. Throughout, we assume that the parameters are such that the steady-state growth rate g^* is positive but not large enough to violate the transversality conditions. This implies that net present values of each firm at all points in time will be finite and enable us to write the maximization problem of a leader that is $n > 0$ steps ahead recursively.

Standard arguments imply that the value function for a firm that is n steps ahead (or $-n$ steps behind) is given by

(14.92)

$$\begin{aligned} r(t)\,V_n(t) - \dot{V}_n(t) &= \max_{z_n(t)}\{[\Pi_n(t) - w^*(t)\,G(z_n(t))] \\ &\quad + z_n(t)\,[V_{n+1}(t) - V_n(t)] + \left(z_{-n}^*(t) + \kappa\right)[V_0(t) - V_n(t)]\}. \end{aligned}$$

In steady state, the net present value of a firm that is n steps ahead, $V_n(t)$, will also grow at a constant rate g^* for all $n \in \mathbb{Z}_+$. Let us then define normalized values as

$$v_n(t) \equiv \frac{V_n(t)}{Y(t)} \tag{14.93}$$

for all n which will be independent of time in steady state, i.e., $v_n(t) = v_n$.

Using (14.93) and the fact that from (14.73), $r(t) = g(t) + \rho$, the steady-state value function (14.92) can be written as:

$$\begin{aligned} \rho v_n &= \max_{z_n}\{\left(1 - \lambda^{-n}\right) - \omega^* G(z_n) + z_n\,[v_{n+1} - v_n]\,, \\ &\quad + \left[z_{-n}^* + \kappa\right][v_0 - v_n]\} \text{ for all } n \geq 1, \end{aligned} \tag{14.94}$$

where z_{-n}^* is the equilibrium value of R&D by a follower that is n steps behind, and ω^* is the steady-state labor share (while z_n is now explicitly chosen to maximize v_n).

Similarly the value for neck-and-neck firms is

$$\rho v_0 = \max_{z_0}\{-\omega^* G(z_0) + z_0\,[v_1 - v_0] + z_0^*\,[v_{-1} - v_0]\}\,, \tag{14.95}$$

while the values for followers are given by

$$\rho v_{-n} = \max_{z_{-n}} \left\{ -\omega^* G\left(z_{-n}\right) + \left[z_{-n} + \kappa\right]\left[v_0 - v_{-n}\right] \right\}.$$

It is clear that these value functions and profit-maximizing R&D decision for followers should not depend on how many steps behind the leader they are, since a single innovation is sufficient to catch-up with the leader. Therefore, we can write

$$\rho v_{-1} = \max_{z_{-1}} \left\{ -\omega^* G\left(z_{-1}\right) + \left[z_{-1} + \kappa\right]\left[v_0 - v_{-1}\right] \right\}, \tag{14.96}$$

where v_{-1} represents the value of any follower (irrespective of how many steps behind it is). The maximization problems involved in the value functions are straightforward and immediately yield the following profit-maximizing R&D decisions

$$z_n^* = \max\left\{ G'^{-1}\left(\frac{\left[v_{n+1} - v_n\right]}{\omega^*} \right), 0 \right\} \tag{14.97}$$

$$z_{-n}^* = \max\left\{ G'^{-1}\left(\frac{\left[v_0 - v_{-n}\right]}{\omega^*} \right), 0 \right\} \tag{14.98}$$

$$z_0^* = \max\left\{ G'^{-1}\left(\frac{\left[v_1 - v_0\right]}{\omega^*} \right), 0 \right\}, \tag{14.99}$$

where $G'^{-1}(\cdot)$ is the inverse of the derivative of the G function, and since G is twice continuously differentiable and strictly concave, G'^{-1} is continuously differentiable and strictly increasing. These equations therefore imply that innovation rates, the z_n^*'s, are increasing in the incremental value of moving to the next step and decreasing in the cost of R&D, as measured by the normalized wage rate, ω^*. Note also that since $G'(0) > 0$, these R&D levels can be equal to zero, which is taken care of by the max operator.

The response of innovation rates, z_n^*, to the increments in values, $v_{n+1} - v_n$, is the key economic force in this model. For example, a policy that reduces the patent protection of leaders that are $n+1$ steps ahead (by increasing κ) will make being $n+1$ steps ahead less profitable, thus reduce $v_{n+1} - v_n$ and z_n^*. This corresponds to the standard *disincentive effect* of relaxing IPR protection. However, relaxing IPR protection may also create a beneficial *composition effect*; this is because, typically, $\{v_{n+1} - v_n\}_{n=0}^{\infty}$ is a decreasing sequence, which implies that z_{n-1}^* is higher than z_n^* for $n \geq 1$ (see Proposition 14.13 below). Weaker patent protection (in the form of shorter patent lengths) will shift more industries into the neck-and-neck state and potentially increase the equilibrium level of R&D in the economy.

Given the equilibrium R&D decisions, the steady-state distribution of industries across states $\boldsymbol{\mu}^*$ has to satisfy the following accounting identities:

$$\left(z_{n+1}^* + z_{-1}^* + \kappa\right)\mu_{n+1}^* = z_n^* \mu_n^* \text{ for } n \geq 1, \tag{14.100}$$

$$\left(z_1^* + z_{-1}^* + \kappa\right)\mu_1^* = 2 z_0^* \mu_0^*, \tag{14.101}$$

$$2 z_0^* \mu_0^* = z_{-1}^* + \kappa. \tag{14.102}$$

The first expression equates exit from state $n+1$ (which takes the form of the leader going one more step ahead or the follower catching-up the leader) to entry into this state (which takes the form of a leader from the state n making one more innovation). The second equation, (14.101), performs the same accounting for state 1, taking into account that entry into this state comes from innovation by either of the two firms that are competing neck-and-neck. Finally, equation (14.102) equates exit from state 0 with entry into this state, which comes from innovation by a follower in any industry with $n \geq 1$.

The labor market clearing condition in steady state can then be written as

$$1 \geq \sum_{n=0}^{\infty} \mu_n^* \left[\frac{1}{\omega^* \lambda^n} + G\left(z_n^*\right) + G\left(z_{-n}^*\right) \right] \text{ and } \omega^* \geq 0, \tag{14.103}$$

with complementary slackness.

The next proposition characterizes the steady-state growth rate in this economy:

Proposition 14.10. *The steady-state growth rate is given by*

$$g^* = \ln \lambda \left[2\mu_0^* z_0^* + \sum_{n=1}^{\infty} \mu_n^* z_n^* \right]. \tag{14.104}$$

Proof. Equations (14.89) and (14.91) imply

$$Y(t) = \frac{w(t)}{\omega(t)} = \frac{Q(t)\,\lambda^{-\sum_{n=0}^{\infty} n\mu_n^*(t)}}{\omega(t)}.$$

Since $\omega(t) = \omega^*$ and $\{\mu_n^*\}_{n=0}^{\infty}$ are constant in steady state, $Y(t)$ grows at the same rate as $Q(t)$. Therefore,

$$g^* = \lim_{\Delta t \to 0} \frac{\ln Q(t+\Delta t) - \ln Q(t)}{\Delta t}.$$

During an interval of length Δt, we have that in the fraction μ_n^* of the industries with technology gap $n \geq 1$ the leaders innovate at a rate $z_n^* \Delta t + o(\Delta t)$ and in the fraction μ_0^* of the industries with technology gap of $n = 0$, both firms innovate, so that the total innovation rate is $2z_0^* \Delta t + o(\Delta t)$). Since each innovation increases productivity by a factor λ, we obtain the preceding equation. Combining these observations, we have

$$\ln Q(t+\Delta t) = \ln Q(t) + \ln \lambda \left[2\mu_0^* z_0^* \Delta t + \sum_{n=1}^{\infty} \mu_n^* z_n^* \Delta t + o(\Delta t) \right].$$

Subtracting $\ln Q(t)$, dividing by Δt and taking the limit as $\Delta t \to 0$ gives (14.104). □

This proposition clarifies that the steady-state growth comes from two sources:

(1) R&D decisions of leaders or of firms in neck-and-neck industries.
(2) The distribution of industries across different technology gaps, $\boldsymbol{\mu}^* \equiv \{\mu_n^*\}_{n=0}^{\infty}$.

The latter channel reflects the composition effect discussed above. This type of composition effect implies that the relationship between competition and growth (or intellectual property rights protection and growth) is more complex than in the models we have seen so far, because such policies will change the equilibrium market structure (i.e., the composition of industries).

DEFINITION 14.1. *A steady-state equilibrium is given by* $\langle \boldsymbol{\mu}^*, \mathbf{v}, \mathbf{z}^*, \omega^*, g^* \rangle$ *such that the distribution of industries* $\boldsymbol{\mu}^*$ *satisfy (14.100), (14.101) and (14.102), the values* $\mathbf{v} \equiv \{v_n\}_{n=-\infty}^{\infty}$ *satisfy (14.94), (14.95) and (14.96), the R&D decisions* $\mathbf{z}^*$ *are given by, (14.97), (14.98) and (14.99), the steady-state labor share* ω^* *satisfies (14.103) and the steady-state growth rate* g^* *is given by (14.104).*

We next provide a characterization of the steady-state equilibrium. The first result is a technical one that is necessary for this characterization.

PROPOSITION 14.11. *In a steady state equilibrium, we have* $v_{-1} \leq v_0$ *and* $\{v_n\}_{n=0}^{\infty}$ *forms a bounded and strictly increasing sequence converging to some positive value* v_∞.

PROOF. Let $\{z_n\}_{n=-1}^{\infty}$ be the R&D decisions of a firm and $\{v_n\}_{n=-1}^{\infty}$ be the sequence of values, taking the decisions of other firms and the industry distributions, $\{z_n^*\}_{n=-1}^{\infty}$, $\{\mu_n^*\}_{n=-1}^{\infty}$, ω^* and g^*, as given. By choosing $z_n = 0$ for all $n \geq -1$, the firm guarantees $v_n \geq 0$ for all $n \geq -1$. Moreover, since flow profits satisfy $\pi_n \leq 1$ for all $n \geq -1$, we have $v_n \leq 1/\rho$ for all $n \geq -1$, establishing that $\{v_n\}_{n=-1}^{\infty}$ is a bounded sequence, with $v_n \in [0, 1/\rho]$ for all $n \geq -1$.

Proof of $v_1 > v_0$: Suppose, first, $v_1 \leq v_0$, then (14.99) implies $z_0^* = 0$, and by the symmetry of the problem in equilibrium, (14.95) implies $v_0 = v_1 = 0$. As a result, from (14.98) we obtain $z_{-1}^* = 0$. Equation (14.94) then implies that when $z_{-1}^* = 0$, $v_1 \geq \left(1 - \lambda^{-1}\right) / (\rho + \kappa) > 0$, yielding a contradiction and proving that $v_1 > v_0$.

Proof of $v_{-1} \leq v_0$: Suppose, to obtain a contradiction, that $v_{-1} > v_0$. Then, (14.98) implies $z_{-1}^* = 0$, which leads to $v_{-1} = \kappa v_0 / (\rho + \kappa)$, contradicting $v_{-1} > v_0$ since $\kappa / (\rho + \kappa) < 1$ (given that $\kappa < \infty$).

Proof of $v_n < v_{n+1}$: Suppose, to obtain a contradiction, that $v_n \geq v_{n+1}$. Now (14.97) implies $z_n^* = 0$, and (14.94) becomes

$$\rho v_n = \left(1 - \lambda^{-n}\right) + z_{-1}^* \left[v_0 - v_n\right] + \kappa \left[v_0 - v_n\right]. \tag{14.105}$$

Also from (14.94), the value for state $n + 1$ satisfies

$$\rho v_{n+1} \geq \left(1 - \lambda^{-n-1}\right) + z_{-1}^* \left[v_0 - v_{n+1}\right] + \kappa \left[v_0 - v_{n+1}\right]. \tag{14.106}$$

Combining the two previous expressions, we obtain

$$\begin{aligned} &\left(1 - \lambda^{-n}\right) + z_{-1}^* \left[v_0 - v_n\right] + \kappa \left[v_0 - v_n\right] \geq \\ &1 - \lambda^{-n-1} + z_{-1}^* \left[v_0 - v_{n+1}\right] + \kappa \left[v_0 - v_{n+1}\right]. \end{aligned}$$

Since $\lambda^{-n-1} < \lambda^{-n}$, this implies $v_n < v_{n+1}$, contradicting the hypothesis that $v_n \geq v_{n+1}$, and establishing the desired result, $v_n < v_{n+1}$.

Consequently, $\{v_n\}_{n=-1}^{\infty}$ is nondecreasing and $\{v_n\}_{n=0}^{\infty}$ is (strictly) increasing. Since a nondecreasing sequence in a compact set must converge, $\{v_n\}_{n=-1}^{\infty}$ converges to its limit point, v_∞, which must be strictly positive, since $\{v_n\}_{n=0}^{\infty}$ is strictly increasing and has a nonnegative initial value. This completes the proof. □

A potential difficulty in the analysis of the current model is that we have to determine R&D levels and values for an infinite number of firms, since the technology gap between the leader and the follower can, in principle, take any value. However, the next result shows that we can restrict attention to a finite sequence of values:

Proposition 14.12. *There exists $n^* \geq 1$ such that $z_n^* = 0$ for all $n \geq n^*$.*

Proof. See Exercise 14.29. □

The next proposition provides the most important economic insights of this model and shows that $\mathbf{z}^* \equiv \{z_n^*\}_{n=0}^{\infty}$ is a decreasing sequence, which implies that technological leaders that are further ahead undertake less R&D. Intuitively, the benefits of further R&D investments are decreasing in the technology gap, since greater values of the technology gap translate into smaller increases in the equilibrium markup (recall (14.85)). The fact that leaders that are sufficiently ahead of their competitors undertake little R&D is the main reason why composition effects play an important role in this model. For example, all else equal, closing the technology gap across a range of industries will increase R&D spending and equilibrium growth (though, as discussed in the previous section, this may not always increase welfare, especially if there is a strong business stealing effect).

Proposition 14.13. *In any steady-state equilibrium, we have $z_{n+1}^* \leq z_n^*$ for all $n \geq 1$ and moreover, $z_{n+1}^* < z_n^*$ if $z_n^* > 0$. Furthermore, $z_0^* > z_1^*$ and $z_0^* \geq z_{-1}^*$.*

Proof. From equation (14.97),

$$\delta_{n+1} \equiv v_{n+1} - v_n < v_n - v_{n-1} \equiv \delta_n \tag{14.107}$$

is sufficient to establish that $z_{n+1}^* \leq z_n^*$.

Let us write:

$$\bar{\rho} v_n = \max_{z_n} \left\{ \left(1 - \lambda^{-n}\right) - \omega^* G\left(z_n\right) + z_n \left[v_{n+1} - v_n\right] + \left(z_{-1}^* + \kappa\right) v_0 \right\}, \tag{14.108}$$

where $\bar{\rho} \equiv \rho + z_{-1}^* + \kappa$. Since z_{n+1}^*, z_n^* and z_{n-1}^* are maximizers of the value functions v_{n+1}, v_n and v_{n-1}, (14.108) implies:

$$\begin{aligned} \bar{\rho} v_{n+1} &= 1 - \lambda^{-n-1} - \omega^* G\left(z_{n+1}^*\right) + z_{n+1}^* \left[v_{n+2} - v_{n+1}\right] + \left(z_{-1}^* + \kappa\right) v_0, \\ \bar{\rho} v_n &\geq 1 - \lambda^{-n} - \omega^* G\left(z_{n+1}^*\right) + z_{n+1}^* \left[v_{n+1} - v_n\right] + \left(z_{-1}^* + \kappa\right) v_0,, \\ \bar{\rho} v_n &\geq 1 - \lambda^{-n} - \omega^* G\left(z_{n-1}^*\right) + z_{n-1}^* \left[v_{n+1} - v_n\right] + \left(z_{-1}^* + \kappa\right) v_0,, \\ \bar{\rho} v_{n-1} &= 1 - \lambda^{-n+1} - \omega^* G\left(z_{n-1}^*\right) + z_{n-1}^* \left[v_n - v_{n-1}\right] + \left(z_{-1}^* + \kappa\right) v_0. \end{aligned} \tag{14.109}$$

Now taking differences with $\bar{\rho} v_n$ and using the definition of δ_n's, we obtain

$$\begin{aligned} \bar{\rho} \delta_{n+1} &\leq \lambda^{-n} \left(1 - \lambda^{-1}\right) + z_{n+1}^* \left(\delta_{n+2} - \delta_{n+1}\right) \\ \bar{\rho} \delta_n &\geq \lambda^{-n+1} \left(1 - \lambda^{-1}\right) + z_{n-1}^* \left(\delta_{n+1} - \delta_n\right). \end{aligned}$$

Therefore,

$$\left(\bar{\rho} + z_{n-1}^*\right) \left(\delta_{n+1} - \delta_n\right) \leq -k_n + z_{n+1}^* \left(\delta_{n+2} - \delta_{n+1}\right),$$

where $k_n \equiv (\lambda - 1)^2 \lambda^{-n-1} > 0$. Now to obtain a contradiction, suppose that $\delta_{n+1} - \delta_n \geq 0$. From the previous equation, this implies $\delta_{n+2} - \delta_{n+1} > 0$ since k_n is strictly positive. Repeating this argument successively, we have that if $\delta_{n'+1} - \delta_{n'} \geq 0$, then $\delta_{n+1} - \delta_n > 0$ for all $n \geq n'$. However, we know from Proposition 14.11 that $\{v_n\}_{n=0}^{\infty}$ is strictly increasing and converges to a constant v_∞. This implies that $\delta_n \downarrow 0$, which contradicts the hypothesis that $\delta_{n+1} - \delta_n \geq 0$ for all $n \geq n' \geq 0$, and establishes that $z_{n+1}^* \leq z_n^*$. To see that the inequality is strict when $z_n^* > 0$, it suffices to note that we have already established (14.107), i.e., $\delta_{n+1} - \delta_n < 0$, thus if equation (14.97) has a positive solution, then we necessarily have $z_{n+1}^* < z_n^*$.

Proof of $z_0^* \geq z_{-1}^*$: (14.95) can be written as

$$\rho v_0 = -\omega^* G\left(z_0^*\right) + z_0^* \left[v_{-1} + v_1 - 2v_0\right]. \tag{14.110}$$

We have $v_0 \geq 0$ from Proposition 14.11. Suppose $v_0 > 0$. Then (14.110) implies $z_0^* > 0$ and

$$\begin{aligned} v_{-1} + v_1 - 2v_0 &> 0 \\ v_1 - v_0 &> v_0 - v_{-1.} \end{aligned} \tag{14.111}$$

This inequality combined with (14.99) and (14.98) yields $z_0^* > z_{-1}^*$. Suppose next that $v_0 = 0$. Inequality (14.111) now holds as a weak inequality and implies that $z_0^* \geq z_{-1}^*$. Moreover, since $G(\cdot)$ is strictly convex and z_0^* is given by (14.99), (14.110) then implies $z_0^* = 0$ and thus $z_{-1}^* = 0$.

Proof of $z_0^* > z_1^*$: See Exercise 14.30. □

This proposition therefore shows that the highest amount of R&D is undertaken in neck-and-neck industries. This explains why composition effects can increase aggregate innovation.

Exercise 14.31 shows how a relaxation of intellectual property rights protection can increase the growth rate in the economy.

So far, we have not provided a closed-form solution for the growth rate in this economy. It turns out that this is generally not possible, because of the endogenous market structure in these types of models. Nevertheless, it can be proved that a steady state equilibrium exists in this economy, though the proof is somewhat more involved and does not generate additional insights for our purposes (see Acemoglu and Akcigit, 2006).

An important feature of this model is that equilibrium markups are endogenous and evolve over time as a function of competition between the firms producing in the same product line. More importantly, when a particular firm is sufficiently ahead of its rival, it undertakes less R&D. Therefore, this model, contrary to the baseline Schumpeterian model and also contrary to all expanding varieties models, implies that greater competition may lead to higher growth rates. Greater competition generated by closing the gap between the followers and leaders induces the leaders to undertake more R&D in order to escape the competition from the followers.

14.5. Taking Stock

This chapter introduced the basic Schumpeterian model of economic growth (i.e., models with "competitive innovations") to emphasize the importance of competition among firms both in the innovation process and in the product market. Schumpeterian growth introduces a process of creative destruction, where new products or machines replace older models, and thus new firms replace incumbent producers.

The baseline model features process innovations leading to quality improvements. The description of economic growth that emerges from this model is, in many ways, more realistic than the expanding variety models. In particular, technological progress does not always correspond to new products or machines complementing existing ones, but involves the creation of higher-quality producers replacing incumbents. Arrow's replacement effect, discussed in Chapter 12, implies that there is a strong incentive for new entrants to undertake research because the new, higher-quality products will replace the products of incumbents, leading to Schumpeterian creative destruction as the engine of economic growth. Even though the description of economic growth in this model is richer than the expanding varieties model, the mathematical structure turns out to be quite similar to the models we studied in the previous chapters. In reduced form, the model again resembles an AK economy. The main difference is that now the growth rate of the economy, through the rate of replacement of old products, affects the value of innovation. Nevertheless, in the baseline version of the model, the effects of various policy interventions are the same as in the expanding product variety model of the previous chapter.

An important insight of Schumpeterian models is that growth comes with potential conflict of interest. The process of creative destruction destroys the monopoly rents of previous incumbents. This raises the possibility that distortionary policies may arise endogenously as a way of protecting the rents of politically powerful incumbents. Models of creative destruction therefore raise the political economy issues that are central for understanding the fundamental causes of economic growth and provide us insights about both the endogenous nature of technology and about the potential resistance to technological change.

Schumpeterian models also enable us to make greater contact with the industrial organization of innovation. In particular, the process of creative destruction implies that the market structure may be evolving endogenously over time. Nevertheless, the baseline Schumpeterian models have a number of major shortcomings, and addressing these is an interesting and important area for future research. An important discrepancy between the baseline models and the data is that, while the models predict all productivity growth to be driven by the process of creative destruction and entry, in the data much of productivity growth is driven by innovation by continuing firms and plants. Section 14.3 provided a first look at how the baseline models can be extended to account for these patterns and to provide a richer framework for the analysis of the industrial organization of innovation. A second important shortcoming of the baseline models is that they predict that markups are constant and there is always a single firm supplying the entire market. These implications can also be relaxed by considering a richer framework, for example, by allowing cumulative or step-by-step innovation and competition between multiple firms that engage in innovation. Section 14.4 showed how the baseline model can be extended to incorporate step-by-step (cumulative) innovations and how such a model leads to endogenously evolving monopoly markups and market structure. Perhaps more interestingly, we have seen that in these models that incorporate different aspects of the industrial organization of innovation, the effects of competition and patent protection on economic growth are potentially quite different from both the baseline model of Schumpeterian growth and from models of expanding varieties. This suggests that Schumpeterian models might provide a useful framework for the analysis of a range of industrial policies, including anti-trust policies, licensing and intellectual property rights policies.

14.6. References and Literature

The baseline model of Schumpeterian growth presented in Section 14.1 is based on the work by Aghion and Howitt (1992). Similar models have also been developed by Segerstrom, Anant and Dinopoulos (1990), Grossman and Helpman (1991a,b). Aghion and Howitt (1998) provide an excellent survey of many Schumpeterian models of economic growth and numerous extensions. The specific modeling assumptions made in the presentation here draw on Acemoglu (1998), which also uses the aggregate production function with proportional quality improvements and costs of production and R&D increasing proportionally with quality. The original Aghion and Howitt (1992) approach is very similar to that used in Section 14.2.

Aghion and Howitt (1992) also discuss uneven growth and potential growth cycles, which were presented in Section 14.2. Uneven growth and cycles are also possible in expanding product or input variety models as shown by Matsuyama (1999, 2001). I only discussed the possibility of such cycles in the context of Schumpeterian growth, since the forces leading to such cycles are more pronounced in these models.

The effect of creative destruction on unemployment is first studied in Aghion and Howitt (1994). The implications of creative destruction for firm-specific investments are discussed in Francois and Roberts (2001) and in Martimort and Verdier (2003).

The model in Section 14.3 draws on Acemoglu (2008) and is a first attempt to introducing productivity growth driven both by incumbents and entrants (see also Barro and Sala-i-Martin, 2004, for a model in which incumbents undertake R&D because they have cost advantage). A related paper is Klette and Kortum (2004). Klette and Kortum construct a richer model of firm and aggregate innovation dynamics based on expanding product varieties. Their key assumption is that firms with more products have an advantage in discovering more new products. With this assumption, their model generates the same patterns of firm growth as the simple model and Section 14.3, and also matches certain additional facts about propensity to patent and the differential survival probabilities of firms by size. One disadvantage of this approach is that the firm size distribution is not driven by the dynamics of continuing plans (in fact, if new products are interpreted as new plants, the Klette-Kortum model predicts that all productivity growth will be driven by entry of new plants, though this may be an extreme interpretation, since some new products may be produced in existing plants). Lentz and Mortensen (2006) extend Klette and Kortum's model by introducing additional sources of heterogeneity and estimate this extended model on Danish data. Klepper (1996) documents various facts about the firm size, entry and exit decisions and innovation, and provides a simple descriptive model that can account for these facts. None of these papers consider a Schumpeterian growth featuring innovation both by incumbents and entrants that can be easily mapped to decomposing the contribution of new and continuing plants (firms) to productivity growth. Other papers related to the model presented in Section 14.3 include Jovanovic (1987), Hopenhayn (1992), Melitz (2003), Rossi-Hansberg and Wright (2003, 2004), and Luttmer (2004, 2007). All of these papers generate realistic firm-size distributions based on heterogeneity of productivity (combined with fixed costs of operation), but do not endogenize the stochastic process for productivity growth or the productivity of firms. The promising area for future research appears to be to develop theoretical and empirical models that can incorporate the heterogeneity emphasized by these models, while at the same time generating the process of productivity growth of continuing plans and new entrants endogenously.

Step-by-step or cumulative innovations have been analyzed in Aghion, Harris and Vickers (1999) and Aghion, Harris, Howitt and Vickers (2001). The model presented here is a simplified version of Acemoglu and Akcigit (2006), which includes a detailed analysis of the implications of intellectual property rights policy and licensing in this class of models. The proof of existence of a steady-state equilibrium under a somewhat more general environment is provided in that paper. The notion of Markov Perfect Equilibrium used in Section 14.4 is a standard equilibrium concept in dynamic games and is a refinement of subgame perfect equilibrium, that restricts strategies to depend only on payoff-relevant state variables. These concepts are discussed in the Appendix Chapter C and in Fudenberg and Tirole (1994).

Blundell (1999), Nickell (1999) and Aghion, Bloom, Blundell, Griffith and Howitt (2005) provide evidence that greater competition may encourage economic growth and technological progress. The latter paper shows that industries where the technology gap between firms is smaller are typically more innovative. Aghion, Harris, Howitt and Vickers (2001) and Aghion, Bloom, Blundell, Griffith and Howitt (2005) show that in step-by-step models of innovation greater competition may increase growth. Aghion, Dewatripont and Ray (2000) provide another reason why competition may encourage growth. In their model competitive pressure improves managerial incentives and efficiency.

14.7. Exercises

Exercise 14.1.
(1) Prove that in the baseline model of Schumpeterian growth in Section 14.1, all R&D will be undertaken by entrants, and there will never be R&D by incumbents. [Hint: rewrite (14.12) by allowing for a choice of R&D investments].
(2) Now suppose that the flow rate of success of R&D is $\phi\eta/q$ for an incumbent as opposed to η/q for an and turned. Show that for any value of ϕ, the incumbent will still choose zero R&D. Explain this result.

Exercise 14.2. The baseline endogenous technological change models, including the model of Schumpeterian growth in this chapter, assume that new products are protected by perpetual patents. This exercise asks you to show that this is not strictly necessary in the logic of these models. Suppose that there is no patent protection for any innovation, but copying an innovator requires a fixed cost $\varepsilon > 0$. Any firm, after paying this cost, has access to the same technology as the innovator. Prove that in this environment there will be no copying and all the results of the model with fully-enforced perpetual patents apply.

Exercise 14.3. Complete the proof of Proposition 14.1. In particular, verify that the equilibrium growth rate is unique, strictly positive and such that the transversality condition (14.15) is satisfied.

Exercise 14.4. Prove Proposition 14.2.

Exercise 14.5. * In the baseline Schumpeterian growth model, instead of (14.3), suppose that the production function of the final good sector is given by

$$Y(t) = \frac{1}{1-\beta}\left[\int_0^1 q(\nu,t)^{\zeta_1} x(\nu,t \mid q)^{1-\beta} d\nu\right] L^{\beta}.$$

Suppose also that producing one unit of an intermediate could of quality q costs ψq^{ζ_2} and that one unit of final good devoted to research on the machine line with quality q generates a flow rate of innovation equal to η/q^{ζ_3}. Characterize the equilibrium of this economy and determine what combinations of the parameters ζ_1, ζ_2 and ζ_3 will ensure balanced growth.

Exercise 14.6. Derive equation (14.24).

Exercise 14.7. Show that condition (14.5) is sufficient to ensure that a firm that innovates will set the unconstrained monopoly price. [Hint: first suppose that the innovator sets the monopoly price $\psi q/(1-\beta)$ for a product of quality q. Then, consider the firm with the next highest quality, $\lambda^{-1}q$. Suppose that this firm sells at marginal cost, $\psi\lambda^{-1}q$. Then find the value of λ such that final good producers are indifferent between buying a machine of quality q at the price $\psi q/(1-\beta)$ versus a machine of quality $\lambda^{-1}q$ at the price $\psi\lambda^{-1}q$.]

Exercise 14.8. Analyze the baseline model of Schumpeterian growth in Section 14.1 assuming that (14.5) is not satisfied.

(1) Show that monopolists will set a limit price.
(2) Characterize the BGP equilibrium growth rate.
(3) Characterize the Pareto optimal allocation and compare it to the equilibrium allocation. How does the comparison differ from the case in which innovations were drastic?
(4) Now consider a hypothetical economy in which the previous highest-quality producer disappears so that the monopolist can charge a markup of $1/(1-\beta)$ instead of the limit price. Show that the BGP growth rate in this hypothetical economy is strictly greater than the growth rate characterized in 2 above. Explain this result.
(5) Let us now go back to the question in Exercise 14.1 and suppose that the incumbent firm has an advantage in R&D as in that exercise. Show that if ϕ is sufficiently high, the incumbent will undertake R&D. Why does this result differ from the one in Exercise 14.1?

Exercise 14.9. Modify the baseline model of Section 14.1 so that the aggregate production function for the final good is

$$Y(t) = \frac{1}{1-\beta}\left[\int_0^1 \left(q(\nu,t)x(\nu,t \mid q)\right)^{1-\beta} d\nu\right] L^{\beta}.$$

All the other features of the model remain unchanged.

(1) Show that with this production function, a BGP does not exist. Explain why this is.
(2) What would you change in the model to ensure the existence of a BGP.

Exercise 14.10. Suppose that there is constant exponential population growth at the rate n. Modify the baseline model of Section 14.1 so that there is no scale effect and the economy grows at the constant rate (with positive growth of income per capita). [Hint: suppose that one unit of final good spent on R&D for improving the machine of quality q leads to flow rate of innovation equal to η/q^{ϕ}, where $\phi > 1$].

Exercise 14.11. Consider a version of the model of Schumpeterian growth in which the x's do not depreciate fully after use (similar to Exercise 13.22 in the previous chapter). Preferences and the rest of the production structure are the same as in the baseline model in Section 14.1.

(1) Define an equilibrium.
(2) Formulate the maximization problem of a monopolist with the highest quality machine.
(3) Show that, contrary to Exercise 13.22, the results are different than those in Section 14.1. Explain why depreciation of machines was not important in the expanding varieties model but it is important in the model of competitive innovations.

Exercise 14.12. Consider a version of the model of Schumpeterian growth in which innovations reduce costs instead of increasing quality. In particular, suppose that the aggregate production function is given by

$$Y(t) = \frac{1}{1-\beta}\left[\int_0^1 x(\nu,t)^{1-\beta} d\nu\right] L^{\beta},$$

and the marginal cost of producing machine variety ν at time t is given by $MC(\nu,t)$. Every innovation reduces this cost by a factor λ.

(1) Define an equilibrium in this economy.
(2) Specify a form of the innovation possibilities frontier that is consistent with balanced economic growth.
(3) Derive the BGP growth rate of the economy and show that there are no transitional dynamics.
(4) Compare the BGP growth rate to the Pareto optimal growth rate of the economy. Can there be excessive innovations?
(5) What are the similarities and differences between this model and the baseline model presented in Section 14.1.

Exercise 14.13. Consider the model in Section 14.2, with R&D performed by workers. Suppose instead that the aggregate production function for the final good is given by

$$Y(t) = \frac{1}{1-\beta}\left[\int_0^1 q(\nu,t)x(\nu,t \mid q)^{1-\beta}d\nu\right]L_E(t)^{\beta},$$

where $L_E(t)$ denotes the number of workers employed in final good production at time t.

(1) Show that in this case, there will only be R&D for the machine with the highest $q(\nu,t)$.
(2) How would you modify the model so that the equilibrium has balanced R&D across sectors?

Exercise 14.14. Consider the model of Schumpeterian growth in Section 14.1, with one difference: conditional on success an innovation generates a random improvement of λ over the previous technology, where the distribution function of λ is $H(\lambda)$ and has support $\left[(1-\beta)^{-(1-\beta)/\beta}, \bar{\lambda}\right]$.

(1) Define an equilibrium in this economy.
(2) Characterize the balanced economic growth path and specify restrictions on parameters so that the transversality condition is satisfied.
(3) Why did we assume that the lower support of λ is $(1-\beta)^{-1}$? How would the analysis change if this were relaxed?
(4) Show that there are no transitional dynamics in this economy.
(5) Compare the BGP growth rate to the Pareto optimal growth rate of the economy. Can there be excessive innovations?

Exercise 14.15. In the model of Section 14.2 show that the economy experiences no growth of output for intervals of average length $1/\eta L_R^*$.

Exercise 14.16. (1) Prove Proposition 14.4, in particular verifying that the allocation described there is unique, that the average growth rate is given by $g^* = \ln\lambda\eta L_R^*$ and that condition (14.27) is necessary and sufficient for the existence of the equilibrium described in the proposition.
(2) Explain why the growth rate features $\ln\lambda$ rather than $\lambda - 1$ as in the model of Section 14.1.

Exercise 14.17. Consider the one-sector Schumpeterian model in discrete time. Suppose as in the model in Section 14.2 that consumers are risk neutral, there is no population growth, and the final good sector has the production function given by (14.25). Assume that the R&D technology is such that $L_R > 0$ workers employed in research will necessarily lead to an innovation, and the number of workers used in research simply determines the quality of the innovation via the function $\lambda(L_R)$, which is strictly increasing, continuously differentiable, strictly concave and satisfies the Inada conditions.

(1) Define an equilibrium in this economy.
(2) Characterize the BGP and specify restrictions on parameters so that the transversality condition is satisfied.
(3) Compare the BGP growth rate to the Pareto optimal growth rate of the economy. Show that the size of innovations is always too small relative to the size of innovations in the Pareto optimal allocation.

Exercise 14.18. * Consider the one-sector Schumpeterian model in discrete time analyzed in the previous exercise. Suppose that when a new innovation arrives a fraction φ of workers employed in the final good production will not be able to adapt to this new technology and will need to remain unemployed for one time period to "retool".

(1) Define an equilibrium in this economy. [Hint: also specify the number of unemployed workers in equilibrium].
(2) Characterize the balanced growth path of this economy and determine the number of unemployed workers in equilibrium.
(3) Show that the economy will experience bursts of unemployment, followed by periods of full employment.
(4) Show that a decline in ρ will increase the average growth rate and the average unemployment rate in the economy.

Exercise 14.19. * Derive equations (14.28)-(14.30).

Exercise 14.20. * Consider the model discussed in subsection 14.2.2.

(1) Choose a functional form for $\eta(\cdot)$ such that equations (14.30) have solutions L_R^1 and $L_R^2 \neq L_R^1$. Explain why, when such solutions exist, there is a perfect foresight equilibrium with two-period endogenous cycles.
(2) Show that even when solutions exist, there also exists a steady-state equilibrium with constant research.
(3) Show that when such solutions do not exist, there exists an equilibrium which exhibits oscillatory transitional dynamics converging to the steady state characterized in 2 above.

Exercise 14.21. * Show that the results of the model in subsection 14.2.2 generalize when there is a single firm undertaking research, thus internalizing the effect of L_R on $\eta(L_R)$.

Exercise 14.22. Suppose that in the model of Section 14.3 incumbents also have access to the radical innovation technology used by entrants. Show that there cannot existing equilibrium in which incumbents undertake positive R&D with this technology. [Hint: use the free entry condition for entrants together with to condition that makes such investments profitable for incumbents and derive a contradiction].

Exercise 14.23. Set up the social planner's problem (of maximizing the utility of the representative household) in Section 14.3.

(1) Show that this maximization problem corresponds to a concave current-value Hamiltonian and derive the unique solution to this problem. Show that this solution involves the consumption of the representative household growing at a constant rate at all points.
(2) Show that the social planner will tend to increase growth because she avoids the monopoly markup over machines.
(3) Show that the social planner will tend to choose lower entry because of the negative externality in the research process.
(4) Give numerical examples in which the growth rate in the Pareto optimal allocation is greater than or less than the decentralized growth rate.

Exercise 14.24. Consider the model of Section 14.3 and suppose that the R&D technology of the incumbents for innovation is such that if an incumbent with a machine of quality q spends an amount zq for incremental innovations, then the flow rate of innovation is $\phi(z)$ (and this innovation again increases the quality of the incumbent's machine to λq). Assume that $\phi(z)$ is strictly increasing, strictly concave, continuously differentiable, and satisfies $\lim_{z\to 0}\phi'(z)=\infty$ and $\lim_{z\to\infty}\phi'(z)=0$.

(1) Focus on steady-state equilibria and conjecture that $V(q)=vq$. Using this conjecture, show that incumbents will choose R&D intensity z^* such that $(\lambda-1)v=\phi'(z^*)$. Combining this equation with the free entry condition for entrants and the equation for growth rate given by (14.55), show that there exists a unique BGP equilibrium (under the conjecture that $V(q)$ is linear).
(2) Is it possible for an equilibrium to involve different levels of z for incumbents with different quality machines?
(3) In light of your answer to 2, what happens if we consider the "limiting case" of this model where $\phi(z)=$ constant?
(4) Show that this equilibrium involves positive R&D both by incumbents and entrants.
(5) Now introduce taxes on R&D by incumbents and entrants at the rates τ_i and τ_e. Show that, in contrast to the results in Proposition 14.7, the effects of both taxes on growth are ambiguous. What happens if $\eta(z)=$constant?

Exercise 14.25. (1) Prove Proposition 14.6 for the case in which $z(\nu,t\mid q)$ could differ across incumbents with different levels of q. Show that the same BGP as in Proposition 14.6 applies and is essentially unique, in the sense that average incumbent R&D effort is always equal to z^*.

(2) Show that the BGP characterized in Proposition 14.6 also applies when the distribution of R&D efforts across incumbents is given by (14.71).

Exercise 14.26. Consider the model of Section 14.3, but modify the production function to $Y(t) = \left[\int_0^1 q(\nu,t)x(\nu,t \mid q)^{1-\beta}d\nu\right] L^{\beta}/(1-\beta)$ and assume that production of an input of quality q requires ψq units of the final good as in the baseline model of Section 14.1. Show that the equilibrium growth rate and the decomposition of productivity growth between incumbents and entrants are identical in this case tothe results in Section 14.3, but there are no firm size dynamics. Explain why this is. Are there dynamics of profits? How does the distribution of profits across firm evolve over time?

Exercise 14.27. * Derive equation (14.78).

Exercise 14.28. * Derive equation (14.91). [Hint: write $\ln Y(t) = \int_0^1 \ln q(\nu,t)\, l(\nu,t)\, d\nu = \int_0^1 \left[\ln q_i(\nu,t) + \ln \frac{Y(t)}{w(t)} \lambda^{-n(\nu)}\right] d\nu$ and rearrange this equation]

Exercise 14.29. * Prove Proposition 14.12.

Exercise 14.30. * Complete the proof of Proposition 14.13, in particular, prove that $z_0^* > z_1^*$ [Hint: use similar arguments to the first part of the proof.]

Exercise 14.31. * Consider a steady-state equilibrium in the model of Section 14.4. Suppose that we have $\kappa = 0$ and

$$G'(0) < \frac{1-\lambda}{\rho}$$

Let

$$z^* \equiv G'^{-1}\left(\frac{1-\lambda}{\rho}\right)$$

and suppose also that

$$G'(0) < \frac{z^*(1-\lambda)/\rho + G(z^*)}{\rho + z^*}.$$

(1) Show that in this case the steady state equilibrium has zero growth.
(2) Show that $\kappa > 0$ will lead to a positive growth rate. Interpret this result and contrast it to the negative effects of relaxing the protection of intellectual property rights in the baseline model of Schumpeterian growth.

Exercise 14.32. * Modify the model presented in Section 14.4 such that followers can now use the innovation of the technological leader and immediately leapfrog the leader, but in this case they have to pay a license fee of ζ to the leader.

(1) Characterize the growth rate of a steady-state equilibrium in this case
(2) Write the value functions.
(3) Explain why licensing can increase the growth rate of the economy in this case, and contrast this result with the one in Exercise 12.9, where licensing was never used in equilibrium. What is the source of the difference between the two sets of results?

EXERCISE 14.33.
(1) What is the effect of competition on the rate of growth of the economy in a standard product variety model of endogenous growth? What about the quality-ladder model? Explain the intuition.
(2) Now consider the following one-period model. There are two Bertrand duopolists, producing a homogeneous good. At the beginning of each period, duopolist 1's marginal cost of production is determined as a draw from the uniform distribution $[0, \bar{c}_1]$ and the marginal cost of the second duopolist is determined as an independent draw from $[0, \bar{c}_2]$. Both cost realizations are observed and then prices are set. Demand is given by $Q = A - P$, with $A > 2 \max\{\bar{c}_1, \bar{c}_2\}$.
   (a) Characterize the equilibrium pricing strategies and calculate expected ex ante profits of the two duopolists.
   (b) Now imagine that both duopolists start with a cost distribution $[0, \bar{c}]$, and can undertake R&D at cost μ. If they do, with probability η, their cost distribution shifts to $[0,\ \bar{c} - \alpha]$ where $\alpha < \bar{c}$. Find the conditions under which one of the duopolists will invest in R&D and the conditions under which both will.
   (c) What happens when $\bar{c}$ declines? Interpreting the decline in $\bar{c}$ as increased competition, discuss the effect of increased competition on innovation incentives. Why is the answer different from that implied by the baseline endogenous technological change models of expanding varieties or Schumpeterian growth?

CHAPTER 15

Directed Technological Change

The previous two chapters introduced the basic models of endogenous technological change. These models provide us with a tractable framework for the analysis of aggregate technological change, but focus on a single type of technological change. Even when there are multiple types of machines, these all play the same role in increasing aggregate productivity. Consequently, technological change in these models is always "neutral" (that is, Hicks-neutral as defined in Chapter 2). There are two important respects in which these models are incomplete. First, technological change in practice is often not neutral: it benefits some factors of production and some agents in the economy more than others. Only in special cases, such as in economies with Cobb-Douglas aggregate production functions, these types of biases can be ignored. The study of why technological change is sometimes biased towards certain factors or sectors is both important for understanding the nature of endogenous technology and also because it clarifies the distributional effects of technological change, which determine which groups will embrace new technologies and which will oppose them. Second, limiting the analysis to only one type of technological change potentially obscures the different competing effects that determine the nature of technological change.

The purpose of this chapter is to extend the models of the last two chapters to consider *directed technological change*, which endogenizes the direction and bias of new technologies that are developed and adopted. Models of directed technological change not only generate new insights about the nature of endogenous technological progress, but also enable us to ask and answer new questions about recent and historical technological developments.

I start with a brief discussion of a range of economic problems in which considering the endogenous bias of technology is important and also present some of the general economic insights that will be important in models of directed technological change. The main results are presented in 15.3. The rest of the chapter generalizes the results and presents a few of their applications. Section 15.6 uses these models to return to the question raised in Chapter 2 concerning why technological change might take a purely labor-augmenting (Harrod-neutral) form. Section 15.8 presents an alternative approach to this question suggested by Jones (2005).

15.1. Importance of Biased Technological Change

To see the potential importance of the biased technological change, let us first review a number of examples:

(1) Perhaps the most important example of biased technological change is the so-called *skill-biased technological change*, which has played an important role in the analysis of recent labor market developments and changes in the wage structure. Figure 15.1 plots a measure of the relative supply of skills (defined as the number of college equivalent workers divided by noncollege equivalents) and a measure of the return to skills, the college premium. It shows that over the past 60 years, the U.S. relative supply of skills has increased rapidly, but there has been no tendency for the returns to college to fall in the face of this large increase in supply—on the contrary, there has been an increase in the college premium over this time period. The standard explanation for this pattern is that new technologies over the post-war period have been *skill-biased*. In fact, at some level this has to be so; if skilled and unskilled workers are imperfect substitutes, an increase in the relative supply of skills, without technological change, will necessarily reduce the skill premium.

    The figure also shows that beginning in the late 1960s, the relative supply of skills increased much more rapidly than before, and the skill premium increased very rapidly beginning precisely in the late 1970s. The standard explanation for this increase is an acceleration in the skill bias of technical change that happens to be coincidental with the significant changes in the relative supply of skills.

    An obvious question is why technological changes have been skill-biased over the past 60 years or even 100 years? Relatedly, why does it appear that skill-biased technological change accelerated starting in the 1970s, precisely when the supply of skills increased rapidly? While some economists are happy to treat the bias of technological change as exogenous, this is not entirely satisfactory. We have seen that understanding the endogenous nature of technology is important for our study of cross-country income differences and the process of modern economic growth. It is unlikely that, while the amount of aggregate technological change is endogenous, the bias of technological change is entirely exogenous. It is therefore important to study the determinants of endogenous bias of technological change and ask why technological change has become more skill-biased in recent decades.

(2) This conclusion is strengthened when we look at the historical process of technological change. In contrast to the developments during the recent decades, technological changes during the late 18th and early 19th centuries appear to have been *unskill-biased*. The artisan shop was replaced by the factory and later by interchangeable

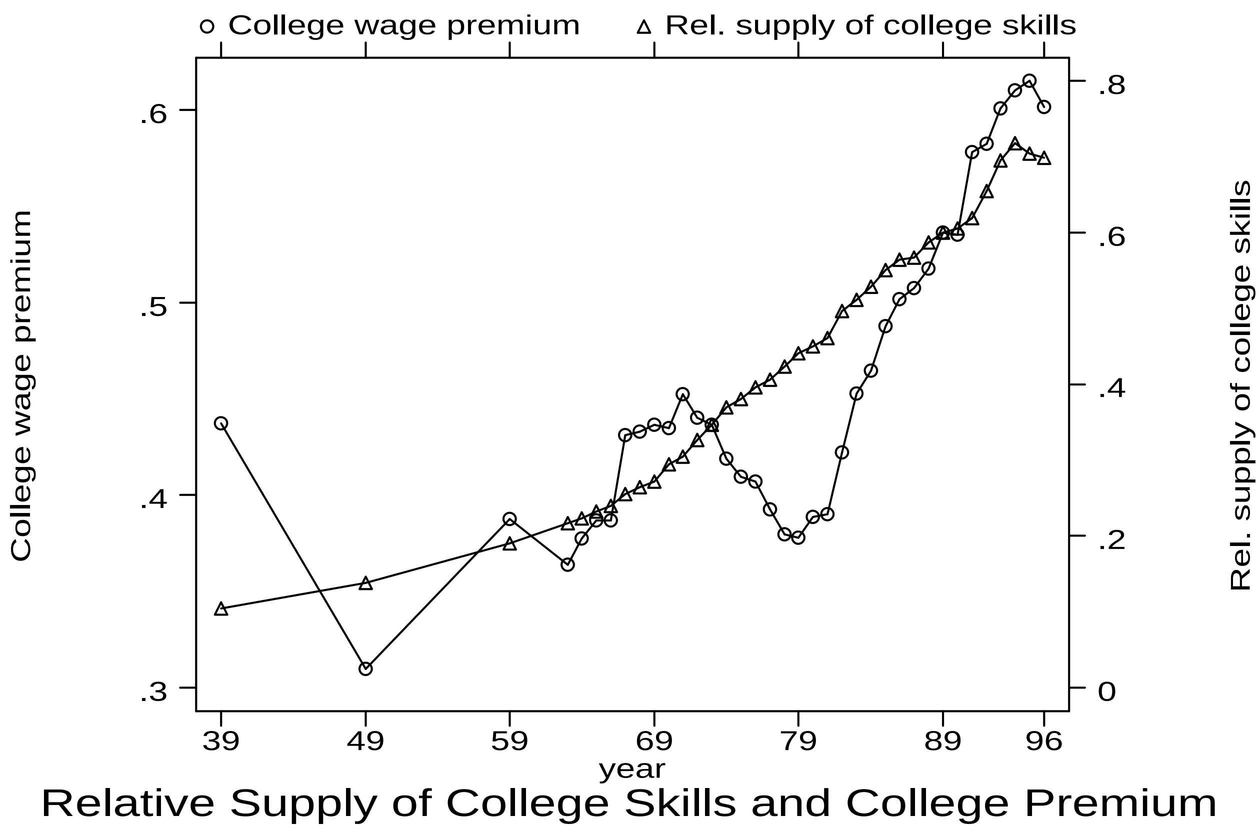

Figure 15.1. Relative supply of college graduates and the college premium in the U.S. labor market.

parts and the assembly line. Products previously manufactured by skilled artisans started to be produced in factories by workers with relatively few skills, and many previously complex tasks were simplified, reducing the demand for skilled workers. Mokyr (1990, p. 137) summarizes this process as follows:

"First in firearms, then in clocks, pumps, locks, mechanical reapers, typewriters, sewing machines, and eventually in engines and bicycles, interchangeable parts technology proved superior and replaced the skilled artisans working with chisel and file."

Even though the types of skills valued in the labor market during the 19th century were different from those supplied by college graduates in today's labor markets, the juxtaposition of technological change biased towards college graduates in the recent past and biased against the most skilled workers of the time in the 19th century is both puzzling and intriguing. It raises the question: why was technological change, that has been generally skill-biased over the 20th century, biased towards unskilled workers in the 19th century?

(3) Beginning in the late 1960s and the early 1970s, both unemployment and the share of labor in national income increased rapidly in a number of continental European countries. During the 1980s, unemployment continued to increase, but the labor share started a steep decline, and in many countries it fell below its initial level. Blanchard (1997) interprets the first phase as the response of these economies to a wage-push by workers, and the second phase as a possible consequence of *capital-biased* technological changes. Is there a connection between capital-biased technological changes in European economies and the wage push preceding it?
(4) As we have seen in Chapters 2 and 8, balanced economic growth is only possible when technological change is asymptotically Harrod-neutral, i.e., purely labor-augmenting. If technological change is not labor-augmenting, we should not expect equilibrium growth to be balanced. But a range of evidence suggests that modern economic growth has been relatively balanced. Is there any reason to expect technological change to be endogenously labor-augmenting?
(5) The past several decades have experienced a large increase in the volume of international trade and a rapid process of globalization. Do we expect globalization to affect the types of technologies that are being developed and used?

We can provide answers to these questions and develop a framework of directed technological change by extending the ideas we have studied in the past few chapters. The main insight is to think of profit incentives as affecting not only the amount but also the direction of technological change. Before presenting detailed models, let us review the basic arguments, which are quite intuitive.

Imagine an economy which has two different factors of production, say L and H (corresponding to unskilled and skilled workers), and two different types of technologies that can complement either one or the other factor. We would expect that whenever the profitability of H-augmenting technologies is greater than the L-augmenting technologies, more of the former type will be developed by profit-maximizing (research) firms. But then, what determines the relative profitability of developing different technologies? The answer to this question summarizes most of the economics in the models of directed technological change. Two potentially counteracting effects shape the relative profitabilities of different types of technologies:

(1) *The price effect*: there will be stronger incentives to develop technologies when the goods produced by these technologies command higher prices.
(2) *The market size effect*: it is more profitable to develop technologies that have a larger market, for the reasons discussed in Chapter 12.

We will see that this market size effect will be powerful enough to outweigh the price effect. In fact, our analysis will show that under fairly general conditions the following two results will hold:

- *Weak Equilibrium (Relative) Bias*: an increase in the relative supply of a factor always induces technological change that is biased in favor of this factor.
- *Strong Equilibrium (Relative) Bias*: if the elasticity of substitution between factors is sufficiently large, an increase in the relative supply of a factor induces sufficiently strong technological change biased towards itself that the endogenous-technology relative demand curve of the economy becomes *upward-sloping.*

To explain these concepts in a little more detail, suppose that the (inverse) relative demand curve takes the form $w_H/w_L = D(H/L, A)$, where w_H/w_L is the relative price of the H factor relative to the L factor, and H/L is the relative supply of the H factor. A is a technology term. A is H-biased if D is increasing in A, so that a higher A increases the relative demand for the H factor. The standard microeconomic theory implies that D is *always* decreasing in H/L. Equilibrium bias concerns the behavior of A as H/L changes, so let us write this as $A(H/L)$ and suppose that it is indeed H-biased. Weak equilibrium bias simply implies that $A(H/L)$ is increasing (nondecreasing) in H/L. Strong equilibrium bias, on the other hand, implies that $A(H/L)$ is sufficiently responsive to an increase in H/L that the total effect of the change in relative supply H/L is to increase w_H/w_L. In other words, let the endogenous-technology relative demand curve be $w_H/w_L = D(H/L, A(H/L)) \equiv \tilde{D}(H/L)$. Then strong equilibrium bias corresponds to this endogenous-technology relative demand curve, $\tilde{D}$, being increasing.

At first, both the weak and the strong equilibrium bias results appear surprising. However, we will see that they are quite intuitive once the logic of directed technological change is understood. Moreover, they have a range of important implications. In particular, subsection 15.3.3 below shows how the weak and the strong relative bias results provide us with potential answers to the questions posed at the beginning of this section.

15.2. Basics and Definitions

Before studying directed technological change, it is useful to clarify the difference between factor-augmenting and factor-biased technological changes, which are sometimes confused in the literature. For this purpose and for much of the analysis in this chapter, we assume that the production side of the economy can be represented by an aggregate production function,

$$Y(t) = F(L(t), H(t), A(t)),$$

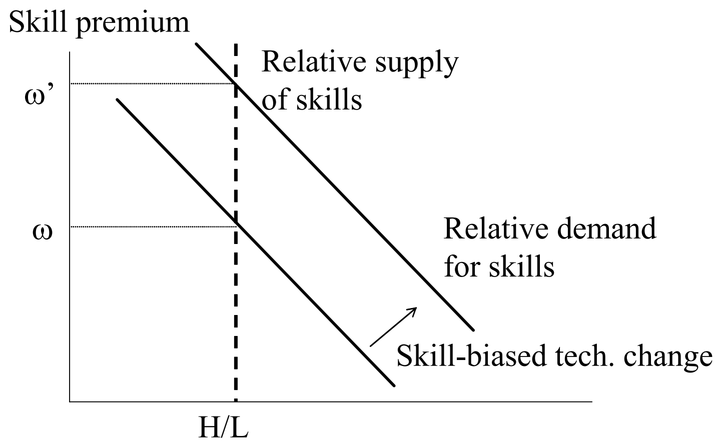

Figure 15.2. The effect of H-biased technological change on relative demand and relative factor prices.

where $L(t)$ is labor, and $H(t)$ denotes another factor of production, which could be skilled labor, capital, land or some intermediate goods, and $A(t)$ represents technology. Without loss of generality imagine that $\partial F/\partial A > 0$, so a greater level of A corresponds to "better technology". Recall that technological change is *L-augmenting* if

$$\frac{\partial F(L,H,A)}{\partial A} \equiv \frac{L}{A}\frac{\partial F(L,H,A)}{\partial L}.$$

This is clearly equivalent to the production function taking the more special form, $F(AL,H)$. In the case where L corresponds to labor and H to capital, this is also equivalent to Harrod-neutral technological change. Conversely, H-augmenting technological change is defined similarly, and corresponds to the production function taking the special form $F(L,AH)$.

Though often equated with factor-augmenting changes, the concept of factor-biased technological change is *very* different. We say that technological change change is *L-biased*, if it increases the *relative* marginal product of factor L compared to factor H. Mathematically, this corresponds to

$$\frac{\partial \frac{\partial F(L,H,A)/\partial L}{\partial F(L,H,A)/\partial H}}{\partial A} \geq 0.$$

Put differently, biased technological change shifts out the relative demand curve for a factor, so that its relative marginal product (relative price) increases at given factor proportions (i.e., given relative quantity of factors). Conversely, technological change is H-biased if this inequality holds in reverse. Figure 15.2 plots the effect of an H-biased (skill-biased) technological change on the relative demand for factor H and on its relative price, the skill premium.

These concepts can be further clarified using the constant elasticity of substitution (CES) production function, which we will use for the rest of this chapter. The CES production function takes the form

$$Y(t) = \left[\gamma_L (A_L(t) L(t))^{\frac{\sigma-1}{\sigma}} + \gamma_H (A_H(t) H(t))^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}},$$

where $A_L(t)$ and $A_H(t)$ are two separate technology terms, the γ_is determine the importance of the two factors in the production function, and $\gamma_L + \gamma_H = 1$. Finally, $\sigma \in (0, \infty)$ is the elasticity of substitution between the two factors. When $\sigma = \infty$, the two factors are perfect substitutes, and the production function is linear. When $\sigma = 1$, the production function is Cobb-Douglas, and when $\sigma = 0$, there is no substitution between the two factors, and the production function is Leontieff. When $\sigma > 1$, we refer to the factors as "gross substitutes," and when $\sigma < 1$, we refer to them as "gross complements". While there are multiple definitions of complementarity in the microeconomics literature, this terminology is useful to distinguish the two cases in which $\sigma < 1$ and $\sigma > 1$, which will have very different implications in the current context.

Clearly, by construction, $A_L(t)$ is L-augmenting, while $A_H(t)$ is H-augmenting. Interestingly, whether technological change that is L-augmenting (or H-augmenting) is L-biased or H-biased depends on the elasticity of substitution, σ. Let us first calculate the relative marginal product of the two factors (see Exercise 15.1):

$$\text{(15.1)} \qquad \frac{MP_H}{MP_L} = \gamma \left(\frac{A_H(t)}{A_L(t)}\right)^{\frac{\sigma-1}{\sigma}} \left(\frac{H(t)}{L(t)}\right)^{-\frac{1}{\sigma}},$$

where $\gamma \equiv \gamma_H / \gamma_L$. The relative marginal product of H is decreasing in its relative abundance, $H(t)/L(t)$. This is simply the consequence of the usual *substitution effect*, leading to a negative relationship between relative supplies and relative marginal products (or prices) and thus to a downward-sloping relative demand curve (see Figure 15.3). The effect of $A_H(t)$ on this relative marginal product depends on σ, however. If $\sigma > 1$, an increase in $A_H(t)$ (relative to $A_L(t)$) increases the relative marginal product of H. In contrast, when $\sigma < 1$, an increase in $A_H(t)$ reduces the relative marginal product of H. Therefore, when the two factors are gross substitutes, H-augmenting technological change is also H-biased. In contrast, when the two factors are gross complements, the relationship is *reversed*, and H-augmenting technical change is now L-biased. Naturally, when $\sigma = 1$, we are in the Cobb-Douglas case, and neither a change in $A_H(t)$ nor in $A_L(t)$ is biased towards any of the factors. Note also for future reference that by virtue of the fact that σ is the elasticity of substitution between the two factors, we have

$$\sigma = -\left(\frac{d\log(MP_H/MP_L)}{d\log(H/L)}\right)^{-1}$$

The intuition for why, when $\sigma < 1$, H-augmenting technical change is L-biased is simple but important: with gross complementarity ($\sigma < 1$), an increase in the productivity of H

increases the demand for labor, L, by more than the demand for H. As a result, the marginal product of labor increases by more than the marginal product of H. This can be seen most clearly in the extreme case where $\sigma \to 0$, so that the two factors become Leontieff. In this case, starting from a situation in which $\gamma_L A_L(t) L(t) = \gamma_H A_H(t) H(t)$, a small increase in $A_H(t)$ will create an "excess of the services" of the H factor (and thus "excess demand" for L), and the price of factor H will fall to 0.

I have so far defined the meaning of H-biased and L-biased technological changes. It is also useful to define two concepts that will play a major role in the remainder of this chapter. There is *weak equilibrium bias* of technology if an increase in the relative supply of H, H/L, induces technological change biased towards H. Mathematically, this is equivalent to:

$$\frac{\partial MP_H/MP_L}{\partial A_H/A_L}\frac{dA_H/A_L}{dH/L} \geq 0.$$

From (15.1), it is clear that this condition will hold if

$$\frac{d\left(A_H(t)/A_L(t)\right)^{\frac{\sigma-1}{\sigma}}}{dH/L} \geq 0,$$

so that in response to the change in relative supplies $A_H(t)/A_L(t)$ changes in a direction that is biased towards the factor that has become more abundant.

On the other hand, there is *strong equilibrium bias* if an increase in H/L induces a sufficiently large change in the bias of technology so that the marginal product of H *relative* to that of L increases following the change in factor supplies. Mathematically, this is equivalent to

$$\frac{dMP_H/MP_L}{dH/L} > 0,$$

where I now use a strict inequality to distinguish strong equilibrium bias from the case in which relative marginal products are independent of relative supplies (e.g., because factors are perfect substitutes). These equations make it clear that the major difference between weak and strong equilibrium bias is whether the relative marginal product of the two factors are evaluated at the initial relative supplies (in the case of weak bias) or at the new relative supplies (in the case of strong bias). Consequently, strong equilibrium bias is a much more demanding concept than weak equilibrium bias.

15.3. Baseline Model of Directed Technological Change

In this section, we present the baseline model of directed technological change, which uses the expanding varieties model of endogenous technological change and the lab equipment specification of the innovation possibilities frontier. The former choice is motivated by the fact that the expanding varieties model is somewhat simpler to work with than the model of Schumpeterian growth introduced in the previous chapter. The lab equipment specification, on the other hand, highlights that none of the results here depend on technological externalities. Section 15.4 will consider a model of directed technological with change knowledge spillovers. Exercise 15.19 shows that all of the results presented here generalize to a model of Schumpeterian growth, thus the assumption of expanding varieties is only adopted for convenience.

The baseline economy has a constant supply of two factors, L and H, and admits a representative household with the standard CRRA preferences given by

$$\int_0^{\infty} \exp(-\rho t) \frac{C(t)^{1-\theta} - 1}{1-\theta} dt, \tag{15.2}$$

where, as usual, $\rho > 0$. The supply side is represented by an aggregate production function combining the outputs of two intermediate sectors with a constant elasticity of substitution:

$$Y(t) = \left[\gamma_L Y_L(t)^{\frac{\varepsilon-1}{\varepsilon}} + \gamma_H Y_H(t)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}, \tag{15.3}$$

where $Y_L(t)$ and $Y_H(t)$ denote the outputs of two intermediate goods. As the indices indicate, the first is L-intensive, while the second is H-intensive. The parameter $\varepsilon \in [0, \infty)$ is the elasticity of substitution between these two intermediate goods. The assumption that (15.3) features a constant elasticity of substitution simplifies the analysis but is not crucial for the results. How relaxing this assumption affects the results is discussed at the end of this chapter.

The resource constraint of the economy at time t takes the form

$$C(t) + X(t) + Z(t) \leq Y(t), \tag{15.4}$$

where, as before, $X(t)$ denotes total spending on machines and $Z(t)$ is aggregate R&D spending.

The two intermediate goods are produced competitively with the following production functions:

$$Y_L(t) = \frac{1}{1-\beta}\left(\int_0^{N_L(t)} x_L(\nu, t)^{1-\beta} d\nu\right) L^{\beta} \tag{15.5}$$

and

$$Y_H(t) = \frac{1}{1-\beta}\left(\int_0^{N_H(t)} x_H(\nu, t)^{1-\beta} d\nu\right) H^{\beta}, \tag{15.6}$$

where $x_L(\nu, t)$ and $x_H(\nu, t)$ denote the quantities of the different types of machines (used in the production of one or the other intermediate good) and $\beta \in (0, 1)$.[1] These machines are again assumed to depreciate after use. The parallel between these production functions and the aggregate production function of the economy in the baseline expanding product varieties model of Chapter 13 is obvious. There are two important differences, however. First, these are now production functions for intermediate goods rather than the final good. Second, the

[1] Note that the range of machines used in the two sectors are different (there are two disjoint sets of machines); we use the index ν to denote either set of machines for notational simplicity.

two production functions (15.5) and (15.6) use different types of machines. The range of machines complementing labor, L, is $[0, N_L(t)]$, while the range of machines complementing factor H is $[0, N_H(t)]$.

Again as in Chapter 13, we assume that all machines in both sectors are supplied by monopolists that have a fully-enforced perpetual patent on the machines. We denote the prices charged by these monopolists at time t by $p_L^x(\nu, t)$ for $\nu \in [0, N_L(t)]$ and $p_H^x(\nu, t)$ for $\nu \in [0, N_H(t)]$. Once invented, each machine can be produced at the fixed marginal cost ψ in terms of the final good, which we again normalize to $\psi \equiv 1 - \beta$. This implies that total resources devoted to machine production at time t are

$$X(t) = (1-\beta)\left(\int_0^{N_L(t)} x_L(\nu, t)\, d\nu + \int_0^{N_H(t)} x_H(\nu, t)\, d\nu\right).$$

The innovation possibilities frontier is assumed to take a form similar to the lab equipment specification in Chapter 13:

$$\dot{N}_L(t) = \eta_L Z_L(t) \text{ and } \dot{N}_H(t) = \eta_H Z_H(t), \tag{15.7}$$

where $Z_L(t)$ is R&D expenditure *directed* at discovering new labor-augmenting machines at time t, while $Z_H(t)$ is R&D expenditure directed at discovering H-augmenting machines. Total R&D spending is the sum of these two, i.e.,

$$Z(t) = Z_L(t) + Z_H(t).$$

The value of a monopolist that discovers one of these machines is again given by the standard formula for the present discounted value of profits:

$$V_f(\nu, t) = \int_t^\infty \exp\left[-\int_t^s r(s')\, ds'\right] \pi_f(\nu, s) ds, \tag{15.8}$$

where $\pi_f(\nu, t) \equiv p_f^x(\nu, t)x_f(\nu, t) - \psi x_f(\nu, t)$ again denotes instantaneous profits for $f = L$ or H, and $r(t)$ is the market interest rate at time t. Once again, it is sometimes more convenient to work with the Hamilton-Jacobi-Bellman version of this value function, which takes the form:

$$r(t) V_f(\nu, t) - \dot{V}_f(\nu, t) = \pi_f(\nu, t). \tag{15.9}$$

Throughout, we normalize the price of the final good at every instant to 1, which is equivalent to setting the ideal price index of the two intermediates equal to one, i.e.,

$$\left[\gamma_L^\varepsilon (p_L(t))^{1-\varepsilon} + \gamma_H^\varepsilon (p_H(t))^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}} = 1 \text{ for all } t, \tag{15.10}$$

where $p_L(t)$ is the price index of Y_L at time t and $p_H(t)$ is the price of Y_H. We also denote the factor prices by $w_L(t)$ and $w_H(t)$.

15.3.1. Characterization of Equilibrium. An allocation in this economy is defined by the following objects: time paths of consumption levels, aggregate spending on machines, and aggregate R&D expenditure $[C(t), X(t), Z(t)]_{t=0}^{\infty}$, time paths of available machine types, $[N_L(t), N_H(t)]_{t=0}^{\infty}$, time paths of prices and quantities of each machine and the net present discounted value of profits from that machine, $[p_L^x(\nu,t), x_L(\nu,t), V_L(\nu,t)]_{\nu\in[0,N_L(t)]}^{\infty}{}_{t=0,}$ and $[p_H^x(\nu,t), x_H(\nu,t), V_H(\nu,t)]_{\nu\in[0,N_H(t)]}^{\infty}{}_{t=0,}$, and time paths of factor prices, $[r(t), w_L(t), w_H(t)]_{t=0}^{\infty}$.

An equilibrium is an allocation in which all existing research firms choose $\left[p_f^x(\nu,t), x_f(\nu,t)\right]_{\nu\in[0,N_f(t)]}^{\infty}{}_{t=0,}$ for $f = L, H$ to maximize profits, the evolution of $[N_L(t), N_H(t)]_{t=0}^{\infty}$ is determined by free entry, the time paths of factor prices, $[r(t), w_L(t), w_H(t)]_{t=0}^{\infty}$, are consistent with market clearing, and the time paths of $[C(t), X(t), Z(t)]_{t=0}^{\infty}$ are consistent with consumer optimization.

To characterize the (unique) equilibrium, let us first consider the maximization problem of producers in the two sectors. Since machines depreciate fully after use, these maximization problems are static and can be written as

$$\max_{L,[x_L(\nu,t)]_{\nu\in[0,N_L(t)]}} p_L(t)\,Y_L(t) - w_L(t)\,L - \int_0^{N_L(t)} p_L^x(\nu,t)\,x_L(\nu,t)\,d\nu, \tag{15.11}$$

and

$$\max_{H,[x_H(\nu,t)]_{\nu\in[0,N_H(t)]}} p_H(t)\,Y_H(t) - w_H(t)\,H - \int_0^{N_H(t)} p_H^x(\nu,t)\,x_H(\nu,t)\,d\nu. \tag{15.12}$$

The main difference from the maximization problem facing final good producers in Chapter 13 is the presence of prices $p_L(t)$ and $p_H(t)$, which reflect the fact that these sectors produce intermediate goods, whereas factor and machine prices are expressed in terms of the numeraire, the final good.

These two maximization problems immediately imply the following demand for machines in the two sectors:

$$x_L(\nu,t) = \left[\frac{p_L(t)}{p_L^x(\nu,t)}\right]^{1/\beta} L \quad \text{for all } \nu \in [0, N_L(t)] \text{ and all } t, \tag{15.13}$$

and

$$x_H(\nu,t) = \left[\frac{p_H(t)}{p_H^x(\nu,t)}\right]^{1/\beta} H \quad \text{for all } \nu \in [0, N_H(t)] \text{ and all } t. \tag{15.14}$$

Similar to the demands for machines in Chapter 13, these are iso-elastic, so the maximization of the net present discounted value of profits implies that each monopolist should set a constant markup over marginal cost and thus a price of

$$p_L^x(\nu,t) = p_H^x(\nu,t) = 1 \text{ for all } \nu \text{ and } t.$$

Substituting these prices into (15.13) and (15.14), we obtain

$$x_L(\nu,t) = p_L(t)^{1/\beta} L \quad \text{for all } \nu \text{ and all } t,$$

and

$$x_H(\nu,t) = p_H(t)^{1/\beta} H \quad \text{for all } \nu \text{ and all } t.$$

Since these quantities do not depend on the identity of the machine, only on the sector that is being served, profits are also independent of the machine type. In particular, we have

$$\pi_L(t) = \beta p_L(t)^{1/\beta} L \text{ and } \pi_H(t) = \beta p_H(t)^{1/\beta} H. \tag{15.15}$$

This implies that the net present discounted values of monopolists only depend on which sector they are supplying and can be denoted by $V_L(t)$ and $V_H(t)$.

Next, combining these with (15.5) and (15.6), we obtain the *derived* production functions for the two intermediate goods:

$$Y_L(t) = \frac{1}{1-\beta} p_L(t)^{\frac{1-\beta}{\beta}} N_L(t) L \tag{15.16}$$

and

$$Y_H(t) = \frac{1}{1-\beta} p_H(t)^{\frac{1-\beta}{\beta}} N_H(t) H. \tag{15.17}$$

These derived production functions are similar to (13.12) in Chapter 13, except for the presence of the price terms.

Finally, the prices of the two intermediate goods are derived from the marginal product conditions of the final good technology, (15.3), which imply

$$\begin{aligned} p(t) &\equiv \frac{p_H(t)}{p_L(t)} = \gamma \left(\frac{Y_H(t)}{Y_L(t)}\right)^{-\frac{1}{\varepsilon}} \\ &= \gamma \left(p(t)^{\frac{1-\beta}{\beta}} \frac{N_H(t) H}{N_L(t) L}\right)^{-\frac{1}{\varepsilon}} \\ &= \gamma^{\frac{\varepsilon\beta}{\sigma}} \left(\frac{N_H(t) H}{N_L(t) L}\right)^{-\frac{\beta}{\sigma}}, \end{aligned} \tag{15.18}$$

where again $\gamma \equiv \gamma_H/\gamma_L$ and

$$\begin{aligned} \sigma &\equiv \varepsilon - (\varepsilon - 1)(1-\beta) \\ &= 1 + (\varepsilon - 1)\beta. \end{aligned}$$

is the (derived) elasticity of substitution between the two factors. The first line of this expression simply defines $p(t)$ as the relative price between the two intermediate goods and uses the fact that the ratio of the marginal productivities of the two intermediate goods must be equal to this relative price. The second line substitutes from (15.16) and (15.17) above.

Using the latter equation, we can also calculate the relative factor prices in this economy as:

$$\begin{aligned}\omega(t) &\equiv \frac{w_H(t)}{w_L(t)} \\ &= p(t)^{1/\beta}\frac{N_H(t)}{N_L(t)} \\ &= \gamma^{\frac{\varepsilon}{\sigma}}\left(\frac{N_H(t)}{N_L(t)}\right)^{\frac{\sigma-1}{\sigma}}\left(\frac{H}{L}\right)^{-\frac{1}{\sigma}}. \end{aligned} \tag{15.19}$$

The first line of (15.19) defines $\omega(t)$ as the relative wage of factor H compared to factor L. The second line uses the definition of marginal product combined with (15.16) and (15.17), and the third line uses (15.18). We refer to σ as the (derived) elasticity of substitution between the two factors, since it is exactly equal to

$$\sigma = -\left(\frac{d\log\omega(t)}{d\log(H/L)}\right)^{-1}.$$

To complete the description of equilibrium in the technology side, we need to impose the following free entry conditions:

$$\eta_L V_L(t) \le 1 \text{ and } \eta_L V_L(t) = 1 \text{ if } Z_L(t) > 0. \tag{15.20}$$

and

$$\eta_H V_H(t) \le 1 \text{ and } \eta_H V_H(t) = 1 \text{ if } Z_H(t) > 0. \tag{15.21}$$

Finally, the consumer side is characterized by the same necessary conditions as usual:

$$\frac{\dot{C}(t)}{C(t)} = \frac{1}{\theta}(r(t)-\rho), \tag{15.22}$$

and

$$\lim_{t\to\infty}\left[\exp\left(-\int_0^t r(s)\,ds\right)(N_L(t)V_L(t)+N_H(t)V_H(t))\right] = 0, \tag{15.23}$$

which uses the fact that $N_L(t)V_L(t)+N_H(t)V_H(t)$ is the total value of corporate assets in this economy.

We are now in a position to characterize a balanced growth path (BGP) equilibrium. Let us define the BGP equilibrium to be one in which consumption grows at the constant rate, g^*, and the relative price $p(t)$ is constant. From (15.10) this implies that $p_L(t)$ and $p_H(t)$ are also constant.

Let V_L and V_H be the BGP net present discounted values of new innovations in the two sectors. Then (15.9) implies that

$$V_L = \frac{\beta p_L^{1/\beta}L}{r^*} \text{ and } V_H = \frac{\beta p_H^{1/\beta}H}{r^*}, \tag{15.24}$$

where r^* is the BGP interest rate, while p_L and p_H are the BGP prices of the two intermediate goods. The comparison of these two values is of crucial importance. As discussed intuitively above, the greater is V_H relative to V_L, the greater are the incentives to develop H-augmenting machines, N_H, rather than N_L. Taking the ratio of these two expressions, we obtain

$$\frac{V_H}{V_L} = \left(\frac{p_H}{p_L}\right)^{\frac{1}{\beta}} \frac{H}{L}.$$

This expression highlights the two effects on the direction of technological change discussed in Section 15.1.

(1) The price effect manifests itself because V_H/V_L is increasing in p_H/p_L. The greater is this relative price, the greater are the incentives to invent technologies complementing the H factor. Since goods produced by relatively scarce factors will be relatively more expensive, the price effect tends to favor technologies complementing scarce factors.

(2) The market size effect is a consequence of the fact that V_H/V_L is increasing in H/L. The market for a technology is the workers (or the other factors) that will be using and working with this technology. Consequently, an increase in the supply of a factor translates into a greater market for the technology complementing that factor. The market size effect encourages innovation for the more abundant factor.

The above discussion is incomplete, however, since prices are endogenous. Combining (15.24) together with (15.18), we can eliminate relative prices and obtain the relative profitability of the technologies as:

$$\frac{V_H}{V_L} = \gamma^{\frac{\varepsilon}{\sigma}} \left(\frac{N_H}{N_L}\right)^{-\frac{1}{\sigma}} \left(\frac{H}{L}\right)^{\frac{\sigma-1}{\sigma}}. \tag{15.25}$$

Note for future reference that an increase in the relative factor supply, H/L, will increase V_H/V_L as long as $\sigma > 1$ and it will reduce it if $\sigma < 1$. This shows that the elasticity of substitution between the factors, σ, regulates whether the price effect dominates the market size effect. Since σ is not a primitive, but a derived parameter, we would like to know when it is greater than 1. It is straightforward to check that

$$\sigma \gtreqless 1 \iff \varepsilon \gtreqless 1.$$

So the two factors will be gross substitutes when the two intermediate goods are gross substitutes in the production of the final good.

Next, using the two free entry conditions (15.20) and (15.21), and assuming that both of them hold as equalities, we obtain the following BGP "technology market clearing" condition:

$$\eta_L V_L = \eta_H V_H. \tag{15.26}$$

Combining this with (15.25), we obtain the following BGP ratio of relative technologies

$$\left(\frac{N_H}{N_L}\right)^* = \eta^{\sigma} \gamma^{\varepsilon} \left(\frac{H}{L}\right)^{\sigma-1}, \tag{15.27}$$

where $\eta \equiv \eta_H/\eta_L$ and the *'s denote that this expression refers to the BGP value. The notable feature here is that relative productivities are determined by the innovation possibilities frontier and the relative supply of the two factors. In this sense, this model totally endogenizes technology. Equation (15.27) contains most of the economics of directed technology. However, before discussing this, it is useful to characterize the BGP growth rate of the economy. This is done in the next proposition:

Proposition 15.1. *Consider the directed technological change model described above. Suppose that*

$$\beta\left[\gamma_H^{\varepsilon}\left(\eta_H H\right)^{\sigma-1}+\gamma_L^{\varepsilon}\left(\eta_L L\right)^{\sigma-1}\right]^{\frac{1}{\sigma-1}} > \rho \text{ and} \tag{15.28}$$

$$(1-\theta)\,\beta\left[\gamma_H^{\varepsilon}\left(\eta_H H\right)^{\sigma-1}+\gamma_L^{\varepsilon}\left(\eta_L L\right)^{\sigma-1}\right]^{\frac{1}{\sigma-1}} < \rho.$$

Then there exists a unique BGP equilibrium in which the relative technologies are given by (15.27), and consumption and output grow at the rate

$$g^{*}=\frac{1}{\theta}\left(\beta\left[\gamma_H^{\varepsilon}\left(\eta_H H\right)^{\sigma-1}+\gamma_L^{\varepsilon}\left(\eta_L L\right)^{\sigma-1}\right]^{\frac{1}{\sigma-1}}-\rho\right). \tag{15.29}$$

Proof. The derivation of (15.29) is provided by the argument preceding the proposition. Exercise 15.2 asks you to check that condition (15.28) ensures that free entry conditions (15.20) and (15.21) must hold, to verify that this is the unique relative equilibrium technology, to calculate the BGP equilibrium growth rate and to verify that the transversality condition is satisfied. □

It can also be verified that there are simple transitional dynamics in this economy whereby starting with technology levels $N_H(0)$ and $N_L(0)$, there always exists a unique equilibrium path and it involves the economy monotonically converging to the BGP equilibrium of Proposition 15.1. This is stated in the next proposition:

Proposition 15.2. *Consider the directed technological change model described above. Starting with any $N_H(0) > 0$ and $N_L(0) > 0$, there exists a unique equilibrium path. If $N_H(0)/N_L(0) < (N_H/N_L)^*$ as given by (15.27), then we have $Z_H(t) > 0$ and $Z_L(t) = 0$ until $N_H(t)/N_L(t) = (N_H/N_L)^*$. If $N_H(0)/N_L(0) < (N_H/N_L)^*$, then $Z_H(t) = 0$ and $Z_L(t) > 0$ until $N_H(t)/N_L(t) = (N_H/N_L)^*$.*

Proof. See Exercise 15.3. □

More interesting than the aggregate growth rate and the transitional dynamics behavior of the economy are the results concerning the direction of technological change and its effects on relative factor prices. These are studied in the next subsection.

15.3.2. Directed Technological Change and Factor Prices. Let us start by studying (15.27). This equation implies that, in BGP, there is a positive relationship between the relative supply of the H factor, H/L, and the relative factor-augmenting technologies, N_H^*/N_L^* only when $\sigma > 1$. In contrast, if the derived elasticity of substitution, σ, is less than 1, the relationship is reversed. This might suggest that, depending on the elasticity of substitution between factors (or between the intermediate goods), changes in factor supplies may induce technological changes that are biased in favor or against the factor that is becoming more abundant. However, this conclusion is not correct. Recall from Section 15.2 that N_H^*/N_L^* refers to the ratio of factor-augmenting technologies, or to the ratio of *physical* productivities. What matters for the bias of technology is *the value of marginal product* of factors, which is affected by changes in relative prices. We have already seen that the relationship between factor-augmenting technologies and factor-biased technologies is reversed precisely when σ is less than 1. Thus, when $\sigma > 1$, an increase in N_H^*/N_L^* is relatively biased towards H, while when $\sigma < 1$, it is a decrease in N_H^*/N_L^* that is relatively biased towards H.

This immediately establishes the following *weak equilibrium bias result*:

Proposition 15.3. *Consider the directed technological change model described above. There is always weak equilibrium (relative) bias in the sense that an increase in H/L always induces relatively H-biased technological change.*

Recall that weak bias was defined in Section 15.2 with a weak inequality, so that the proposition is also correct when $\sigma = 1$, even though in this case it can be verified easily from (15.27) that N_H^*/N_L^* does not depend on H/L.

Proposition 15.3 is the basis of the discussion about induced biased technological change in Section 15.1, and already gives us a range of insights about how changes in the relative supplies of skilled workers may be at the root of the skill-biased technological change. These implications are further discussed in the next subsection.

The results of this proposition reflect the strength of the market size effect discussed above. Recall that the price effect creates a force favoring factors that become relatively scarce. In contrast, the market size effect, which is related to the non-rivalry of ideas discussed in Chapter 12, suggests that technologies should change in a way that favors factors that are becoming relatively abundant. Proposition 15.3 shows that the market size effect always dominates the price effect.

Proposition 15.3 is only informative about the direction of the induced technological change, but does not specify whether this induced effect will be strong enough to make the endogenous-technology relative demand curve for factors upward-sloping. Recall that in basic producer theory, all demand curves, and thus relative demand curves, are downward-sloping as well. However, as hinted in Section 15.1, directed technological change can lead to the seemingly paradoxical result that relative demand curves can be upward-sloping once the endogeneity of technology is taken into account. To obtain this result, let us substitute for $(N_H/N_L)^*$ from (15.27) into the expression for the relative wage given technologies, (15.19), and obtain the following BGP relative factor price ratio (see Exercise 15.4):

$$\omega^* \equiv \left(\frac{w_H}{w_L}\right)^* = \eta^{\sigma-1}\gamma^{\varepsilon}\left(\frac{H}{L}\right)^{\sigma-2}. \tag{15.30}$$

Inspection of this equation immediately establishes conditions for *strong equilibrium (relative) bias.*

Proposition 15.4. *Consider the directed technological change model described above. Then if $\sigma > 2$, there is strong equilibrium (relative) bias in the sense that an increase in H/L raises the relative marginal product and the relative wage of the factor H compared to factor L.*

Figure 15.3 illustrates the results of Propositions 15.3 and 15.4, referring to H as skilled labor and L as unskilled labor as in the first application discussed in Section 15.1.

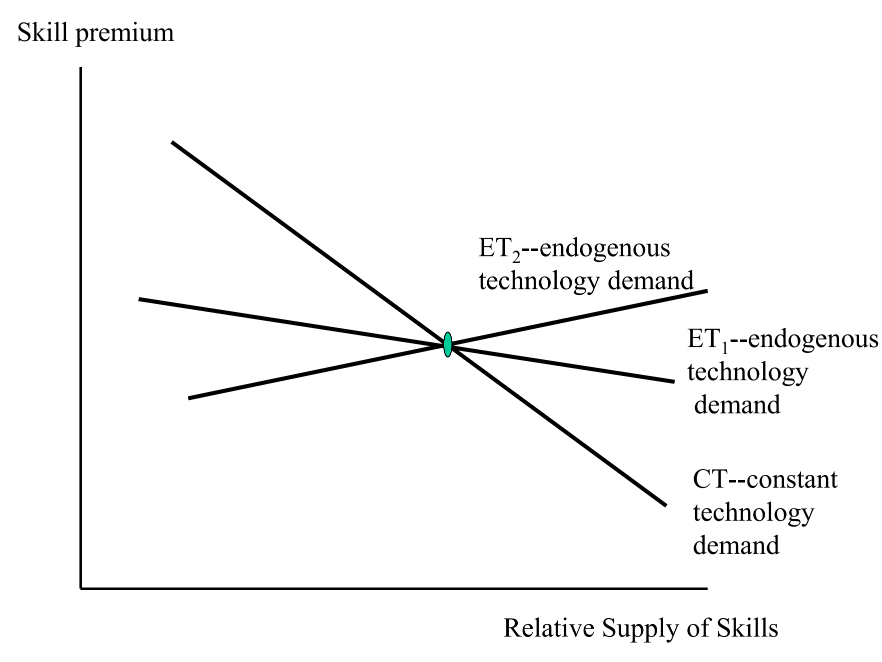

Figure 15.3. The relationship between the relative supply of skills and the skill premium in the model of directed technical change.

The curve marked with CT corresponds to the constant-technology relative demand from equation (15.19). It is always downward-sloping because it holds the relative technologies, N_H/N_L, constant, and thus only features the usual substitution effect. The fact that this curve is downward-sloping follows from basic producer theory. The curve marked as ET_1 applies when technology is endogenous, but the condition in Proposition 15.4, that $\sigma > 2$, is not satisfied. We know from Proposition 15.3 that even in this case an increase in H/L will induce skill-biased (H-biased) technological change. This implies that when H/L is higher than its initial level, the induced-technology effect will shift the constant-technology demand curve CT to the right (i.e., as technology changes, it will be another CT curve, above the original one, that will apply). When it is below, the same effect will shift CT to the left. Consequently, the locus of points that the endogenous-technology demand, ET_1, is shallower than CT. [This can be also verified comparing the relative demand functions with constant and endogenous technology, (15.19) and (15.30), and noting that $\sigma - 2$ is never less than $-1/\sigma$]. There is an obvious analogy between this result and Samuelson's LeChatelier principle, which states that long-run demand curves, which apply when all factors can adjust, must be more elastic than the short-run demand curves which hold some factors constant. We can think of the endogenous-technology demand curve as adjusting the "factors of production" corresponding to technology. However, the analogy is imperfect because the effects here are caused by general equilibrium changes, while the LeChatelier principle and the basic producer theory focus on partial equilibrium effects. In fact, in basic producer theory, with or without the LeChatelier effects, all demand curves *must be downward-sloping*, whereas here ET_2, which applies when the conditions of Proposition 15.4 hold, is upward-sloping; higher levels of relative supply of skills correspond to higher skill premia.

A complementary intuition for this result can be obtained by going back to the importance of non-rivalry of ideas as discussed in Chapter 12. Here, as in the basic endogenous technology models of the last two chapters, the non-rivalry of ideas leads to an aggregate production function that exhibits increasing returns to scale (in all factors including technologies). It is the increasing returns to scale in the production possibilities set of the economy that leads to potentially upward-sloping relative demand curves. Put differently, the market size effect, which results from the non-rivalry of ideas and is at the root of aggregate increasing returns, can create sufficiently strong induced technological change to increase the relative marginal product and the relative price of the factor that has become more abundant.

15.3.3. Implications. The results of Propositions 15.3 and 15.4 are not only of theoretical interest, but also shed light on a range of important empirical patterns. As already discussed above, one of the most interesting applications is to changes in the skill premium. For this application, suppose that H stands for college-educated workers. In the U.S. labor market, the skill premium has shown no tendency to decline despite a very large increase in the supply of college educated workers. On the contrary, following a brief period of decline during the 1970s in the face of the very large increase in the supply of college-educated workers, the skill (college) premium has increased very sharply throughout the 1980s and the 1990s, to reach a level not experienced in the postwar era. Figure 15.1 above showed these general patterns.

In the labor economics and parts of the macroeconomics literature, the most popular explanation for these patterns is skill-biased technological change. For example, computers or new IT technologies are argued to favor skilled workers relative to unskilled workers. But why should the economy adopt and develop more skill-biased technologies throughout the past 30 years, or more generally throughout the entire 20th century? This question becomes more relevant once we remember that during the 19th century many of the technologies that were fueling economic growth, such as the factory system and the major spinning and weaving innovations, were unskill-biased rather than skill-biased.

Thus, in summary, we have the following stylized facts:

(1) Secular skill-biased technological change increasing the demand for skills throughout the 20th century.
(2) Possible acceleration in skill-biased technological change over the past 25 years.
(3) A range of important technologies biased against skill workers during the 19th century.

The current model, in particular, Propositions 15.3 and 15.4, gives us a way to think about these issues. In particular, when $\sigma > 2$, the long-run (endogenous-technology) relationship between the relative supply of skills and the skill premium is positive. With an upward-sloping relative demand curve, or simply with the degree of skill bias endogenized, we have a natural explanation for all of the patterns mentioned above.

(1) According to Propositions 15.3 and 15.4, the increase in the number of skilled workers that has taken place throughout the 20th century should cause steady skill-biased technical change. Therefore, models of directed technological change offer a natural explanation for the secular skill-biased technological developments of the past century.
(2) Acceleration in the increase in the number of skilled workers over the past 25 years, shown in Figure 15.1, should induce an acceleration in skill-biased technological change. We will also discuss below how this class of models might account for the dynamics of factor prices in the face of endogenously changing technologies.
(3) Can the framework also explain the prevalence of skill-replacing/labor-biased technological change in the late 18th and 19th centuries? While we know less about both changes in relative supplies and technological developments during these historical periods, available evidence suggests that there were large increases in the number of

unskilled workers available to be employed in the factories during this time periods. Bairoch (1988, p. 245), for example, describes this rapid expansion of the supply of unskilled labor as follows:

"... between 1740 and 1840 the population of England ... went up from 6 million to 15.7 million. ... while the agricultural labor force represented 60-70% of the total work force in 1740, by 1840 it represented only 22%."

Habakkuk's well-known account of 19th-century technological development (pp. 136-137) also emphasizes the increase in the supply of unskilled labor in English cities, and attributes it to five sources. First, "technical changes in agriculture increased the supply of labor available to industry" (p. 137). Second, "population was increasing very rapidly" (p. 136). Third, labor reserves of rural industry came to the cities. Fourth, "there was a large influx of labor from Ireland" (p. 137). Finally, enclosures released substantial labor from agriculture.[2] According to our model of directed technological change, this increase in the relative supply of unskilled labor should have encouraged unskill-biased technical change, and this is consistent with the patterns discussed above.

In addition to accounting for the recent skill-biased technological developments and for the historical technologies that appear to have been biased towards unskilled workers, this framework also gives a potential interpretation for the dynamics of the college premium during the 1970s and 1980s. It is reasonable to presume that the equilibrium skill bias of technologies, N_H/N_L, is a sluggish variable determined by the slow buildup and development of new technologies (as the analysis of transitional dynamics in Proposition 15.2 shows). In this case, a rapid increase in the supply of skills would first reduce the skill premium as the economy would be moving along a constant technology (constant N_H/N_L) curve as shown in Figure 15.4. After a while technology would start adjusting, and the economy would move back to the upward sloping relative demand curve, with a relatively sharp increase in the college premium. This approach can therefore explain both the decline in the college premium during the 1970s and its subsequent large surge, and relates both of these phenomena to the large increase in the supply of skilled workers.

If on the other hand we have $\sigma < 2$, the long-run relative demand curve will be downward sloping, though again it will be shallower than the short-run relative demand curve. Following the increase in the relative supply of skills there will again be an initial decline in the college premium, and as technology starts adjusting the skill premium will increase. But it will end

[2]Although there is general agreement that the supply of unskilled labor in British cities increased between the 18th and 19th centuries, there is disagreement among economic historians about the importance of various factors. For example, Habakkuk's emphasis on the importance of enclosures is challenged by Allen (1992, 2004). The exact contribution of the enclosures movement to the increase in the supply of unskilled labor is secondary for the argument here.

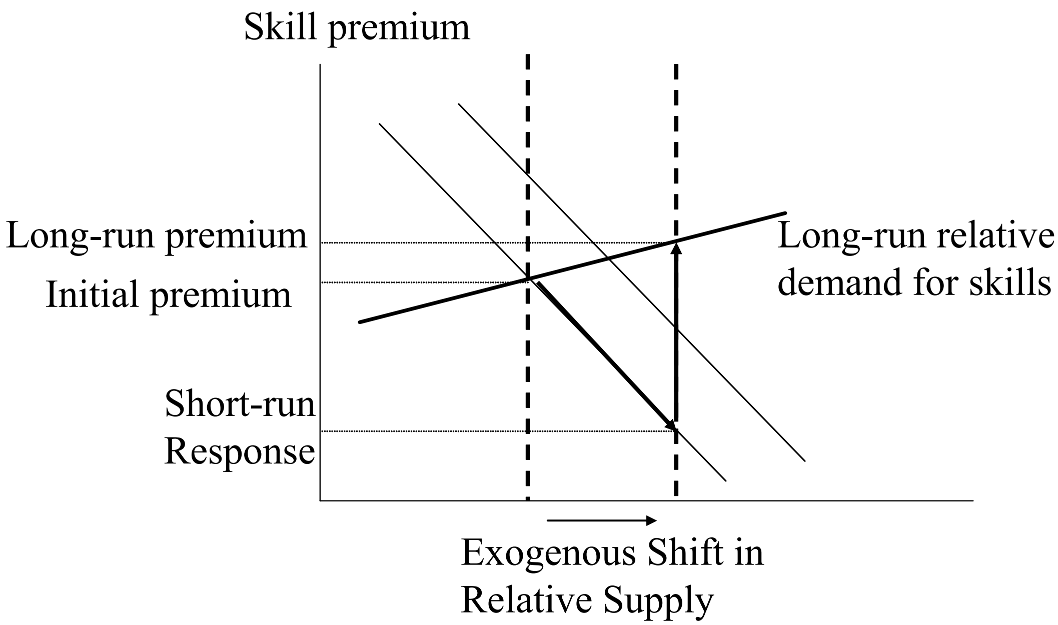

Figure 15.4. Dynamics of the skill premium in response to an exogenous increase in the relative supply of skills, with an upward-sloping endogenous-technology relative demand curve.

up below its initial level. To explain the larger increase in the college premium in the 1980s, in this case we would need some exogenous skill-biased technical change. Figure 15.5 draws this case.

Consequently, a model of directed technological change can shed light both on the secular skill bias of technology and on the relatively short-run changes in technology-induced factor prices. We will study other implications of these results below. However, before doing this, a couple of further issues need to be discussed. First, Proposition 15.4 shows that upward-sloping relative demand curves arise only when $\sigma > 2$. In the context of substitution between skilled and unskilled workers, an elasticity of substitution much higher than 2 is unlikely. Most estimates put the elasticity of substitution between 1.4 and 2. One would like to understand whether $\sigma > 2$ is a feature of the specific model discussed here and how different assumptions about the technology of production or the innovation possibilities frontier affect this result. This issue will be discussed in Section 15.4. Second, we would like to understand the relationship between the market size effect and the scale effects, in particular, whether the results on induced technological change are an artifact of the scale effect (which many economists do not view as an attractive feature of endogenous technological change models). Section 15.5 shows that this is not the case and exactly the same results apply

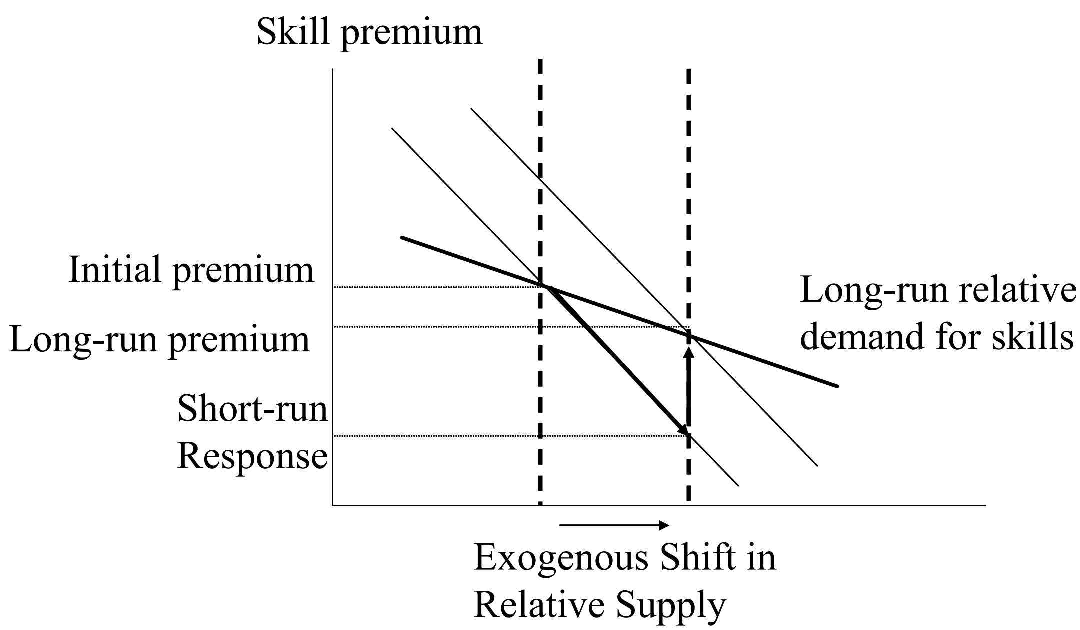

Figure 15.5. Dynamics of the skill premium in response to an increase in the relative supply of skills, with a downward-sloping endogenous-technology relative demand curve.

when scale effects are removed. Third, we would like to apply these ideas to investigate whether there are reasons for technological change to be endogenously labor-augmenting in the neoclassical growth model. This will be investigated in Section 15.6. Finally, it is also useful to contrast equilibrium allocation to the Pareto optimal allocation. We will start with this latter comparison in the next subsection.

15.3.4. Pareto Optimal Allocations. The analysis of Pareto optimal allocation is very similar to the analysis of optimal growth in Chapter 13. For this reason, we will present only a sketch of the argument. As in that analysis, it is straightforward to see that the social planner would not charge a markup on machines, thus we have

$$x_L^S(\nu,t) = (1-\beta)^{-1/\beta} p_L(t)^{1/\beta} L \text{ and } x_H^S(\nu,t) = (1-\beta)^{-1/\beta} p_H(t)^{1/\beta} H.$$

Combining these with the production function and some algebra establish that net output, which can be used for consumption or research, is equal to (see Exercise 15.6):

$$Y^S(t) = (1-\beta)^{-1/\beta}\beta\left[\gamma_L^{\varepsilon}\left(N_L^S(t)L\right)^{\frac{\sigma-1}{\sigma}} + \gamma_H^{\varepsilon}\left(N_H^S(t)H\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}. \tag{15.31}$$

In view of this, the current-value Hamiltonian for the social planner can be written as

$$\mathcal{H}\left(N_L^S, N_H^S, Z_L^S, Z_H^S, C^S, \mu_L, \mu_H\right) = \frac{C^S(t)^{1-\theta}-1}{1-\theta} + \mu_L(t)\eta_L Z_L^S(t) + \mu_H(t)\eta_H Z_H^S(t),$$

subject to

$$C^S(t) = (1-\beta)^{-1/\beta}\left[\gamma_L^\varepsilon\left(N_L^S(t)\,L\right)^{\frac{\sigma-1}{\sigma}} + \gamma_H^\varepsilon\left(N_H^S(t)\,H\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} - Z_L^S(t) - Z_H^S(t)\,.$$

The necessary conditions for this problem give the following characterization of the Pareto optimal allocation in this economy.

Proposition 15.5. *The stationary solution of the Pareto optimal allocation involves relative technologies given by (15.27) as in the decentralized equilibrium. The stationary growth rate is higher than the equilibrium growth rate and is given by*

$$g^S = \frac{1}{\theta}\left((1-\beta)^{-1/\beta}\beta\left[(1-\gamma)^\varepsilon(\eta_H H)^{\sigma-1} + \gamma^\varepsilon(\eta_L L)^{\sigma-1}\right]^{\frac{1}{\sigma-1}} - \rho\right) > g^*,$$

where g^* is the BGP growth rate given in (15.29).

Proof. See Exercise 15.7. □

15.4. Directed Technological Change with Knowledge Spillovers

I now consider the directed technological change model of the previous section with knowledge spillovers. This exercise has three purposes. First, it will show how the main results on the direction of technological change can be generalized to a model using the other common specification of the innovation possibilities frontier. Second, this analysis will show that the strong bias result in Proposition 15.4 can hold under somewhat weaker conditions. Third, this formulation will be essential for our study of labor-augmenting technological change in Section 15.6.

The lab equipment specification of the innovation possibilities frontier is special in one respect: it does not allow for *state dependence*. State dependence refers to the phenomenon in which the path of past innovations affects the relative costs of different types of innovations. The lab equipment specification implied that R&D spending always leads to the same increase in the number of L-augmenting and H-augmenting machines. We will now introduce a specification with knowledge spillovers, which allows for state dependence. Recall that, as discussed in Section 13.2 in Chapter 13, when there are scarce factors used for R&D, then growth cannot be sustained by continuously increasing the amount of these factors allocated to R&D. Therefore, in order to achieve sustained growth, these factors need to become more and more productive over time, because of spillovers from past research. Here for simplicity, let us assume that R&D is carried out by scientists and that there is a constant supply of scientists equal to S (Exercise 15.18 shows that the results are identical when workers can be employed in the R&D sector). With only one sector, the analysis in Section 13.2 in Chapter 13 indicates that sustained endogenous growth requires $\dot{N}/N$ to be proportional to S. With two sectors, instead, there is a variety of specifications with different degrees of state dependence, because productivity in each sector can depend on the state of knowledge in both sectors. A flexible formulation is the following:

(15.32)
$$\dot{N}_L(t) = \eta_L N_L(t)^{(1+\delta)/2} N_H(t)^{(1-\delta)/2} S_L(t) \text{ and } \dot{N}_H(t) = \eta_H N_L(t)^{(1-\delta)/2} N_H(t)^{(1+\delta)/2} S_H(t),$$

where $\delta \leq 1$, and $S_L(t)$ is the number of scientists working to produce L-augmenting machines, while $S_H(t)$ denotes the number of scientists working on H-augmenting machines. Clearly, market clearing for scientists requires that

$$S_L(t) + S_H(t) \leq S. \tag{15.33}$$

In this specification, δ measures the degree of state-dependence: when $\delta = 0$, there is no state-dependence—$\left(\partial \dot{N}_H / \partial S_H\right) / \left(\partial \dot{N}_L / \partial S_L\right) = \eta_H / \eta_L$ irrespective of the levels of N_L and N_H—because both N_L and N_H creates spillovers for current research in both sectors. In this case, the results are identical to those in the previous subsection. In contrast, when $\delta = 1$, there is an extreme amount of state-dependence. In this case, $\left(\partial \dot{N}_H / \partial S_H\right) / \left(\partial \dot{N}_L / \partial S_L\right) = \eta_H N_H / \eta_L N_L$, so an increase in the stock of L-augmenting machines today makes future labor-complementary innovations cheaper, but has no effect on the cost of H-augmenting innovations. This discussion clarifies the role of the parameter δ and the meaning of state dependence. In some sense, state dependence adds another layer of "increasing returns," this time not for the entire economy, but for specific technology lines. In particular, a significant amount of state dependence implies that when N_H is high relative to N_L, it becomes more profitable to undertake more N_H-type innovations.

With this formulation of the innovation possibilities frontier, the free entry conditions become (see Exercise 15.8):

$$\begin{aligned} \eta_L N_L(t)^{(1+\delta)/2} N_H(t)^{(1-\delta)/2} V_L(t) &\leq w_S(t) \\ \text{and } \eta_L N_L(t)^{(1+\delta)/2} N_H(t)^{(1-\delta)/2} V_L(t) &= w_S(t) \text{ if } S_L(t) > 0. \end{aligned} \tag{15.34}$$

and

$$\begin{aligned} \eta_H N_L(t)^{(1-\delta)/2} N_H(t)^{(1+\delta)/2} V_H(t) &\leq w_S(t) \\ \text{and } \eta_H N_L(t)^{(1-\delta)/2} N_H(t)^{(1+\delta)/2} V_H(t) &= w_S(t) \text{ if } S_H(t) > 0, \end{aligned} \tag{15.35}$$

where $w_S(t)$ denotes the wage of a scientist at time t. When both of these free entry conditions hold, BGP technology market clearing implies

$$\eta_L N_L(t)^{\delta} \pi_L = \eta_H N_H(t)^{\delta} \pi_H, \tag{15.36}$$

where δ captures the importance of state-dependence in the technology market clearing condition, and profits are not conditioned on time, since they refer to the BGP values, which are constant as in the previous section (recall (15.15)). When $\delta = 0$, this condition is identical to (15.26) in the previous section. Therefore, as claimed above, all of the results concerning the direction of technological change would be identical to those from the lab equipment specification.

This is no longer true when $\delta > 0$. To characterize the results in this case, let us combine condition (15.36) with equations (15.15) and (15.18), we obtain the equilibrium relative technology as (see Exercise 15.9):

$$\left(\frac{N_H}{N_L}\right)^* = \eta^{\frac{\sigma}{1-\delta\sigma}} \gamma^{\frac{\varepsilon}{1-\delta\sigma}} \left(\frac{H}{L}\right)^{\frac{\sigma-1}{1-\delta\sigma}}, \tag{15.37}$$

where recall that $\gamma \equiv \gamma_H/\gamma_L$ and $\eta \equiv \eta_H/\eta_L$. This expression shows that the relationship between the relative factor supplies and relative physical productivities now depends on δ. This is intuitive: as long as $\delta > 0$, an increase in N_H reduces the relative costs of H-augmenting innovations, so for technology market equilibrium to be restored, π_L needs to fall relative to π_H. Substituting (15.37) into the expression for relative factor prices for given technologies, which is still (15.19), yields the following long-run (endogenous-technology) relationship between relative factor prices and relative factor supplies:

$$\omega^* \equiv \left(\frac{w_H}{w_L}\right)^* = \eta^{\frac{\sigma-1}{1-\delta\sigma}} \gamma^{\frac{(1-\delta)\varepsilon}{1-\delta\sigma}} \left(\frac{H}{L}\right)^{\frac{\sigma-2+\delta}{1-\delta\sigma}}. \tag{15.38}$$

It can be verified that when $\delta = 0$, so that there is no state-dependence in R&D, both of the previous expressions are identical to their counterparts in the previous section.

The growth rate of this economy is determined by the number of scientists. In BGP, both sectors grow at the same rate, so we need $\dot{N}_L(t)/N_L(t) = \dot{N}_H(t)/N_H(t)$, or

$$\eta_H N_H(t)^{\delta-1} S_H(t) = \eta_L N_L(t)^{\delta-1} S_L(t).$$

Combining this equation with (15.33) and (15.37), we obtain the following BGP condition for the allocation of researchers between the two different types of technologies,

$$\eta^{\frac{1-\sigma}{1-\delta\sigma}} \left(\frac{1-\gamma}{\gamma}\right)^{-\frac{\varepsilon(1-\delta)}{1-\delta\sigma}} \left(\frac{H}{L}\right)^{-\frac{(\sigma-1)(1-\delta)}{1-\delta\sigma}} = \frac{S_L^*}{S-S_L^*}, \tag{15.39}$$

and the BGP growth rate (15.40) below. Notice that given H/L, the BGP researcher allocations, S_L^* and S_H^*, are uniquely determined. We summarize these results with the following proposition:

Proposition 15.6. *Consider the directed technological change model with knowledge spillovers and state dependence in the innovation possibilities frontier. Suppose that*

$$(1-\theta)\frac{\eta_L\eta_H (N_H/N_L)^{(\delta-1)/2}}{\eta_H (N_H/N_L)^{(\delta-1)} + \eta_L} S < \rho,$$

where N_H/N_L is given by (15.37). Then there exists a unique BGP equilibrium in which the relative technologies are given by (15.37), and consumption and output grow at the rate

$$g^* = \frac{\eta_L\eta_H (N_H/N_L)^{(\delta-1)/2}}{\eta_H (N_H/N_L)^{(\delta-1)} + \eta_L} S. \tag{15.40}$$

Proof. See Exercise 15.10. □

In contrast to the model with the lab equipments technology, transitional dynamics do not always take the economy to the BGP equilibrium, however. This is because of the additional increasing returns to scale mentioned above. With a high degree of state dependence, when $N_H(0)$ is very high relative to $N_L(0)$, it may no longer be profitable for firms to undertake further R&D directed at labor-augmenting (L-augmenting) technologies. Whether this is so or not depends on a comparison of the degree of state dependence, δ, and the elasticity of substitution, σ. The latter matters because it regulates how prices change as there is an abundance of one type of technology relative to another, and thus determines the strength of the price effect on the direction of technological change. The next proposition analyzes the transitional dynamics in this case.

Proposition 15.7. *Consider the directed technological change model with knowledge spillovers and state dependence in the innovation possibilities frontier. Suppose that*

$$\sigma < 1/\delta.$$

Then, starting with any $N_H(0) > 0$ and $N_L(0) > 0$, there exists a unique equilibrium path. If $N_H(0)/N_L(0) < (N_H/N_L)^*$ as given by (15.37), then we have $Z_H(t) > 0$ and $Z_L(t) = 0$ until $N_H(t)/N_L(t) = (N_H/N_L)^*$. $N_H(0)/N_L(0) < (N_H/N_L)^*$, then $Z_H(t) = 0$ and $Z_L(t) > 0$ until $N_H(t)/N_L(t) = (N_H/N_L)^*$.

If

$$\sigma > 1/\delta,$$

then starting with $N_H(0)/N_L(0) > (N_H/N_L)^*$, the economy tends to $N_H(t)/N_L(t) \to \infty$ as $t \to \infty$, and starting with $N_H(0)/N_L(0) < (N_H/N_L)^*$, it tends to $N_H(t)/N_L(t) \to 0$ as $t \to \infty$.

Proof. See Exercise 15.12. □

Of greater interest for our focus are the results on the direction of technological change. Our first result on weak equilibrium bias immediately generalizes from the previous section:

Proposition 15.8. *Consider the directed technological change model with knowledge spillovers and state dependence in the innovation possibilities frontier. Then there is always weak equilibrium (relative) bias in the sense that an increase in H/L always induces relatively H-biased technological change.*

Proof. See Exercise 15.13. □

While the results regarding weak bias have not changed, inspection of (15.38) shows that it is now easier to obtain *strong equilibrium (relative) bias.*

Proposition 15.9. *Consider the directed technological change model with knowledge spillovers and state dependence in the innovation possibilities frontier. Then if*

$$\sigma > 2 - \delta,$$

there is strong equilibrium (relative) bias in the sense that an increase in H/L raises the relative marginal product and the relative wage of the H factor compared to the L factor.

Intuitively, the additional increasing returns to scale coming from state dependence makes strong bias easier to obtain, because the induced technology effect is stronger. When a particular factor, say H, becomes more abundant, this encourages an increase in N_H relative to N_L (in the case where $\sigma > 1$). However, from state dependence, this makes further increases in N_H more profitable, thus has a larger effect on N_H/N_L. Since with $\sigma > 1$ greater values of N_H/N_L tend to increase the relative price of factor H compared to L, this tends to make the strong bias result easier to obtain.

Returning to our discussion of the implications of the strong bias results for the behavior of the skill premium in the U.S. market, Proposition 15.9 implies that values of the elasticity of substitution between skilled and unskilled labor significantly less than 2 may be sufficient to generate strong equilibrium bias. How much lower than 2 the elasticity of substitution can be depends on the parameter δ. Unfortunately, this parameter is not easy to measure in practice, even though existing evidence suggests that there is some amount of state dependence in the R&D technologies. For example, this is confirmed by the empirical finding that most patents developed in a particular industry build upon and cite previous patents in the same industry.

15.5. Directed Technological Change without Scale Effects

We will now see that the market size effect and its implications for the direction of technological change are independent of whether or not there are scale effects. The market size effect here refers to the relative market sizes of the users of two different types of technologies, not necessarily to the scale of the entire economy, whereas the scale effect concerns the impact of the size of the population on the equilibrium growth rate. The results in this section show that it is possible to entirely separate the market size effect responsible for the weak and strong endogenous bias results and the scale effect.

Suppose that we are in the case with knowledge-based R&D model of the previous section, but only with limited spillovers from past research. In particular, suppose that equation (15.32) is modified to

$$\dot{N}_L = \eta_L N_L^{\lambda} S_L \text{ and } \dot{N}_H = \eta_H N_H^{\lambda} S_H, \tag{15.41}$$

where $\lambda \in (0, 1]$. In the case where $\lambda = 1$, we have the knowledge-based R&D formulation of the previous section, but with no state dependence. When $\lambda < 1$, the extent of spillovers from past research are limited, and this economy will not have steady growth in the absence of population growth.

Let us also modify the baseline environment by assuming that total population, in particular, the population of scientists, grows at the exponential rate n. With a similar arguments to that in Section 13.3 in Chapter 13, it can be verified that aggregate output in this economy will grow at the rate (see Exercise 15.15):

$$g^* = \frac{n}{1-\lambda}. \tag{15.42}$$

Consequently, even with limited knowledge spillovers there will be income per capita growth at the rate $n/(1-\lambda)$. As usual, this is because of the amplification to the externalities provided by population growth. It can also be verified that when $\lambda = 1$, there is no balanced growth and output would reach infinity in finite time (see Exercise 15.16).

The important point for the focus here concerns the market size effect on the direction of technical change. To investigate this issue, note that the technology market clearing condition implied by (15.41) is (see Exercise 15.17):

$$\eta_L N_L^{\lambda} \pi_L = \eta_H N_H^{\lambda} \pi_H, \tag{15.43}$$

which is parallel to (15.36). Exactly the same analysis as above implies that equilibrium relative technology can be derived as

$$\left(\frac{N_H}{N_L}\right)^* = \eta^{\frac{\sigma}{1-\lambda\sigma}} \gamma^{\frac{\varepsilon}{1-\lambda\sigma}} \left(\frac{H}{L}\right)^{\frac{\sigma-1}{1-\lambda\sigma}}. \tag{15.44}$$

Now combining this with (15.19)—which still determines the relative factor prices given technology—we obtain

$$\omega^* \equiv \left(\frac{w_H}{w_L}\right)^* = \eta^{\frac{\sigma-1}{1-\lambda\sigma}} \gamma^{\frac{(1-\lambda)\varepsilon}{1-\lambda\sigma}} \left(\frac{H}{L}\right)^{\frac{\sigma-2+\lambda}{1-\lambda\sigma}}. \tag{15.45}$$

This equation shows that even without scale effects we obtain exactly the same results as before. Specifically:

Proposition 15.10. *Consider the directed technological change model with no scale effects described above. Then there is always weak equilibrium (relative) bias, meaning that an increase in H/L always induces relatively H-biased technological change.*

Proof. See Exercise 15.8. □

Proposition 15.11. *Consider the directed technological change model with no scale effects described above. If*

$$\sigma > 2 - \lambda,$$

then there is strong equilibrium (relative) bias in the sense that an increase in H/L raises the relative marginal product and the relative wage of the H factor compared to the L factor.

15.6. Endogenous Labor-Augmenting Technological Change

One of the advantages of the models of directed technical change is that they allow us to investigate why technological change might be purely labor-augmenting as required for balanced growth. We will see that models of directed technological change create a natural reason for technology to be more labor-augmenting than capital-augmenting. However, under most circumstances, the resulting equilibrium is not purely labor-augmenting and as a result, a BGP fails to exist. However, in one important special case, the model delivers long-run purely labor-augmenting technological changes exactly as in the neoclassical growth model, thus providing a rationale for one of the strong assumptions of the standard growth models.

In thinking about labor-augmenting technological change, it is useful to consider a two-factor model with H corresponding to capital, that is, $H(t) = K(t)$, in the aggregate production function (15.3). Given the focus on capital, throughout the section we use N_L and N_K to denote the varieties of machines in the two sectors. Let us also simplify the discussion by assuming that there is no depreciation of capital. Note also that in this case the price of the second factor, $K(t)$, is the same as the interest rate, $r(t)$, since investing in the capital stock of the economy is a way of transferring consumption from one instant to another.

Let us first note that in the context of capital-labor substitution, the empirical evidence suggests that an elasticity of substitution of $\sigma < 1$ is much more plausible (whereas in the case of substitution between skilled and unskilled labor, the evidence suggested that $\sigma > 1$). An elasticity less than 1 is not only consistent with the available empirical evidence, but it is also economically plausible. For example, with the CES production function an elasticity of substitution between capital and labor greater than 1 would imply that production is possible without labor or without capital, which appears counterintuitive.

Now, recall that when $\sigma < 1$, factor-augmenting and factor-biased technologies are reversed. Therefore, labor-augmenting technological change corresponds to capital-biased technological change. Then the question becomes: under what circumstances would the economy generate relatively capital-biased technological change? And also, when will the equilibrium technology be sufficiently capital biased that it corresponds to Harrod-neutral technological change? What distinguishes capital from labor is the fact that it accumulates. In other words, most growth models feature some type of capital-deepening, with $K(t)/L$ increasing as the economy grows. This implies that in contrast to our analysis so far, where the focus was on the effect of one-time changes in relative supplies, our interest will now be on the implications of continuous changes in the relative supply of capital on technological change. In light of this observation, the answer to the first question above is straightforward: capital deepening, combined with Proposition 15.3, implies that technological change should be more labor-augmenting than capital-augmenting.

The next proposition summarizes the main idea of the previous paragraph. For simplicity, this proposition treats the increase in $K(t)/L$ as a sequence of one-time increases (full equilibrium dynamics are investigated in the next two propositions).

Proposition 15.12. *In the baseline model of directed technological change with $H(t) = K(t)$ as capital, if $K(t)/L$ is increasing over time and $\sigma < 1$, then $N_L(t)/N_K(t)$ will also increase over time, i.e., technological change will be "labor-augmenting".*

Proof. Equation (15.27) or equation (15.37) together with $\sigma < 1$ implies that an increase in $K(t)/L$ will raise $N_L(t)/N_K(t)$. □

This result already gives us important economic insights. The reasoning of directed technological change indicates that there are natural reasons for technology to be more labor-augmenting than capital-augmenting. While this is encouraging, the next proposition shows that the results are not easy to reconcile with the fact that technological change should be *purely* labor-augmenting (Harrod neutral). To state this result in the simplest possible way and to facilitate the analysis in the rest of this section, let us simplify the analysis and suppose that capital accumulates at an exogenous rate, i.e.,

$$\frac{\dot{K}(t)}{K(t)} = s_K > 0. \tag{15.46}$$

Then the next proposition shows a negative result on the possibility of purely labor-augmenting technological change.

Proposition 15.13. *Consider the baseline model of directed technological change with the knowledge spillovers specification and state dependence. Suppose that $\delta < 1$ and capital accumulates according to (15.46). Then there exists no BGP.*

Proof. See Exercise 15.22. □

Intuitively, even though technological change is more labor-augmenting than capital-augmenting, there is still capital-augmenting technological change in equilibrium. This, combined with capital accumulation, is inconsistent with balanced growth. In fact, even a more negative result can be proved (see again Exercise 15.22): in any asymptotic equilibrium, the interest rate cannot be constant, thus consumption and output growth cannot be constant.

In contrast to these negative results, there is a special case that justifies the basic structure of the neoclassical growth model. This takes place when there is extreme state dependence, that is, $\delta = 1$. This case is, in many ways, quite natural and posits that spillovers are limited to the same class of technologies, so that $\dot{N}_L(t)/N_L(t) = \eta_L S_L(t)$ and $\dot{N}_H(t)/N_H(t) = \eta_H S_H(t)$. In this case, it can be verified that technology market equilibrium implies the following relationship in BGP (see Exercise 15.23):

$$\frac{r(t)\,K(t)}{w_L(t)\,L} = \eta^{-1}. \tag{15.47}$$

Thus, directed technological change implies that in the long-run the share of capital is constant in national income. Long-run constant factor shares (combined with capital deepening) means that asymptotically all technological change must be purely-labor-augmenting. More specifically, recall from (15.19) that

$$\frac{r(t)}{w_L(t)} = \gamma^{\frac{\varepsilon}{\sigma}} \left(\frac{N_K(t)}{N_L(t)}\right)^{\frac{\sigma-1}{\sigma}} \left(\frac{K(t)}{L}\right)^{-\frac{1}{\sigma}},$$

where $\gamma \equiv \gamma_K/\gamma_L$ and γ_K replaces γ_H in the production function (15.3). Consequently,

$$\frac{r(t)\,K(t)}{w_L(t)\,L(t)} = \gamma^{\frac{\varepsilon}{\sigma}} \left(\frac{N_K(t)}{N_L(t)}\right)^{\frac{\sigma-1}{\sigma}} \left(\frac{K(t)}{L}\right)^{\frac{\sigma-1}{\sigma}}.$$

In this case, (15.47) combined with (15.46) implies that

$$\frac{\dot{N}_L(t)}{N_L(t)} - \frac{\dot{N}_K(t)}{N_K(t)} = s_K. \tag{15.48}$$

Moreover, it can be verified that the equilibrium interest rate is given by (see Exercise 15.24):

$$r(t) = \beta\gamma_K N_K(t) \left[\gamma_L \left(\frac{N_L(t)\,L}{N_K(t)\,K(t)}\right)^{\frac{\sigma-1}{\sigma}} + \gamma_L)\right]^{\frac{1}{\sigma-1}}. \tag{15.49}$$

Let us now define a constant growth path as one in which consumption grows at a constant rate. From (15.22), this is only possible if $r(t)$ is constant and equal to some r^*. Equation (15.48) then implies that $(N_L(t)\,L)\,/\,(N_K(t)\,K(t))$ is constant, thus $N_K(t)$ must also be constant. Therefore, equation (15.48) implies that for the economy to ultimately converge to a constant growth path, long-run technological change must be purely labor-augmenting. This is summarized in the following proposition:

Proposition 15.14. *Consider the baseline model of directed technological change with the two factors corresponding to labor and capital. Suppose that the innovation possibilities frontier is given by the knowledge spillovers specification and extreme state dependence, i.e., $\delta = 1$ and that capital accumulates according to (15.46). Then there exists a constant growth path allocation in which there is only labor-augmenting technological change, the interest rate is constant and consumption and output grow at constant rates. Moreover, there cannot be any other constant growth path allocations.*

Proof. Part of the proof is provided by the argument preceding the proposition. Exercise 15.25 asks you to complete the proof and show that no other constant constant growth path allocation can exist. □

Notice that Proposition 15.14 does not imply that all technological change must be Harrod neutral (purely labor-augmenting). Along the transition path, there can be (and in fact there will be) capital-augmenting technological change. However, in the long run (that is, asymptotically or as $t \to \infty$), all technological change will be labor-augmenting.

It can also be verified that the constant growth path allocation with purely labor-augmenting technological change is globally stable if $\sigma < 1$ (see Exercise 15.26). This is reasonable, especially in view of the results in Proposition 15.7, which indicated that the stability of equilibrium dynamics in the model with the knowledge spillovers requires $\sigma < 1/\delta$. Since here we have extreme state dependence, $\delta = 1$, stability requires $\sigma < 1$. Intuitively, if capital and labor were gross substitutes ($\sigma > 1$), the equilibrium would involve rapid accumulation of capital and capital-augmenting technological change, leading to an asymptotically increasing growth rate of consumption. However, when capital and labor are gross complements ($\sigma < 1$), capital accumulation would increase the price of labor and the profits from labor-augmenting technologies. This will then encourage further labor-augmenting technological change. These strong price effects are responsible for the stability of the constant growth path allocation in Proposition 15.14. Intuitively, we can interpret this as an elasticity of substitution less than 1 inducing the economy to strive towards a balanced allocation of effective capital and labor units (where "effective" here refers to capital and labor units augmented with their complementary technologies). Since capital accumulates at a constant rate, a balanced allocation implies that the productivity of labor should increase faster, in particular, the economy should converge to an equilibrium path with purely labor-augmenting technological progress.

The results in Proposition 15.14 are important, since they provide a justification for the assumption in the Solow and neoclassical growth models that long-run technological change is purely labor-augmenting. Naturally, whether or not this is the case in practice is an empirical matter and is an interesting topic of future empirical research.

15.7. Generalizations and Other Applications

The results presented so far rely on a range of specific assumptions that are inherent in endogenous technological change models (e.g., Dixit-Stiglitz preferences and linear structure to ensure sustained growth). One may naturally wonder whether the results on weak and strong equilibrium bias generalize to situations in which these assumptions are relaxed. The answer is broadly yes. Acemoglu (2007) shows that, as long as only factor-augmenting techological changes are possible, the main results presented here also apply in an environment in which production and cost functions take more general forms. In particular, in this general environment, there is always weak (relative) equilibrium bias in response to increases in relative supplies and there will be strong equilibrium bias when the elasticity of substitution is sufficiently high. However, once we allow for a richer menu of technological changes, these results do not necessarily hold. Nevertheless, the essence of the results appears to be much more general. Acemoglu (2007) defines the complementary notions of weak and strong absolute equilibrium bias, which refer to whether the equilibrium price of a factor change as the supply of that factor changes (rather than the price of a factor relative to the price of another factor, which is what we have focused on in this chapter). Standard producer theory again suggests that an increase in the supply of a factor should reduce price. Acemoglu (2007) shows that under very weak regularity assumptions, there is always weak absolute equilibrium bias, in the sense that an increase in the supply of a factor always induces technological change biased in favor of that factor. Moreover, under plausible assumptions, this effect can be strong enough that the price of the factor that has become more abundant can increase. In this case, there is strong absolute equilibrium bias and the (general equilibrium) demand curves for factors are upward sloping. Since these results require additional notation and somewhat different arguments, I will not present them here.

It is also useful to briefly discuss a number of other important applications of the models of directed technological change. To save space, these are not discussed in the text and are left as exercises for the reader. In particular, Exercise 15.20 shows how this model can be used to shed light on the famous Habakkuk hypothesis in economic history, which relates the rapid technological progress in 19th-century United States to relative labor scarcity. Despite the importance of this hypothesis in economic history, there have been no compelling models of this process. This exercise shows why neoclassical models may have difficulties in explaining these patterns and how a model of directed technological change can account for this phenomenon as long as the elasticity of substitution is less than 1.

Exercise 15.21 shows the effects of international trade on the direction of technological change. It highlights that international trade will often affect the direction in which new technologies are developed, and this often works through the price effect emphasized above.

Exercise 15.27 returns to the discussion of the technological change and unemployment experiences of continental European countries we started with. It shows how a "wage push shock" can first increase equilibrium unemployment, and then induce endogenous capital-biased technological change, which reduces the demand for employment, further increasing unemployment.

Finally, Exercise 15.28 shows how the relative supply of factors can be endogenized, so that the two-way causality between relative supplies and relative technology can be studied.

15.8. An Alternative Approach to Labor-Augmenting Technological Change

The models presented so far in this chapter are all based on the basic directed technological change framework developed in Acemoglu (1998, 2002). Section 15.6 showed how this approach can be used to provide conditions under which technological change will be endogenously labor-augmenting (recall that this type of technological progress is necessary for balanced growth). An alternative approach to this problem is suggested in the recent paper by Jones (2005). I now briefly discussed this alternative approach.

The models developed so far treat the different types of technologies (e.g., N_L and N_H in the previous sections) as state variables. Thus short-run production functions correspond to the production possibilities sets for given state variables, while long-run production functions are derived from the production possibilities sets as the state variables themselves also adjust. Jones proposes a different approach, building on a classic paper by Houthakker (1995). Houthakker suggested that the aggregate production function should be derived as the "upper envelope" of different techniques (or "activities"). Each technique or activity corresponds to a particular way of combining capital and labor (thus to a Leontief production function of these two factors of production). However, when a producer has access to multiple ways of combining capital and labor, the resulting envelope will be different than Leontief. In a remarkable result, Houthakker showed that if the distribution of techniques is given by the Pareto distribution (which we already encountered in the previous chapter), this upper envelope will correspond to a Cobb-Douglas production function. Houthakker thus suggested a justification for Cobb-Douglas production functions based on "activity analysis".

Jones builds on and extend these insights. He argues that the long-run production function should be viewed as the upper envelope of different ideas generated over time. At a given point in time, the set of ideas that the society has access to is fixed and these ideas determine the short-run production function of the economy. In the long run, however, the society generates more ideas (either exogenously or via R&D) and the long-run production function is obtained as the upper envelope of this expanding set of ideas. Using a combination of Pareto distribution and Leontief production possibilities for a given idea, Jones shows that there will be a major difference between short-run and long-run production functions. In particular, as in Houthakker's analysis, the long-run production function will take a Cobb-Douglas form and will imply a constant share of capital in national income. However, this is not necessarily the case for short-run production functions. Then with an argument similar to that in Section 15.6, the economy will adjust from the short-run to long-run production functions by undergoing a form of labor-augmenting technological change.

I now provide a brief sketch of Jones's model, focusing on the main economic insights. As pointed out above, the key building block of Jones's model are "ideas". An idea is a technique for combining capital and labor to produce output. At any given point in time, the economy will have access to a set of ideas. Let us denote the set of possible ideas by $\mathcal{I} \subset \mathbb{R}$ and the set of ideas available at time t by $\mathcal{I}(t) \subset \mathcal{I}$. Each idea $i \in \mathcal{I}$ is represented by a vector (a_i, b_i). The essence of the model is to construct the production possibilities of the economy from the set of available ideas. To do this, we first need to specify how a given idea is used for production. Let us suppose that there is a single final good, Y, which can be produced using any idea $i \in \mathcal{I}$ with a Leontief production function given by

$$Y(t) = \min\{b_i K(t), a_i L(t)\}, \tag{15.50}$$

where $K(t)$ and $L(t)$ are the amounts of capital and labor in the economy. In general, the economy may use multiple ideas, thus $K(t)$ and $L(t)$ should be indexed by i to denote the amount of capital and labor allocated to idea i. However, I will follow Jones and assume that at any point in time the economy will use a single idea. This is a restrictive assumption, but it simplifies the model and its exposition significantly (see Exercise 15.30). The production function (15.50) makes it clear that a_i corresponds to labor-augmenting productivity of idea i and b_i is its capital-augmenting productivity.

Recall from Section 15.6 that the standard model delivers purely labor-augmenting technological change only under the (special) assumption of extreme state dependence or $\delta = 1$ (cfr. Proposition 15.14). In this model, we also have to make a special assumption, which, following Houthakker, is that ideas are independently and identically drawn from a Pareto distribution. In particular, we assume that each component of the idea, a_i and b_i, are drawn (independently) from two separate Pareto distributions. Recall from Section 14.3 in the previous chapter that y has a Pareto distribution if its distribution function is given by $G(y) = 1 - \Gamma y^{-\alpha}$ for some parameter α. The assumption that a_i and b_i are independently drawn from Pareto distributions then implies that

$$\begin{aligned}
\Pr[a_i \leq a] &= 1 - \left(\frac{a}{\gamma_a}\right)^{-\alpha}, \text{ for } a \geq \gamma_a > 0 \\
\Pr[b_i \leq b] &= 1 - \left(\frac{b}{\gamma_b}\right)^{-\beta}, \text{ for } b \geq \gamma_b > 0
\end{aligned}$$

where γ_a, γ_b, α and β are strictly positive constants, and $\alpha + \beta > 1$ (see Exercise 15.32 for the importance of this last inequality).

This Pareto assumption will play a crucial role in the result of this model. It is therefore appropriate to understand what the special features of the Pareto distribution are and why this distribution plays an important role in many different areas of economics. The Pareto distribution has two related special features. One is that its tails are relatively thick (and for this reason, the variance of a variable that is distributed Pareto is infinite, see Exercise 15.31). The second special feature is that if y has a Pareto distribution, then its expected value conditional on being greater than some y', $\mathbb{E}[y \mid y \geq y']$, is proportional to y'. Thus loosely speaking, the Pareto distribution has a certain degree of "proportionality" built into it. The expectation of something better in the future is proportional to what has been achieved today. This makes it quite convenient in the modeling of growth-related processes.

Now, given this structure, let us define the function

$$\mathbb{G}(b,a) \equiv \Pr\left[a_i \geq a \text{ and } b_i \geq b\ \right] = \left(\frac{b}{\gamma_b}\right)^{-\beta}\left(\frac{a}{\gamma_a}\right)^{-\alpha} \tag{15.51}$$

as the joint probability $a_i \geq a$ and $b_i \geq b$. Denote the level of aggregate output that can be produced using technique i with capital K and labor L by $\tilde{Y}_i(K,L)$. Before we know the realizations of a_i and b_i for idea i, this level of output is a random variable. Since the production function is Leontief, the distribution of $\tilde{Y}_i$ can be represented by the distribution function of this variable,

$$\begin{aligned} H(y) \equiv \Pr\left[\tilde{Y}_i \leq y\ \right] &= 1 - \Pr\left[a_i L \geq y \text{ and } b_i K \geq y\ \right] \\ &= 1 - \mathbb{G}\left(\frac{y}{K}, \frac{y}{L}\right) \\ &= 1 - \gamma K^{\beta} L^{\alpha} y^{-(\alpha+\beta)}, \end{aligned} \tag{15.52}$$

where the second line follows from the definition of the function $\mathbb{G}$ and the third line from (15.51) with $\gamma \equiv \gamma_a^{\alpha}\gamma_b^{\beta}$. This implies that the distribution of $\tilde{Y}_i$ is also Pareto (provided that $y \geq \min\{\gamma_b K, \gamma_a L\}$).

Let us next turn to the "global" production function, which describes the maximum amount of output that can be produced using any of the available techniques. In other words, let $\tilde{Y}(K,L) = \max_{i\in\mathcal{I}(t)} \tilde{Y}_i\left(K,L\right)$. Let $N\left(t\right)$ denote the total number of production techniques (ideas) that are available in the set $\mathcal{I}\left(t\right)$ (at time t). This equivalently implies that by time t there have been N distinct ideas that have been discovered. Since, by assumption, these N ideas are drawn independently, the "global" production function can be alternatively written as

$$\tilde{Y}\left(t;N\left(t\right)\right) = F(K\left(t\right), L\left(t\right); N\left(t\right)) \equiv \max_{i=1,\ldots,N(t)} \min\left\{b_i K\left(t\right), a_i L\left(t\right)\right\}, \tag{15.53}$$

where $\tilde{Y}\left(t;N\left(t\right)\right)$ is also a random variable. Since the realization of the N ideas are random, output at time t, $\tilde{Y}\left(t;N\left(t\right)\right)$, conditional on capital $K\left(t\right)$ and $L\left(t\right)$, is a random variable and we are interested in determining its distribution. Here the fact that the N draws of ideas are independent simplifies the analysis. In particular, the probability that the realization of $\tilde{Y}\left(t;N\left(t\right)\right)$ is less than y is equal to the probability that each of the N ideas will produce less than y. Therefore,

$$\begin{aligned} \Pr\left[\tilde{Y}\left(t;N\left(t\right)\right) \leq y\ \right] &= H(y)^{N(t)}. \\ &= \left(1 - \gamma K\left(t\right)^{\beta} L\left(t\right)^{\alpha} y^{-(\alpha+\beta)}\right)^{N(t)}. \end{aligned} \tag{15.54}$$

Equation (15.54) makes it clear that as the number of ideas N gets large, the probability that Y is less than any level of y will go to zero. This is simply a restatement of the fact that output will grow without bound, which here follows from the fact that the Pareto distribution has unbounded support. Therefore, we cannot simply determine the distribution of output as $N(t) \to \infty$. Instead, we have to look at aggregate output normalized by an appropriate variable, such as its "expected value" (and apply a type of reasoning similar to the Central Limit Theorem). Given the Pareto distribution, the normalizing factor turns out to be $n(t) \equiv \left(\gamma N(t) K(t)^{\beta} L(t)^{\alpha}\right)^{\frac{1}{\alpha+\beta}}$, so that we can write

$$\begin{aligned}
\Pr\left[\tilde{Y}(t;N(t)) \leq \left(\gamma N(t) K(t)^{\beta} L(t)^{\alpha}\right)^{\frac{1}{\alpha+\beta}} y\right] &= \left(1-\gamma K(t)^{\beta} L(t)^{\alpha} (n(t)\,y)^{-(\alpha+\beta)}\right)^{N(t)} \\
&= \left(1-\frac{y^{-(\alpha+\beta)}}{N(t)}\right)^{N(t)}, \qquad (15.55)
\end{aligned}$$

where the second line, (15.55), makes it clear that $n(t) \equiv \left(\gamma N(t) K(t)^{\beta} L(t)^{\alpha}\right)^{\frac{1}{\alpha+\beta}}$ was indeed the correct normalizing factor. Now recalling that $\lim_{N\to\infty}(1-x/N)^N = \exp(-x)$, we obtain

$$\lim_{N(t)\to\infty} \Pr\left[\tilde{Y}(t;N(t)) \leq \left(\gamma N(t) K(t)^{\beta} L(t)^{\alpha}\right)^{\frac{1}{\alpha+\beta}} y\right] = \exp(-y^{-(\alpha+\beta)}) \qquad (15.56)$$

for $y > 0$. Equation (15.56) gives the famous Fréchet distribution, which is one of the three limiting distributions for extreme values.[3] More specifically, this implies that

$$\frac{\tilde{Y}(t;N(t))}{(\gamma N(t) K(t)^{\beta} L(t)^{\alpha})^{1/\alpha+\beta}} \sim \text{Fréchet}(\alpha+\beta), \qquad (15.57)$$

so that the global production function, appropriately normalized, converges asymptotically to a Fréchet distribution. This means that as $N(t)$ becomes large (which will happen naturally as $t \to \infty$ and more ideas are discovered), the long-run or the "global" production function behaves approximately like

$$\tilde{Y}(t;N(t)) \approx \varepsilon(t)\left(\gamma N(t) K(t)^{\beta} L(t)^{\alpha}\right)^{\frac{1}{\alpha+\beta}} \qquad (15.58)$$

where $\varepsilon(t)$ is a random variable drawn from a Fréchet distribution. The intuition for this result is similar to Houthakker's result that aggregation over different units producing with techniques drawn independently from a Pareto distribution leads to a Cobb-Douglas production function The implications are quite different, however. In particular, since the long-run production function behaves approximately as Cobb-Douglas, it implies that factor shares must be constant in the long run. However, the short-run production function (for a finite number of ideas) is not Cobb-Douglas. Therefore, as $N(t)$ increases, the production function evolves endogenously towards the Cobb-Douglas limit with constant factor shares, and as in the analysis in Section 15.6, this means that technological change must ultimately become purely labor-augmenting. Therefore, Jones's model shows that ideas related to Houthakker's

[3] In particular, a basic result in statistics shows that regardless of the distribution $F(y)$, if we take N independent draws from F and look at the probability distribution of the highest draw, then as $N \to \infty$, this distribution converges to one of the following three distributions: the Weibull, the Gumbel or the Frechet. In fact, the Frechet distribution is the most common one. See, for example, Billingsley (1995, Section 14).

derivation of a static production function also imply that the short-run production function will evolve endogenously on average with labor-augmenting technological change dominating the limiting behavior and making sure that the economy, in the long run, acts as if it has a Cobb-Douglas production function.

Although this is an interesting idea, and as we have already seen in Section 14.3, the Pareto distribution appears in many important contexts and has various desirable properties, it is not clear whether it provides a compelling reason for technological change to be labor-augmenting in the long run. Labor-augmenting technological change should be an equilibrium outcome (resulting from the research and innovation incentives of firms and workers). The directed technological change models emphasized how these incentives play out under various scenarios. In the current model, the Cobb-Douglas production function arises purely from aggregation. There is no equilibrium interactions, price or market size effects. Related to this, the unit of analysis is unclear. The same argument can be applied to a single firm, to an industry, or to a region. Thus if we are happy with this argument for the economy as a whole, we may also wish to apply it to firms, industries, and regions, concluding that the long-run production function of every unit of production or every firm, industry and region should be Cobb-Douglas. However, existing evidence suggests that there are considerable differences in the production functions across industries and they can not be well-approximated by Cobb-Douglas production functions (see the overview of the evidence on industry and aggregate production functions in Acemoglu 2003a). This suggests that the potentially promising approach related to aggregation of different activities or "ideas "used in Houthakker and Jones's papers should be combined with some type of equilibrium structure, which will delineate at what level the aggregation should take place and why it may apply to (some) economies, but not necessarily to single firms or industries. This appears to be another interesting area for future research.

15.9. Taking Stock

This chapter introduced the basic models of directed technological change. These approaches differ from the endogenous technological change models of the previous two chapters because they not only determine the rate of aggregate technological change, but also endogenize the direction and bias of technological change. The bias of technological change is potentially important for the distributional consequences of the introduction of new technologies (i.e., who will be the losers and who will be the winners from technological progress?). Therefore, the bias of technological change will play an important role in our study of political economy of growth.

Equally important, models of directed technological change enable us to investigate a range of new questions. These include the sources of skill-biased technological change over the past 100 years, the causes of acceleration in skill-biased technological change during more recent decades, the causes of unskilled-biased technological developments during the 19th century, the impact of international trade on the direction of technological change, the relationship between labor market institutions and the types of technologies that are developed and adopted, and last but not least, an investigation of why technological change in neoclassical-type models may be largely labor-augmenting.

We have seen that a relatively simple class of directed technological change models can shed light on all of these questions. These models are quite tractable and allow closed-form solutions for equilibrium relative technologies and long-run growth rates. Their implications for the empirical questions mentioned above stem from two important, and perhaps at first surprising, results, which we can refer to as *weak equilibrium bias* and *strong equilibrium bias* results. The first states that under fairly weak assumptions an increase in the relative supply of a factor *always* induces endogenous changes in technology that are relatively biased towards that factor. Consequently, any increase in the ratio of skilled to unskilled workers or in the capital-labor ratio will have major implications for the relative productivity of these factors. The more surprising result is the strong equilibrium bias one, which states that contrary to basic producer theory, (relative) demand curves can slope up. In particular, if the elasticity of substitution between factors is sufficiently high, a greater relative supply of a factor causes sufficiently strong induced technological change to make the resulting relative price of the more abundant factor increase. In other words, the long run (endogenous-technology) relative demand curve becomes upward-sloping. The possibility that relative demand curves may be upward-sloping not only has a range of important empirical implications, but also illustrates the strength of endogenous technological change models, since such a result is not possible in the basic producer theory with exogenous technology.

The chapter has concluded with a number of applications of these ideas to a range of empirically important areas. Models of directed technological change are very much in their infancy and there are many theoretical dimensions in which further developments are possible. Perhaps more importantly, there are also numerous applications of these ideas.

Finally, this chapter has also been an important step in our investigation of the causes of cross-country income differences and sources of modern economic growth. Its main lesson for us is in clarifying the determinants of the nature of technological progress. Technology should not be thought of as a black box, but the outcome of decisions by firms, individuals and other agents in the economy. This implies that profit incentives will play a major role in both the aggregate rate of technological progress and also in the biases of the technologies that are being developed and adopted. Models of directed technological change illustrate this reasoning in a sharp way and show a range of its implications.

15.10. References and Literature

Models of directed technological change were developed in Acemoglu (1998, 2002a, 2003a,b, 2007a), Kiley (1999), and Acemoglu and Zilibotti (2001). These papers use the term *directed technical change*, but here we used the related term *directed technological change*, to emphasize continuity with the models of endogenous technological change studied in the previous chapters. The framework presented here builds on Acemoglu (2002a). A somewhat more general framework, with much less functional form restrictions, is presented in Acemoglu (2007a).

Other papers modeling the direction of technological change include Caselli and Coleman (2004), Xu (2001), Gancia (2003), Thoenig and Verdier (2003), Ragot (2003), Duranton (2004), Benabou (2005), and Jones (2005).

Models of directed technological change are closely related to the earlier literature on induced innovation. The induced innovation literature was started indirectly by Hicks, who in *The Theory of Wages* (1932), argued

> "A change in the relative prices of the factors of production is itself a spur to invention, and to invention of a particular kind—directed to economizing the use of a factor which has become relatively expensive." (pp. 124-5).

Hicks' reasoning, that technical change would attempt to economize on the more expensive factor, was criticized by Salter (1960) who pointed out that there was no particular reason for saving on the more expensive factor—firms would welcome all cost reductions. Moreover, the concept of "more expensive factor" did not make much sense, since all factors were supposed to be paid their marginal product. An important paper by Kennedy (1964) introduced the concept of "innovation possibilities frontier" and argued that it is the form of this frontier—rather than the shape of a given neoclassical production function—that determines the factor distribution of income. Kennedy, furthermore, argued that induced innovations would push the economy to an equilibrium with a constant relative factor share (see also Samuelson, 1965, and Drandakis and Phelps, 1965). Around the same time, Habakkuk (1962) published his important treatise, *American and British Technology in the Nineteenth Century: the Search for Labor Saving Inventions*, where he argued that labor scarcity and the search for labor saving inventions were central determinants of technological progress. The flavor of Habakkuk's argument was one of induced innovations: labor scarcity increased wages, which in turn encouraged labor-saving technical change. Nevertheless, neither Habakkuk nor the induced innovation provided any micro-founded model of technological change or technology adoption. For example, in Kennedy's specification the production function at the firm level exhibited increasing returns to scale because, in addition to factor quantities, firms could choose "technology quantities," but this increasing returns to scale was not taken into account in the analysis. Similar problems are present in the other earlier works as well. It was also not clear who undertook the R&D activities and how they were financed and priced. These shortcomings reduced the interest in this literature, and there was little research for almost 30 years, with the exception of some empirical work, such as that by Hayami and Ruttan (1970) on technical change in American and Japanese agriculture.

The analysis in Acemoglu (1998) and the subsequent work in this area, instead, starts from the explicit micro-foundations of the endogenous technological change models discussed in the previous two chapters. The presence of monopolistic competition avoids the problems that the induced innovations literature had with increasing returns to scale.

Acemoglu (2002 and 2003b) show that the specific way in which endogenous technological change is modeled does not affect the major results on the direction of technological change. This is also illustrated in Exercises 15.19 and 15.29. In addition, even though the focus here has been on technological progress, Acemoglu (2007a) shows that all of the results generalize to models of technology adoption as well. Acemoglu (2007a) also introduces the alternative concept of *weak absolute bias* and *strong absolute bias*, which look at the marginal product of a factor rather than relative marginal product. He shows that there are even more general theorems on weak and strong absolute bias. I sometimes referred to weak relative bias and strong relative bias to distinguish the results here from the absolute bias results. The results in Acemoglu (2007a) also show that the constant elasticity of substitution aggregator of the output of the two sectors is unnecessary for the results. Nevertheless, I have kept the constant elasticity of substitution structure to simplify the exposition.

Changes in the US wage inequality over the past 60 years are surveyed in Katz and Autor (1999), Autor, Katz and Krueger (1998) and Acemoglu (2002b). The latter paper also discusses how models on directed technological change can provide a good explanation for changes in wage inequality over the past 100 years and also changes in the direction of technological change in the US and UK economies over the past 200 years. There are many studies estimating the elasticity of substitution between skilled and unskilled workers. The estimates are typically between 1.4 and 2. See, for example, Katz and Murphy (1992), Krusell, Ohanian, Rios-Rull and Violante (1999), and Angrist (1995). A number of estimates are summarized and discussed in Hamermesh (1993) and Acemoglu (2002b).

Evidence that 19th century technologies were generally labor-complementary (unskilled-biased) is provided by, among others, James and Skinner (1985) and Mokyr (1990), while Goldin and Katz (1998) argued the same for a range of important early 20th century technologies.

Blanchard (1997) discusses the persistence of European unemployment and argues that the phase during the 1990s can only be understood by changes in technology reducing demand for high-cost labor. This is the basis of Exercise 15.27 below. Caballero and Hammour (1999) provide an alternative and complementary explanation to that suggested here.

Acemoglu and Zilibotti (2001) discuss implications of directed technological change for cross-country income differences. We have not dwelled on this topic here, since this will be discussed in greater detail in Chapter 18.4 in the context of appropriate technologies.

Acemoglu (2003b) suggested that increased international trade can cause endogenous skill-biased technological change. Exercise 15.21 is based on this idea. Variants of this story have been developed by Xu (2001), Gancia (2003), Thoenig and Verdier (2003).

The model of long-run purely labor-augmenting technological change presented in Section 15.6 was first proposed in Acemoglu (2003a), and the model presented here is a simplified version of the model in the paper. Similar ideas were discussed informally in Kennedy (1964) and a recent paper by Funk (2002) provides microfoundations for the ideas by Kennedy. Section 15.8 builds on Jones (2005), which in turn uses ideas that first appeared in Houthakker (1955). A related paper to Jones (2005) is Lagos (2001), who also uses an approach similar to Houthakker's (1955) and shows how aggregation of productivity across heterogeneous production units can determine aggregate total factor productivity in the economy.

The assumption that the elasticity of substitution between capital and labor is less than 1 receives support from a variety of different empirical strategies. The evidence is summarized in Acemoglu (2003a).

The Habakkuk hypothesis has been widely debated in the economic history literature. It was first formulated by Habakkuk (1962), though Rothbarth (1946) had anticipated these ideas almost two decades earlier. David (1975) contains a detailed discussion of the Habakkuk hypothesis and potential theoretical explanations. Recent work by Allen (2005) argues for the importance of the Habakkuk hypothesis for understanding the British Industrial Revolution. Exercise 15.20 shows how models of directed technological change can clarify the conditions necessary for this hypothesis to apply.

15.11. Exercises

Exercise 15.1. Derive equation (15.1).

Exercise 15.2. Complete the proof of Proposition 15.1. In particular, verify that in any BGP, (15.27) must hold and derive the equilibrium growth rate as given by (15.29). Also prove that (15.28) ensures that the two free entry conditions (15.20) and (15.21) must hold as equalities. Finally, check that this condition is also sufficient to guarantee that the transversality condition is satisfied. [Hint: calculate the equilibrium interest rate and then use (15.22)].

Exercise 15.3. Prove Proposition 15.2. [Hint: use (15.9) to show that when $N_H(0)/N_L(0)$ does not satisfy (15.27), (15.20) and (15.21) cannot both hold as equalities].

Exercise 15.4. Derive equation 15.30.

Exercise 15.5. Explain why in Proposition 15.1 the effect of γ on the BGP growth rate, (15.29), is ambiguous. When is this effect positive? Provide an intuition.

Exercise 15.6. Derive equation 15.31.

Exercise 15.7. Prove Proposition 15.5. [Hint: first substitute for $C(t)$ from the constraint. Then show that $\mu_H(t)/\mu_L(t) = (\eta_H(t)/\eta_L(t))^{-1}$. Then use the necessary conditions with $\dot{\mu}_H(t) = \dot{\mu}_L(t)$].

Exercise 15.8. Derive the free entry conditions (15.34) and (15.35). Provide an intuition for these conditions.

Exercise 15.9. Derive equation (15.37).

Exercise 15.10. Prove Proposition 15.6. In particular, check that there is a unique BGP and that the BGP growth rate satisfies the transversality condition.

Exercise 15.11. In the model of Section 15.4, show that an increase in η_H will raise the number of scientists working in H-augmenting technologies in the BGP, S_H^*, when $\sigma > 1$ (and $\sigma < 1/\delta$) and reduce it when $\sigma < 1$. Interpret this result.

Exercise 15.12.

(1) Prove Proposition 15.7. In particular, use equation (15.9) and show that when (15.37) is not satisfied, both free entry conditions cannot hold simultaneously. Then show that if $\sigma < 1/\delta$, there will be greater incentives to undertake research for the technology that is relatively scarce, and the opposite holds when $\sigma > 1/\delta$.

(2) Interpret the economic significance of the condition $\sigma < 1/\delta$. [Hint: relate this to the fact that When $\sigma < 1/\delta$, $\partial\left(N_H^\delta V_H/N_L^\delta V_L\right)/\partial\left(N_H/N_L\right) < 0$, but the inequality is reversed when $\sigma > 1/\delta$].

Exercise 15.13. Prove Proposition 15.8.

Exercise 15.14. Characterize the Pareto optimal allocation in the model with knowledge spillovers and state dependence (Section 15.4). Show that the relative technology ratio in the stationary Pareto optimal allocation no longer coincides with the BGP equilibrium. Explain why this result differs from that in Section 15.3.

Exercise 15.15. Derive equation (15.42).

Exercise 15.16. Show that in the model of Section 15.5, if $\lambda = 1$, there exists no BGP.

Exercise 15.17. Derive equations (15.43) and (15.44).

Exercise 15.18. Generalize the model of Section 15.4 so that there are no scientists and the R&D sector also uses workers. Thus the labor market clearing condition is

$$L^E(t) + L_L^R(t) + L_H^R(t) \leq 0,$$

where $L^E(t)$ is employment in the final good sector and $L_L^R(t)$ and $L_H^R(t)$ denote the employment in the two R&D sectors.

(1) Define an equilibrium in this economy.

(2) Specify the free entry conditions for each machine variety.
(3) Characterize the BGP equilibrium, show that it is uniquely defined and determine conditions such that the growth rate is positive and the transversality condition is satisfied.
(4) Show that the equivalents of Propositions 15.3 and 15.4 hold in this environment.
(5) Characterize the transitional dynamics and show that they are similar to those in Proposition 15.2.
(6) Characterize the Pareto optimal allocation in this economy and show that the Pareto optimal ratio of technologies in the stationary equilibrium are also given by (15.27).

Exercise 15.19. Consider a version of the baseline directed technological change model introduced above with the only difference that technological change is driven by quality improvements rather than expanding machine varieties. In particular, let us suppose that the intermediate goods are produced with the production functions:

$$\begin{aligned} Y_L(t) &= \frac{1}{1-\beta}\left[\int_0^1 q_L(\nu,t)x_L(\nu,t\mid q)^{1-\beta}d\nu\right]L^\beta, \text{ and} \\ Y_H(t) &= \frac{1}{1-\beta}\left[\int_0^1 q_H(\nu,t)x_H(\nu,t\mid q)^{1-\beta}d\nu\right]H^\beta. \end{aligned}$$

Producing a machine of quality q costs ψq, where we again normalize $\psi \equiv 1-\beta$. R&D of amount $Z_f(\nu,t)$ directed at a particular machine of quality $q_f(\nu,t)$ leads to an innovation at the flow rate $\eta_f Z_f(\nu,t)/q_f(\nu,t)$ and leads to an improved machine of quality $\lambda q_f(\nu,t)$, where $f = L$ or H, and $\lambda \geq (1-\beta)^{-(1-\beta)/\beta}$, so that firms that undertake an innovation can charge the unconstrained monopoly price.

(1) Define an equilibrium in this economy.
(2) Specify the free entry conditions for each machine variety.
(3) Characterize the BGP equilibrium, show that it is uniquely defined and determine conditions such that the growth rate is positive and the transversality condition is satisfied.
(4) Show that the relative technologies in the BGP equilibrium are given by (15.27).
(5) Show that the equivalents of Propositions 15.3 and 15.4 hold in this environment.
(6) Characterize the transitional dynamics and show that they are similar to those in Proposition 15.2.
(7) Characterize the Pareto optimal allocation in this economy and show that the Pareto optimal ratio of technologies in the stationary equilibrium are also given by (15.27).
(8) What are the pros and cons of this model relative to the baseline model studied in Section 15.3.

Exercise 15.20. As a potential application of the models of directed technological change, consider the famous Habakkuk hypothesis, which claims that technology adoption in the U.S.

economy during the 19th century was faster than in Britain because of relative labor scarcity in the former (which increased wages and encouraged technology adoption).

(1) First, consider a neoclassical-type model with two factors, labor and technology, $F(A, L)$, where F exhibits constant returns to scale. Show that an increase in wages, either caused by a decline in labor supply or an exogenous increase in wages because of the minimum wage, cannot increase A.
(2) Next, consider the model here with H interpreted as land and assume that N_H is fixed (so that there is only R&D for increasing N_L). Show that if $\sigma > 1$, the opposite of the Habakkuk hypothesis obtains. If in contrast, $\sigma < 1$, the model delivers results consistent with the Habakkuk hypothesis. Interpret this result and explain why the implications are different from the neoclassical model considered in 1 above.

Exercise 15.21. Consider the baseline model of directed technological change in Section 15.3 and assume that it is in steady state.

(1) Show that in steady state the relative price of the two intermediate goods, p, is proportional to $(H/L)^{\beta}$.
(2) Now assume that the economy opens up to world trade, and faces a relative price of intermediate goods $p' < p$. Derive the implications of this for the endogenous changes in technology. Explain why the results are different from those in the text. [Hint: relate your results to the price effect].

Exercise 15.22. (1) Prove Proposition 15.13. In particular, show that in any BGP equilibrium (15.37) must hold, and that this equation is inconsistent with capital accumulation.
(2) * Prove that there exists no equilibrium allocation in which consumption grows at the constant rate. [Hint: show that a relationship similar to (15.37) must hold, and this will lead to an increase in $N_K(t)$, which then implies that the interest rate cannot be constant].

Exercise 15.23. Derive equation (15.47).

Exercise 15.24. Derive equation (15.49).

Exercise 15.25. Complete the proof of Proposition 15.14 and show that there cannot exist any other constant growth path equilibrium.

Exercise 15.26. * Show that if $\sigma < 1$, the constant growth path equilibrium in Proposition 15.14 is globally stable. Show that if $\sigma > 1$, it is unstable. Relate your results to Proposition 15.7.

Exercise 15.27. Now let us use the results of Proposition 15.14 to revisit the discussion of the experiences of continental European economies provided in Blanchard (1997). Consider the model of Section 15.6. Discuss how a wage push, in the form of a wage floor above the market clearing level will first cause unemployment and then if $\sigma < 1$, it will cause capital-biased technological change. Can this model shed light on the persistent unemployment dynamics in continental Europe? [Hint: distinguish two cases: (i) the minimum wage floor is constant; (ii) the minimum wage floor increases at the same rate as the growth of the economy].

EXERCISE 15.28. * The analysis in the text has treated the supply of the two factors as exogenous and looked at the impact of relative supplies on factor prices. Clearly, factor prices can also affect relative supplies. In this exercise, we look at the joint determination of relative supplies and technologies.

Let us focus on a model with the two factors corresponding to skilled and unskilled labor. Suppose a continuum υ of unskilled agents are born every period, and each faces a flow rate of death equal to υ, so that population is constant at 1 (as in Section 9.8 above). Each agent chooses upon birth whether to acquire education to become a skilled worker. For agent x it takes K_x periods to become skilled, and during this time, he earns no labor income. The distribution of K_x is given by the distribution function $\Gamma(K)$ which is the only source of heterogeneity in this economy. The rest of the setup is the same as in the text. Suppose that $\Gamma(K)$ has no mass points. Define a BGP as a situation in which H/L and the skill premium remain constant.

(1) Show first that in BGP, there is a single-crossing property: if an individual with cost of education K_x chooses schooling, another with $K_{x'} < K_x$ must also acquire skills. Conclude from this that there exists a cutoff level of talent, $\bar{K}$, such that all $K_x > \bar{K}$ do not get education.
(2) Show that, along BGP relative supplies take the form:
$$\frac{H}{L} = \frac{\Gamma(\bar{K})}{1 - \Gamma(\bar{K})}.$$
Explain why such a simple expression would not hold away from the BGP.
(3) How would you determine $\bar{K}$? [Hint: agent with talent $\bar{K}$ has to be indifferent between acquiring skills and not].

Show that the relative supply of skills as a function of the skill premium must satisfy
$$\frac{H}{L} = \frac{\Gamma\left(\ln \omega / \left(r^* + \upsilon - g^*\right)\right)}{1 - \Gamma\left(\ln \omega / \left(r^* + \upsilon - g^*\right)\right)},$$
where r^* and g^* refer to the BGP interest-rate and growth rate.
(4) Determine the BGP skill premium by combining this equation with (15.27) and (15.30). Can there be multiple equilibria? Explain the intuition.

EXERCISE 15.29. * Consider an economy with a constant population and risk neutral consumers discounting the future at the rate r. Utility is defined over the final good, which is produced as

$$Y(t) = \left[\int_0^n y(\nu,t)^{\frac{\varepsilon-1}{\varepsilon}} d\nu\right]^{\frac{\varepsilon}{\varepsilon-1}}$$

where $\varepsilon > 1$ and intermediate $y(\nu,t)$ can be produced using either skilled or unskilled labor. In particular, when a new intermediate is invented, it is first produced using skilled labor only, with the production function $y(\nu,t) = h(\nu,t)$, and eventually, another firm may find a way to produce this good using unskilled labor with the production function $y(\nu,t) = l(\nu,t)$. Assume that when there exist n goods in the economy and m goods can be produced using unskilled labor, we have

$$\dot{n}(t) = b_n X_n(t) \text{ and } \dot{m}(t) = b_m X_m(t)$$

where $X_n(t)$ and $X_m(t)$ are expenditures on R&D to invent new goods and to transform existing goods to be produced by unskilled labor. A firm that invents a new good becomes the monopolist producer, but can be displaced by a new monopolist who finds a way of producing the good using unskilled labor.

(1) Denote the unskilled wage by $w(t)$ and the skilled wage by $v(t)$. Show that, as long as $v(t)$ is sufficiently larger than $w(t)$, the instantaneous profits of a monopolist producing skill-intensive and labor-intensive goods are

$$\pi_h(t) = \frac{1}{\varepsilon-1}\frac{v(t)H}{n(t)-m(t)} \text{ and } \pi_l(t) = \frac{1}{\varepsilon-1}\frac{w(t)L}{m(t)}$$

where L is the total supply of unskilled labor and H is the total supply of skilled labor. Interpret these equations. Why is the condition that $v(t)$ is sufficiently larger than $w(t)$ necessary?

(2) Define a balanced growth path as an allocation where n and m grow at the same rate g (and output and wages grow at the rate $g/(\varepsilon-1)$). Assume moreover that a firm that undertakes R&D to replace the skill-intensive good has an equal probability of replacing any of the existing $n-m$ skill-intensive goods. Show that the balanced growth path has to satisfy the following condition

$$\frac{vH}{(1-\mu)(r+\lambda-(1-\mu)\lambda/\mu)} = \frac{wL}{(r-(1-\mu)\lambda/\mu)\mu}$$

where $\mu \equiv m/n$ and $\lambda \equiv \dot{m}/(n-m) = g\mu/(1-\mu)$. [Hint: Note that a monopolist producing a labor-intensive good will never be replaced, and its profits will grow at the rate g (because equilibrium wages are growing). A monopolist producing a skill-intensive good faces a constant flow rate of being replaced, and while it survives, its profits grow at the rate g.]

(3) Using consumer demands over varieties (i.e., the fact that $y(\nu,t)/y(\nu',t) = (p(\nu,t)/p(\nu',t))^{-1/\varepsilon}$), characterize the balanced growth path level of μ. What is the effect of an increase in H/L on μ? Interpret.

EXERCISE 15.30. Consider the model presented in Section 15.8.

(1) Show that if capital and labor are allocated in competitive markets, in general more than one technique will be used in equilibrium. [Hint: construct an example in which there are three ideas $i = 1, 2$ and 3, such that when only one can be used, it will be $i = 1$, but output can be increased by allocating some of labor and capital to ideas 2 and 3].
(2) * Show that in this case the exact aggregation result used in Section 15.8 does not apply.

EXERCISE 15.31. Suppose that y has a Pareto distribution given by $G(y) = 1 - By^{-\alpha}$. Show that the variance of y is infinite.

EXERCISE 15.32. Suppose that y has a Pareto distribution given by $G(y) = 1 - By^{-\alpha}$. Show that

$$\mathbb{E}\left[y \mid y \geq y'\right] = \frac{\alpha}{\alpha - 1} \frac{y'}{C^{-\alpha}}.$$

What happens if $\alpha < 1$?

Introduction to Modern Economic Growth: Parts 6-9

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Contents

Part 5

Stochastic Growth

This part of the book focuses on stochastic growth models and provides a brief introduction to basic tools of stochastic dynamic optimization. Stochastic growth models are useful for two related reasons. First, a range of interesting growth problems involve either aggregate uncertainty or nontrivial individual level uncertainty interacting with investment decisions and the growth process. Some of these models will be discussed in Chapter 17. Second, the stochastic neoclassical growth model has a wide range of applications in macroeconomics and in other areas of dynamic economic analysis. Various aspects of the stochastic neoclassical growth model will be discussed in the next two chapters. The study of stochastic models requires us to extend the dynamic optimization tools of Chapters 6 and 7 to an environment in which either returns or constraints are uncertain (governed by probability distributions).[4] Unfortunately, dynamic optimization under uncertainty is considerably harder than the non-stochastic optimization. The generalization of continuous-time methods to stochastic optimization requires fairly advanced tools from measure theory and stochastic differential equations. While continuous-time stochastic optimization methods are very powerful, they are not used widely in macroeconomics and economic growth, so I have decided to focus on discrete-time stochastic models. Thus the next chapter will include the most straightforward generalization of the discrete-time dynamic programming techniques presented in Chapter 6 to stochastic environments. Unfortunately, a rigorous development of stochastic dynamic programming also requires further mathematical investment than is typically necessary in most macroeconomics and economic growth courses. To avoid a heavy dose of new mathematical tools, in particular a lengthy detour into measure theory at this stage of the book, the next chapter develops the basics of stochastic dynamic programming without measure theory. I will then include a few pointers about how the results in this chapter can be extended and made more rigorous.

[4]Throughout, I do not draw a distinction between risk and uncertainty along the lines of the work by Frank Knight, who identified risk with situations in which there is a known probability distribution of events and uncertainty with situations in which such a probability distribution cannot be specified. While "Knightian uncertainty" may be important in a range of situations, given the set of models being studied here there is little cost of following the standard practice of using the word "uncertainty" interchangeably with "risk".

CHAPTER 16

Stochastic Dynamic Programming

This chapter provides an introduction to basic stochastic dynamic programming. To avoid the use of measure theory in the main body of the text, I will first focus on economies in which stochastic variables take finitely many values. This will enable us to use Markov chains, instead of general Markov processes, to represent uncertainty. Since many commonly-used stochastic processes, such as those based on normal or uniform distributions, fall outside this class, I will then indicate how the results can be generalized to situations in which stochastic variables can be represented by continuous, or mixture of continuous and discrete, random variables. Throughout my purpose is to provide a basic understanding of the tools of stochastic dynamic programming and how they can be used in dynamic macroeconomic models. For this reason, I will make a number of judicious choices rather than attempting to provide the most general results.

16.1. Dynamic Programming with Expectations

We use a notation similar to that in Chapter 6. Let us first introduce the *stochastic* (random) variable $z(t) \in \mathcal{Z} \equiv \{z_1, ..., z_N\}$. Note that the set $\mathcal{Z}$ is finite and thus compact, which will simplify the analysis considerably. Let the instantaneous payoff at time t be $U(x(t), x(t+1), z(t))$, where $x(t) \in X \subset \mathbb{R}^K$ for some $K \geq 1$ and $U : X \times X \times \mathcal{Z} \to \mathbb{R}$. This extends the payoff function in Chapter 6, which took the form $U(x(t), x(t+1))$, by making payoffs directly a function of the stochastic variable $z(t)$. As usual, returns will be discounted by some discount factor $\beta \in (0, 1)$. The initial value $x(0)$ is given. We will again think of $x(t)$ as the *state variable* (state vector) and of $x(t+1)$ as the *control variable* (control vector) at time t.

An additional difference from Problem A1 in Chapter 6 is that the constraint on $x(t+1)$ is no longer of the form $x(t+1) \in G(x(t))$. Instead, the constraint also incorporates the stochastic variable $z(t)$ and is written as

$$x(t+1) \in G(x(t), z(t)),$$

where again $G(x, z)$ is a set-valued mapping or a correspondence

$$G : X \times \mathcal{Z} \rightrightarrows X.$$

We assume that the stochastic variable $z(t)$ follows a (first-order) *Markov chain*.[1] The important property implied by the Markov chain assumption is that the current value of $z(t)$ only depends on its last period value, $z(t-1)$. Mathematically, this can be expressed as

$$\Pr\left[z(t)=z_j \mid z(0),...,z(t-1)\right] \equiv \Pr\left[z(t)=z_j \mid z(t-1)\right].$$

The simplest example of an economic model with uncertainty represented by a Markov chain would be one in which the stochastic variable takes finitely many values and is independently distributed over time. In this case, clearly, $\Pr\left[z(t)=z_j \mid z(0),...,z(t-1)\right]=\Pr\left[z(t)=z_j\right]$ and the Markov property is trivially satisfied. More generally, however, Markov chains enable us to model economic environments in which stochastic shocks are correlated over time. Markov chains are widely used in the theory of probability, in research in stochastic processes and in various areas of dynamic economic analysis. While the theory of Markov chains is relatively straightforward, not much of this theory is necessary for the basic treatment of stochastic dynamic programming here.

The Markov property not only simplifies the mathematical structure of economic models but also allows us to use relatively simple notation for the probability distribution of the random variable $z(t)$. We can also represent a Markov chain as

$$\Pr\left[z(t)=z_j \mid z(t-1)=z_{j'}\right] \equiv q_{jj'},$$

for any any $j, j'=1,...,N$, where $q_{jj'}\geq 0$ for all j, j' and

$$\sum_{j=1}^{N} q_{jj'}=1 \text{ for each } j'=1,...,N.$$

Here $q_{jj'}$ is also referred to as a *transition probability*, meaning the probability of the stochastic state z transitioning from $z_{j'}$ to z_j. I will make use of this notation in some of the proofs in the next section.

To see how this particular way of introducing stochastic elements into dynamic optimization is useful in economic problems, let us start with a simple example, which is also useful for introducing some additional notation.

Example 16.1. Recall the optimal growth problem, where the objective is to maximize

$$\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t u(c(t)).$$

As usual, $c(t)$ denotes per capita consumption at time t and $u(\cdot)$ is the instantaneous utility function. The maximand in this problem differs from those we have seen so far only because of the presence of the expectations operator, $\mathbb{E}_0$, which stands for expectations conditional on information available at time $t=0$. We take expectations here because the future values

[1]We adopt the standard terminology that $z(t)$ follows a Markov chain when it takes finitely many values and that it follows a general Markov process when it has a continuous distribution or a mixture of the continuous and discrete distribution.

of consumption per capita is stochastic (as they will depend on the realization of future z's). In particular, suppose that the production function (per capita) takes the form

$$y(t) = f(k(t), z(t)),$$

where $k(t)$ again denotes the capital-labor ratio and $z(t) \in \mathcal{Z} \equiv \{z_1, ..., z_N\}$ represents a stochastic variable that affects how much output will be produced with a given amount of inputs. We continue to assume that $z(t)$ follows a Markov chain. The most natural interpretation of $z(t)$ in this context is as a stochastic TFP term, so one might be tempted to write $y(t) = z(t) f(k(t))$ and in the next chapter I will sometimes impose this form, but there is no mathematical or economic gain from doing so here. Consequently, the constraint facing the maximization problem at time t takes the form

$$k(t+1) = f(k(t), z(t)) + (1-\delta) k(t) - c(t), \tag{16.1}$$

$k(t) \geq 0$ and given $k(0)$, with δ again representing the depreciation rate. This formulation implies that at the time consumption $c(t)$ is chosen, the random variable $z(t)$ has been realized, thus $c(t)$ is a random variable depending on the realization of $z(t)$. In fact, more generally, $c(t)$ may depend on the entire history of the random variables. For this reason, let us define

$$z^t \equiv (z(0), z(1), ...z(t))$$

as the *history* of variable $z(t)$ up to date t. Let $\mathcal{Z}^t \equiv \mathcal{Z} \times ... \times \mathcal{Z}$ (the t-times product), so that $z^t \in \mathcal{Z}^t$. For given $k(0)$, the level of consumption at time t can be most generally written as

$$c(t) = \tilde{c}\left[z^t\right],$$

which simply states that consumption at time t will be a function of the entire sequence of random variables observed up to that point. Clearly, consumption at time t cannot depend on future realizations of the random variable—those values have not been realized yet. Therefore, a consumption plan that depends on future realizations of the stochastic variable z would not be feasible. A function of the form $c(t) = \tilde{c}\left[z^t\right]$ is thus natural. Nevertheless, not all functions $\tilde{c}\left[z^t\right]$ could be admissible as feasible plans, because they may violate the resource constraint. We will shortly return to additional restrictions to ensure feasibility. There is also no point in making consumption a function of the history of capital stocks at this stage, since those are endogenously determined by the choice of past consumption levels and by the realization of past stochastic variables. (When we turn to the recursive formulation of this problem, we will write consumption as a function of the current capital stock and the current value of the stochastic variable). Let $x(t) = k(t)$, so that

$$\begin{aligned} x(t+1) &= k(t+1) \\ &= f(k(t), z(t)) + (1-\delta) k(t) - \tilde{c}\left[z^t\right] \\ &\equiv \tilde{k}\left[z^t\right], \end{aligned}$$

where the second line simply uses the resource constraint with equality and the third line defines the function $\tilde{k}\left[z^t\right]$. With this notation, feasibility is easier to express, since

$$k(t+1) \equiv \tilde{k}\left[z^t\right]$$

by definition depends only on the history of the stochastic shocks up to time t and not on $z(t+1)$. In addition, feasibility requires that the function $\tilde{k}[\cdot]$ satisfies

$$\tilde{k}\left[z^t\right] \leq f(\tilde{k}\left[z^{t-1}\right], z(t)) + (1-\delta)\tilde{k}\left[z^{t-1}\right] \text{ for all } z^{t-1} \in \mathcal{Z}^{t-1} \text{ and } z(t) \in \mathcal{Z}.$$

The maximization problem can then be expressed as

$$\max_{\left\{\tilde{c}[z^t], \tilde{k}[z^t]\right\}_{t=0}^{\infty}} \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t u\left(\tilde{c}\left[z^t\right]\right) \mid z(0)\right]$$

subject to the constraint

$$\tilde{k}\left[z^t\right] \leq f(\tilde{k}\left[z^{t-1}\right], z(t)) + (1-\delta)\tilde{k}\left[z^{t-1}\right] - \tilde{c}\left[z^t\right] \text{ for all } z^{t-1} \in \mathcal{Z}^{t-1} \text{ and } z(t) \in \mathcal{Z},$$

and starting with the initial conditions $\tilde{k}\left[z^{-1}\right] = k(0)$ and $z(0)$. We can also write this maximization problem using the instantaneous payoff function $U(x(t), x(t+1), z(t))$ introduced above. In this case, the maximization problem would take the form

$$\max_{\left\{\tilde{k}[z^t]\right\}_{t=0}^{\infty}} \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t U\left(\tilde{k}\left[z^{t-1}\right], \tilde{k}\left[z^t\right], z(t)\right),$$

where now $U(x(t), x(t+1), z(t)) = u(f(k(t), z(t)) - k(t+1) + (1-\delta) k(t))$. Notice the timing convention here: $\tilde{k}\left[z^{t-1}\right]$ is the value of the capital stock at time t, which is inherited from the investments at time $t-1$ and thus depends on the history of stochastic shocks up to time $t-1$, z^{t-1}, whereas $\tilde{k}\left[z^t\right]$ is the choice of capital stock for next period made at time t given the history of stochastic shocks up to time t, z^t.

This example can also be used to give us a first glimpse of how to express the same maximization problem recursively. Since $z(t)$ follows a Markov chain, the current value of $z(t)$ contains both the information about the available resources for consumption and future capital stock and the information regarding the stochastic distribution of $z(t+1)$. Thus we might naturally expect the policy function determining the capital stock at the next date to take the form

$$k(t+1) = \pi(k(t), z(t)). \tag{16.2}$$

With the same reasoning, the recursive characterization would naturally take the form

$$V(k,z) = \sup_{y \in [0, f(k,z)+(1-\delta)k]} \left\{ u\left(f(k,z) + (1-\delta)k - y\right) + \beta \mathbb{E}\left[V\left(y, z'\right) \mid z\right] \right\}, \tag{16.3}$$

where $\mathbb{E}[\cdot \mid z]$ denotes the expectation conditional on the current value of z and incorporates the fact that the random variable z is a Markov chain. Let us suppose that this program has a solution, meaning that there exists a feasible plan that achieves the value $V(k,z)$ starting with capital-labor ratio k and stochastic variable z. Then the set of the next date's capital stock that achieve this maximum value can be represented by a correspondence $\Pi(k,z) \subset X$ for each $k \in \mathbb{R}_+$ and $z \in \mathcal{Z}$. For any $\pi(k,z) \in \Pi(k,z)$,

$$V(k,z) = u\left(f(k,z) + (1-\delta)k - \pi(k,z)\right) + \beta \mathbb{E}\left[V\left(\pi(k,z), z'\right) \mid z\right].$$

When the correspondence $\Pi(k,z)$ is single valued, then $\pi(k,z)$ would be uniquely defined and the optimal choice of next period's capital stock can be represented as in (16.2).

Example 16.1 already indicates how we could write a stochastic optimization problem in a sequential form and also gives us a hint about how to express such a problem recursively. We now do this more systematically. Let a *plan* be denoted by $\tilde{x}\left[z^t\right]$. This plan specifies the value of the vector $x \in \mathbb{R}^K$ for time $t+1$, i.e., $x(t+1) = \tilde{x}\left[z^t\right]$, for any $z^t \in \mathcal{Z}^t$. Using the same notation as in Chapter 6, the sequence problem takes the form

Problem B1 :

$$\begin{aligned} V^*\left(x(0), z(0)\right) &= \sup_{\left\{\tilde{x}[z^t]\right\}_{t=-1}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U\left(\tilde{x}\left[z^{t-1}\right], \tilde{x}\left[z^t\right], z(t)\right) \\ &\quad \text{subject to} \\ \tilde{x}\left[z^t\right] &\in G(\tilde{x}\left[z^{t-1}\right], z(t)), \qquad \text{for all } t \geq 0 \\ &\quad \tilde{x}\left[z^{-1}\right] = x(0) \text{ given,} \end{aligned}$$

where expectations at time $t=0$, denoted by $\mathbb{E}_0$, are taken over the possible infinite sequences of $(z(0), z(1), z(2), z(3), ...)$. In this problem, as in the rest of this and the next chapter, we adopt the convention that $\tilde{x}\left[z^{-1}\right] = x(0)$ and write the maximization problem with respect to the sequence $\left\{\tilde{x}\left[z^t\right]\right\}_{t=-1}^{\infty}$ (which starts at $t=-1$ and the value $\tilde{x}\left[z^{-1}\right] = x(0)$ is introduced as an additional constraint). The function V^* is conditioned on $x(0) \in \mathbb{R}^K$, since this is the initial value of the vector x, taken as given, and also on $z(0)$, since the choice of $x(1)$ is made after $z(0)$ is observed. Finally, the first constraint in Problem B1 ensures that the sequence $\left\{\tilde{x}\left[z^t\right]\right\}_{t=-1}^{\infty}$ is feasible.

Similar to (16.3) in Example 16.1, the functional equation corresponding to the recursive formulation of this problem can be written as:

Problem B2 :

$$V(x,z) = \sup_{y \in G(x,z)} \left\{ U(x,y,z) + \beta \mathbb{E}\left[V(y,z') \mid z \right] \right\}, \text{ for all } x \in X \text{ and } z \in \mathcal{Z} \tag{16.4}$$

where $V : X \times \mathcal{Z} \to \mathbb{R}$ is a real-valued function and $y \in G(x,z)$ represents the constraint on next period's state vector as a function of the realization of the stochastic variable z. Problem B2 is a direct generalization of the Bellman equation in Problem A2 of Chapter 6 to a stochastic dynamic programming setup. One can also write Problem B2 as

$$V(x,z) = \sup_{y \in G(x,z)} \left\{ U(x,y,z) + \beta \int V(y,z') Q\left(z, dz'\right) \right\}, \text{ for all } x \in X \text{ and } z \in \mathcal{Z},$$

where $\int f(z') Q(z_0, dz')$ denotes the Lebesgue integral of the function f with respect to the Markov process for z given last period's value of z as z_0. This notation is useful in emphasizing that an expectation is nothing but a Lebesgue integral (and thus contains regular summation as a special case). Remembering the equivalence between expectations and integrals is important both for a proper appreciation of the theory and also for recognizing where some of the difficulties in the use of stochastic methods may lie.[2] There is typically little gain in rigor or insight in using the explicit Lebesgue integral instead of the expectation and I will not do so unless absolutely necessary.

As in Chapter 6, we would like to establish conditions under which the solutions to Problems B1 and B2 coincide. Let us first introduce the set of feasible *plans* starting with an initial value $x(t)$ and a value of the stochastic variable $z(t)$ as

$$\mathbf{\Phi}(x(t), z(t)) = \{\{\tilde{x}[z^s]\}_{s=t-1}^{\infty} : \tilde{x}[z^s] \in G(\tilde{x}\left(z^{s-1}\right), z(s)), \text{ for } s = t-1, t, t+1, ...\}.$$

We denote a generic element of $\Phi(x(0), z(0))$ by $\mathbf{x} \equiv \left\{\tilde{x}\left[z^t\right]\right\}_{t=-1}^{\infty}$. In contrast to Chapter 6, the elements of $\Phi(x(0), z(0))$ are not infinite sequences of vectors in $\mathbb{R}^K$, but infinite sequences of feasible plans $\tilde{x}\left[z^t\right]$ that assign a value $x \in \mathbb{R}^K$ for any history $z^t \in \mathcal{Z}^t$ for any $t = 0, 1, ...$ We are interested in (i) when the solution $V(x,z)$ to the Problem B2 coincides with the solution $V^*(x,z)$; and (ii) when the set of maximizing plans $\Pi(x,z) \subset \Phi(x,z)$ also generates an optimal feasible plan for Problem B1 (presuming that both problems have feasible plans attaining their supremums). Recall that the set of maximizing plans $\Pi(x,z)$ is defined such that for any $\pi(x,z) \in \Pi(x,z)$, we have

$$V(x,z) = U(x, \pi(x,z), z) + \beta \mathbb{E}\left[V(\pi(x,z), z') \mid z \right]. \tag{16.5}$$

[2] In particular, potential difficulties arise when one needs to exchange limits and expectations; in contrast, there is no problem in differentiating a functional under the integral or the expectations sign as long as the integrand is differentiable.

We now introduce analogs of Assumption 6.1-6.5 from Chapter 6 and the appropriate generalizations of Theorems 6.1-6.6.

Assumption 16.1. $G(x,z)$ is nonempty for all $x \in X$ and $z \in \mathcal{Z}$. Moreover, for all $x(0) \in X$, $z(0) \in \mathcal{Z}$, and $\mathbf{x} \in \Phi(x(0), z(0))$, $\lim_{n\to\infty} \mathbb{E}\left[\sum_{t=0}^{n} \beta^t U(\tilde{x}\left[z^{t-1}\right], \tilde{x}\left[z^t\right], z(t)) \mid z(0)\right]$ exists and is finite.

Assumption 16.2. X is a compact subset of $\mathbb{R}^K$, G is nonempty, compact-valued and continuous. Moreover, let $\mathbf{X}_G = \{(x,y,z) \in X \times X \times \mathcal{Z} : y \in G(x,z)\}$ and suppose that $U : \mathbf{X}_G \to \mathbb{R}$ is continuous.

Observe that Assumption 16.1 only imposes the compactness of X, since $\mathcal{Z}$ is already compact in view of the fact that it consists of a finite number of elements. Moreover, the continuity of U in (x,y,z) is equivalent to its continuity in (x,y), since $\mathcal{Z}$ is a finite set, so we can endow it with the discrete topology, so that continuity is automatically guaranteed. As in Chapter 6, these assumptions enable us to establish a number of useful results about the equivalence between Problems B1 and B2 and the solution to the dynamic optimization problems specified above. I state these results without proof here, and provide some of the proofs in Section 16.2 and leave the rest to exercises.

Our first result is a generalization of Theorem 6.1 from Chapter 6.

Theorem 16.1. ***(Equivalence of Values)*** *Suppose Assumptions 16.1 and 16.2 hold. Then for any $x \in X$ and any $z \in \mathcal{Z}$, any $V^*(x,z)$ defined in Problem B1 is a solution to Problem B2. Moreover, any solution $V(x,z)$ to Problem B2 that satisfies $\lim_{t\to\infty} \beta^t \mathbb{E}\left[V\left(\tilde{x}\left[z^{t-1}\right], z(t)\right)\right] = 0$ for any $\left\{\tilde{x}\left[z^t\right]\right\}_{t=-1}^{\infty} \in \Phi(x(0), z(0))$, and any $\tilde{x}\left[z^{-1}\right] = x(0) \in X$ and $z \in \mathcal{Z}$ is a solution to Problem B1, so that $V^*(x,z) = V(x,z)$ for any $x \in X$ and any $z \in \mathcal{Z}$.*

The next theorem establishes the *principle of optimality* for stochastic problems. As in Chapter 6, the principle of optimality enables us to break the returns from an optimal plan into two parts, the current return and the continuation return, which now corresponds to expected returns.

Theorem 16.2. ***(Principle of Optimality)*** *Suppose Assumptions 16.1 and 16.2 hold. For $x(0) \in X$ and $z(0) \in \mathcal{Z}$, let $\mathbf{x}^* \equiv \left\{\tilde{x}^*\left[z^t\right]\right\}_{t=-1}^{\infty} \in \Phi(x(0), z(0))$ be a feasible plan that attains $V^*(x(0), z(0))$ in Problem B1. Then we have that*

$$V^*(\tilde{x}^*\left[z^{t-1}\right], z(t)) = U(\tilde{x}^*\left[z^{t-1}\right], \tilde{x}^*\left[z^t\right], z(t)) + \beta \mathbb{E}\left[V^*(\tilde{x}^*\left(z^t\right), z(t+1)) \mid z(t)\right] \tag{16.6}$$

for $t = 0, 1, ...$.

Moreover, if any $\mathbf{x}^* \in \Phi(x(0), z(0))$ satisfies (16.6), then it attains the optimal value in Problem B1.

The next result establishes the uniqueness of the value function and existence of solutions.

Theorem 16.3. ***(Existence of Solutions)*** *Suppose that Assumptions 16.1 and 16.2 hold. Then the unique function* $V : X \times \mathcal{Z} \to \mathbb{R}$ *that satisfies (16.4) is continuous and bounded in* x *for each* $z \in \mathcal{Z}$. *Moreover, an optimal plan* $\mathbf{x}^* \in \Phi(x(0), z(0))$ *exists for any* $x(0) \in X$ *and any* $z(0) \in \mathcal{Z}$.

The remaining results, as their analogs in Chapter 6 use further assumptions to establish concavity, monotonicity and the differentiability of the value function.

Assumption 16.3. U is strictly concave, in the sense that for any $\alpha \in (0,1)$ and any (x, y, z), $(x', y', z) \in \mathbf{X}_G$, we have

$$U\left(\alpha x + (1-\alpha)x', \alpha y + (1-\alpha)y', z\right) \geq \alpha U(x,y,z) + (1-\alpha)U(x',y',z),$$

and if $x \neq x'$,

$$U\left(\alpha x + (1-\alpha)x', \alpha y + (1-\alpha)y', z\right) > \alpha U(x,y,z) + (1-\alpha)U(x',y',z).$$

Moreover, $G(x,z)$ is convex in x in the sense that for any $z \in \mathcal{Z}$, any $\alpha \in [0,1]$, and any $x, x' \in X$, whenever $y \in G(x,z)$ and $y' \in G(x', z)$, then

$$\alpha y + (1-\alpha)y' \in G\left(\alpha x + (1-\alpha)x', z\right).$$

Assumption 16.4. For each $y \in X$ and $z \in \mathcal{Z}$, $U(\cdot, y, z)$ is strictly increasing in its first K arguments, and G is monotone in the sense that $x \leq x'$ implies $G(x,z) \subset G(x', z)$ for each $z \in \mathcal{Z}$.

Assumption 16.5. $U(x,y,z)$ is continuously differentiable in x in the interior of its domain $\mathbf{X}_G$.

Theorem 16.4. ***(Concavity of the Value Function)*** *Suppose that Assumptions 16.1, 16.2 and 16.3 hold. Then the unique function* V *that satisfies (16.4) is strictly concave in* x *for each* $z \in \mathcal{Z}$. *Moreover, the optimal plan can be expressed as* $\tilde{x}^*\left[z^t\right] = \pi\left(x^*(t), z(t)\right)$, *where the policy function* $\pi : X \times \mathcal{Z} \to X$ *is continuous in* x *for each* $z \in \mathcal{Z}$.

Theorem 16.5. ***(Monotonicity of the Value Function I)*** *Suppose that Assumptions 16.1, 16.2 and 16.4 hold and let* $V : X \times \mathcal{Z} \to \mathbb{R}$ *be the unique solution to (16.4). Then for each* $z \in \mathcal{Z}$, V *is strictly increasing in* x.

Theorem 16.6. ***(Differentiability of the Value Function)*** *Suppose that Assumptions 16.1, 16.2, 16.3 and 16.5 hold. Let* π *be the policy function defined above and assume that* $x' \in \text{Int}X$ *and* $\pi(x', z) \in \text{Int}G(x', z)$ *at* $z \in \mathcal{Z}$, *then* $V(x,z)$ *is continuously differentiable at* (x', z), *with derivative given by*

$$D_x V\left(x', z\right) = D_x U\left(x', \pi\left(x', z\right), z\right). \tag{16.7}$$

These theorems have exact analogs in Chapter 6. Since the value function now also depends on the stochastic variable z, an additional monotonicity result can also be obtained. For this, let us introduce the following additional assumption:

Assumption 16.6. (i) G is monotone in z in the sense that $z \leq z'$ implies $G(x,z) \subset G(x,z')$ for each any $x \in X$ and $z, z' \in \mathcal{Z}$ such that $z \leq z'$.

(ii) For each $(x,y,z) \in \mathbf{X}_G$, $U(x,y,z)$ is strictly increasing in z.

(iii) The Markov chain for z is monotone in the sense that for any nondecreasing function $f{:}\mathcal{Z} \to \mathbb{R}$, $\mathbb{E}\left[f\left(z'\right) \mid z\right]$ is also nondecreasing in z.

To interpret the last part of this assumption, suppose that $z_j \leq z_{j'}$ whenever $j < j'$. Then this condition will be satisfied if and only if we have that for any $\bar{j} = 1, ..., N$ and any $j'' > j'$, $\sum_{j=\bar{j}}^{N} q_{jj''} \geq \sum_{j=\bar{j}}^{N} q_{jj'}$ (see Exercise 16.1).

Theorem 16.7. ***(Monotonicity of the Value Function II)*** *Suppose that Assumptions 16.1, 16.2 and 16.6 hold and let $V : X \times \mathcal{Z} \to \mathbb{R}$ be the unique solution to (16.4). Then for each $x \in X$, V is strictly increasing in z.*

16.2. Proofs of the Stochastic Dynamic Programming Theorems*

This section provides proofs for the main theorems provided in the previous section, Theorems 16.1-16.3. The proofs for theorems 16.5-16.7. are left as exercises.

First, for any feasible $\mathbf{x} \equiv \left\{\tilde{x}\left[z^t\right]\right\}_{t=-1}^{\infty}$, and any initial conditions $x\left(0\right) \in X$ and $z\left(0\right) \in \mathcal{Z}$, define

$$\bar{\mathbf{U}}(\mathbf{x}, z\left(0\right)) \equiv \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t U\left(\tilde{x}\left[z^{t-1}\right], \tilde{x}\left[z^t\right], z\left(t\right)\right) \mid z\left(0\right)\right]$$

and note that for any $x\left(0\right) \in X$ and $z\left(0\right) \in \mathcal{Z}$,

$$V^*(x\left(0\right), z\left(0\right)) = \sup_{\mathbf{x} \in \mathbf{\Phi}(x(0), z(0))} \bar{\mathbf{U}}(\mathbf{x}, z\left(0\right)).$$

In view of Assumption 16.1, which ensures that all values are bounded, it follows by definition that

$$V^*(x\left(0\right), z\left(0\right)) \geq \bar{\mathbf{U}}(\mathbf{x}, z\left(0\right)) \text{ for all } \mathbf{x} \in \mathbf{\Phi}(x\left(0\right), z\left(0\right)) \tag{16.8}$$

and

$$\text{for any } \varepsilon > 0, \text{ there exists } \mathbf{x}' \in \mathbf{\Phi}(x\left(0\right), z\left(0\right)) \text{ s.t. } V^*(x\left(0\right), z\left(0\right)) \leq \bar{\mathbf{U}}(\mathbf{x}', z\left(0\right)) + \varepsilon \tag{16.9}$$

The conditions for $V\left(\cdot,\cdot\right)$ to be a solution to Problem B2 are similar. For any $x\left(0\right) \in X$ and $z\left(0\right) \in \mathcal{Z}$,

$$V(x\left(0\right), z\left(0\right)) \geq U(x\left(0\right), y, z) + \beta \mathbb{E}\left[V(y, z\left(1\right)) \mid z\left(0\right)\right], \quad \text{all } y \in G(x\left(0\right), z\left(0\right)), \tag{16.10}$$

and

$$\text{(16.11)} \qquad \begin{aligned} \text{for any } \varepsilon &> 0, \text{ there exists } y' \in G\left(x\left(0\right), z\left(0\right)\right) \\ \text{s.t.} V\left(x\left(0\right), z\left(0\right)\right) &\leq U\left(x\left(0\right), y', z\left(0\right)\right) + \beta \mathbb{E}\left[V\left(y, z\left(1\right)\right) \mid z\left(0\right)\right] + \varepsilon. \end{aligned}$$

The following lemma is as straightforward generalization of Lemma 6.1 in 6

Lemma 16.1. *Suppose that Assumption 16.1 holds. Then for any $x(0) \in X$, any $z(0) \in \mathcal{Z}$, any $\mathbf{x} \equiv \left\{\tilde{x}\left[z^t\right]\right\}_{t=-1}^{\infty} \in \Phi(x(0), z(0))$, we have that*

$$\bar{\mathbf{U}}\left(\mathbf{x}, z\left(0\right)\right) = U\left(x\left(0\right), \tilde{x}\left[z^0\right], z\left(0\right)\right) + \beta \mathbb{E}\left[\bar{\mathbf{U}}\left(\left\{\tilde{x}\left[z^t\right]\right\}_{t=0}^{\infty}, z\left(1\right)\right) \mid z\left(0\right)\right].$$

Proof. See Exercise 16.2. □

Proof of Theorem 16.1. If $\beta = 0$, Problems B1 and B2 are identical, thus the result follows immediately.

Suppose that $\beta > 0$ and take an arbitrary $x(0) \in X$ and an arbitrary $z(0) \in \mathcal{Z}$. First, note that U continuous over $X \times X \times \mathcal{Z}$ (with the finite set $\mathcal{Z}$ endowed with the natural discrete topology). Then by the same argument as in the proof of Theorem 6.1, Assumptions 16.1 and 16.2 imply that the objective function in Problem B1 is continuous in the product topology (recall Theorem A.11) and the constraint set is compact (this again follows from Theorem A.12 and Fact A.2 in Appendix Chapter A as in the proof of Theorem 6.1). Therefore, by Weierstrass's Theorem, Theorem A.9, a solution to this maximization problem exists and thus $V^*(x(0), z(0))$ is well defined. Moreover, again Berge's Maximum Theorem, Theorem A.13, implies that $V^*(x(0), z(0))$ is continuous and thus bounded over the compact set $X \times \mathcal{Z}$. Now consider some $x(1) \in G(x(0), z(0))$. Another application of Theorem A.9 implies that there exists $\mathbf{x}' \equiv \left\{\tilde{x}'\left[z^t\right]\right\}_{t=0}^{\infty} \in \Phi(x(1), z(1))$ attaining $V^*(x(1), z(1))$ for any $z(1) \in \mathcal{Z}$ (and with $\tilde{x}'\left[z^0\right] = x(1)$). This implies that

$$\mathbb{E}\left[V^*\left(x\left(1\right), z\left(1\right)\right) \mid z\left(0\right)\right] = \sum_{j=1}^{N} q_{jj'} V^*\left(x\left(1\right), z_j\right)$$

for j' defined by $z(0) = z_{j'}$. Next, since $(x(0), \mathbf{x}') \in \Phi(x(0), z(0))$ and $V^*(x(0), z(0))$ is the supremum in Problem B1 starting with $x(0)$ and $z(0) \in \mathcal{Z}$, Lemma 16.1 implies that

$$\begin{aligned} V^*\left(x\left(0\right), z\left(0\right)\right) &\geq U\left(x\left(0\right), \tilde{x}'\left[z^0\right], z\left(0\right)\right) + \beta \mathbb{E}\left[\bar{\mathbf{U}}\left(\left\{\tilde{x}'\left[z^t\right]\right\}_{t=0}^{\infty}, z\left(1\right)\right) \mid z\left(0\right)\right], \\ &= U\left(x\left(0\right), \tilde{x}'\left[z^0\right], z\left(0\right)\right) + \beta \mathbb{E}\left[V^*\left(x\left(1\right), z\left(1\right)\right) \mid z\left(0\right)\right], \end{aligned}$$

and establishes (16.10).

Next, take an arbitrary $\varepsilon > 0$. By (16.9), there exists $\mathbf{x}'_{\varepsilon} = \left(x(0), \tilde{x}'_{\varepsilon}\left[z^0\right], \tilde{x}'_{\varepsilon}\left[z^1\right] ...\right) \in \Phi(x(0), z(0))$ such that

$$\bar{\mathbf{U}}\left(\mathbf{x}'_{\varepsilon}, z\left(0\right)\right) \geq V^*\left(x\left(0\right), z\left(0\right)\right) - \varepsilon.$$

By the feasibility of $\mathbf{x}'_\varepsilon$, we have $\mathbf{x}''_\varepsilon = \left(\tilde{x}'_\varepsilon\left[z^0\right], \tilde{x}'_\varepsilon\left[z^1\right], ...\right) \in \Phi\left(\tilde{x}'_\varepsilon\left[z^0\right], z(1)\right)$ for any $z(1) \in \mathcal{Z}$. Moreover, also by definition $V^*\left(\tilde{x}'_\varepsilon\left[z^0\right], z(1)\right)$ is the supremum in Problem B1 starting with the initial conditions $\tilde{x}'_\varepsilon\left[z^0\right]$ and $z(1)$. Then Lemma 16.1 implies that for any $\varepsilon > 0$,

$$\begin{aligned} V^*(x(0), z(0)) - \varepsilon &\leq U\left(x(0), \tilde{x}'_\varepsilon\left[z^0\right], z(0)\right) + \beta\mathbb{E}\left[\bar{\mathbf{U}}\left(\left\{\tilde{x}\left[z^t\right]\right\}_{t=0}^\infty, z(1)\right) \mid z(0)\right] \\ &= U\left(x(0), \tilde{x}'_\varepsilon\left[z^0\right], z(0)\right) + \beta\mathbb{E}\left[V^*\left(\tilde{x}'_\varepsilon\left[z^0\right], z(1)\right) \mid z(0)\right], \end{aligned}$$

so that (16.11) is satisfied. This establishes that any solution to Problem B1 satisfies (16.10) and (16.11), and is thus a solution to Problem B2.

To establish the converse, note that (16.10) implies that for any $\tilde{x}\left[z^0\right] \in G(x(0), z(0))$,

$$V(x(0), z(0)) \geq U\left(x(0), \tilde{x}\left[z^0\right], z(0)\right) + \beta\mathbb{E}\left[V\left(\tilde{x}\left[z^0\right], z(1)\right) \mid z(0)\right].$$

Now substituting recursively for $V\left(\tilde{x}\left[z^0\right], z(1)\right)$, $V\left(\tilde{x}\left[z^1\right], z(2)\right)$, etc., and taking expectations, we have

$$V(x(0), z(0)) \geq \mathbb{E}\left[\sum_{t=0}^n U\left(\tilde{x}\left[z^{t-1}\right], \tilde{x}\left[z^t\right], z(t)\right) \mid z(0)\right] + \beta^{n+1}\mathbb{E}\left[V\left(\tilde{x}\left[z^n\right], z(n+1)\right) \mid z(0)\right].$$

By definition $\lim_{n\to\infty} \mathbb{E}\left[\sum_{t=0}^n U\left(\tilde{x}\left[z^{t-1}\right], \tilde{x}\left[z^t\right], z(t)\right) \mid z(0)\right] = \bar{\mathbf{U}}(\mathbf{x}, z(0))$ and by the hypothesis of the theorem $\lim_{n\to\infty} \beta^{n+1}\mathbb{E}\left[V\left(\tilde{x}\left[z^n\right], z(n+1)\right) \mid z(0)\right] = 0$, so that (16.8) is verified.

Next, let $\varepsilon > 0$ be a positive scalar. From (16.11), we have that for any $\varepsilon' = \varepsilon(1-\beta) > 0$, there exists $\tilde{x}_\varepsilon\left[z^0\right] \in G(x(0), z(0))$ such that

$$V(x(0), z(0)) \leq U\left(x(0), \tilde{x}_\varepsilon\left[z^0\right]\right) + \beta\mathbb{E}V\left(\tilde{x}_\varepsilon\left[z^0\right], z(1) \mid z(0)\right) + \varepsilon'.$$

Let $\tilde{x}_\varepsilon\left[z^t\right] \in G\left(\tilde{x}_\varepsilon\left[z^{t-1}\right], z(t)\right)$, with $\tilde{x}_\varepsilon\left[z^{-1}\right] = x(0)$, and define $\mathbf{x}_\varepsilon \equiv \left(x(0), \tilde{x}_\varepsilon\left[z^0\right], \tilde{x}_\varepsilon\left[z^1\right], \tilde{x}_\varepsilon\left[z^2\right] ...\right)$. Again substituting recursively for $V\left(\tilde{x}_\varepsilon\left[z^1\right]\right)$, $V\left(\tilde{x}_\varepsilon\left[z^t\right]\right)$, etc. and taking expectations, we obtain

$$\begin{aligned} V(x(0), z(0)) &\leq \mathbb{E}\left[\sum_{t=0}^n U\left(\tilde{x}_\varepsilon\left[z^{t-1}\right], \tilde{x}_\varepsilon\left[z^t\right], z(t)\right) \mid z(0)\right] \\ &\quad + \beta^{n+1}\mathbb{E}\left[V\left(\tilde{x}_\varepsilon\left[z^n\right], z(n+1)\right) \mid z(0)\right] + \varepsilon' + \varepsilon'\beta + ... + \varepsilon'\beta^n \\ &\leq \bar{\mathbf{U}}(\mathbf{x}_\varepsilon, z(0)) + \varepsilon, \end{aligned}$$

where the last step follows using the fact that $\varepsilon = \varepsilon' \sum_{t=0}^\infty \beta^t$ and that as $\lim_{n\to\infty} \mathbb{E}\left[\sum_{t=0}^n U\left(\tilde{x}_\varepsilon\left[z^{t-1}\right], \tilde{x}_\varepsilon\left[z^t\right], z(t)\right) \mid z(0)\right] = \bar{\mathbf{U}}(\mathbf{x}_\varepsilon, z(0))$. This establishes that V satisfies (16.9) and completes the proof. □

Proof of Theorem 16.2. Suppose that $\mathbf{x}^* \equiv \left(x(0), \tilde{x}^*\left[z^0\right], \tilde{x}^*\left[z^1\right], \tilde{x}^*\left[z^2\right], ...\right) \in \Phi(x(0), z(0))$ is a feasible plan attaining the solution to Problem B1. Let $\mathbf{x}_t^* \equiv \left(\tilde{x}^*\left[z^{t-1}\right], \tilde{x}^*\left[z^t\right], \tilde{x}^*\left[z^{t+1}\right], ...\right)$ be the continuation of this plan from time t.

We first show that for any $t \geq 0$, $\mathbf{x}_t^*$ attains the supremum starting from $\tilde{x}^*\left[z^{t-1}\right]$ and any $z\left(t\right) \in \mathcal{Z}$, that is,

$$\bar{\mathbf{U}}(\mathbf{x}_t^*, z\left(t\right)) = V^*\left(\tilde{x}^*\left[z^{t-1}\right], z\left(t\right)\right). \tag{16.12}$$

The proof is by induction. The hypothesis is trivially satisfied for $t = 0$ since, by definition, $\mathbf{x}_0^* = \mathbf{x}^*$ attains $V^*\left(x\left(0\right), z\left(0\right)\right)$.

Next suppose that the statement is true for t, so that $\mathbf{x}_t^*$ attains the supremum starting from $\tilde{x}^*\left[z^{t-1}\right]$ and any $z\left(t\right) \in \mathcal{Z}$, or equivalently (16.12) holds for t and for $z\left(t\right) \in \mathcal{Z}$. Now using this relationship we will establish that (16.12) holds and $\mathbf{x}_{t+1}^*$ attains the supremum starting from $\tilde{x}^*\left[z^{t}\right]$ and any $z\left(t+1\right) \in \mathcal{Z}$. Equation (16.12) implies that

$$\begin{aligned} V^*(\tilde{x}^*\left[z^{t-1}\right], z\left(t\right)) &= \bar{\mathbf{U}}\left(\mathbf{x}_t^*, z\left(t\right)\right) \\ &= U\left(\tilde{x}^*\left[z^{t-1}\right], \tilde{x}^*\left[z^{t}\right], z\left(t\right)\right) \\ &\quad + \beta\mathbb{E}\left[\bar{\mathbf{U}}(\mathbf{x}_{t+1}^*, z\left(t+1\right)) \mid z\left(t\right)\right]. \end{aligned} \tag{16.13}$$

Let $\mathbf{x}_{t+1} = \left(\tilde{x}^*\left[z^{t}\right], \tilde{x}\left[z^{t+1}\right], ...\right) \in \Phi\left(\tilde{x}^*\left[z^{t}\right], z\left(t+1\right)\right)$ be any feasible plan starting with state vector $\tilde{x}^*\left[z^{t}\right]$ and stochastic variable $z\left(t+1\right)$. By definition, $\mathbf{x}_t = \left(\tilde{x}^*\left[z^{t-1}\right], \mathbf{x}_{t+1}\right) \in \Phi\left(\tilde{x}^*\left[z^{t-1}\right], z\left(t\right)\right)$. Since, by the induction hypothesis, $V^*\left(\tilde{x}^*\left[z^{t-1}\right], z\left(t\right)\right)$ is the supremum starting with $\tilde{x}^*\left[z^{t-1}\right]$ and $z\left(t\right)$, we have

$$\begin{aligned} V^*\left(\tilde{x}^*\left[z^{t-1}\right], z\left(t\right)\right) &\geq \bar{\mathbf{U}}(\mathbf{x}_t, z\left(t\right)) \\ &= U\left(\tilde{x}^*\left[z^{t-1}\right], \tilde{x}^*\left[z^{t}\right], z\left(t\right)\right) + \beta\mathbb{E}\left[\bar{\mathbf{U}}(\mathbf{x}_{t+1}, z\left(t+1\right)) \mid z\left(t\right)\right] \end{aligned}$$

for any $\mathbf{x}_{t+1}$. Combining this inequality with (16.13), we obtain that

$$\begin{aligned} \mathbb{E}\left[V^*(\tilde{x}^*\left[z^{t}\right], z\left(t+1\right)) \mid z\left(t\right)\right] &= \mathbb{E}\left[\bar{\mathbf{U}}(\mathbf{x}_{t+1}^*, z\left(t+1\right)) \mid z\left(t\right)\right] \\ &\geq \mathbb{E}\left[\bar{\mathbf{U}}(\mathbf{x}_{t+1}, z\left(t+1\right)) \mid z\left(t\right)\right] \end{aligned} \tag{16.14}$$

for all $\mathbf{x}_{t+1} \in \Phi\left(\tilde{x}^*\left[z^{t}\right], z\left(t+1\right)\right)$. Next, we complete the proof that $\mathbf{x}_{t+1}^*$ attains the supremum starting from $\tilde{x}^*\left[z^{t}\right]$ and any $z\left(t\right) \in \mathcal{Z}$ and equation (16.12) holds starting from $\tilde{x}^*\left[z^{t}\right]$ and any $z\left(t\right) \in \mathcal{Z}$. Suppose, to obtain a contradiction, that this is not the case. Then there exists $\hat{\mathbf{x}}_{t+1} \in \Phi\left(\tilde{x}^*\left[z^{t}\right], z\left(t+1\right)\right)$ for some $z\left(t+1\right) = \hat{z}$ such that

$$\bar{\mathbf{U}}(\mathbf{x}_{t+1}^*, \hat{z}) < \bar{\mathbf{U}}(\hat{\mathbf{x}}_{t+1}, \hat{z}).$$

Then construct the sequence $\hat{\mathbf{x}}_{t+1}^* = \mathbf{x}_{t+1}^*$ if $z\left(t\right) \neq \hat{z}$ and $\hat{\mathbf{x}}_{t+1}^* = \hat{\mathbf{x}}_{t+1}$ if $z\left(t\right) = \hat{z}$. Since $\mathbf{x}_{t+1}^* \in \Phi\left(\tilde{x}^*\left[z^{t}\right], \hat{z}\right)$ and $\hat{\mathbf{x}}_{t+1} \in \Phi\left(\tilde{x}^*\left[z^{t}\right], \hat{z}\right)$, we also have $\hat{\mathbf{x}}_{t+1}^* \in \Phi\left(\tilde{x}^*\left[z^{t}\right], \hat{z}\right)$. Then without loss of generality taking $\hat{z} = z_1$,

$$\begin{aligned}\mathbb{E}\left[\bar{\mathbf{U}}(\hat{\mathbf{x}}^*_{t+1}, z(t+1)) \mid z(t)\right] &= \sum_{j=1}^{N} q_{jj'}\bar{\mathbf{U}}(\hat{\mathbf{x}}^*_{t+1}, z_j) \\ &= q_{1j'}\bar{\mathbf{U}}(\hat{\mathbf{x}}_{t+1}, z_j) + \sum_{j=2}^{N} q_{jj'}\bar{\mathbf{U}}(\mathbf{x}^*_{t+1}, z_j) \\ &> q_{1j'}\bar{\mathbf{U}}(\mathbf{x}^*_{t+1}, z_j) + \sum_{j=2}^{N} q_{jj'}\bar{\mathbf{U}}(\mathbf{x}^*_{t+1}, z_j) \\ &= \mathbb{E}\left[\bar{\mathbf{U}}(\mathbf{x}^*_{t+1}, z(t+1)) \mid z(t)\right],\end{aligned}$$

contradicting (16.14) and completing the induction step, which establishes that $\mathbf{x}^*_{t+1}$ attains the supremum starting from $\tilde{x}^*\left[z^t\right]$ and any $z(t+1) \in \mathcal{Z}$.

Equation (16.12) then implies that

$$\begin{aligned}V^*\left(\tilde{x}^*\left[z^{t-1}\right], z(t)\right) &= \bar{\mathbf{U}}(\mathbf{x}^*_t, z(t)) \\ &= U\left(\tilde{x}^*\left[z^{t-1}\right], \tilde{x}^*\left[z^t\right], z(t)\right) + \beta\mathbb{E}\left[\bar{\mathbf{U}}(\mathbf{x}^*_{t+1}, z(t+1)) \mid z(t)\right] \\ &= U\left(\tilde{x}^*\left[z^{t-1}\right], \tilde{x}^*\left[z^t\right], z(t)\right) + \beta\mathbb{E}\left[V^*(\tilde{x}^*\left(z^t\right), z(t+1)) \mid z(t)\right],\end{aligned}$$

establishing (16.6) and thus completing the proof of the first part of the theorem.

Now suppose that (16.6) holds for $\mathbf{x}^* \in \Phi(x(0), z(0))$. Then substituting repeatedly for $\mathbf{x}^*$, we obtain

$$V^*(x(0), z(0)) = \sum_{t=0}^{n} \beta^t U\left(\tilde{x}^*\left[z^{t-1}\right], \tilde{x}^*\left[z^t\right], z(t)\right) + \beta^{n+1}\mathbb{E}\left[V^*(\tilde{x}^*(z^n), z(n+1)) \mid z(0)\right].$$

In view of the fact that V^* is bounded, we have that $\lim_{n\to\infty} \beta^{n+1}\mathbb{E}\left[V^*(\tilde{x}^*(z^n), z(n+1)) \mid z(0)\right] = 0$ and thus

$$\begin{aligned}\bar{\mathbf{U}}(\mathbf{x}^*, z(0)) &= \lim_{n\to\infty} \sum_{t=0}^{n} \beta^t U\left(\tilde{x}^*\left[z^{t-1}\right], \tilde{x}^*\left[z^t\right], z(t)\right) \\ &= V^*(x(0), z(0)),\end{aligned}$$

thus $\mathbf{x}^*$ attains the optimal value in Problem B1. This completes the proof of the second part of the theorem. □

Proof of Theorem 16.3. Consider Problem B2. In view of Assumptions 16.1 and 16.2, there exis some $M < \infty$, such that $|U(x, y, z)| < M$ for all $(x, y, z) \in \mathbf{X}_G$. This immediately implies that $|V^*(x, z)| \leq M/(1-\beta)$, all $x \in X$ and all $z \in \mathcal{Z}$. Consequently, consider the function $V^*(\cdot, \cdot) \in \mathbf{C}(X \times \mathcal{Z})$, where $\mathbf{C}(X \times \mathcal{Z})$ denotes the set of continuous functions defined on $X \times \mathcal{Z}$, where X is endowed with the sup norm, $\|f\| = \sup_{x\in X} |f(x)|$ and $\mathcal{Z}$ is endowed with the discrete topology. Moreover, all functions in $\mathbf{C}(X \times \mathcal{Z})$ are bounded because they are continuous and both X and $\mathcal{Z}$ are compact.

Now define the operator T

$$TV(x,z) = \max_{y \in G(x,z)} \left\{ U(x,y,z) + \beta \mathbb{E}\left[V\left(y, z'\right) \mid z \right] \right\}. \tag{16.15}$$

Suppose that $V(x,z)$ is continuous and bounded. Then $\mathbb{E}[V(y,z') \mid z]$ is also continuous and bounded, since it is simply given by

$$\mathbb{E}\left[V\left(y,z'\right) \mid z\right] \equiv \sum_{j=1}^{N} q_{jj'} V\left(y, z_j\right),$$

with j' defined such that $z = z_{j'}$. Moreover, $U(x,y,z)$ is also continuous and bounded over $\mathbf{X}_G$. A fixed point of the operator T, $V(x,z) = TV(x,z)$, will then be a solution to Problem B2 for given $z \in \mathcal{Z}$. We first prove that such a fixed point (solution) exists. The maximization problem on the right-hand side of (16.15) is one of maximizing a continuous function over a compact set, and by Weierstrass's Theorem, it has a solution. Consequently, T is well defined. It can be verified straightforwardly that it satisfies Blackwell's sufficient conditions for a contraction (Theorem 6.9 from Chapter 6). Therefore, applying Theorem 6.7, a unique fixed point $V \in \mathbf{C}(X \times \mathcal{Z})$ to (16.15) exists and this is also the unique solution to Problem B2. Now consider the maximization in Problem B2. Since U and V are continuous and $G(x,z)$ is compact-valued, we can apply Weierstrass's Theorem, Theorem A.9, once more to conclude that $y \in G(x,z)$ achieving the maximum exists. This defines the set of maximizers $\Pi(x,z) \subset \Phi(x,z)$ for Problem B2. Let $\mathbf{x}^* \equiv \left(x(0), \tilde{x}^*\left[z^0\right], \tilde{x}^*\left[z^1\right], \tilde{x}^*\left[z^2\right], ...\right) \in \Phi(x(0), z(0))$ with $\tilde{x}^*\left[z^t\right] \in \Pi\left(\tilde{x}^*\left[z^{t-1}\right], z(t)\right)$ for all $t \geq 0$ and each $z(t) \in \mathcal{Z}$. Then from Theorems 16.1 and 16.2, $\mathbf{x}^*$ is also an optimal plan for Problem B1. □

Finally, the proofs of Theorems 16.4-16.6 are similar to those of Theorems 6.4-6.6 from Chapter 6, and are left as exercises (see Exercises 16.3-16.5). The proof of Theorem 16.7 is similar to 16.5 and is left to Exercise 16.6.

16.3. Stochastic Euler Equations

In Chapter 6, Euler equations and transversality conditions played a central role. In the present context, instead of the standard Euler equations, we have to work with stochastic Euler equations. While this is not conceptually any more involved than the standard Euler equations, stochastic Euler equations are not always easy to manipulate. Sometimes, as in the permanent income hypothesis model we will study in Section 16.5, the stochastic Euler equation itself may contain enough economics to be useful. In other instances, our interest will be with the characterization of optimal plans. Although this is typically a non-trivial task, the combination of stochastic Euler equations and the appropriate transversality condition can sometimes be used to determine certain qualitative features of optimal plans.

Let us follow the treatment in Chapter 6 and also build on the results from Section 16.1. Let us use $*$'s to denote optimal values and D for gradients. Then using Assumption 16.5 and Theorem 16.6, we can write the necessary conditions for an interior optimal plan as

$$D_y U(x, y^*, z) + \beta \mathbb{E}\left[D_x V\left(y^*, z'\right) \mid z\right] = 0, \tag{16.16}$$

where $x \in \mathbb{R}^K$ is the current value of the state vector, $z \in \mathcal{Z}$ is the current value of the stochastic variable, and $D_x V(y^*, z')$ denotes the gradient of the value function evaluated at next period's state vector y^*. Now using the stochastic equivalent of the Envelope Theorem for dynamic programming and differentiating (16.5) with respect to the state vector, x, we obtain:

$$D_x V(x, z) = D_x U(x, y^*, z). \tag{16.17}$$

Here there are no expectations, since this equation is conditioned on the realization of $z \in \mathcal{Z}$. Note that y^* here is a shorthand for $\pi(x, z)$. Now using this notation and combining these two equations, we obtain the canonical form of the stochastic Euler equation

$$D_y U(x, \pi(x, z), z) + \beta \mathbb{E}\left[D_x U\left(\pi(x, z), \pi\left(\pi(x, z), z'\right), z'\right) \mid z\right] = 0,$$

where as in Chapter 6, $D_x U$ represents the gradient vector of U with respect to its first K arguments, and $D_y U$ represents its gradient with respect to the second set of K arguments. Writing this equation in the notation more congruent with the sequence version of the problem, the stochastic Euler equation takes the form

$$D_y U(\tilde{x}^*\left[z^{t-1}\right], \tilde{x}^*\left[z^t\right], z(t)) + \beta \mathbb{E}\left[D_x U\left(\tilde{x}^*\left[z^t\right], \tilde{x}^*\left[z^{t+1}\right], z(t+1)\right) \mid z(t)\right] = 0, \tag{16.18}$$

for $z^{t-1} \in \mathcal{Z}^{t-1}$.

How do we write the transversality condition in this case? The transversality condition essentially requires the discounted marginal return from the state variable to tend to zero as the planning horizon goes to infinity. In a stochastic environment, we clearly have to look at expected returns. The question is what information to condition upon. One idea might be to condition on the information available at date $t = 0$, i.e., $z(0) \in \mathcal{Z}$. However, this transversality condition would cover variations from the viewpoint of time $t = 0$. More generally, we would need the transversality condition associated with this stochastic Euler equation to take the form

$$\lim_{t \to \infty} \beta^t \mathbb{E}\left[D_x U(\tilde{x}^*\left[z^{s+t-1}\right], \tilde{x}^*\left[z^{s+t}\right], z(s+t)) \cdot \tilde{x}^*\left[z^{s+t-1}\right] \mid z(s)\right] = 0 \tag{16.19}$$

for all $z(s) \in \mathcal{Z}$ and $z^{s-1} \in \mathcal{Z}^{s-1}$.

The next theorem generalizes Theorem 6.10 from Chapter 6 to an environment with uncertainty. In particular, it shows that the transversality condition together with the transformed Euler equations in (16.18) are sufficient to characterize an optimal solution to Problem A1 and therefore to Problem A2.

THEOREM 16.8. ***(Euler Equations and the Transversality Condition)*** *Let $X \subset \mathbb{R}_+^K$ and suppose that Assumptions 16.1-16.5 hold. Then the sequence of feasible plans $\{\tilde{x}^*[z^t]\}_{t=-1}^{\infty}$, with $\tilde{x}^*[z^t] \in \text{Int}G(\tilde{x}^*[z^{t-1}], z(t))$ for each $z(t) \in \mathcal{Z}$ and each $t = 0, 1, \ldots,$ is optimal for Problem B1 given $x(0)$ and $z(0) \in \mathcal{Z}$ if it satisfies (16.18) and (16.19).*

PROOF. Consider an arbitrary $x(0) \in X$ and $z(0) \in \mathcal{Z}$, and let $\mathbf{x}^* \equiv \{\tilde{x}^*[z^t]\}_{t=-1}^{\infty} \in \Phi(x(0), z(0))$ be a feasible plan satisfying (16.18) and (16.19). We first show that $\mathbf{x}^*$ yields a higher value than any other $\mathbf{x} \equiv \{\tilde{x}[z^t]\}_{t=-1}^{\infty} \in \Phi(x(0), z(0))$. For any $\mathbf{x} \in \Phi(x(0), z(0))$ and any $z^\infty \in \mathcal{Z}^\infty$ define

$$\Delta_{\mathbf{x}}(z^\infty) \equiv \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t [U(\tilde{x}^*[z^{t-1}], \tilde{x}^*[z^t], z(t)) - U(\tilde{x}[z^{t-1}], \tilde{x}[z^t], z(t))]$$

as the difference of the *realized* objective function between the feasible sequences $\mathbf{x}^*$ and $\mathbf{x}$.

From Assumptions 16.2 and 16.5, U is continuous, concave, and differentiable, so that for any $z^\infty \in \mathcal{Z}^\infty$ and any $\mathbf{x} \in \Phi(x(0), z(0))$

$$\begin{aligned} \Delta_{\mathbf{x}}(z^\infty) \geq & \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t [D_x U(\tilde{x}^*[z^{t-1}], \tilde{x}^*[z^t], z(t)) \cdot (\tilde{x}^*[z^{t-1}] - \tilde{x}[z^{t-1}]) \\ & + D_y U(\tilde{x}^*[z^{t-1}], \tilde{x}^*[z^t], z(t)) \cdot (\tilde{x}^*[z^t] - \tilde{x}[z^t])]. \end{aligned}$$

Since this is true for any $z^\infty \in \mathcal{Z}^\infty$, we can take expectations on both sides to obtain

$$\begin{aligned} & \mathbb{E}[\Delta_{\mathbf{x}}(z^\infty) \mid z(s)] \\ \geq & \lim_{T\to\infty} \mathbb{E}\left[\sum_{t=0}^{T} \beta^t [D_x U(\tilde{x}^*[z^{t-1}], \tilde{x}^*[z^t], z(t)) \cdot (\tilde{x}^*[z^{t-1}] - \tilde{x}[z^{t-1}]) \mid z(s)\right] \\ & + \lim_{T\to\infty} \mathbb{E}\left[\sum_{t=0}^{T} \beta^t D_y U(\tilde{x}^*[z^{t-1}], \tilde{x}^*[z^t], z(t)) \cdot (\tilde{x}^*[z^t] - \tilde{x}[z^t]) \mid z(s)\right] \end{aligned}$$

for any $z(s) \in \mathcal{Z}$. Rearranging the previous expression, we obtain

$$\begin{aligned} & \mathbb{E}[\Delta_{\mathbf{x}}(z^\infty) \mid z(s)] \geq \\ & \lim_{T\to\infty} \mathbb{E}\left[\sum_{t=0}^{T} \beta^t D_y U(\tilde{x}^*[z^{t-1}], \tilde{x}^*[z^t], z(t)) \cdot (\tilde{x}^*[z^t] - \tilde{x}[z^t]) \mid z(s)\right] \\ & \lim_{T\to\infty} \mathbb{E}\left[\sum_{t=0}^{T} \beta^{t+1} D_x U(\tilde{x}^*[z^t], \tilde{x}^*[z^{t+1}], z(t+1)) \cdot (\tilde{x}^*[z^t] - \tilde{x}[z^t]) \mid z(s)\right] \\ & - \lim_{T\to\infty} \mathbb{E}[\beta^{T+1} D_x U(\tilde{x}^*[z^T], \tilde{x}^*[z^{T+1}], z(T+1)) \cdot \tilde{x}^*[z^T] \mid z(s)] \\ & + \lim_{T\to\infty} \mathbb{E}[\beta^{T+1} D_x U(\tilde{x}[z^T], \tilde{x}[z^{T+1}], z(T+1)) \cdot \tilde{x}[z^T] \mid z(s)]. \end{aligned}$$

Since $\mathbf{x}^* \equiv \{\tilde{x}^*[z^t]\}_{t=-1}^{\infty}$ satisfies (16.18), the terms in first and second lines are all equal to zero. Moreover, since $\mathbf{x}^* \equiv \{\tilde{x}^*[z^t]\}_{t=-1}^{\infty}$ satisfies (16.19), the third line is also equal to zero. Finally, since U is increasing in x, $D_x U \geq 0$, and $x \geq 0$, the fourth line is nonnegative, establishing that $\mathbb{E}[\Delta_{\mathbf{x}}(z^{\infty}) \mid z(s)] \geq 0$ for any $\mathbf{x} \in \Phi(x(0), z(0))$ and any $z(s) \in \mathcal{Z}$. Consequently, $\mathbf{x}^*$ yields higher value than any feasible $\mathbf{x} \in \Phi(x(0), z(0))$, and is therefore optimal. □

16.4. Generalization to Markov Processes*

What happens if z does not take on finitely many values? For example, z may be represented by a general Markov process, taking values in a compact metric space. The simplest example would be a one-dimensional stochastic variable $z(t)$ given by the process $z(t) = \rho z(t-1) + \sigma\varepsilon(t)$, where $\varepsilon(t)$ has a standard normal distribution. At some level, most of the results we care about generalize to such cases. At another level, however, greater care needs to be taken in formulating these problems both in the sequence form of Problem B1 and in the recursive form of Problem B2. The main difficulty in this case arises in ensuring that there exist appropriately defined feasible plans, which now need to be "measurable" with respect to the information set available at the time. Unfortunately, to state the appropriate theorems in a rigorous manner requires a lengthy detour into measure theory. Instead, I will assume that both $\mathcal{Z}$ and X are compact and that the function $\tilde{x}[z^t]$ introduced in Section 16.1 is "well-defined"—in particular, finite-valued and measurable. Under these assumptions and again representing all integrals with the expectations, we can state the main theorems for stochastic dynamic programming with general Markov processes without proof.

Let us first define $\mathcal{Z}$ as a compact subset of $\mathbb{R}$, which includes $\mathcal{Z}$ consisting of finite number of elements and $\mathcal{Z}$ corresponding to an interval as special cases. Let $z(t) \in \mathcal{Z}$ represent the uncertainty in this environment, and suppose that its probability distribution can be represented as a Markov process, i.e.,

$$\Pr[z(t) \mid z(0), ..., z(t-1)] \equiv \Pr[z(t) \mid z(t-1)].$$

Let us also use the notation $z^t \equiv (z(0), z(1), ..., z(t))$ to represent the history of the realizations of the stochastic variable. The objective function and the constraint sets are represented as in Section 16.1, so that $\tilde{x}[z^t]$ again denotes a *feasible plan.* Let the set of feasible plans after history z^t be denoted by $\Phi\big(\tilde{x}[z^{t-1}], z(t)\big)$. The set of feasible plans starting with $z(0) \equiv z^0$ is then $\Phi\big(x(0), z^0\big)$. Also whenever there exists a function V that is a solution to Problem B2, let us define $\Pi(x, z) \subset \Phi(x, z)$ such that any $\pi(x, z) \in \Pi(x, z)$ satisfies

$$V(x, z) = U(x, \pi(x, z), z) + \beta\mathbb{E}\left[V(\pi(x, y), z') \mid z\right].$$

Finally, to state the appropriate theorems, let us refer to the same assumptions as in Section 16.1, except that these assumptions now require the relevant functions to be *measurable* in the appropriate sense and the correspondence $\Phi\big(x(t), z^t\big)$ to always admit *a measurable* *selection* for all $x(t) \in X$ and $z^t \in \mathcal{Z}^t$. For this reason, I will refer to these assumptions with a * (i.e., instead of Assumption 16.2, I will refer to Assumption 16.2*).

Theorem 16.9. ***(Existence of Solutions)*** *Suppose that $\Phi\big(x(0), z^0\big)$ is nonempty for all $z^0 \in \mathcal{Z}$ and all $x(0) \in X$. Suppose also that for any $\mathbf{x} \in \Phi\big(x(0), z^0\big)$, $\mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t U\left(\tilde{x}\left[z^{t-1}\right], \tilde{x}\left[z^t\right], z(t)\right) \mid z(0)\right]$ is well-defined and finite-valued. Then any solution $V(x,z)$ to Problem B2 coincides with the solution $V^*(x,z)$ to Problem B1. Moreover, if $\Pi(x,z)$ is non-empty for all $(x,z) \in X \times \mathcal{Z}$, then any $\pi(x,z) \in \Pi(x,z)$ achieves $V^*(x,z)$.*

Notice that this theorem already imposes stronger requirements than Assumption 16.1 and hence there is no need to refer to Assumption 16.1.

Theorem 16.10. ***(Continuity of Value Functions)*** *Suppose the hypotheses in Theorem 16.9 are satisfied and Assumption 16.2* holds. Then there exists a unique function $V : X \times \mathcal{Z} \to \mathbb{R}$ that satisfies (16.4). Moreover, V is continuous and bounded. Finally, an optimal plan $\mathbf{x}^* \in \Phi(x(0), z(0))$ exists for any $x(0) \in X$ and any $z(0) \in \mathcal{Z}$.*

Theorem 16.11. ***(Concavity of Value Functions)*** *Suppose the hypotheses in Theorem 16.9 are satisfied and Assumptions 16.2* and 16.3* hold. Then the unique function V that satisfies (16.4) is strictly concave in x for each $z \in \mathcal{Z}$. Moreover, the optimal plan can be expressed as $\tilde{x}^*\left[z^t\right] = \pi(x(t), z(t))$, where the policy function $\pi : X \times \mathcal{Z} \to X$ is continuous in x for each $z \in \mathcal{Z}$.*

Theorem 16.12. ***(Monotonicity of Value Functions)*** *Suppose the hypotheses in Theorem 16.9 are satisfied and Assumptions 16.2* and 16.4* hold. Then the unique value function $V : X \times \mathcal{Z} \to \mathbb{R}$ that satisfies (16.4) is strictly increasing in x for each $z \in \mathcal{Z}$.*

Theorem 16.13. ***(Differentiability of Value Functions)*** *Suppose the hypotheses in Theorem 16.9 are satisfied and Assumptions 16.2*, 16.3* and 16.5* hold. Let π be the policy function defined above and assume that $x' \in \text{Int}X$ and $\pi(x', z) \in \text{Int}G(x', z)$ for each $z \in \mathcal{Z}$, then $V(x,z)$ is continuously differentiable at x', with derivative given by*

$$D_x V\left(x', z\right) = D_x U\left(x', \pi\left(x', z\right), z\right). \tag{16.20}$$

Given the hypotheses of Theorem 16.9, the proofs of these theorems are not difficult, though they are long and require a little care. Somewhat more general versions of these theorems can be found in Stokey, Lucas and Prescott (1989, Chapter 9), who first develop the necessary measure theory and some of the theory of general Markov processes to state more rigorous and complete versions of these theorems.

16.5. Applications of Stochastic Dynamic Programming

We now present a number of applications of the methods of stochastic dynamic programming. Some of the most important applications, related to stochastic growth and growth with incomplete markets, are left for next chapter. In each application, I try to point out how formulating the problem recursively and using stochastic dynamic programming methods simplify the analysis.

16.5.1. The Permanent Income Hypothesis. One of the most important applications of stochastic dynamic optimization is to the consumption smoothing problem of the consumer facing an uncertain income stream. This problem was first discussed by Irving Fisher (1930) and then received its first systematic analysis in Milton Friedman's classic book on consumption theory (1956). With Robert Hall's (1978) seminal paper on dynamic consumption behavior it became one of the most celebrated macroeconomic models. Here I present a simple version of this problem with linear-quadratic preferences and characterize the solution using the sequence formulation of the problem and also stochastic dynamic programming.

Consider a consumer maximizing discounted lifetime utility

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u\left(c\left(t\right)\right),$$

with $c\left(t\right) \geq 0$ as usual denoting consumption. To start with we assume that $u\left(\cdot\right)$ is strictly increasing, continuously differentiable and concave and denote its derivative by $u'\left(\cdot\right)$. We will shortly look at the case in which $u\left(\cdot\right)$ is given by a quadratic.

The consumer can borrow and lend freely at a constant interest rate $r > 0$, thus his lifetime budget constraint takes the form

$$\sum_{t=0}^{\infty} \frac{1}{\left(1+r\right)^t} c\left(t\right) \leq \sum_{t=0}^{\infty} \frac{1}{\left(1+r\right)^t} w\left(t\right) + a\left(0\right), \tag{16.21}$$

where $a\left(0\right)$ denotes his initial assets and $w\left(t\right)$ is his labor income. We assume that $w\left(t\right)$ is random and takes values from the set $\mathcal{W} \equiv \{w_1, ..., w_N\}$. This corresponds to potential labor income fluctuations due to aggregate or idiosyncratic shocks facing the individual. To simplify the analysis, let us suppose that $w\left(t\right)$ is distributed independently over time and the probability that $w\left(t\right) = w_j$ is q_j (naturally with $\sum_{j=1}^{N} q_j = 1$). Consequently, the lifetime budget constraint (16.21) has to be interpreted as a stochastic constraint. We therefore require this constraint to hold *almost surely*. This implies that the constraint has to hold with probability 1. The reader may wonder why this particular concept from measure theory has crept into our discussion, since $w\left(t\right)$ still takes finitely many values. The reason is that even when $w\left(t\right)$ takes only finitely many values, the probability distribution for the infinite sequence of random variables $w^{\infty} \equiv \left(w\left(0\right), w\left(1\right), ...\right)$ is equivalent to a continuous probability distribution. Nevertheless, for our purposes this is also a technicality and not much more than the requirement that the lifetime budget constraint (16.21) should hold almost surely is necessary for our analysis.

Leaving technicalities aside, the fact that the lifetime budget constraint is stochastic has important economic implications. In particular, although we have not introduced an explicit borrowing constraint, the fact that the lifetime budget constraint must hold with probability 1 imposes *endogenous borrowing constraints*. For example, suppose that $w_1 = 0$ and $q_1 > 0$ (so that this state corresponds to unemployment and zero labor income). Then there is a positive probability that the individual will receive zero income for any sequence of periods of length $T < \infty$. Then if the individual ever chooses a negative asset holding, $a(t) < 0$, there will be a positive probability of violating his lifetime budget constraint, even if he were to choose zero consumption in all future periods. Therefore, there is an endogenous borrowing constraint, which takes the form

$$a(t) \geq -\sum_{s=0}^{\infty} \frac{1}{(1+r)^s} w_1 \equiv -b_1,$$

with w_1 denoting the minimum value of w within the set $\mathcal{W}$ and the last relationship defining b_1.

Let us first solve this problem treating it as a sequence problem, that is, the problem of choosing a sequence of feasible plans $\left\{\tilde{c}\left[w^t\right]\right\}_{t=0}^{\infty}$. This can be done simply by forming a Lagrangian. Even though there is a single lifetime budget constraint (16.21), it would be incorrect to treat the problem as if there were a unique Lagrange multiplier λ. This is because consumption plans are made conditional on the realizations of events up to a certain date. In particular, consumption at time t will be conditioned on the history of shocks up to that date, $w^t \equiv (w(0), w(1), ..., w(t))$, and in fact we use the notation $\tilde{c}\left[w^t\right]$ to emphasize that consumption at date t is a mapping from the history of income realizations, w^t. At that point, since there is also more information about how much the individual has earned and how much he has spent, it is also natural to think that the Lagrange multiplier, which represents the marginal utility of money, is also a random variable and can depend only on the realizations of the shocks up to date t, w^t. We therefore write this multiplier as $\tilde{\lambda}\left[w^t\right]$.

The first-order conditions for this problem immediately give

$$\beta^t u'\left(\tilde{c}\left[w^t\right]\right) = \frac{1}{(1+r)^t} \tilde{\lambda}\left[w^t\right], \tag{16.22}$$

which requires the (discounted) marginal utility of consumption after history w^t to be equated to the (discounted) marginal utility of income after history w^t, $\tilde{\lambda}\left[w^t\right]$. While economically interpretable, this first-order condition is not particularly useful unless we know the law of motion of the marginal utility of income, $\tilde{\lambda}\left[w^t\right]$. This law of motion is not straightforward to derive with this formulation. An alternative formulation of the sequence problem, where prices for all possible claims to consumption contingent on any realization of history are introduced, is much more tractable and gives similar results to the recursive approach below. I will introduce this contingent-claims formulation in the analysis of the competitive equilibrium of the neoclassical growth model under uncertainty in the next chapter.

Instead, if we formulate the same problem recursively, sharper results can be easily obtained. Using the tools of this chapter, let us write this problem recursively. First, instead of the lifetime budget constraint, the flow budget constraint of the individual can be written as

$$a' = (1+r)(a+w-c),$$

where a' refers to next period's asset holdings. Conversely, this implies $c = a+w-(1+r)^{-1}a'$. Then the value function of the individual, conditioned on current asset holding a and current realization of the income shock w, can be written as

$$V(a,w) = \max_{a' \in [-b_1,(1+r)(a+w)]} \left\{ u\left(a+w-(1+r)^{-1}a'\right) + \beta \mathbb{E} V\left(a', w'\right) \right\},$$

where I have made use of the fact that w is distributed independently across periods, so the expectation of the continuation value is not conditioned on the current realization of w. Now as in Example 6.5 in Chapter 6, where we studied the non-stochastic version of this problem, we need to restrict the set of feasible asset levels to be able to apply Theorems 16.1-16.6 from Section 16.1. In particular, let us take $\bar{a} \equiv a(0) + w_N/r$, where w_N is the highest level of labor income. We can then impose that $a(t) \in [0,\bar{a}]$ and then again verify the conditions under which this has no effect on the solution (in particular the condition for $a(t)$ to be always in the interior of the set, see Exercise 16.10).

The first-order condition for the maximization problem gives

$$\frac{1}{1+r}u'(c(t)) = \beta \mathbb{E}_t \frac{\partial V(a(t+1), w(t+1))}{\partial a}. \tag{16.23}$$

Noting that $\partial V(a', w')/\partial a$ is also the marginal utility of income, this equation is very similar to (16.22). The additional mileage now comes from the envelope condition from Theorem 16.6, which implies that

$$\frac{\partial V(a(t), w(t))}{\partial a} = u'(c(t)).$$

Combining this equation with (16.23), we obtain the famous stochastic Euler equation of stochastic permanent income hypothesis:

$$u'(c(t)) = \beta(1+r)\mathbb{E}_t u'(c(t+1)). \tag{16.24}$$

The notable feature here is that on the right-hand side we have the expectation of the marginal utility of consumption at date $t+1$. We thus have a simple stochastic Euler equation.

This equation becomes even simpler and perhaps more insightful when we assume that the utility function is quadratic, for example, taking the form

$$u(c) = \phi c - \frac{1}{2}c^2,$$

with ϕ sufficiently large that in the relevant range $u(\cdot)$ is increasing in c. Using this quadratic form with (16.24), we obtain Hall's famous stochastic equation that

$$c(t) = (1-\kappa)\phi + \kappa \mathbb{E}_t c(t+1), \tag{16.25}$$

where $\kappa \equiv \beta(1+r)$. A striking prediction of this equation is that variables, such as current or past income, should not predict future consumption growth. A large empirical literature investigates whether or not this is the case in aggregate or individual data, focusing on *excess sensitivity* tests. If future consumption growth depends on current income, this is interpreted as evidence for excess sensitivity, rejecting (16.25). This rejection is often considered as evidence in favor of credit constraints, which prevent individuals from freely borrowing and lending (subject to the endogenous borrowing constraint derived above). Nevertheless, excess sensitivity can also emerge when the utility function is not quadratic (see, for example, Zeldes, 1989, Caballero, 1990).

Equation (16.25) takes an even simpler form when $\beta = (1+r)^{-1}$, i.e., when the discount factor is the inverse of the gross interest rate. In this case, $\kappa = 1$ and $c(t) = \mathbb{E}_t c(t+1)$ or $\mathbb{E}_t \Delta c(t+1) = 0$, so that the expected value of future consumption should be the same as today's consumption. This last property is sometimes referred to as the "martingale" property, since a random variable $z(t)$ is a martingale with respect to some information set Ω_t if $\mathbb{E}[z(t+1) \mid \Omega_t] = z(t)$. It is a submartingale, if $\mathbb{E}[z(t+1) \mid \Omega_t] \geq z(t)$ and supermartingale if $\mathbb{E}[z(t+1) \mid \Omega_t] \leq z(t)$. Thus whether consumption is a martingale, submartingales or supermartingale depends on the interest rate relative to the discount factor. Exercises 16.7 and 16.10 further discuss the implications of this equation.

16.5.2. Search for Ideas. This subsection provides another example of an economic problem where dynamic programming techniques are very useful. This example also provides us with an alternative and complementary way of thinking about the endogeneity of technology to that offered by the models presented in Part 4.

Consider the problem of a single entrepreneur, with risk-neutral objective function

$$\sum_{t=0}^{\infty} \beta^t c(t).$$

This entrepreneur's consumption is given by the income he generates in that period (there is no saving or borrowing). The entrepreneur can produce income equal to

$$y(t) = a'(t)$$

at time t, where $a'(t)$ is the quality of the technique he has available for production.[3] At $t = 0$, the entrepreneur starts with $a(0) = 0$. From then on, at each date, he can either

[3] The use of a here for the quality of ideas, rather than as asset holdings of individual before, should cause no confusion.

engage in production using one of the techniques he has already discovered, or spend that period searching for a new technique. Let us assume that each period in which he engages in such a search, he gets an independent draw from a time-invariant distribution function $H(a)$ defined over a bounded interval $[0, \bar{a}]$.

Therefore, the decision of the entrepreneur at each date is whether to search for a new technique or to produce with one of the techniques he has discovered so far. The consumption decision of the entrepreneur is trivial, since there is no saving or borrowing, and he has to consume his current income, $c(t) = y(t)$.

This problem introduces a slightly different perspective on some of the ideas already discussed in the book. In particular, as in the endogenous technological change models we have studied so far, the entrepreneur has a non-trivial choice which affects the technology available to him; by searching more, which is a costly activity in terms of foregone production, he can potentially improve the set of techniques available to him. Moreover, this economic decision is related to the standard tradeoffs in technology choices; whether to produce with what he has available today or make an "investment" in one more round of search with the hope of discovering something better. This type of economic tradeoff is complementary to the incentives to invest in new technology in the models of endogenous technology we have already seen in the previous part of the book.

For now, our main objective is to demonstrate how dynamic programming techniques can be used to analyze this problem. Let us first try to write the maximization problem facing the entrepreneur as a sequence problem. We begin with the class of decision rules of the agent. In particular, let $\mathbf{a}^t \in \mathbf{A}^t \equiv [0, \bar{a}]^t$ be a sequence of techniques observed by the entrepreneur over the past t periods, with $a(s) = 0$, if at time s, the entrepreneur engaged in production. We write $\mathbf{a}^t = (a(0), ..., a(t))$. Then a decision rule for this individual would be

$$q(t) : \mathbf{A}^t \to \{a(t)\} \cup \{\text{search}\},$$

which denotes the action of the agent at time t, which is either to produce with the current technique he has discovered, $a(t)$, or to choose $q(t) =$"search" and spend that period searching for or researching a new technique. Let $\mathcal{P}_t$ be the set of functions from $\mathbf{A}^t$ into $a(t) \cup \{\text{search}\}$, and $\mathcal{P}^\infty$ the set of infinite sequences of such functions. The most general way of expressing the problem of the individual would be as follows. Let $\mathbb{E}$ be the expectations operator. Then the individual's problem is

$$\max_{\{q(t)\}_{t=0}^\infty \in \mathcal{P}^\infty} \mathbb{E} \sum_{t=0}^\infty \beta^t c(t)$$

subject to $c(t) = 0$ if $q(t) =$"search" and $c(t) = a'$ if $q(t) = a'$ for $a(s) = a'$ for some $s \leq t$. Naturally, written in this way, the problem looks complicated, even daunting. The point of writing it in this way is to show that in certain classes of models, the dynamic programming formulation will be quite tractable even when the sequence problem may look quite complicated.

To demonstrate this, we now write this optimization problem recursively using dynamic programming techniques. Let us simplify the formulation of the recursive form of this problem by making two observations (which will both be proved in Exercise 16.11). First, because the problem is stationary we can discard all of the techniques that the individual has sampled except the last one and thus write the problem simply conditioning on the last period's stochastic state. In particular, denote the value of an agent who has just sampled a technique $a \in [0, \bar{a}]$ by $V(a)$. Second, we suppose that once the individual starts producing at some technique a', he will continue to do so forever, instead of going back to searching again at some future date. This is also intuitive due to the stationarity of the problem; if the individual is willing to accept production at technique a' rather than searching more at time t, he would also do so at time $t+1$. This last observation implies that if the individual accepts production at some technique a' at date t, he will consume $c(s) = a'$ for all $s \geq t$. Consequently, we obtain the value on accepting technique a' as

$$V^{accept}(a') = \frac{a'}{1-\beta}.$$

Therefore, we can write

$$\begin{aligned} V(a') &= \max_{q \in \{0,1\}} qV^{accept}(a') + (1-q)\beta\mathbb{E}V \\ &= \max\left\{V^{accept}(a'), \beta\mathbb{E}V\right\} \\ &= \max\left\{\frac{a'}{1-\beta}, \beta\mathbb{E}V\right\}, \end{aligned} \tag{16.26}$$

where q is the acceptance decision, with $q = 1$ corresponding to acceptance, and

$$\mathbb{E}V = \int_0^{\bar{a}} V(a)\, dH(a) \tag{16.27}$$

is the expected continuation value of not producing at the available techniques. The expression in (16.26) follows from the fact that the individual will choose whichever option, starting production or continuing to search, gives him higher utility. That the value of continuing to search is given by (16.27) follows by definition. At the next date, the individual will have value $V(a)$ as given by (16.26) when he draws a from the distribution $H(a)$, and thus integrating over this expression gives $\mathbb{E}V$. The integral is written as a Lebesgue integral, since we have not assumed that $H(a)$ has a continuous density.

A SLIGHT DIGRESSION*. Even though the special structure of the search problem enables a direct solution, it is also useful to see that optimal policies can be derived by applying the techniques developed in Section 6.3 in Chapter 6. For this, combine the two previous equations and write

$$V(a') = \max\left\{\frac{a'}{1-\beta}, \beta\int_0^{\bar{a}} V(a)\,dH(a)\right\}, \quad (16.28)$$
$$= TV(a'),$$

where the second line defines the mapping T. Now (16.28) is in a form to which we can apply the above theorems. Blackwell's sufficiency theorem (Theorem 6.9) applies directly and implies that T is a contraction since it is monotonic and satisfies discounting.

Next, let $V \in \mathbf{C}([0,\bar{a}])$, i.e., the set of real-valued continuous (hence bounded) functions defined over the set $[0,\bar{a}]$, which is a complete metric space with the sup norm. Then the Contraction Mapping Theorem, Theorem 6.7 from Chapter 6, immediately implies that a unique value function $V(a)$ exists in this space. Thus the dynamic programming formulation of the sequential search problem immediately leads to the existence of an optimal solution (and thus optimal strategies, which will be characterized below).

Moreover, Theorem 6.8 also applies by taking S' to be the space of nondecreasing continuous functions over $[0,\bar{a}]$, which is a closed subspace of $\mathbf{C}([0,\bar{a}])$. Therefore, $V(a)$ is nondecreasing. In fact, using Theorem 6.8 we could also prove that $V(a)$ is piecewise linear with first a flat portion and then an increasing portion. Let the space of such functions be S'', which is another subspace of $\mathbf{C}([0,\bar{a}])$, but is not closed. Nevertheless, now the second part of Theorem 6.8 applies, since starting with any nondecreasing function $V(a)$, $TV(a)$ will be a piecewise linear function starting with a flat portion. Therefore, the theorem implies that the unique fixed point, $V(a)$, must have this property too. ■

The digression above used Theorem 6.8 from Chapter 6 to argue that $V(a)$ would take a piecewise linear form. In fact, in this case, this property can also be deduced directly from (16.28), since $V(a)$ is a maximum of two functions, one of them flat and the other one linear. Therefore $V(a)$ must be piecewise linear, with first a flat portion.

Our next task is to determine the optimal policy using the recursive formulation of Problem B2. The fact that $V(a)$ is linear (and strictly increasing) after a flat portion immediately tells us that the optimal policy will take a *cutoff rule*, meaning that there will exist a cutoff technology level R such that all techniques above R are accepted and production starts, while those $a < R$ are turned down and the entrepreneur continues to search. This cutoff rule property follows because $V(a)$ is strictly increasing after some level, thus if some technology a' is accepted, all technologies with $a > a'$ will also be accepted.

Moreover, this cutoff rule must satisfy the following equation

$$\frac{R}{1-\beta} = \int_0^{\bar{a}} \beta V(a)\,dH(a), \quad (16.29)$$

so that the individual is just indifferent between accepting the technology $a = R$ and waiting for one more period. Next we also have that since $a < R$ are turned down, for all $a < R$

$$\begin{aligned} V(a) &= \beta \int_0^{\bar{a}} V(a)\, dH(a) \\ &= \frac{R}{1-\beta}, \end{aligned}$$

and for all $a \geq R$, we have

$$V(a) = \frac{a}{1-\beta}.$$

Using these observations, we obtain

$$\int_0^{\bar{a}} V(a)\, dH(a) = \frac{RH(R)}{1-\beta} + \int_{a \geq R} \frac{a}{1-\beta} dH(a).$$

Combining this equation with (16.29), we have

$$\frac{R}{1-\beta} = \beta \left[\frac{RH(R)}{1-\beta} + \int_{a \geq R} \frac{a}{1-\beta} dH(a) \right]. \tag{16.30}$$

Manipulating this equation, we obtain

$$R = \frac{\beta}{1 - \beta H(R)} \int_R^{\bar{a}} a dH(a),$$

which is a convenient way of expressing the cutoff rule R. Equation (16.30) can be rewritten in a more useful way as follows:

$$\frac{R}{1-\beta} = \beta \left[\int_{a<R} \frac{R}{1-\beta} dH(a) + \int_{a \geq R} \frac{a}{1-\beta} dH(a) \right].$$

Now subtracting $\beta R / (1-\beta) = \beta R \int_{a<R} dH(a) / (1-\beta) + \beta R \int_{a \geq R} dH(a) / (1-\beta)$ from both sides, we obtain

$$R = \frac{\beta}{1-\beta} \left[\int_R^{\bar{a}} (a - R)\, dH(a) \right], \tag{16.31}$$

which is an important way of characterizing the cutoff rule. The left-hand side is best understood as the cost of foregoing production with a technology of R, while the right-hand side is the expected benefit of one more round of search. At the cutoff threshold, these two terms have to be equal, since the entrepreneur is indifferent between starting production and continuing search.

Let us now define the right-hand side of equation (16.31), the expected benefit of one more search, as

$$\gamma(R) \equiv \frac{\beta}{1-\beta} \left[\int_R^{\bar{a}} (a - R)\, dH(a) \right].$$

Suppose also that H has a continuous density, denoted by h. Then we have

$$\begin{aligned} \gamma'(R) &= -\frac{\beta}{1-\beta} (R - R)\, h(R) - \frac{\beta}{1-\beta} \left[\int_R^{\bar{a}} dH(a) \right] \\ &= -\frac{\beta}{1-\beta} [1 - H(R)] < 0 \end{aligned}$$

This implies that equation (16.31) has a unique solution. It can be easily verified that a higher β, by making the entrepreneur more patient, increases the cutoff threshold R.

16.5.3. Other Applications. There are numerous other applications of stochastic dynamic programming. In addition to the three growth models we will study in the next chapter, the following are noteworthy.

(1) *Asset Pricing*: following Lucas (1978), we can consider an economy in which a set of identical agents trade claims on stochastic returns of a set of given assets ("trees"). Each agent solves a consumption smoothing problem similar to that in subsection 16.5.1, with the major difference that he or she has to save in assets with stochastic returns rather than at a constant interest rate. Market clearing will be achieved when the total supply of assets is equal to total demand. This implies that in equilibrium the prices have to be such that each agent is happy to hold the appropriate amount of claims on the returns from these assets. Given the marginal utility of consumption derived from the recursive formulation, these assets can be priced. Exercise 16.13 considers this case.

(2) *Investment under Uncertainty*: the model of investment under adjustment costs discussed in Section 7.8 of Chapter 7 has much wider application in macroeconomics and industrial organization once augmented by the possibility that firms are uncertain about future demand and/or productivity. Exercise 16.14 considers this case.

(3) *Optimal Stopping Problems*: the search model discussed in the previous subsection is an example of an optimal stopping problem. More general optimal stopping problems can also be set up and analyzed as stochastic dynamic programming problems. Exercise 16.15 considers an example of such a stopping problem.

16.6. Taking Stock

The material in this section is technical in nature and is useful for its applications more than for its own sake. At the level at which it has been presented here, it has widespread applications in macroeconomics and economic growth. The stochastic neoclassical growth model, which we will see in the next chapter, makes heavy use of the methods developed here and is the workhorse model of modern macroeconomics.

In addition to presenting the basic tools of stochastic dynamic programming, this chapter has presented two important economic models. The first, the stochastic permanent income hypothesis model is one of the most famous macroeconomic models and has led both to a large theoretical and empirical literature. The early empirical literature focused on excess sensitivity tests as discussed in subsection 16.5.1 using aggregate data. The more recent literature focuses on micro and panel data in order to derive sharper results about the behavior of individual consumption. The other substantial model introduced in this chapter is the search for ideas model in subsection 16.5.2, which is adapted from McCall's (1978) labor market search model. McCall's model is the basis of much of the modern equilibrium theory of unemployment. While the model here has been cast in terms of searching for ideas, the reader can easily adapt it to unemployment and use it as an introduction to equilibrium unemployment theory (see Exercise 16.12). In addition, some of the other applications, mentioned above and treated in exercises, including the asset pricing model based on Lucas (1978) and the model of investment under uncertainty, are widely used models in other areas of macroeconomics.

16.7. References and Literature

Most of the references from Chapter 6 are relevant for stochastic dynamic programming as well. The reader may want to look at Howard (1960), Blackwell (1965) and Puterman (1994), for advanced treatments. The most complete treatment of discounted stochastic dynamic programming problems with economic applications is in Stokey, Lucas and Prescott (1989). This chapter covered almost the same material as Stokey, Lucas and Prescott, though at a slightly less technical level. In particular, I presented all the major results of stochastic dynamic programming without introducing measure theory and general Markov processes. A thorough study of stochastic dynamic programming requires a nontrivial investment in these methods. Stokey, Lucas and Prescott (1989) provide both a good introduction and the relevant references. Puterman (1994) provides a more advanced treatment. A quick understanding of some of the measure-theoretic issues can be obtained from the very easy-to-read book by Williams (1991), which also contains an excellent introductory treatment of martingales.

The best survey of work on consumption is still Deaton (1991). A survey of recent work can be found in Browning and Crossley (2001). Exercise 16.10 is based on Chamberlain and Wilson (2000) and the reader is referred to this paper for some of the subtle mathematical issues that arise in determining the limiting behavior of the stochastic consumption distribution when the discount factor is equal to the inverse of the gross interest rate. The search for ideas example in subsection 16.5.2 is adapted from McCall's (1978) labor market search model. Kortum (1994) provides the first search-theoretical model of technology choice that I am aware of. Kortum's model is significantly more advanced, but also more insightful than the model presented in subsection 16.5.2. Ljungqvist and Sargent (2005) contains an excellent exposition of the basic McCall model. Pissarides (2001) and Rogerson, Shimer and Wright (2004) provide excellent surveys of recent work in search theory applied to labor market problems.

16.8. Exercises

Exercise 16.1. Show that Assumption 16.6 (iii) is satisfied if and only if for any $j'' > j'$ and any $\bar{j} = 1, ..., N$, we have that $\sum_{j=\bar{j}}^{N} q_{jj''} \geq \sum_{j=\bar{j}}^{N} q_{jj'}$. What does this imply about the relationship between the conditional distribution of z given $z_{j''}$ and given $z_{j'}$?

Exercise 16.2. * Prove Lemma 16.1.

Exercise 16.3. * Prove Theorem 16.4.

Exercise 16.4. * Prove Theorem 16.5.

Exercise 16.5. * Prove Theorem 16.6.

Exercise 16.6. * Prove Theorem 16.7.

Exercise 16.7. Consider the stochastic permanent income hypothesis model studied in Section 16.5 and suppose that $u(c)$ is not quadratic. Explain the conditions under which the excess sensitivity tests described in that section would fail even when the stochastic Euler equation (16.24) holds. [Hint: you may want to consider the constant relative risk aversion preferences for concreteness].

Exercise 16.8.

(1) Consider the stochastic permanent income hypothesis model studied in Section 16.5 and assume that the interest rate r is no longer constant, but is equal to $r(t) > 0$ at time t. Derive the equivalent of (16.24) in this case. Show that excess sensitivity tests can be applied in this case as well.

(2) Now suppose that $r(t)$ is a random variable taking one of finitely many values, $r_1, ..., r_J$, and to simplify the analysis, suppose that the realizations of the interest rate are independent over time. Derive the equivalent of (16.24) in this case. Show that excess sensitivity tests can be applied in this case as well.

Exercise 16.9. Consider the stochastic permanent income hypothesis model studied in Section 16.5. Suppose that instead of being distributed independently, $w(t)$ follows a Markov chain. Show that (16.24) still holds. Now suppose that $u(c)$ takes a quadratic form and assume that the econometrician incorrectly believes that $w(t)$ is independently distributed, so that the individual has superior information relative to the econometrician. Show that a regression of consumption growth on past income realizations will still lead to a zero coefficient (thus the excess sensitivity test will not reject). [Hint: make use of the law of iterated expectations, which states that if Ω is an information set that is finer than Ω' and z is a random variable, then $\mathbb{E}[\mathbb{E}[z \mid \Omega] \mid \Omega'] = \mathbb{E}[z \mid \Omega']$].

Exercise 16.10. In the stochastic permanent income hypothesis model studied in Section 16.5, suppose that $c(t) \geq 0$, $u(\cdot)$ is twice continuously differentiable, everywhere strictly concave and strictly increasing, and $u''(\cdot)$ is increasing. Suppose also that $w(t)$ has a non-degenerate probability distribution.

(1) Show that consumption can never converge to a constant level.

(2) * Prove that if $u(\cdot)$ takes the CRRA form and $\beta < (1+r)^{-1}$, then there exists some $\bar{a} < \infty$ such that $a(t) \in (0, \bar{a})$ for all t.
(3) * Prove that when $\beta \leq (1+r)^{-1}$, there exists no $\bar{a} < \infty$ such that $a(t) \in (0, \bar{a})$ for all t. [Hint: consider the case where $\beta = (1+r)^{-1}$ and take the stochastic sequence where $w(t) = w_N$ for an arbitrarily large number of periods, which is a positive probability sequence. Then generalize this argument to the case where $\beta \leq (1+r)^{-1}$].
(4) * Prove that when $\beta \leq (1+r)^{-1}$, marginal utility of consumption follows a (non-degenerate) submartingale and therefore consumption must converge to infinity. [Hint: note that in this case (16.24) implies $u'(c(t)) \geq \mathbb{E}u'(c(t+1))$ and use this equation to argue that consumption must be increasing "on average"].

Exercise 16.11. Consider the model of searching for ideas introduced in subsection 16.5.2. Suppose that the entrepreneur can use any of the techniques he has discovered in the past to produce at any point in time and also then stop production at any point and go back to searching.

(1) Prove that if the entrepreneur has turned down production at some technique a' at date t, he will never accept technique a' at date $t+s$, for $s > 0$ (i.e., he will not accept it for any possible realization of events between dates t and $t+s$).
(2) Prove that if the entrepreneur accepts technique a' at date t, he will continue to produce with this technique for all dates $s \geq t$ rather than stopping production and going back to searching.
(3) Using 1 and 2, show that the maximization problem of the entrepreneur can be formulated as in the text without loss of any generality.
(4) Now suppose that when not producing, the entrepreneur receives income b. Write the recursive formulation for this case and show that as b increases, the cutoff threshold R increases.

Exercise 16.12. Formulate the problem in subsection 16.5.2 as one of an unemployed worker sampling wages from an exogenously given stationary wage distribution $H(w)$. The objective of the worker is to maximize the net present discounted value of his income stream. Assume that once the worker accepts a job he can work at that wage forever.

(1) Formulate the dynamic maximization problem of the worker recursively assuming that once the worker finds a job he will never quit.
(2) Prove that the worker will never quit a job that he has accepted.
(3) Prove that the worker will use the reservation wage R for deciding what job to accept.
(4) Calculate the expected duration of unemployment for the worker.

(5) Show that if the wages in the wage distribution $H(w)$ are offered by firms and all workers are identical, the wage offers of all firms other than those offering $w = R$ are not profit-maximizing. What does this observation imply about the McCall search model?

Exercise 16.13. Consider an economy populated by identical households each with preferences given by $\mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t u(c(t))\right]$, where $u(\cdot)$ is strictly increasing, strictly concave and twice continuously differentiable. Normalize the measure of agents in the economy to 1. Each household has a claim to a single tree, which delivers $z(t)$ units of consumption good at time t. Assume that $z(t)$ is a random variable taking values from the set $\mathcal{Z} \equiv \{z_1, ..., z_N\}$ and is distributed according to a Markov chain (all trees have exactly the same output, so there is no gain in diversification). Each household can sell any fraction of its trees or buy fractions of new trees, though cannot sell trees short (i.e., negative holdings are not allowed). Suppose that the price of a tree when the current realization of $z(t)$ is z is given by the function $p : \mathcal{Z} \to \mathbb{R}_+$. There are no other assets to transfer resources across periods.

(1) Show that for a given price function $p(z)$, the flow budget constraint of a representative household can be written as

$$c(t) + p(z(t))\, x(t+1) \leq [z(t) + p(z(t))]\, x(t),$$

where $x(t)$ denotes the tree holdings of the household at time t. Interpret this constraint.

(2) Show that for a given price function $p(z)$, the maximization problem of the representative household subject to the flow budget constraint and the constraint that $c(t) \geq 0$, $x(t) \geq 0$ can be written in a recursive form as follows

$$V(x, z) = \sup_{y \in \left[0, p(z)^{-1}(z+p(z))x\right]} \left\{ u\left((z + p(z))\, x - p(z)\, y\right) + \beta \mathbb{E}\left[V\left(y, z'\right) \mid z\right] \right\}.$$

(3) Use the results from Section 16.1 to show that $V(x, y)$ has a solution, is increasing in both of its arguments and strictly concave, and is differentiable in x in the interior of its domain.

(4) Derive the stochastic Euler equations for this maximization problem.

(5) Now impose market clearing, which implies that $x(t) = 1$ for all t. Explain why this condition is necessary and sufficient for market clearing.

(6) Under market clearing, derive $p(z)$ the equilibrium prices of trees as a function of the current realization of z.

Exercise 16.14. Consider a discrete stochastic version of the investment model from Section 7.8, where a firm maximizes the net present discounted value of its profits, with discount factor given by $(1 + r)^{-1}$ and instantaneous returns given by

$$f(k(t), z(t)) - i(t) - \phi(i(t)).$$

Here $f(k(t), z(t))$ is the revenue or profit of the firm as a function of its capital stock, $k(t)$, and a stochastic variable, representing productivity or demand, $z(t)$. As in Section 7.8, $i(t)$ is investment and $\phi(i(t))$ represents adjustment costs.

(1) Assume that $z(t)$ has a distribution represented by a Markov chain. Formulate the sequence version of the maximization problem of the firm.
(2) Formulate the recursive version of the maximization problem of the firm.
(3) Provide conditions under which the two problems have the same solutions.
(4) Derive the stochastic Euler equation for the investment decision of the firm and compare the results to those in Section 7.8.

Exercise 16.15. Consider a general stopping problem, where the objective of the individual is to maximize $\mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t u(y(t))\right]$. We assume that the individual faces a stream of random variables represented by $z(t)$ and assume that $z(t)$ follows a Markov chain. At any t, the individual can "stop" the process. Let $y(t) = 0$ while the individual has not stopped and $y(t) = z(s)$ if the individual has stopped the process at some $s \leq t$.

(1) Formulate the problem of the individual as a stochastic dynamic programming problem and show that there exists some R^* such that the individual will stop the process at time t if $z(t) \geq R^*$.
(2) Now assume that $z(t)$ has a distribution at time t given by $H(z \mid \zeta(t))$ and $\zeta(t)$ follows a Markov chain with values in the finite set $\mathcal{Z}$. Formulate the problem of the individual as a stochastic dynamic programming problem. Prove that there exists a function $R^* : \mathcal{Z} \to \mathbb{R}_+$ such that the individual will stop the process when $z(t) \geq R^*(\zeta(t))$ when the current state is $\zeta(t)$. Explain why the stopping rule is no longer constant. What does this result imply for the job acceptance decisions of unemployed workers studied in Exercise 16.12 when the distribution of wages is different during periods of recession?

CHAPTER 17

Stochastic Growth Models

In this chapter, I present four models of stochastic growth emphasizing different aspects of the interaction between growth and uncertainty. The first is the baseline neoclassical growth model (with complete markets) augmented with stochastic productivity shocks, first studied by Brock and Mirman (1972). This model is not only an important generalization of the baseline neoclassical growth of Chapter 8 but also provides the starting point of the influential *Real Business Cycle* models, which have been used widely to study a range of short- and medium-run macroeconomics questions. I present this model and some of its implications in the next three sections. The baseline neoclassical growth model incorporates complete markets in the sense that households and firms can trade using any Arrow-Debreu commodity. In the presence of uncertainty, this implies that a full set of *contingent claims* is traded competitively. For example, an individual can buy an asset that will pay one unit of the final good after a pre-specified history. The presence of complete markets—or the set of contingent claims—implies that individuals can fully insure themselves against idiosyncratic risks. The source of interesting uncertainty in these models is aggregate shocks. For this reason, the standard neoclassical growth model under uncertainty does not even introduce idiosyncratic shocks (had they been present, they could have been easily diversified away). This shows the importance of contingent claims in the basic neoclassical model under uncertainty. Moreover, trading in contingent claims is not only sufficient, but it is essentially also necessary for the representative household assumption to hold in environments with uncertainty. This is illustrated in Section 17.4, which considers a model where households cannot use contingent claims and can only trade in riskless bonds. This model, which builds on Bewley's seminal work in the 1970s and the 1980s, explicitly prevents risk-sharing across households and thus features "incomplete markets"—in particular, one of the most relevant type of market incompleteness for macroeconomic questions, which prevents the sharing or diversification of idiosyncratic risk. Households face a stochastic stream of labor income and can only achieve consumption smoothing via "self-insurance," that is, by borrowing and lending at a market interest rate. Like the overlapping generations model of Chapter 9, the Bewley model does *not* admit a representative consumer. The Bewley model is not only important in illustrating the role of contingent claims in models under uncertainty, but also because it is a tractable model for the study of a range of macroeconomic questions related to risk, income fluctuations and policy. Consequently, over the past decade or so, it has become a workhorse model for macroeconomic analysis.

The last two sections, Sections 17.5 and 17.6 turn to stochastic overlapping generations models. The first presents a simple extension of the canonical overlapping generations model that includes stochastic elements.

Section 17.6 shows how stochastic growth models can be useful in understanding the process of takeoff from low growth and to sustained growth, which we discussed in Chapter 1. A notable feature of the long-run experience of many societies is that the early stages of economic development were characterized by slow or no growth in income per capita and by frequent economic crises. The process of takeoff not only led to faster growth but also to a more steady (less variable) growth process. An investigation of these issues requires a model of stochastic growth. Section 17.6 presents a model that provides a unified framework for the analysis of the variability of economic performance and take off. The key feature is the tradeoff between investment in risky activities and safer activities with lower returns. At the early stages of development, societies do not have enough resources to invest in sufficiently many activities to achieve diversification and are thus forced to bear considerable risk. As a way of reducing this risk, they also invest in low-return safe activities, such as a storage or safe technology and low-yield agricultural products. The result is an equilibrium process that features a lengthy period of slow or no growth associated with high levels of variability in economic performance. The growth is truly stochastic and an economy can escape this stage of development and takeoff into sustained growth only when its risky investments are successful for a number of periods. When this happens, the economy achieves better diversification and also better risk management through more developed financial markets. Better diversification reduces risk and also enables the economy to channel its investments in higher return activities, increasing its productivity and growth rate. Thus this simple model of stochastic growth presents a stylistic account of the process of takeoff from low and variable growth and to sustained and steady growth. The model I will use to illustrate these ideas features both a simple form of stochastic growth and also endogenously incomplete markets. I will therefore use this model to show how some simple ideas from Markov processes can be used to characterize the stochastic equilibrium path of a dynamic economy and also to highlight potential inefficiencies resulting from models with endogenous incomplete markets. Finally, this model will give us a first glimpse of the relationship between financial development and economic growth, a topic to which we will return in Chapter 20.

17.1. The Brock-Mirman Model

The first systematic analysis of economic growth with stochastic shocks was undertaken by Brock and Mirman in their 1972 paper. Brock and Mirman focused on the optimal growth problem and solved for the social planner's maximization problem in a dynamic neoclassical environment with uncertainty. Since, with competitive and complete markets, the First and Second Welfare Theorems still hold, the equilibrium growth path is identical to the optimal growth path. Nevertheless, the analysis of equilibrium growth is more involved and also introduces a number of new concepts. I start with the Brock-Mirman model approach here and then discuss competitive equilibrium growth under uncertainty in the next section.

The economy is similar to the baseline neoclassical growth model of Chapter 8, except that the production technology is now given by

$$Y(t) = F(K(t), L(t), z(t)), \tag{17.1}$$

where $z(t)$ denotes a stochastic *aggregate* productivity term affecting how productive a given combination of capital and labor will be in producing the unique final good of the economy. Let us suppose that $z(t)$ follows a monotone Markov chain (as defined in Assumption 16.6) with values in the set $\mathcal{Z} \equiv \{z_1, ..., z_N\}$. Many applications of the neoclassical growth model under uncertainty also assume that the stochastic shock is a labor-augmenting productivity term, so that the aggregate production function takes the form $Y(t) = F(K(t), z(t) L(t))$, though for the analysis here, we do not need to impose this additional restriction. Throughout we assume that the production function F satisfies Assumptions 1 and 2, and define per capita output and the per capita production function as

$$\begin{aligned} y(t) &\equiv \frac{Y(t)}{L(t)} \\ &\equiv f(k(t), z(t)), \end{aligned}$$

with $k(t) \equiv K(t)/L(t)$ once again corresponding to the capital-labor ratio. A fraction δ of the existing capital stock depreciates at each date. Finally, we also suppose that the numbers $z_1, ..., z_N$ are arranged in ascending order and that $j > j'$ implies $f(k, z_j) > f(k, z_{j'})$ for all $k \in \mathbb{R}_+$. This assumption implies that higher values of the stochastic shock z correspond to greater productivity at *all* capital-labor ratios.

On the preference side, the economy admits a representative household with instantaneous utility function $u(c)$ that satisfies the standard assumptions laid out in Assumption 3. The representative household supplies one unit of labor inelastically, so that $K(t)$ and $k(t)$ can be used interchangeably (and there is no reason to distinguish total consumption $C(t)$ from per capita consumption $c(t)$). Finally, consumption and saving decisions at time t are made after observing the realization of the stochastic shock for time t, $z(t)$.

The sequence version of the expected utility maximization problem of a social planner in this economy can be written as

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c(t)) \tag{17.2}$$

subject to

$$k(t+1) = f(k(t), z(t)) + (1-\delta) k(t) - c(t) \text{ and } k(t) \geq 0, \tag{17.3}$$

with given $k(0) > 0$. To characterize the optimal growth path using the sequence problem we would need to define feasible plans, in particular, the mappings $\tilde{k}\left[z^t\right]$ and $\tilde{c}\left[z^t\right]$ introduced in the previous chapter, with $z^t \equiv (z(0), ..., z(t))$ again standing for the history of (aggregate) shocks up to date t. Rather than going through these steps again, let us directly look at the recursive version of this program, which can be written as

$$V(k, z) = \max_{k' \in [0, f(k,z)+(1-\delta)k]} \left\{ u\left(f(k,z) + (1-\delta) k - k'\right) + \beta \mathbb{E}\left[V\left(k', z'\right) \mid z\right] \right\}. \tag{17.4}$$

The main theorems from the previous chapter immediately apply to this problem and yield the following result:

Proposition 17.1. *In the stochastic optimal growth problem described above, the value function $V(k, z)$ is uniquely defined, strictly increasing in both of its arguments, strictly concave in k and differentiable in $k > 0$. Moreover, there exists a uniquely defined policy function $\pi(k, z)$ such that the capital stock at date $t+1$ is given by $k(t+1) = \pi(k(t), z(t))$.*

Proof. The proof simply involves verifying that Assumptions 16.1-16.6 from the previous chapter are satisfied, so that Theorems 16.1-16.7 can be applied. To do this, first define $\bar{k}$ such that $\bar{k} = f\left(\bar{k}, z_N\right) + (1-\delta)\bar{k}$, and show that starting with $k(0) \in \left(0, \bar{k}\right)$, the capital-labor ratio will always remain within the compact set $\left(0, \bar{k}\right)$. □

In addition, we have:

Proposition 17.2. *In the stochastic optimal growth problem described above, the policy function for next period's capital stock, $\pi(k, z)$, is strictly increasing in both of its arguments.*

Proof. From Assumption 3 u is differentiable and from Proposition 17.1 V is differentiable in k. Moreover, by the same argument as in the proof of Proposition 17.1, $k \in \left(0, \bar{k}\right)$, thus we are in the interior of the domain of the objective function. Thus, the value function V is differentiable in its first argument and we have

$$u'\left(f(k,z) + (1-\delta) k - k'\right) - \beta \mathbb{E}\left[V'\left(k', z'\right) \mid z\right] = 0,$$

where V' denotes the derivative of the $V(k, z)$ function with respect to its first argument. Since from Proposition 17.1 V is strictly concave in k, this equation can hold when the level of k or z increases only if k' also increases. For example, an increase in k reduces the first-term (because u is strictly concave), hence an increase in k' is necessary to increase the first term and to reduce the second term (by the concavity of V). The argument for the implications of an increase in z is similar. □

It is also straightforward to derive the stochastic Euler equations corresponding to the neoclassical growth model with uncertainty. For this purpose, let us first define the policy function for consumption as

$$\pi^c(k,z) \equiv f(k,z) + (1-\delta)k - \pi(k,z),$$

where $\pi(k,z)$ is the optimal policy function for next date's capital stock determined in Proposition 17.1. Using this notation, the stochastic Euler equation can be written as

$$u'(\pi^c(k,z)) = \beta \mathbb{E}\left[\left(f'\left(\pi(k,z),z'\right) + (1-\delta)\right) u'\left(\pi^c\left(\pi(k,z),z'\right)\right) \mid z\right], \tag{17.5}$$

where f' denotes the derivative of the per capita production function with respect to the capital-labor ratio, k. In this form, the Euler equation looks complicated. A slightly different way of expressing this equation makes it both simpler and more intuitive:

$$u'(c(t)) = \beta \mathbb{E}_t\left[p(t+1)\, u'(c(t+1))\right], \tag{17.6}$$

where $\mathbb{E}_t$ denotes the expectation conditional on information available at time t and $p(t+1)$ is the stochastic marginal product of capital (including undepreciated capital) at date $t+1$. This form of writing the stochastic Euler equation is also useful for comparison with the competitive equilibrium because $p(t+1)$ corresponds to the stochastic (date $t+1$) dividends paid out by one unit of capital invested at time t.

Although Proposition 17.1 characterizes the form of the value function and policy functions, it has two shortcomings. First, it does not provide us with an analog of the "Turnpike Theorem" of the non-stochastic neoclassical growth model. In particular, it does not characterize the long-run behavior of the neoclassical growth model under uncertainty. Second, while the characterization provides a number of qualitative results about the value and the policy functions, it does not deliver comparative static results.

A full analysis of the long run behavior of the stochastic growth model would take us too far afield into the analysis of Markov processes. Nevertheless, a few simple observations are useful to appreciate the salient features of the stochastic law of motion of the capital-labor ratio in this model. The capital stock at date $t+1$ is given by the policy function π, thus we have

$$k(t+1) = \pi(k(t), z(t)), \tag{17.7}$$

which defines a general Markov process, since before the realization of $z(t)$, $k(t+1)$ is a random variable, with its law of motion governed by the last period's value of $k(t)$ and the realization of $z(t)$. If $z(t)$ has a non-degenerate distribution, $k(t)$ does not typically converge to a single value (see Exercise 17.4). Instead, we may hope that it will converge to an *invariant limiting distribution.* It can indeed be verified that this is the case. The Markov process (17.7) defines a sufficiently well-behaved stochastic process that starting with any $k(0)$, it converges to a unique invariant limiting distribution, meaning that when we look at sufficiently faraway horizons, the distribution of k should be independent of $k(0)$. Moreover, the average value of $k(t)$ in this invariant limiting distribution will be the same as the time average of $\{k(t)\}_{t=0}^{T}$ as $T \to \infty$ (so that the stochastic process for the capital stock is "ergodic"). Consequently, a "steady-state" equilibrium now corresponds not to specific values of the capital-labor ratio and output per capita but to invariant limiting distributions. If the stochastic variable $z(t)$ takes values within a sufficiently small set, this limiting invariant distribution would hover around some particular values, which we may wish to refer to as "quasi-steady-state" values of the capital-labor ratio, because even though the equilibrium capital-labor ratio may not converge to this value, it will have a tendency to return to a neighborhood thereof. But in general the range of the limiting distribution could be quite wide.

To obtain a better understanding of the behavior of the neoclassical growth model under uncertainty, we next consider a simple example, which allows us to obtain a closed-form solution for the policy function π.

Example 17.1. Suppose that $u(c) = \log c$, $F(K, L, z) = zK^{\alpha}L^{1-\alpha}$, and $\delta = 1$. We continue to assume that z follows a Markov chain over the set $\mathcal{Z} \equiv \{z_1, ..., z_N\}$, with transition probabilities denoted by $q_{jj'}$. Let $k \equiv K/L$. The stochastic Euler equation (17.5) implies

$$\frac{1}{zk^{\alpha} - \pi(k, z)} = \beta \mathbb{E}\left[\left. \frac{\alpha z' \pi(k, z)^{\alpha - 1}}{z' \pi(k, z)^{\alpha} - \pi(\pi(k, z), z')} \right| z \right], \tag{17.8}$$

which is a relatively simple functional equation in a single function $\pi(\cdot, \cdot)$. Though simple, this functional equation would still be difficult to solve unless we had some idea about what the solution looked like. Here, fortunately, the method of "guessing and verifying" the solution of the functional equation becomes handy. Let us conjecture that

$$\pi(k, z) = B_0 + B_1 z k^{\alpha}.$$

Substituting this guess into (17.8), we obtain

$$\frac{1}{(1 - B_1) zk^{\alpha} - B_0} = \beta \mathbb{E}\left[\left. \frac{\alpha z' (B_0 + B_1 z k^{\alpha})^{\alpha - 1}}{z' (B_0 + B_1 z k^{\alpha})^{\alpha} - B_0 - B_1 z' (B_0 + B_1 z k^{\alpha})^{\alpha}} \right| z \right]. \tag{17.9}$$

It is straightforward to check that this equation cannot be satisfied for any $B_0 \neq 0$ (see Exercise 17.5). Thus imposing $B_0 = 0$ and writing out the expectation explicitly with $z = z_{j'}$, this expression becomes

$$\frac{1}{(1 - B_1) z_{j'} k^{\alpha}} = \beta \sum_{j=1}^{N} q_{jj'} \frac{\alpha z_j \left(B_1 z_{j'} k^{\alpha}\right)^{\alpha - 1}}{z_j \left(B_1 z_{j'} k^{\alpha}\right)^{\alpha} - B_1 z_j \left(B_1 z_{j'} k^{\alpha}\right)^{\alpha}}.$$

Simplifying each term within the summation, we obtain

$$\frac{1}{(1 - B_1) z_{j'} k^{\alpha}} = \beta \sum_{j=1}^{N} q_{jj'} \frac{\alpha}{B_1 (1 - B_1) z_{j'} k^{\alpha}}.$$

Now taking $z_{j'}$ and k out of the summation and using the fact that, by definition, $\sum_{j=1}^{N} q_{jj'} = 1$, we can cancel the remaining terms and obtain

$$B_1 = \alpha\beta,$$

so that irrespective of the exact Markov chain for z, the optimal policy rule is

$$\pi(k, z) = \alpha\beta z k^{\alpha}.$$

The reader can verify that this is identical to the result in Example 6.4 in Chapter 6, with z there corresponding to a non-stochastic productivity term. Consequently, in this case the stochastic elements have not changed the form of the optimal policy function. Exercise 17.6 shows that the same result applies when z follows a general Markov process rather than a Markov chain.

Using this example, we can fully analyze the stochastic behavior of the capital-labor ratio and output per capita. In fact, the stochastic behavior of the capital-labor ratio in this economy is identical to that of the overlapping generations model analyzed in Section 17.5 and Figure 17.1 in that section applies exactly to this example. A more detailed discussion of these issues is left to Exercise 17.7. Unfortunately, Example 17.1 is one of the few instances of the neoclassical growth model that admit closed-form solutions. In particular, if the depreciation rate of the capital stock δ is not equal to 1, the neoclassical growth model under uncertainty does not admit an explicit form characterization (see Exercise 17.8).

17.2. Equilibrium Growth under Uncertainty

Let us now consider the competitive equilibria of the neoclassical growth model under uncertainty. The environment is identical to that in the previous section and z corresponds to an aggregate productivity shock affecting all production units and all households. We continue to assume that z follows a Markov chain. Defining the Arrow-Debreu commodities in the standard way, so that goods indexed by different realizations of the history z^t correspond to different commodities, we have an economy with a countable infinity of commodities. The Second Welfare Theorem, Theorem 5.7 from Chapter 5, applies and implies that the optimal growth path characterized in the previous section can be decentralized as a competitive equilibrium (see Exercise 17.10). Moreover, since we are focusing on an economy with a representative household, this allocation is a competitive equilibrium without any redistribution of endowments. These observations justify the frequent focus on social planner's problems in analyses of stochastic growth models in the literature.

Here I will briefly discuss the explicit characterization of competitive equilibria of this economy both to show the equivalence between the optimal growth problem and the equilibrium growth problem under complete markets and also to introduce a number of important ideas related to the pricing of various contingent claims in competitive equilibrium under uncertainty. The reference to complete markets in this context implies that, in principle, any commodity, including any contingent claim, can be traded competitively. Nevertheless, as shown by our analysis in Section 5.8 in Chapter 5, in practice there is no need to specify or trade all of these commodities and a subset of the available commodities is sufficient to provide all the necessary trading opportunities to households and firms. The analysis in this section will also show what subsets of commodities or contingent claims are typically sufficient to ensure an equilibrium with complete markets.

Preferences and technology are as in the previous section. Recall that the economy admits a representative household and that the production side of the economy can be represented by a representative firm (Theorem 5.4). Let us first consider the problem of the representative household. This household will maximize the objective function given by (17.2) subject to the lifetime budget constraint (written from the viewpoint of time $t = 0$). The analysis in Section 5.8 in Chapter 5 shows that there is no loss of generality in considering the competitive equilibrium from the viewpoint of time $t = 0$ relative to formulating it with sequential trading constraints. It is nonetheless useful to spell out this equivalence by showing how the baseline neoclassical growth model under uncertainty can be analyzed both assuming that all trades take place at date $t = 0$ and also under sequential trading.

To write the lifetime budget constraint of the household, let $\mathcal{Z}^t$ be the set of all possible histories of the stochastic variable z^t up to date t and $\mathcal{Z}^\infty$ be the set of infinite histories. With a slight abuse of notation, I will write $z^t \in \mathcal{Z}^\infty$ to denote a possible history of length t. For any z^t, let $p_0\left[z^t\right]$ be the price of the unique final good at time t in terms of the final good of date 0 following a history z^t, $c\left[z^t\right]$ be the time t consumption of the household following history z^t, and $w_0\left[z^t\right]$ be the wage rate and thus total labor earnings of the household, in terms of the final good dated 0 following history z^t. Using this notation, the household's lifetime budget constraint can be written as

$$\sum_{t=0}^{\infty} \sum_{z^t \in \mathcal{Z}^\infty} p_0\left[z^t\right] c\left[z^t\right] \leq \sum_{t=0}^{\infty} \sum_{z^t \in \mathcal{Z}^\infty} w_0\left[z^t\right] + k(0). \tag{17.10}$$

A number of features about this lifetime budget constraint are worth noting.[1] First, there are no expectations. This is because we are considering an economy with complete markets, which implies that the household is making all of his (lifetime) trades in the initial period of the economy $t = 0$ at a well-defined price vector for all Arrow-Debreu commodities. Consequently, the lifetime budget constraint applies in exactly the same way as a static budget constraint

[1]Here $c\left[z^t\right]$ can be interpreted as a policy mapping from possible histories of the stochastic variable to consumption levels, which was defined as $\tilde{c}\left[z^t\right]$ in the previous chapter. I use the simpler expression $c\left[z^t\right]$ in this chapter both to simplify notation and also to emphasize the slightly different interpretation of this object in the present context as "contingent claims" on consumption after history z^t placed at date $t = 0$.

in the standard theory of general equilibrium. More explicitly, the household buys claims to different "contingent" consumption bundles. These bundles are contingent in the sense that they are conditioned on the history of the aggregate state variable (stochastic shock) z^t and thus whether they are realized and delivered depends on the realization of the sequence of the stochastic shock. For example, $c\left[z^t\right]$ denotes units of final good allocated to consumption at time t if history z^t is realized. If a different history is realized, then this claim will not be realized. This way of writing the lifetime budget constraint reiterates the importance of thinking in terms of Arrow-Debreu commodities.

Second, with this interpretation the left-hand side is simply the total expenditure of the individual taking the prices of all possible claims, i.e., the entire set of $p_0\left[z^t\right]$s, as given. The right-hand side has a similar interpretation, except that it denotes the labor earnings of the household rather than his expenditures. The last term on the right-hand side is the value of the initial capital stock per capita, which is part of the household's initial wealth. As noted above, the price of the final good at date $t = 0$ is normalized to 1. As in the standard neoclassical growth model, capital is in terms of the final good, thus has a price of 1 as well.[2]

Finally, the right-hand side of (17.10) could also include profits accruing to the individuals (as in Definition 5.1 in Chapter 5). The fact that the aggregate production function exhibits constant returns to scale combined with the presence of competitive markets implies that equilibrium profits will be equal to 0. This enables us to omit the additional term referring to profits in the representative household's budget constraint without loss of any generality.

The objective function of the household at time $t = 0$ can also be written somewhat more explicitly than (17.2) as follows:

$$\sum_{t=0}^{\infty} \beta^t \sum_{z^t \in \mathcal{Z}^\infty} q\left[z^t \mid z^0\right] u\left(c\left[z^t\right]\right), \tag{17.11}$$

where $q\left[z^t \mid z^0\right]$ is the probability at time 0 that the history z^t will be realized at time t. I have written this in the form of a conditional probability to create continuity between the treatment based on trades at date $t = 0$ and sequential trading. Notice that there is no longer the expectations operator in this objective function because the explicit summation over all possible events weighted by their probabilities has been introduced instead.

Although we will shortly formulate the problem of the representative household recursively, we can already understand the structure of the equilibrium by looking at the sequence problem of maximizing (17.11) subject to (17.10). Assuming that the interior solution exists,

[2]Recall that the initial value of the aggregate stochastic variable $z(0)$ is also taken as given. This is important for enabling us to normalize the price of the initial period capital to 1. As we will see shortly there will be a different price sequence for capital purchases conditional on the realization of the current stochastic variable. Had $z(0)$ not been known, the price of initial capital stock would also have to be conditioned on the value of $z(0)$.

the first-order conditions of this problem is

$$\beta^t q\left[z^t \mid z^0\right] u'\left(c\left[z^t\right]\right) = \lambda p_0\left[z^t\right] \tag{17.12}$$

for all t and all z^t, where λ is the Lagrange multiplier on (17.10) and now does in fact correspond to the marginal utility of income at date $t = 0$ (see Exercise 17.11 on why a single multiplier for the lifetime budget constraint is sufficient in this case). Combining this first-order condition for two different date t histories z^t and $\hat{z}^t$, we obtain

$$\frac{u'\left(c\left[\hat{z}^t\right]\right)}{u'\left(c\left[z^t\right]\right)} = \frac{p_0\left[\hat{z}^t\right]/q\left[\hat{z}^t \mid z^0\right]}{p_0\left[z^t\right]/q\left[z^t \mid z^0\right]},$$

which shows that the right-hand side is the relative price of consumption claims conditional on histories z^t and $\hat{z}^t$. Combining this first-order condition for histories z^t and z^{t+1} such that $z^{t+1} = \left(z^t, z\left(t+1\right)\right)$, we obtain

$$\frac{\beta u'\left(c\left[z^{t+1}\right]\right)}{u'\left(c\left[z^t\right]\right)} = \frac{p_0\left[z^{t+1}\right]/q\left[z^{t+1} \mid z^0\right]}{p_0\left[z^t\right]/q\left[z^t \mid z^0\right]},$$

so that the right-hand side now corresponds to the contingent interest rate between date t and $t+1$ conditional on z^t (and contingent on the realization of z^{t+1}). While these expressions are intuitive, they cannot be used to characterize equilibrium consumption or investment sequences until we know more about the prices $p_0\left[z^t\right]$. We will be able to derive these prices from the profit maximization problem of the representative firm.

Let us consider the value of the firm at date $t = 0$. To do this, let us define one more price sequence, $R_0\left[z^t\right]$ corresponding to the price of one unit of capital after the state has been revealed as z^t and also denote the capital and labor employment levels of the representative firm after history z^t by $K^e\left[z^t\right]$ and $L\left[z^t\right]$. Notice that I have introduced the additional superscript "e" for the capital to distinguish the capital employed by the firm after history z^t from the capital that is saved by the households after history z^t. These two objects are quite different as we will see.

The value of the firm can then be written as

$$\sum_{t=0}^{\infty}\beta^t \sum_{z^t \in \mathcal{Z}^\infty}\left\{p_0\left[z^t\right]\left(F\left(K^e\left[z^t\right], L\left[z^t\right], z\left(t\right)\right) + \left(1-\delta\right)K^e\left[z^t\right]\right) - R_0\left[z^t\right]K^e\left[z^t\right] - w_0\left[z^t\right]L\left[z^t\right]\right\},$$

where recall that $R_0\left[z^t\right]$ is the price of capital after history z^t and $w_0\left[z^t\right]$ is the wage rate conditional on history z^t. Profit maximization by the firm implies

$$\begin{aligned} p_0\left[z^t\right]\left(\frac{\partial F\left(K^e\left[z^t\right], L\left[z^t\right], z\left(t\right)\right)}{\partial K^e} + \left(1-\delta\right)\right) &= R_0\left[z^t\right] \\ p_0\left[z^t\right]\frac{\partial F\left(K^e\left[z^t\right], L\left[z^t\right], z\left(t\right)\right)}{\partial L} &= w_0\left[z^t\right]. \end{aligned}$$

Using constant returns to scale and expressing everything in per capita terms, these first-order conditions can be written as

$$p_0\left[z^t\right]\left(f'\left(k^e\left[z^t\right],z\left(t\right)\right)+\left(1-\delta\right)\right) = R_0\left[z^t\right] \tag{17.13}$$
$$p_0\left[z^t\right]\left(f\left(k^e\left[z^t\right],z\left(t\right)\right)-k^e\left[z^t\right]f'\left(k^e\left[z^t\right],z\left(t\right)\right)\right) = w_0\left[z^t\right],$$

where f' denotes the derivative of the per capita production function with respect to the capital-labor ratio $k^e \equiv K^e/L$. The first equation relates the price of the final good to the price of capital goods and to the marginal productivity of capital, while the second equation determines the wage rate in terms of the price of the final good and the marginal (physical) product of labor. Equation (17.13) can also be interpreted as stating that the price of a unit of capital good after history z^t, $R_0\left[z^t\right]$, is equal to the value of dividends paid out by this unit of capital inclusive of undepreciated capital, that is, the price of the final good, $p_0\left[z^t\right]$, times the marginal product of capital $f'\left(k^e\left[z^t\right],z\left(t\right)\right)$ plus the $\left(1-\delta\right)$ fraction of the capital that is not depreciated and paid back to the holder of the capital good in terms of date $t+1$ final good. An alternative way of formulating the competitive equilibrium and writing (17.13) is to assume that capital goods are *rented*—not purchased—by firms, thus introducing a rental price sequence for capital goods. Exercise 17.12 shows that this alternative formulation leads to identical results. This is not surprising because, with complete markets, buying one unit of capital today and selling contingent claims on $1-\delta$ units of capital tomorrow is equivalent to renting. Whether one uses the formulation in which capital goods are purchased or rented by firms is then just a matter of convenience and emphasis.

To complete the characterization of a competitive equilibrium, we need to impose market clearing. For labor, this is straightforward and requires

$$L\left[z^t\right]=1 \text{ for all } z^t. \tag{17.14}$$

To write the market clearing condition for capital, recall that per capita production after history z^t is given by $f\left(k^e\left[z^t\right],z\left(t\right)\right)+\left(1-\delta\right)k^e\left[z^t\right]$, and this is divided between consumption $c\left[z^t\right]$ and savings $s\left[z^t\right]$. The capital used at time $t+1$ (after history z^{t+1}) must be equal to $s\left[z^t\right]$, since this is the amount of capital available at the beginning of date $t+1$. Savings $s\left[z^t\right]$ is therefore the equivalent of capital stock choice of the planner for the next period, $k\left[z^{t-1}\right]$, in terms of the terminology in the previous subsection. Market clearing for capital implies that for any $z^{t+1}=\left(z^t,z\left(t+1\right)\right)$,

$$k^e\left[z^{t+1}\right]=s\left[z^t\right], \tag{17.15}$$

because the amount of available capital at time t is fixed irrespective of the realization of $z\left(t+1\right)$. The capital market clearing condition can then be written as

$$c\left[z^t\right]+s\left[z^t\right]\leq f\left(s\left[z^{t-1}\right],z\left(t\right)\right)+\left(1-\delta\right)s\left[z^{t-1}\right] \tag{17.16}$$

for any $z^{t+1} = \left(z^t, z(t+1)\right)$.

The capital market clearing condition also implies a particular *no arbitrage* condition linking the price of capital conditional on z^{t+1} ($R_0\left[z^{t+1}\right]$) to the price of the final good at time t ($p_0\left[z^t\right]$). In particular, consider the following riskless arbitrage; buy one unit of the final good after history z^t to be used as capital at time $t+1$ and simultaneously sell claims on capital goods for each $z^{t+1} = \left(z^t, z(t+1)\right)$. These combined transactions carry no risk, since the one unit of the final good bought after history z^t will cover the obligation to pay one unit of capital good after any history $z^{t+1} = \left(z^t, z(t+1)\right)$. Consequently, this transaction should not make or lose money, which implies the no arbitrage condition

$$p_0\left[z^t\right] = \sum_{z(t+1)\in\mathcal{Z}} R_0\left[\left(z^t, z(t+1)\right)\right]. \tag{17.17}$$

A competitive equilibrium is defined in a standard manner as feasible policies determining consumption and capital levels, $\left\{c\left[z^t\right], s\left[z^t\right], k^e\left[z^{t+1}\right]\right\}_{z^t\in\mathcal{Z}^t}$, and price sequences, $\left\{p_0\left[z^t\right], R_0\left[z^t\right], w_0\left[z^t\right]\right\}_{z^t\in\mathcal{Z}^t}$, such that households maximize utility (i.e., satisfy (17.12)), firms maximize profits (i.e., satisfy (17.13) and (17.17)), and labor and capital markets clear (i.e., (17.14), (17.15), and (17.16) are satisfied).

To characterize the equilibrium path, let us substitute from (17.13) and (17.17) into the first-order condition for consumption given by (17.12) and rearrange to obtain

$$u'\left(c\left[z^t\right]\right) = \sum_{z(t+1)\in\mathcal{Z}} \frac{\lambda p_0\left[z^{t+1}\right]}{\beta^t q\left[z^t \mid z^0\right]}\left(f'\left(k\left[z^{t+1}\right], z(t+1)\right) + (1-\delta)\right). \tag{17.18}$$

Next using (17.12) for $t+1$, we also have

$$\begin{aligned}\beta u'\left(c\left[z^{t+1}\right]\right) &= \frac{\lambda p_0\left[z^{t+1}\right]}{\beta^t q\left[z^{t+1} \mid z^0\right]} \\ &= \frac{\lambda p_0\left[z^{t+1}\right]}{\beta^t q\left[z^{t+1} \mid z^t\right] q\left[z^t \mid z^0\right]},\end{aligned}$$

where the second line simply uses the fact that, by the law of iterated expectations, $q\left[z^{t+1} \mid z^0\right] \equiv q\left[z^{t+1} \mid z^t\right] q\left[z^t \mid z^0\right]$. Substituting this into (17.18), we obtain

$$\begin{aligned}u'\left(c\left[z^t\right]\right) &= \beta \sum_{z(t+1)\in\mathcal{Z}} q\left[z^{t+1} \mid z^t\right] u'\left(c\left[z^{t+1}\right]\right)\left(f'\left(k\left[z^{t+1}\right], z(t+1)\right) + (1-\delta)\right) \\ &= \beta\mathbb{E}\left[u'\left(c\left[z^{t+1}\right]\right)\left(f'\left(k\left[z^{t+1}\right], z(t+1)\right) + (1-\delta)\right) \mid z^t\right],\end{aligned}$$

which is identical to (17.6). Given this first-order condition, it is straightforward to prove the equivalence between the optimal growth path and the competitive growth path (in this instance using the sequence approach).

Proposition 17.3. *In the above-described economy, optimal and competitive growth path coincide.*

Proof. See Exercise 17.13. □

Complementary insights can be obtained by considering the equilibrium problem in its equivalent form with sequential trading rather than all trades taking place at the initial date $t = 0$. To do this, we will write the budget constraint of the representative household somewhat differently. First, normalize the price of the final good at each date to 1 (recall the discussion in Section 5.8 in Chapter 5). The $a\left[z^t\right]$s now correspond to Basic Arrow securities that pay out only in specific states on nature. More explicitly, $a\left[z^t\right]$ denotes the assets of the household in terms of the final good at date t conditional on history z^t. We interpret $\left\{a\left[z^t\right]\right\}_{z^t \in \mathcal{Z}^t}$ as a set of contingent claims that the household has purchased that will pay $a\left[z^t\right]$ units of the final good at date t when history z^t is realized. We also denote the price of claim to one unit of $a\left[z^t\right]$ at time $t-1$ after history z^{t-1} by $\bar{p}\left[z(t) \mid z^{t-1}\right]$, where naturally $z^t = \left(z^{t-1}, z(t)\right)$. The amount of these claims purchased by the household is denoted by $a\left[\left(z^{t-1}, z(t)\right)\right]$. Consequently, the flow budget constraint of the household can be written as

$$c\left[z^t\right] + \sum_{z(t+1) \in \mathcal{Z}} \bar{p}\left[z(t+1) \mid z^t\right] a\left[\left(z^{t-1}, z(t)\right)\right] \leq w\left[z^t\right] + a\left[z^t\right],$$

where $w\left[z^t\right]$ is the equilibrium wage rate after history z^t in terms of final goods dated t, so the right-hand side is the total amount the resources available to the household after history z^t, which will be spent on consumption $c\left[z^t\right]$ and for purchasing contingent claims to final good at the next date, $a\left[\left(z^t, z(t+1)\right)\right]$. The total expenditure on these is equal to $\sum_{z(t+1) \in \mathcal{Z}} \bar{p}\left[z(t+1) \mid z^t\right] a\left[\left(z^{t-1}, z(t)\right)\right]$. With this formulation, we can once again write the sequence version of the optimization problem of the household. To save space, let us directly go to the recursive formulation, leaving the sequence version of the household's problem with sequential trading to Exercise 17.14.

Preparing for the recursive formulation, let a denote the current asset holdings of the household (in terms of the notation above, you can think of this as the realization of the current assets after some history z^t has been realized). Then the flow budget constraint of the household can be written as

$$c + \sum_{z' \in \mathcal{Z}} \bar{p}\left[z' \mid z\right] a'\left[z' \mid z\right] \leq w + a,$$

where the function $\bar{p}\left[z' \mid z\right]$ summarizes the prices of contingent claims (for next date's state z' given current state z) and $a'\left[z' \mid z\right]$ denotes the corresponding asset holdings. Let $V(a, z)$ be the value function of the household when it holds a units of the final good as assets and the current realization of the stochastic variable is z. The choice variables of the household are contingent asset holdings for the next date, denoted by $a'\left[z' \mid z\right]$, and consumption today denoted by $c[a, z]$. Let us also denote the probability that next period's stochastic variable will be equal to z' conditional on today's value being z by $q\left[z' \mid z\right]$. Then taking the sequence of equilibrium prices $\bar{p}$ as given, the value function of the representative household can be written as

(17.19)

$$V(a,z) = \sup_{\{a'[z'|z]\}_{z'\in\mathcal{Z}}} \left\{ u\left(a + w - \sum_{z'\in\mathcal{Z}} \bar{p}\left[z' \mid z\right] a'\left[z' \mid z\right] \right) + \beta \sum_{z'\in\mathcal{Z}} q\left[z' \mid z\right] V\left(a'\left[z' \mid z\right], z'\right) \right\}.$$

Theorems 16.1-16.7 from the previous chapter can again be applied to this value function (see Exercise 17.15). The first-order condition for current consumption can now be written as

$$\bar{p}\left[z' \mid z\right] u'\left(c\left[a,z\right]\right) = \beta q\left[z' \mid z\right] \frac{\partial V\left(a'\left[z' \mid z\right], z'\right)}{\partial a}$$

for any $z' \in \mathcal{Z}$ with $c[a,z]$ denoting the optimal consumption conditional on asset holdings a and stochastic variable z. Since this equation refers to the representative household and asset holdings, in the form of capital, are decided before next period's stochastic shock z' is realized, the capital market clearing condition once again requires

$$a'\left[z' \mid z\right] = a'\left[z\right],$$

so that, in the aggregate, the same amount of assets will be present in all states at the next date. This implies that the first-order condition for consumption can be alternatively written as

$$\bar{p}\left[z' \mid z\right] u'\left(c\left[a,z\right]\right) = \beta q\left[z' \mid z\right] \frac{\partial V\left(a'\left[z\right], z'\right)}{\partial a}. \tag{17.20}$$

With a similar reasoning to before, the no arbitrage condition implies

$$\sum_{z'\in\mathcal{Z}} \bar{p}\left[z' \mid z\right] R\left[z' \mid z\right] = 1, \tag{17.21}$$

where $R[z' \mid z]$ is the price of capital goods when the current state is z' and last period's state was z. Intuitively, the cost of one unit of the final good now, which is 1, has to be equal to the return that the individual will obtain by carrying this good to the next period and selling it as a capital good then. When tomorrow's state is z', the gross rate of return in terms of tomorrow's goods is $R[z' \mid z]$ and the relative price of tomorrow's goods in terms of today's goods is $\bar{p}[z' \mid z]$. Summing over all possible states z' tomorrow must then have total return of 1 to ensure no arbitrage (see Exercise 17.16). Let us now combine (17.20) with the envelope condition

$$\frac{\partial V(a,z)}{\partial a} = u'\left(c\left[a,z\right]\right),$$

and then multiply both sides of (17.20) by $R[z' \mid z]$ and sum over all $z' \in \mathcal{Z}$ to obtain the first-order condition of the household as

$$\begin{aligned} u'\left(c\left[a,z\right]\right) &= \beta \sum_{z'\in\mathcal{Z}} q\left[z' \mid z\right] R\left[z' \mid z\right] u'\left(c\left[a', z'\right]\right). \\ &= \beta \mathbb{E}\left[R\left[z' \mid z\right] u'\left(c\left[a', z'\right]\right) \mid z\right]. \end{aligned}$$

Next the market clearing condition for capital, combined with the fact that the only asset in the economy is capital, implies that

$$a = k.$$

Therefore, this first-order condition can be written as

$$u'\left(c\left[k,z\right]\right) = \beta\mathbb{E}\left[R\left[z' \mid z\right]u'\left(c\left[k',z'\right]\right) \mid z\right]$$

which is identical to (17.6). This again shows the equivalence between the social planner's problem and the competitive equilibrium path.

Given the equivalence between the social planner's problem (the optimal growth problem) and the characterization of the equilibrium and the fact that the former is considerably simpler, much of the literature focuses on the social planner's problem when this can be done. The social planner's problem characterizes the equilibrium path of all the real variables and the comparison of the analysis in this section to that in the previous section shows that various different prices are also straightforward to obtain from the Lagrange multiplier of the social planner's problem. Here the most important prices are those for capital goods, i.e., the $R\left[z' \mid z\right]$s. Equations (17.5) and (17.6) show that these can be easily obtained from the marginal product of capital in the optimal growth path.

17.3. Application: Real Business Cycle Models

One of the most important applications of the neoclassical growth model under uncertainty over the past 25 years has been to the analysis of short- and medium-run fluctuations. The approach, pioneered by Kydland and Prescott's seminal (1982) paper and referred to as the *Real Business Cycle* theory, uses the neoclassical growth model with aggregate productivity shocks in order to provide a framework for the analysis of macroeconomic fluctuations. The Real Business Cycle (RBC) theory has been one of the most active research areas of macroeconomics in the 1990s and also one of the most controversial. On the one hand, its conceptual simplicity and its relative success in matching certain moments of employment, consumption and investment fluctuations for a given (appropriately chosen) sequence of aggregate productivity shocks have attracted a large following. On the other and, the absence of monetary factors and demand shocks, the traditional pillars of Keynesian economics and previous research on macroeconomic fluctuations, has generated a ferocious opposition and much debate on the merits of this theory. The merits of the RBC theory are not relevant for our focus here and would take us too far afield from the key questions of economic growth. Nevertheless, a brief exposition of the canonical RBC model is useful for two purposes. First, it constitutes one of the most important applications of the neoclassical growth model under uncertainty and has become another one of the workhorse models for macroeconomic research over the past 25 years. Second, it illustrates how the introduction of labor supply choices into the neoclassical growth model under uncertainty generates new insights. So far I have assumed, except in Exercise 8.17 in Chapter 8, that labor is supplied inelastically and this choice has enabled us to focus on the first-order issues related to economic growth. Because the issue of labor supply is central to a number of questions in macroeconomics, a brief analysis of the neoclassical growth model with labor supply is also useful.

The economic environment is identical to that in the previous two sections, with the only difference that the instantaneous utility function of the representative household now takes the form

$$u(C, L),$$

where C denotes consumption and L labor supply. I use uppercase letters for consistency with what will come below. I assume that u is jointly concave and continuously differentiable in both of its arguments and strictly increasing in C and strictly decreasing in L. I also assume that L has to lie in some convex compact set $\left[0, \bar{L}\right]$.

Given the equivalence between the optimal growth and the equilibrium growth problems, I will focus on the optimal growth formulation here and set up the social planner's problem. This problem can be expressed as the maximization of

$$\mathbb{E}\sum_{t=0}^{\infty}\beta^t u\left(C\left(t\right), L\left(t\right)\right)$$

subject to the flow resource constraint

$$K\left(t+1\right) \leq F\left(K\left(t\right), L\left(t\right), z\left(t\right)\right) + \left(1-\delta\right)K\left(t\right) - C\left(t\right),$$

where $z(t)$ again represents an aggregate productivity shock following a monotone Markov chain.

The social planner's problem can be written recursively as

(17.22)

$$V\left(K, z\right) = \sup_{\substack{L\in\left[0,\bar{L}\right] \\ K'\in\left[0,F\left(K,L,z\right)+\left(1-\delta\right)K\right]}} \left\{u\left(F\left(K, L, z\right) + \left(1-\delta\right)K - K', L\right) + \beta\mathbb{E}\left[V\left(K', z'\right) \mid z\right]\right\}.$$

The following proposition is again a direct consequence of Theorems 16.1-16.7:

Proposition 17.4. *The value function $V(K, z)$ defined in (17.22) is continuous and strictly concave in K, strictly increasing in K and z, and differentiable in $K > 0$. There exist uniquely defined policy functions $\pi^k(K, z)$ and $\pi^l(K, z)$ that determine the level of capital stock chosen for next period and the level of labor supply as a function of the current capital stock K and the stochastic variable z.*

Proof. See Exercise 17.17. □

Clearly, once again assuming an interior solution, the relevant prices can be obtained from the appropriate multipliers and the standard first-order conditions characterize the form of the equilibrium. In particular, the two key first-order conditions determine the evolution of consumption over time and the equilibrium level of labor supply. Denoting the derivatives of the u function with respect to its first and second arguments by u_c and u_l, the derivatives of the F function by F_k and F_l, and defining the policy function for consumption as

$$\pi^c(K,z) \equiv F\left(K, \pi^l(K,z), z\right) + (1-\delta)K - \pi^k(K,z),$$

these take the form

(17.23)

$$\begin{aligned} u_c\left(\pi^c(K,z), \pi^l(K,z)\right) &= \beta\mathbb{E}\left[R\left(\pi^k(K,z), z'\right) u_c\left(\pi^c\left(\pi^k(K,z), z'\right), \pi^l\left(\pi^k(K,z), z'\right)\right) \mid z\right], \\ w(K,z)\, u_c\left(\pi^c(K,z), \pi^l(K,z)\right) &= -u_l\left(\pi^c(K,z), \pi^l(K,z)\right). \end{aligned}$$

where

$$\begin{aligned} R(K,z) &= F_k(K,z) + (1-\delta) \\ w(K,z) &= F_l(K,z) \end{aligned}$$

denote the gross rate of return to capital and the equilibrium wage rate. Notice that the first condition in (17.23) is essentially identical to (17.5), whereas the second is a static condition determining the level of equilibrium (or optimal) labor supply. The second condition does not feature expectations, since it is conditional on the current value of the capital stock, K, and the current realization of the aggregate productivity variable, z.

Why is this framework useful for the analysis of macroeconomic fluctuations? The answer lies in the fact that estimates of total factor productivity, along the lines described in Chapter 3, indicate that it is procyclical—that is, it is higher in periods during which output is above trend and unemployment is low. So let us think of a period in which z is low. Clearly, if there is no offsetting change in labor supply, output will be low, so we can think of this period as a "recession". Moreover, under standard assumptions, the wage rate $w(K,z)$ and equilibrium labor supply will decline.[3] Thus there will be low employment as well as low output. Thus a negative productivity shock generates two of the important characteristics of a recessionary period. In addition, if the Markov chain (or more generally the Markov process) governing the behavior of z exhibits persistence, output will be low the following period as well, so output and employment will exhibit *persistent fluctuations*. Finally, provided that the aggregate production function $F(K,L,z)$ takes a form such that low output is also associated with low marginal product of capital, the expectation of future low output will typically reduce

[3] It is useful to note, however, that there is no agreement as to whether wages are indeed procyclical. Average wages do not seem to be procyclical over the business cycle, but this may be because of "selection bias," the fact that the composition of the labor force changes over the business cycle as those who lose their jobs during recessions are typically different from the "average" worker. Depending on how one corrects for this potential source of bias, wages appear to be either mildly procyclical or acyclical. See, for example, Bils (1985), Barsky, Parker and Solon (1994), and Abraham and Haltiwanger (1995).

savings and thus future levels of capital stock (though this does also depend on the form of the utility function, which regulates the desire for consumption smoothing and the balance between income and substitution effects).

This brief discussion suggests that the neoclassical growth model under uncertainty with labor supply choices and with aggregate productivity shocks may generate some of the major qualitative features of macroeconomic fluctuations. The RBC literature also argues that this model, under suitable assumptions, generates the *major* quantitative features of the business cycles, such as the correlations between output, investment, and employment. A large part of the debate on the merits of the RBC model focused on: (i) whether the model did indeed match these moments in the data; (ii) whether these were the right empirical objects to look at (for example, as opposed to persistence in employment or output at different frequencies); and (iii) whether a framework in which the driving force of fluctuations is exogenous changes in aggregate productivity is sidestepping the more interesting question of why there are shocks that cause recessions and booms. It is fair to say that, while the RBC debate is not as active today as it was in the 1990s, there has not been a complete agreement on these questions. In the meantime, many extensions of the standard RBC model have improved over the bare-bones version presented here.

The model presented here considered the neoclassical growth model without exogenous technological progress. Exercise 17.18 introduces exogenous technological progress into this model and shows that the analysis is essentially unchanged. The next example considers a very simple case of the RBC model that can be solved in closed form (though the price of doing so is that some of the interesting features of the model are lost).

Example 17.2. Consider an example economy similar to that studied in Example 17.1. In particular, suppose that $u(C,L) = \log C - \gamma L$, $F(K,L,z) = zK^{\alpha}L^{1-\alpha}$, and $\delta = 1$. We continue to assume that z follows a monotone Markov chain over the set $\mathcal{Z} \equiv \{z_1, ..., z_N\}$, with transition probabilities denoted by $q_{jj'}$. As in that example, let us conjecture that

$$\pi^k(K,z) = BzK^{\alpha}L^{1-\alpha}.$$

Then with these functional forms, the stochastic Euler equation for consumption (17.23) implies

$$\frac{1}{(1-B)\,zK^{\alpha}L^{1-\alpha}} = \beta\mathbb{E}\left[\left.\frac{\alpha z'\left(BzK^{\alpha}L^{1-\alpha}\right)^{-(1-\alpha)}(L')^{1-\alpha}}{(1-B)\,z'\left(BzK^{\alpha}L^{1-\alpha}\right)^{\alpha}(L')^{1-\alpha}}\right| z\right],$$

where L' denotes next period's labor supply. Canceling constants within the expectations and taking terms that do not involve z' out of the expectations operator this equation simplifies to

$$\frac{1}{zK^{\alpha}L^{1-\alpha}} = \beta\mathbb{E}\left[\alpha\left.\left(BzK^{\alpha}L^{1-\alpha}\right)^{-1}\right| z\right],$$

which yields

$$B = \alpha\beta.$$

The resulting policy function for the capital stock is therefore

$$\pi^k (K, z) = \alpha\beta z K^\alpha L^{1-\alpha},$$

which is identical to that in Example 17.1. Next, considering the first-order condition for labor, we obtain

$$\frac{(1-\alpha) z K^\alpha L^{-\alpha}}{(1-B) z K^\alpha L^{1-\alpha}} = \gamma.$$

The resulting policy function for labor as

$$\pi^l (K, z) = \frac{(1-\alpha)}{\gamma (1-\alpha\beta)},$$

which implies that labor supply is constant. This is because with the preferences as specified here, the income and the substitution effects cancel out, thus the increase in wages induced by a change in aggregate productivity has no effect on labor supply. Exercise 17.19 shows that the same result obtains whenever the utility function takes the form of $U(C, L) = \log C + h(L)$ for some decreasing and concave function h. Overall, this simple version of the RBC model with a closed-form solution therefore replicates the covariation in output and investment, but does not generate labor fluctuations.

17.4. Growth with Incomplete Markets: The Bewley Model

We now turn to a fundamentally different model of economic growth, where the economy does not admit a representative household and idiosyncratic risks are not diversified. This model was first introduced and studied by Truman Bewley (1977, 1980, 1986). It has subsequently been revived, extended and applied to a variety of new questions including the structure of optimal fiscal policy, business cycle fluctuations and asset pricing in Aiyagari (1994, 1997) and Krusell and Smith (1998, 1999). Many economists believe that, to a first approximation, such a structure provides a better approximation to reality than the complete markets neoclassical growth model. Unfortunately, this model, which I will refer to as the Bewley economy, is considerably more complicated than the baseline neoclassical growth model. As we will discuss below, however, the assumption that there is no insurance for individual income fluctuations—except through "self-insurance," that is, the process of accumulating assets to be used in a rainy day—is extreme and may limit the applications of the current model in the growth context.

The economy is populated by a continuum 1 of households and the set of households is denoted by $\mathcal{H}$. Each household has preferences given by (17.2) and supplies labor inelastically. Suppose also that the second derivative of this utility function, $u''(\cdot)$, is increasing. Differently from the baseline neoclassical growth model, the efficiency units that each household supplies vary over time. In particular, each household $h \in \mathcal{H}$ has a labor endowment of $z^h(t)$ at time t, where $z^h(t)$ is an independent draw from the set $\mathcal{Z} \equiv [z_{\min}, z_{\max}]$, where $0 < z_{\min} < z_{\max} < \infty$, so that the minimum labor endowment is $z_{\min}$. We assume that the labor endowment of each household is identically and independently distributed with distribution function $G(z)$ defined over $[z_{\min}, z_{\max}]$.

The production side of the economy is the same as in the canonical neoclassical growth model under certainty and is represented by an aggregate production function satisfying Assumptions 1 and 2, as in (17.1). The only difference is that $L(t)$ is now the sum (integral) of the heterogeneous labor endowments of all the agents and is written as

$$L(t) = \int_{h \in \mathcal{H}} z^h(t)\, dh.$$

Appealing to a law of large numbers type argument, we assume that $L(t)$ is constant at each date and we normalize it to 1. Thus output per capita in the economy can be expressed as

$$y(t) = f(k(t)),$$

with $k(t) = K(t)$. Notice that there is no longer any aggregate productivity shock. The only source of uncertainty is at the individual level (i.e., it is idiosyncratic). Consequently, while individual households will experience fluctuations in their labor income and consumption, we can imagine a *stationary* equilibrium in which aggregates are constant over time. Throughout this section I focus on such a stationary equilibrium. In particular, in a stationary equilibrium the wage rate w and the gross rate of return on capital R will be constant (though of course their levels will be determined endogenously to ensure equilibrium). Let us first take these prices as given and look at the behavior of a typical household $h \in \mathcal{H}$ (I am using the language "typical" household, since, though not representative, this household faces an identical problem to all other households in the economy). This household will solve the following maximization problem: maximize (17.2) subject to the flow budget constraint

$$a^h(t+1) \leq Ra^h(t) + wz^h(t) - c^h(t)$$

for all t, where $a^h(t)$ is the asset holding of household $h \in \mathcal{H}$ at time t. Consumption cannot be negative, so $c^h(t) \geq 0$. In addition, though we do not impose any exogenous borrowing constraints, with the same reasoning as in the model of the permanent income hypothesis in subsection 16.5.1 in the previous chapter, the requirement that the individual should satisfy its lifetime budget constraint in all histories imposes the endogenous borrowing constraint

$$\begin{aligned} a^h(t) &\geq -\frac{z_{\min}}{R-1} \\ &\equiv -b, \end{aligned}$$

for all t. We can then write the maximization problem of household $h \in \mathcal{H}$ recursively as

$$V^h(a,z) = \sup_{a' \in [-b, Ra+wz]} \left\{ u\left(Ra + wz - a'\right) + \beta \mathbb{E}\left[V^h\left(a', z'\right) \mid z\right]\right\}. \tag{17.24}$$

Standard dynamic programming arguments then establish the following proposition:

Proposition 17.5. *The value function $V^h(a,z)$ defined in (17.24) is uniquely defined, continuous and strictly concave in a, strictly increasing in a and z, and differentiable in $a \in (-b, Ra + wz)$. Moreover, the policy function that determines next period's asset holding $\pi(a,z)$ is uniquely defined and continuous in a.*

Proof. See Exercise 17.20. □

Moreover, as in Proposition 17.2, we have the additional result on the form of the policy function:

Proposition 17.6. *The policy function $\pi(a,z)$ derived in Proposition 17.5 is strictly increasing in a and z.*

Proof. See Exercise 17.21. □

The total amount of capital stock in the economy can be obtained by aggregating the asset holdings of all households in the economy, thus in a stationary equilibrium we have

$$\begin{aligned} k(t+1) &= \int_{h\in\mathcal{H}} a^h(t)\, dh \\ &= \int_{h\in\mathcal{H}} \pi\left(a^h(t), z^h(t)\right) dh. \end{aligned}$$

This equation integrates over all households taking their asset holdings and the realization of their stochastic shock as given. It states that both the average of current asset holdings and also the average of tomorrow's asset holdings must be equal by the definition of a stationary equilibrium. To understand this condition, recall that as in the neoclassical growth model, the policy function $a' = \pi(a,z)$ defines a general Markov process. Under fairly weak assumptions this Markov process will admit a unique invariant distribution. If this were not the case, the economy could have multiple stationary equilibria or even there might be problems of non-existence. For our purposes here, we can ignore this complication and assume the existence of a unique invariant distribution, which we denote by $\Gamma(a)$, so that the stationary equilibrium capital-labor ratio is given by

$$k^* = \int\int \pi(a,z)\, d\Gamma(a)\, dG(z),$$

which uses the fact that z is distributed identically and independently across households and over time.

Next turning to the production side, we have the same factor prices as the neoclassical growth model under certainty, i.e.,

$$
\begin{aligned}
R &= f'(k^*) + (1-\delta) \\
w &= f(k^*) - k^* f'(k^*).
\end{aligned}
$$

Recall from Chapters 6 and 8 that the neoclassical growth model with complete markets and no uncertainty implies that there exists a unique steady state in which $\beta R = 1$, i.e.,

$$
f'(k^{**}) = \beta^{-1} - (1-\delta), \tag{17.25}
$$

where k^{**} refers to the capital-labor ratio of the neoclassical growth model under certainty.

Perhaps the most interesting implication of the Bewley economy is that this is no longer true. In particular, we have

Proposition 17.7. *In any stationary equilibrium of the Bewley economy, we have that the stationary equilibrium capital-labor ratio k^* is such that*

$$
f'(k^*) < \beta^{-1} - (1-\delta) \tag{17.26}
$$

and

$$
k^* > k^{**}, \tag{17.27}
$$

where k^{**} is the capital-labor ratio of the neoclassical growth model under certainty.

Proof. (Sketch) Suppose $f'(k^*) \geq \beta^{-1} - (1-\delta)$. Then the result in Exercise 16.10 from the previous chapter implies that each household's expected consumption is strictly increasing. This implies that average consumption in the population, which is deterministic, is strictly increasing and would tend to infinity. This is not possible in view of Assumption 2, which implies that aggregate resources must always be finite. This establishes (17.26). Given this result, (17.27) immediately follows from (17.25) and from the strict concavity of $f(\cdot)$ (Assumption 1). □

Intuitively, the interest rate in the incomplete markets economy is "depressed" relative to the neoclassical growth model with certainty because each household has an additional self-insurance (or precautionary) incentive to save. These additional savings increase the capital-labor ratio and reduce the equilibrium interest rate. Interestingly, therefore, the Bewley economy, like the overlapping generations model of Chapter 9, leads to a higher capital intensity of production than the standard neoclassical growth model. Observe that in both cases, the lack of a representative household plays an important role in this result.

While the Bewley model is an important workhorse for macroeconomic analysis, two features, that may be viewed as potential shortcomings, are worth noting here. First, as we already remarked in the context of the overlapping generations model, the source of inefficiency coming from overaccumulation of capital is unlikely to be important for explaining income per capita differences across countries. Thus the Bewley model is not interesting because of the greater capital-labor ratio that it generates. Instead, it is important as an illustration of how an economy might exhibit a stationary equilibrium in which aggregates are constant while individual households have uncertain and fluctuating consumption and income profiles. It also emphasizes issues of individual risk in the context of a relatively familiar neoclassical growth setup. Issues of individual risk bearing are important in the context of economic development as shown in Section 17.6 below and also in Chapter 21. Second, the incomplete markets assumption in this model may be extreme. In practice, when their incomes are low, individuals do receive transfers, either because they have entered into some form of private insurance or because of government-provided social insurance. Instead, the current model exogenously assumes that there are no insurance possibilities. Much more satisfactory would be models in which the lack of insurance opportunities are derived from microfoundations (for example, from moral hazard or adverse selection) or models in which the set of active markets is determined endogenously. While models of limited insurance due to moral hazard or adverse selection are beyond the scope of this book, we will return to an economic growth model with endogenously incomplete markets in Section 17.6.

17.5. The Overlapping Generations Model with Uncertainty

Let us now briefly consider the canonical overlapping generations model from Section 9.3 in Chapter 9. Time is discrete and runs to infinity. Each individual lives for two periods. Suppose as in Section 9.3 that the utility of a household in generation t is given by

$$U(t) = \log c_1(t) + \beta \log c_2(t+1). \tag{17.28}$$

There is a constant rate of population growth equal to n, so that

$$L(t) = (1+n)^t L(0), \tag{17.29}$$

where $L(0)$ is the size of the first generation. As in Section 9.3, we assume that the aggregate production technology is Cobb-Douglas but now also includes an aggregate stochastic shock z, which is assumed to follow a Markov process. Consequently, total output at time t is given by

$$Y(t) = z(t) K(t)^{\alpha} L(t)^{1-\alpha}.$$

Expressing this in per capita terms

$$y(t) = z(t) k(t)^{\alpha}.$$

To simplify the notation, we also assume that capital depreciates fully, i.e., $\delta = 1$. Factor prices clearly only depend on the current values of z and the capital-labor ratio k, and can be expressed as

$$R(k,z) = \alpha z k^{\alpha-1} \tag{17.30}$$
$$w(k,z) = (1-\alpha) z k^{\alpha}.$$

The consumption Euler equation for an individual of generation t, then can be expressed as

$$\frac{c_2(t+1)}{c_1(t)} = \beta R(t+1) = \beta R(k,z),$$

with $R(k,z)$ given by (17.30). The total amount of savings at time t is then given by $s(t) = s(k(t), z(t))$ such that

$$s(k,z) = \frac{\beta}{1+\beta} w(k,z), \tag{17.31}$$

which, as in the canonical overlapping generations model of Section 9.3 in Chapter 9 and also as in the baseline Solow growth model of Chapter 2, corresponds to a constant savings rate now equal to $\beta/(1+\beta)$.

Now combining (17.31) with (17.29) and the fact that $\delta = 1$, next date's capital stock $k(t+1)$ can be written as

$$\begin{aligned} k(t+1) &= \pi(k,z) \\ &= \frac{s(k,z)}{(1+n)} \\ &= \frac{\beta(1-\alpha) z k^{\alpha}}{(1+n)(1+\beta)}. \end{aligned} \tag{17.32}$$

Clearly, if $z = \bar{z}$, this equation would have a unique steady state with capital-labor ratio given by

$$k^* = \left[\frac{\beta(1-\alpha)\bar{z}}{(1+n)(1+\beta)}\right]^{\frac{1}{1-\alpha}}. \tag{17.33}$$

However, when z has a non-degenerates distribution, (17.32) defines a stochastic first-order difference equation. As in the neoclassical growth model under uncertainty, the long-run equilibrium of this model will correspond to an invariant distribution of the capital stock. In this particular case, however, since (17.32) is very tractable, we can obtain more insights about the behavior of the economy with a diagrammatic analysis. Suppose that z is distributed between $[z_{\min}, z_{\max}]$, then the behavior of the economy can be analyzed by plotting *the stochastic correspondence* associated with (17.32), which is done in Figure 17.1. The stochastic correspondence plots the entire range of possible values of $k(t+1)$ for a given value of $k(t)$. The upper thick curve corresponds to the realization of $z_{\max}$, while the lower thick curve corresponds to the realization of $z_{\min}$. The dotted curve in the middle is for $z = \bar{z}$. Observe that the curves for both $z_{\min}$ and $z_{\max}$ start above the 45^0 line, which is a consequence of the Inada condition implied by the Cobb-Douglas technology—the marginal product of capital is arbitrarily large when the capital stock is close to zero. The stochastic correspondence enables a simple analysis of the dynamics of stochastic models as in this case. For example, Figure 17.1 plots a particular sample path of capital-labor ratio in this economy, where starting with $k(0)$, the economy first receives a fairly favorable productivity shock moving to $k(1)$. Following this, there is another moderately favorable productivity realization and the capital-labor ratio increases to $k(2)$. In the following period, however, the realization of the stochastic variable is quite bad and the capital-labor ratio and thus output per capita decline sharply. This figure illustrates the type of dynamics that can emerge. Similar methods will be used in the next section in a somewhat richer model.

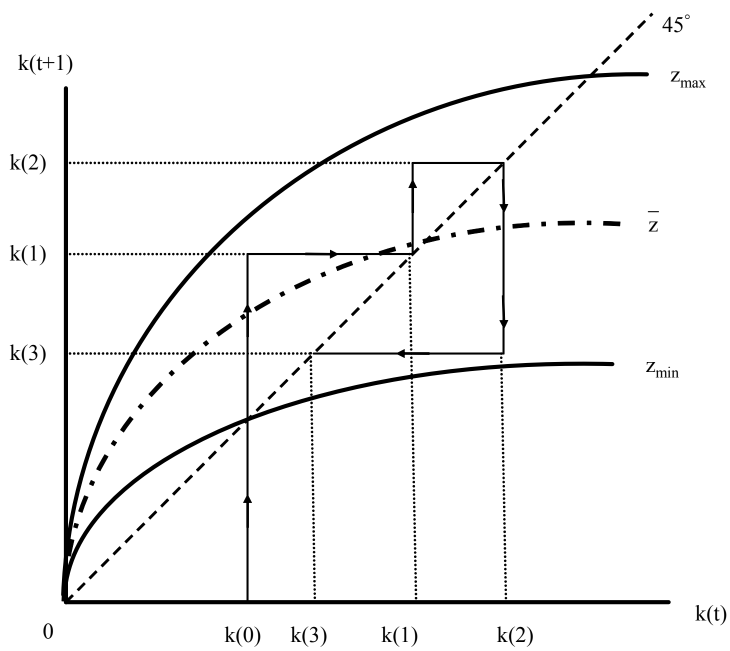

Figure 17.1. This figure shows the stochastic correspondence of the overlapping generations model. Any value for next period's capital-labor ratio within the two curves marked $z_{\min}$ and $z_{\max}$ has positive probability. The path $k(0) \to k(1) \to k(2) \to k(3)$ illustrates a particular realization of the stochastic path of the equilibrium capital-labor ratio.

Another noteworthy feature of this model is that, together with the stochastic Solow model discussed in Exercise 17.3 and the specific form of the neoclassical growth model in Example 17.1, it provides a much simpler model of stochastic growth than the neoclassical growth under uncertainty. In the overlapping generations model and the Solow model, this is because saving decisions are "myopic" and remain unaffected by the distribution of stochastic shocks or even their realizations. Thus for the analysis of a range of macroeconomic questions, these more myopic models or the simple neoclassical model of Example 17.1 might provide a tractable attractive alternative to the full neoclassical growth model under uncertainty.

17.6. Risk, Diversification and Growth

In this section, I present a stochastic model of long-run growth based on Acemoglu and Zilibotti (1997). This model is useful for two distinct purposes. First, because it is simpler than the baseline neoclassical growth model under uncertainty, it will provide a complete characterization of the stochastic dynamics of growth and show how simple ideas from the theory of Markov processes can be used in the context of the study of economic growth. Second and more important, this model will enable the analysis of a number of important topics in the theory of long-run growth. In particular, I have so far focused on models with balanced growth and relatively well-behaved transitional dynamics. The experience of economic growth over the past few thousand years has been much less "orderly" than implied by these models, however. Until about 200 years ago, growth in income per capita was relatively rare. Sustained growth in income per capita is a relatively recent phenomenon. Before this "takeoff" into sustained growth, societies experienced periods of growth followed by large slumps and crises. Acemoglu and Zilibotti (1997), Imbs and Wacziarg (2003), and Koren and Tenreyro (2007) document that even today richer countries have much more stable growth performances than less developed economies, which suffer from much higher variability in their growth rates. In many ways, this pattern of relatively risky growth and low productivity followed by a process of capital-deepening, financial development and better risk management is a major characteristic of the history of economic growth. The famous economic historian Fernand Braudel describes the start of economic growth in Western Europe as follows:

> "The advance occurred very slowly over a long period and was broken by sharp recessions. The right road was reached and thereafter never abandoned, only during the eighteenth century, and then only by a few privileged countries. Thus, before 1750 or even 1800 the march of progress could still be affected by unexpected events, even disasters." F. Braudel (1973, p. xi).

In the model I will present here, these patterns arise endogenously because the extent to which the economy can diversify risks by investing in imperfectly correlated activities is limited by the amount of capital it possesses. As the amount of capital increases, the economy achieves better diversification and faces fewer risks. The resulting equilibrium process thus generates greater variability and risk at the early stages of development and these risks are significantly reduced after the economy manages to "take off" into sustained growth. Moreover, the desire of individual households to avoid risk makes them invest in lower return less risky activities during the early stages of development, thus the growth rate of the economy is also endogenously limited during this pre-takeoff stage. In addition, in this economic development goes hand-in-hand with financial development as greater availability of capital enables better risk sharing through asset markets. Finally, because the model is one of endogenously incomplete markets, it also enables us to show that price-taking behavior by itself is not sufficient to guarantee Pareto optimality, and the form of inefficiency of the equilibrium in this economy will be interesting both on substantive and methodological grounds.

17.6.1. The Environment. We consider an overlapping generations economy. Each generation lives for two periods. There is no population growth and the size of each generation is normalized to 1. The production sector consists of two sectors. The first sector produces final goods with the Cobb-Douglas production function

$$Y(t) = K(t)^{\alpha} L(t)^{1-\alpha}, \tag{17.34}$$

where as usual $L(t)$ is total labor and $K(t)$ is the total capital stock available at time t. Capital depreciates fully after use (i.e., $\delta = 1$ in terms of our previous notation).

The second sector transforms savings at time $t-1$ into capital to be used for production at time t. This sector consists of a continuum $[0,1]$ of intermediates, and stochastic elements only affect this sector. In particular, let us represent possible states of nature also with the unit interval and assume that intermediate sector $j \in [0,1]$ pays a positive return only in state j and nothing in any other state. This formulation implies that investing in a sector is equivalent to buying a Basic Arrow Security that only pays in one state of nature. Since there is a continuum of sectors, the probability that a single sector will have positive payoff is 0, but if an individual invests in some subset J of $[0,1]$, then there will be positive returns with probability equal to the (Lebesgue) measure on the set J. Thus each intermediate sector is a risky activity but an individual (and in particular, the representative household in the economy) can diversify risks by investing in multiple sectors. In particular, if one were to invest in all of the sectors, then one would receive positive returns with probability 1. What makes the economic interactions in this model non-trivial is that investing in all sectors will not be possible at every date because of potential *nonconvexities*. More specifically, we assume that each sector has a *minimum size requirement*, denoted by $M(j)$ and positive returns will be realized only if aggregate investment in that sector exceeds $M(j)$.

In light of this description, let $I(j,t)$ be the aggregate investment in intermediate sector j at time t. We assume that this investment will generate date $t+1$ capital equal to $QI(j,t)$ if state j is realized *and* $I(j,t) \geq M(j)$, and nothing otherwise. Thus aggregate investment in the intermediate sector exceeding the minimum size requirement is *necessary* for any positive returns.

In addition to the risky intermediate sectors, we assume that there is also a safe intermediate sector which transforms one unit of savings at date t into q units of date $t+1$ capital. The crucial assumption is that

$$q < Q, \tag{17.35}$$

so that the safe option is also less productive.

The requirement that $I(j,t) \geq M(j)$ combined with the fact that the amount of capital obtained from savings $I(j,t)$ in state j is equal to $QI(j,t)$ implies that all intermediate sectors have linear technologies, but only after the minimum size requirement, $M(j)$, is met. For any $I(j,t) < M(j)$, the output is equal to 0. In order to simplify the exposition and the computations, let us adopt a simple distribution of minimum size requirements by intermediate sector:

$$M(j) = \max\left\{0, \frac{D}{(1-\gamma)}(j-\gamma)\right\}. \tag{17.36}$$

This equation implies that intermediate sectors $j \leq \gamma$ have no minimum size requirement, so aggregate investments of any size can be made in the sectors. For the remaining sectors, the minimum size requirement increases linearly. Figure 17.2 shows the minimum size requirements with the thick line. This figure will be used to illustrate the determination of the set of open sectors once the equilibrium investments are specified.

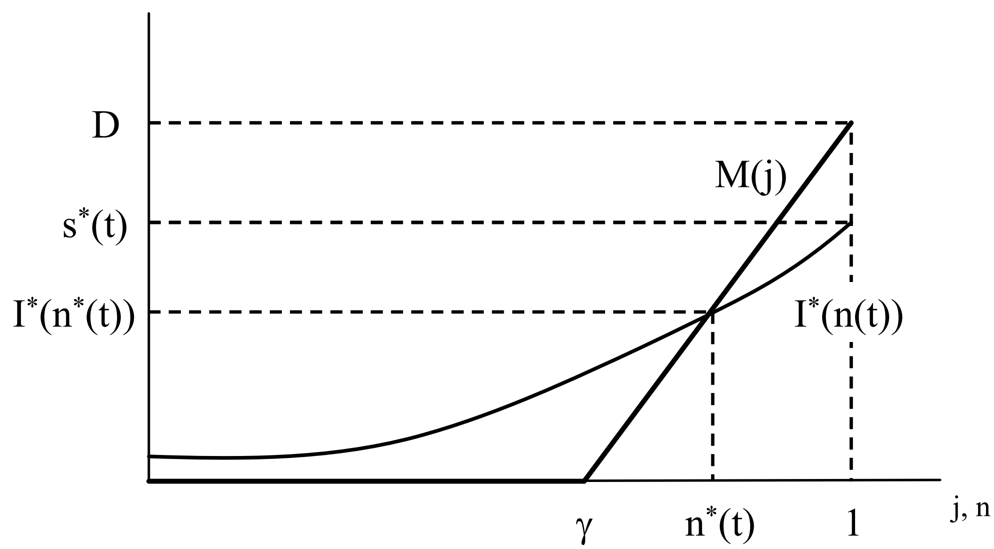

Figure 17.2. Minimum size requirements, $M(j)$, of different sectors and demand for assets, $I^*(n)$.

It is worth noting that there are three important features introduced so far.

(1) Risky investments have a higher expected return than the safe investment, which is captured by the assumption that $Q > q$.
(2) The output of the risky investments (of the intermediate sectors) are imperfectly correlated so that there is safety in numbers.

(3) The mathematical formulation here implies a simple relationship between investments and returns. As already hinted above, if an individual holds a portfolio consisting of an equi-proportional investment I in all sectors $j \in \bar{J} \subseteq [0,1]$, and the (Lebesgue) measure of the set $\bar{J}$ is p, then the portfolio pays the return QI with probability p, and nothing with probability $1-p$.

The first two features imply that if the aggregate production set of this economy had been convex, for example because $D = 0$, all agents would invest an equal amount in all intermediate sectors and manage to diversify all risks without sacrificing any of the high returns. However, in the presence of nonconvexities, as captured by the minimum size requirements, there is a tradeoff between insurance and high productivity.

Let us next turn to the preferences of households. Recall that each generation has size normalized to 1. Consider a household from a generation born at time t. The preferences of this household are given by

$$\mathbb{E}_t U(c_1(t), c_2(t+1)) = \log(c_1(t)) + \beta \int_0^1 \log c_2(j, t+1)\, dj, \tag{17.37}$$

where $c_1(t)$ is the consumption of final goods in the first period of this household's life (which is at time t) and c_2 refers to consumption in the second period of this household. $\mathbb{E}_t$ refers to the expectations operator, because second period consumption is risky. This is spelled out on the right-hand side of (17.37), with $c_2(j, t+1)$ denoting consumption in state j at time t. The integral replaces the expectation using the fact that all states are equally likely. As in the canonical overlapping generations model, each household has 1 unit of labor when young and no labor endowment when old. Thus the total supply of labor in the economy is 1. Moreover, in the second period of their lives, each individual consumes the return from their savings. For future reference, the set of young households at time t is denoted by $\mathcal{H}_t$ and Figure 17.3 depicts the life cycle and the various decisions of a typical household, emphasizing that uncertainty affects the return on their savings and thus the amount of capital they will have in the second period of their lives.

The aggregate capital stock depends on the realization of the state of nature, which determines how much of the investments in different intermediate sectors at time t is turned into capital. The realization of the capital stock at time $t+1$ therefore depends on the realization of the state of nature as well as the composition of investment of young agents. In particular, in state j, the aggregate stock of capital is

$$K(j, t+1) = \int_{h \in \mathcal{H}_t} (qX^h(t) + QI^h(j,t)) dh$$

where $I^h(j,t)$ is the amount of savings invested by (young) agent $h \in \mathcal{H}_t$ in sector j at time t, and $X^h(t)$ is the amount invested in the safe intermediate sector.

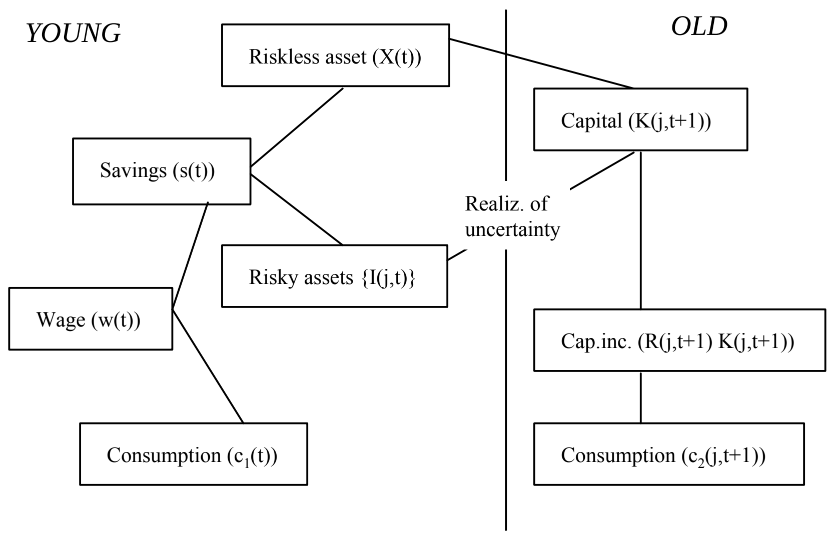

FIGURE 17.3. Life cycle of a typical household.

Since the capital stock is potentially random, so will be output and factor prices. In particular, both labor and capital are assumed to be traded in competitive markets, so the equilibrium factor prices will be given by their marginal products. Since the total capital stock in state j at time $t+1$ is $K(j,t+1)$ and the total supply of labor is equal to 1, these prices are given by

$$\begin{aligned} w(j,t+1) &= (1-\alpha)K(j,t+1)^{\alpha} \\ &= (1-\alpha)\left(\int_{h\in\mathcal{H}_t}(qX^h(t)+QI^h(j,t))dh\right)^{\alpha}. \end{aligned} \tag{17.38}$$

and

$$\begin{aligned} R(j,t+1) &= \alpha K(j,t+1)^{\alpha-1} \\ &= \alpha\left(\int_{h\in\mathcal{H}_t}(qX^h(t)+QI^h(j,t))dh\right)^{\alpha-1}. \end{aligned} \tag{17.39}$$

To complete the description of the environment, we also specify the market structure of the intermediate sector. We assume that households make investments in different intermediate sectors through *financial intermediaries*. There is free entry into financial intermediation (either by a large number of firms or by the households themselves). Any intermediary can form costlessly and mediate funds for a particular sector, that is, it can collect funds, invest them in a particular intermediate sector and give corresponding Basic Arrow Securities to its investors. The important requirement is that, to be able to invest, any financial intermediary should raise enough funds to cover the minimum size requirement. For now, we assume that each financial intermediary can operate only a single sector.[4] Thus we are ruling out the formation of a grand financial intermediary managing all investments. We return to this issue in subsection 17.6.5. We denote the price charged for a security associated with intermediate sector j at time t by $P(j,t)$. Although the decentralized equilibrium in this economy will be defined and analyzed in the next section, we can already make a few useful observations about the prices for the securities, the $P(j,t)$s. Clearly $P(j,t) < 1$ is not possible, since one unit of the security requires one unit of the final good, so $P(j,t) < 1$ would lose money. What about $P(j,t) > 1$? This is also ruled out by free entry. Imagine that a particular intermediary offers security j at some price $P(j,t) > 1$ and raises enough funds so that the total investment in this intermediate sector $I(j,t)$ is greater than the minimum size requirement $M(j)$. But in that case, some other intermediary can also enter, offer a lower price for the security, and attract all the funds that were otherwise received by the first intermediary. This argument shows that $P(j,t) > 1$ is not possible either, so that equilibrium behavior will force $P(j,t) = 1$ for all securities that are being supplied. However, we will see that securities for not all intermediate sectors will be supplied in equilibrium.

17.6.2. Equilibrium. We now characterize the equilibrium of the economy described in the previous subsection. Recall the two observations from the previous paragraph. First, not all intermediate sectors will be *open* at each date, meaning that there will be securities for only a subset of the intermediate sectors at any date. Let the set of intermediate sectors that are open at date t be denoted by $J(t)$. Second, by the argument at the end of the previous subsection, for any $j \in J(t)$ free entry implies that $P(j,t) = 1$. These two observations enable us to write the problem of a representative household $h \in \mathcal{H}_t$ taking prices and the set of available securities at time t as given. This problem takes the following form:

$$\max_{s(t),X(t),[I(j,t)]_{0\leq j\leq 1}} \log c(t) + \beta \int_0^1 \log c(j,t+1)\,dj, \tag{17.40}$$

subject to:

$$X(t) + \int_0^1 I(j,t)\,dj = s(t), \tag{17.41}$$

$$c(j,t+1) = R(j,t+1)\left(qX(t) + QI(j,t)\right), \tag{17.42}$$

$$I(j,t) = 0, \quad \forall j \notin J(t), \tag{17.43}$$

[4]To simplify the notation and the argument, I am sacrificing mathematical rigor here. Since there is a continuum of sectors, all (equilibrium) statements should be accompanied with the qualifier "*almost everywhere*". This implies that investment in a single sector (or in fact a countable subset of the $[0,1]$ sectors) may deviate from optimality. In addition, a fully rigorous analysis would require each financial intermediary to deal with a set of intermediate sectors of measure ε and then consider the limit $\varepsilon \to 0$. Throughout, I will ignore these qualifications and impose that investment in each sector needs to be consistent with equilibrium and assume that each intermediary controls a single sector.

$$c(t) + s(t) \leq w(t), \tag{17.44}$$

where I have suppressed the superscript h to simplify the notation. Here equation (17.40) is the expected utility (objective function) of the representative household. Equations (17.41)-(17.44) are the constraints on this maximization problem. The first one, (17.41), requires that the investment in the safe sector and the sum of the investments in all other securities are equal to the total savings of the individual, $s(t)$. Equation (17.42) expresses consumption of the individual in state j at time $t+1$. Two features are worth noting. First, recall households supply labor only when young and consume capital income when old. This implies that second period consumption for the household is equal to its capital holdings times the rate of return to capital, $R(j, t+1)$ given by (17.39). This rate of return is conditioned on the state j (at time $t+1$) since the amount of capital and thus the marginal product of capital will differ across states. Second, the amount of capital available to the household is equal to what it receives from the safe investment, $qX(t)$, plus the return from the Basic Arrow Security for state j, $QI(j, t)$. Equation (17.43) encapsulates a major constraint on household behavior: it emphasizes that the household cannot invest in any security that is not being supplied in the market. In particular, recall that $I(j, t) \geq M(t)$ is necessary for an intermediate sector to be open and this may not be possible for all sectors in $[0, 1]$, so some subset of the sectors in $[0, 1]$ may not be open and thus there will not be securities for the sectors that are *not* traded in equilibrium. Naturally, the household cannot invest in these non-traded securities and the constraint (17.43) ensures this. Finally, (17.44) requires the sum of consumption and savings to be less than or equal to the income of the individual, which only consists of his wage income, given by (17.38).

We are now in a position to define an equilibrium. We do this in two steps. A *static equilibrium* is an equilibrium for time t, taking the amount of capital available at time t, $K(t)$, and thus the wage $w(t)$ as given. A time t tuple $\left\langle s^*(t), X^*(t), [I^*(j,t)]_{0 \leq j \leq J^*(t)}, J^*(t), [P^*(j,t)]_{0 \leq j \leq J^*(t)}, w^*(j,t), R^*(j,t) \right\rangle$ is a static equilibrium if $s^*(t), X^*(t), [I^*(j,t)]_{0 \leq j \leq 1}$ solve the maximization of (17.40) subject to (17.41)-(17.44) for given $[P^*(j,t)]_{0 \leq j \leq J^*(t)}, J^*(t)$,$w^*(j,t)$ and $R^*(j,t)$; $w^*(j,t)$ and $R^*(j,t)$ are given by (17.38) and by (17.39); and $J^*(t)$ and $[P^*(j,t)]_{0 \leq j \leq J^*(t)}$ are such that for all $j \in J^*(t)$, $P^*(j,t) = 1$ and the set $J^*(t)$ is determined by free entry in the sense that if some $j' \notin J^*(t)$ were offered for a price $P(j', t) \geq 1$, then the solution to the modified maximization problem (17.40) subject to (17.41)-(17.44) would involve $I(j', t) < M(j)$; in other words, there is no more room for one more intermediate sector to open and attract sufficient funds to cover the minimum size requirement.

A *dynamic equilibrium* is a sequence of static equilibria linked to each other through (17.38) given the realization of the state $j(t)$ at each $t = 1, 2, ...$

Because preferences in (17.40) are logarithmic, the saving rate of all households will be constant as in the canonical overlapping generations model. Consequently, we obtain the following saving rule regardless of the risk-return tradeoff:

$$s^*(t) \equiv s^*(w(t)) = \frac{\beta}{1+\beta} w(t). \tag{17.45}$$

Given this result, a household's optimization problem can be broken into two parts: first, the amount of savings is determined, and then an optimal portfolio is chosen. This decomposition of the optimization problem is particularly useful because of two observations (with proofs left as exercises):

(1) For any $j, j' \in J(t)$, we have that $I^*(j,t) = I^*(j',t)$. Intuitively, since each individual is facing the same price for all of the traded *symmetric* Basic Arrow Securities, he would want to purchase an equal amount of each and thus achieving a *balanced portfolio* (see Exercise 17.23).

(2) The set of open projects at time t will take the form $J^*(t) = [0, n^*(t)]$ for some $n(t) \in [0,1]$. Intuitively, when only a subset of projects can be opened in equilibrium, intermediate sectors with small minimum size requirements will be opened before those with greater minimum size requirements. Consequently, if an intermediate sector j^* is open, all sectors $j \leq j^*$ must also be open (see Exercise 17.24).

The previous two observations also imply that we can divide the states of nature at time t into two sets: states in $[0, n(t)]$ that are "good" in the sense that the society is lucky and its risky investments have delivered positive returns, and states in $(n(t), 1]$ that are "bad" in the sense that the society is unlucky and its risky investments have zero returns. Clearly, the rate of return to capital (and the wage rate) will take different values in these two sets of states. We denote the rate of return to capital when a good state is realized by $R^G(t+1)$ and when a bad state is realized by $R^B(t+1)$—these returns are dated $t+1$, because they are paid out at time $t+1$. In light of this structure, the maximization problem of a representative household can be written in the much simpler form:

(17.46)

$$\max_{X(t), I(t)} \quad n^*(t) \log\left[R^G(t+1)(qX(t) + QI(t))\right] + (1 - n^*(t)) \log\left[R^B(t+1) qX(t)\right],$$

subject to:

$$X(t) + n^*(t) I(t) \leq s^*(t) \tag{17.47}$$

where $n^*(t)$, $R^G(t+1)$ and $R^B(t+1)$ are taken as given by the representative household, and $s^*(t)$ is given by (17.45). Clearly, from (17.39)

$$R^B(t+1) = \alpha (qX(t))^{\alpha - 1}$$

is the marginal product of capital in the "bad" state, when the realized state is $j > n^* (t)$ and no risky investment pays off, and

$$R^G (t) = \alpha (qX (t) + QI (t))^{\alpha - 1}$$

applies in the "good state", i.e. when the realized state is $j [0, n^* (t)]$.

Straightforward maximization of (17.46) subject to (17.47) yields the unique solution to the household's problem as:

$$X^* (t) = \frac{(1 - n^* (t))Q}{Q - qn^* (t)} s^* (t) , \tag{17.48}$$

and

$$I^* (j, t) = \begin{cases} I^* (n^* (t)) \equiv \frac{Q-q}{Q-qn(t)} s^* (t) , & \text{for } j \leq n^* (t) \\ 0 & \text{for } j > n^* (t) \end{cases} . \tag{17.49}$$

Notably, equation (17.49) implies that the demand for *each asset* (or investment in each intermediate sector) grows as the measure of open sectors increases, i.e., $I^* (n)$ is strictly increasing in n. This is because when more securities are available, the risk-diversification opportunities improve and consumers become willing to reduce their investments in the safe asset and increase their investments in risky projects. This represents an important economic force. What holds back investment in the higher productivity sectors in this economy is the fact that they are riskier than the safe sector. But since there is "safety in numbers," that is, a first-order benefit from diversification, when there are financial assets traded on more sectors, each household is willing to invest more in risky assets in total. This complementarity between the set of traded assets and investments will play an important role in the dynamics of economic development below.

Equations (17.45), (17.48) and (17.49) completely characterize the utility-maximizing behavior of the representative household given the set of intermediate sectors that are active. To completely characterize the equilibrium, we need to find the set of sectors that are active. We know that this is equivalent to finding a threshold sector $n^* (t)$ such that all $j \leq n^* (t)$ can meet their minimum size requirements while no additional sector can enter and raise enough funds to meet its minimum size requirements. Diagrammatically, this can be done by plotting the level of investment for each sector in a balanced portfolio, $I^* (n^* (t))$ given by (17.49), together with the minimum size requirement, $M (j)$ given by (17.36). The first curve can be loosely interpreted as the "demand for assets" in the financial market and the curve for (17.36) can be thought of as corresponding to the "supply of assets". The two curves and their intersection is plotted in Figure 17.2. The figure shows a unique intersection between the two curves. However, because both curves are upward-sloping, more than one intersection is possible in general. It can be verified that the condition $Q \geq (2 - \gamma) q$ is sufficient to ensure a unique intersection (see Exercise 17.25). If this condition is violated, there might be multiple solutions, corresponding to multiple equilibria. These equilibria would involve different number of active sectors. When there are only a few active sectors, households invest a large fraction of their resources in the safe asset, and in equilibrium only a few risky sectors can be operated. In contrast, when there is a significant number of active risky sectors, each household invests a large fraction of its resources in risky assets. This enables more sectors to be open and creates better risk diversification for all households. When such multiple equilibria exist, the equilibrium with more active sectors gives higher ex ante utility to all households. While interesting for illustrating the forces at work, one would expect that financial intermediaries might be successful in avoiding this type of coordination failures. Motivated by this reasoning, let us focus on on the part of the parameter space where $Q \geq (2-\gamma)\,q$. In that case, the static equilibrium is uniquely defined and the following proposition summarizes this equilibrium.

Proposition 17.8. *Suppose that $Q \geq (2-\gamma)\,q$ and that $K(t)$ is given. Then there exists a unique time t equilibrium in which all sectors $j \leq n^*(t) = n^*[K(t)]$ are open and those $j > n^*[K(t)]$ are shut, where*

$$n^*[K(t)] = \frac{(Q+q\gamma) - \left\{(Q+q\gamma)^2 - 4q\left[D^{-1}(Q-q)(1-\gamma)\Gamma K(t)^{\alpha} + \gamma Q\right]\right\}^{1/2}}{2q} \tag{17.50}$$

if $K(t) \leq D^{1/\alpha}\Gamma^{-1/\alpha}$, and $n^*[K(t)] = 1$ if $K(t) > D^{1/\alpha}\Gamma^{-1/\alpha}$ with Γ defined as $\Gamma \equiv (1-\alpha)\beta(1+\beta)^{-1}$. In this equilibrium,

$$s^*(t) = \frac{\beta}{1+\beta}(1-\alpha)K(t),$$

and $X^*(t)$ and $I^*(j,t)$ are given by (17.48) and (17.49) with $n^*(t) = n^*[K(t)]$.

Proof. See Exercise 17.26. □

An important feature is that the equilibrium threshold sector $n^*[K]$ is increasing in K. When there is more capital, the economy is able to open more intermediate sectors. This again contributes to the complementarity in the behavior mentioned above, since equation (17.49), in turn, implies that when there are more open sectors, investment in each sector will increase.

17.6.3. Equilibrium Dynamics. We next turn to the characterization of equilibrium dynamics. Given the static equilibrium in Proposition 17.8, it is straightforward to characterize the full stochastic equilibrium process. The law of motion for the capital stock, $K(t)$, will be given by a simple Markov process. Recall that investments in risky sectors will be successful with probability $n^*[K(t)]$ when the capital stock is $K(t)$, and it will be unsuccessful with the complementary probability $1-n^*[K(t)]$. This implies the following stochastic law of motion for the capital stock:

$$K(t+1) = \begin{cases} \frac{q(1-n^*[K(t)])}{Q-qn^*[K(t)]} Q\Gamma K(t)^\alpha & \text{with probability } 1-n^*[K(t)] \\ Q\Gamma K(t)^\alpha & \text{with probability } n^*[K(t)] \end{cases} \tag{17.51}$$

where $n^*[K(t)]$ is given by equation (17.50) and recall that $\Gamma \equiv (1-\alpha)\beta(1+\beta)^{-1}$. Notice that the first line of (17.51) is always less than the second line, which reflects the fact that the second line refers to the case in which the investments in the intermediate sectors have been successful.

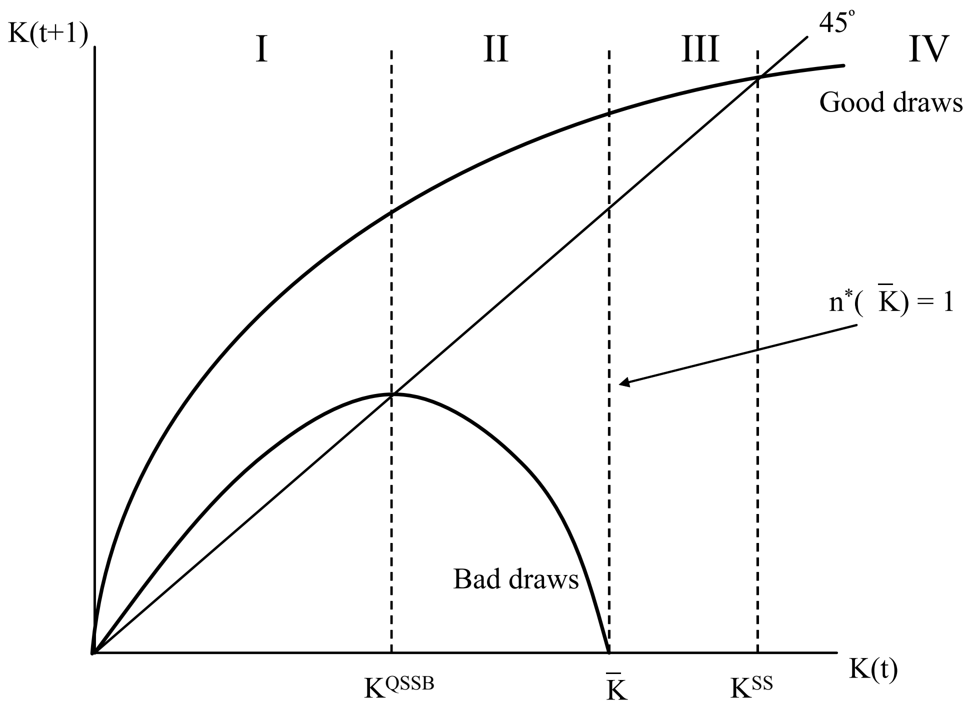

Figure 17.4. The stochastic correspondence of the capital stock.

Equation (17.51) is a particularly simple Markov process, since given $K(t)$, $K(t+1)$ can only take two values. However, it is a Markov process not a Markov chain, since for different values of $K(t)$, the possible values of $K(t+1)$ belong to the entire $\mathbb{R}_+$. A diagrammatic analysis of this Markov process is particularly illuminating. Consider Figure 17.4, which plots the stochastic correspondence of the Markov process in (17.51) and is thus similar to Figure 17.1 in the previous section. The main difference is that in Figure 17.1, any value between the two curves for $z_{\min}$ and $z_{\max}$ were possible. In contrast, here, only values exactly on the two curves plotted in the figure are possible. The first curve corresponds to $Q\Gamma K(t)^\alpha$. This is the value of the capital stock that would result if households followed their equilibrium investment strategies given in (17.48) and (17.49), and at each date, the economy turned out to be lucky, so that their investments always had positive return. The second, inverse U-shaped curve corresponds to $q(1-n^*[K(t)])Q\Gamma K(t)^\alpha / (Q - qn^*[K(t)])$ and thus applies if the economy is unlucky at each date. Both curves start above the 45^0 line near zero for the same reason as that given for the similar pattern in Figure 17.1 (i.e., because the aggregate production function (17.34) satisfies the Inada conditions). The economy will be on the upper curve with probability $n^*[K(t)]$ and on the lower curve with probability $1-n^*[K(t)]$. This implies that not only do the probabilities of success and failure change with the aggregate capital stock but so does average productivity. To quantify this variation in average productivity, let us define *expected total factor productivity* (TFP) conditional on the proportion of intermediate sectors that are open:

$$\sigma^e(n^*[K(t)])) = (1-n^*[K(t)])\frac{q(1-n^*[K(t)])}{Q-qn^*[K(t)]}Q + n^*[K(t)]\,Q. \tag{17.52}$$

Straightforward differentiation establishes that $\sigma^e(n^*[K(t)])$ is strictly increasing in $n^*[K(t)]$. This implies that as the economy develops and manages to open more intermediate sectors, its productivity will endogenously increase. Since $n^*[K]$ is increasing in K, this implies that average productivity is increasing in the capital stock of the economy. We record this as a proposition for future reference:

Proposition 17.9. *The expected total factor productivity of the economy $\sigma^e(n^*[K]))$ is increasing in n^* and thus increasing in K.*

Inspection of Figure 17.4 also suggests that the following two levels of capital stock are special and useful in the analysis.

- **(i):** K^{QSSB} refers to the "quasi steady state" of an economy which *always has unlucky draws*. An economy would converge towards this quasi steady state if it follows the optimal investments characterized above but the sectors chosen by the households never have positive pay-off due to bad luck .
- **(ii):** K^{QSSG} refers to the "quasi steady state" of an economy which *always receives good news*, meaning that it is always on the upper curve in Figure 17.4.

These two capital stock levels are plotted in the figure and are also easy to compute as:

$$K^{QSSB} = \left[\frac{q\left(1-n^*\left[K^{QSSB}\right]\right)}{Q-qn^*\left[K^{QSSB}\right]}Q\Gamma\right]^{\frac{1}{1-\alpha}} \quad \text{and} \quad K^{QSSG} = (Q\Gamma)^{\frac{1}{1-\alpha}}. \tag{17.53}$$

The form of K^{QSSG} is particularly noteworthy, since it refers to the case in which the economy never faces any risk and thus acts very much like a standard neoclassical growth model. In particular, if, in equilibrium, $n^*\left[K^{QSSG}\right] = 1$, then K^{QSSG} in fact becomes a proper steady state and the economy would stay at this level of capital stock once it reaches it. This is because once the economy accumulates sufficient capital to open all intermediate sectors, it would eliminate all risk and would always be on the upper curve in Figure 17.4.

Equations (17.50) and (17.53) show that the condition for this good steady state to exist, i.e., for $n^*\left[K^{QSSG}\right]=1$, is that the saving level corresponding to K^{QSSG} be sufficient to ensure a balanced portfolio of investments, of at least D, in all the intermediate sectors. It is straightforward to show that the following condition is sufficient to ensure this

$$D<\Gamma^{\frac{1}{1-\alpha}}Q^{\frac{\alpha}{1-\alpha}}. \tag{17.54}$$

Thus when (17.54) is satisfied, the good quasi steady state will indeed generate sufficient capital to open all sectors and eliminate all the risk, thus becoming a proper steady state. In this case, we denote K^{QSSG} by K^{SS}. Under the assumption that (17.54) is satisfied, Figure 17.4 draws $n^*\left[K^{SS}\right]$. Now returning to this figure, we can get a better sense of the stochastic dynamics of this equilibrium. The figure divides the range of capital stocks into four regions. In region I, the capital stock is low enough so that both the curve conditional on good draws and bad draws are above the 45° line, so that in this range the economy will grow regardless of whether it experiences good or bad productivity realizations. Next comes region II, which in many ways is the most interesting one. Here the economy grows if it receives positive shocks but suffers a crisis if its investments are unsuccessful. Between these two regions lies the bad quasi steady state K^{QSSB}. The figure justifies the terminology of calling this level of capital stock a "quasi steady state," since when $K<K^{QSSB}$, the economy will definitely grow towards K^{QSSB}. When $K>K^{QSSB}$, the economy may grow or contract. Nevertheless, as noted above, because $n^*[K]$ is increasing in K, in the right-hand side neighborhood of K^{QSSB}, the economy has the highest probability of contracting (recall that to the left of K^{QSSB}, negative shocks do not lead to a contraction).

For most parameter values, the economy tends to spend a long time in region II. Acemoglu and Zilibotti (1997) provide examples where the number of periods in which the economy is in regions I and II could be arbitrarily large. However, if the economy were to receive a sequence of good news, it would ultimately exit from region II and enter region III. The level of capital stock that divides these two regions, $\bar{K}$, is defined such that $n^*\left[\bar{K}\right]=1$. This means that once the economy reaches the capital stock of $\bar{K}$, it has enough capital to open all the sectors. Consequently, in region III all risk is diversified and the dynamics are exactly the same as those of the canonical overlapping generations model without uncertainty. Finally, starting anywhere in region III the economy travels towards the steady state K^{SS}, which stands between regions III and IV. Region IV, on the other hand, has so much capital that even with the positive shocks, the economy will contract. Naturally, unless it starts there, the economy will never enter region IV.

This discussion, combined with Figure 17.4, gives a fairly complete characterization of the stochastic equilibrium growth path. In particular, an economy that starts with a low enough capital stock will first experience some growth, but then spend a long time fluctuating between successful periods and periods of severe crises. Eventually, a string of good news will take the economy to a level of capital stock such that much (here all) of the risks can be diversified. At this level, we can think of the economy as achieving *takeoff* as in Rostow's account discussed in Chapter 1. The economy is experiencing a takeoff in two senses. First, after takeoff it successfully diversifies all risk, so that growth from this point onwards progresses steadily rather than being subject to significant fluctuations as in region II. Second, Proposition 17.9 implies that the aggregate (labor and total factor) productivity will increase after this level of capital. Thus takeoff comes with a decline in the fluctuations in economic activity and an increase in productivity.

In addition, as the economy develops by accumulating more capital, it achieves both higher productivity and better diversification, and manages its risks better. This takes the form of more sectors being open, which equivalently corresponds to more financial intermediaries being active. Thus in this model financial and economic development go hand-in-hand. In this respect, it is important to emphasize that in the model it is neither economic growth that causes financial development nor financial development that causes economic growth. Both are determined jointly and affect each other along the equilibrium path.

A natural question is whether the economy will necessarily reach region III and then region IV. The next proposition answers this question.

Proposition 17.10. *Suppose that condition (17.54) holds, then the stochastic process* $\{K(t)\}_{t=1}^{\infty}$ *converges to the point* K^{SS} *with probability 1.*

Proof. See Exercise 17.27. □

This proposition establishes that the variability of growth in the economy will *eventually* decline (and in fact disappear). But one might wish to know whether the amplitudes of economic fluctuations are systematically related to the level of the capital stock or output in the economy. This is particularly relevant, since, as already discussed, both cross-sectional and time-series comparisons suggest that poorer nations suffer from greater economic variability. To answer this question, the natural variable to look at is the conditional variance of TFP (whose expected value was defined in (17.52) above). Define $\sigma(n^*[K(t)])$ as a random variable that takes the values $q(1-n^*[K(t)])Q/(Q-qn^*[K(t)])$ and Q with respective probabilities $(1-n^*[K(t)])$ and $n^*[K(t)]$. The expectation of this random variable is $\sigma^e(n^*[K(t)]))$ as defined in (17.52). Then, taking logs, we can rewrite (17.51) as

$$\triangle \log(K(t+1)) = \log\Gamma - (1-\alpha)\log(K(t)) + \log(\sigma(n^*[K(t)])). \tag{17.55}$$

It is clear from this equation that capital (and output) growth volatility, after removing the deterministic "convergence effects" due to the standard neoclassical effects, are determined by the stochastic component σ. Denoting the (conditional) variance of $\sigma(n^* [K(t)]$ given $K(t)$ by V_n, we can state the following proposition.

Proposition 17.11. *Let* $\mathcal{V}_n \equiv Var(\sigma(n^*) \mid n^*) = n^*(1-n^*) [Q(Q-q)/(Q-qn^*)]^2$*. Then we have that*

- *If* $\gamma \geq Q/(2Q-q)$*, then* $\partial\mathcal{V}_n/\partial K \leq 0$ *for all* $K \geq 0$*.*
- *If* $\gamma < Q/(2Q-q)$*, then there exists* $\tilde{K}$ *defined such that* $n^*(\tilde{K}) = Q/(2Q-q) < 1$ *and.*

$$\frac{\partial\mathcal{V}_n}{\partial K} \leq 0 \text{ for all } K \geq \tilde{K}$$
$$\frac{\partial\mathcal{V}_n}{\partial K_t} > 0 \text{ for all } K < \tilde{K}.$$

Proof. See Exercise 17.28. □

The behavior of the variability of growth in this proposition results from the counteracting effects of two forces; first, as the economy develops, more savings are invested in risky assets; and second, as more sectors are opened, idiosyncratic risks are better diversified. The proposition shows that if $\gamma \geq Q/(2Q-q)$, the second effect always dominates and thus the richer economies are less risky. If $\gamma < Q/(2Q-q)$, then the first effect dominates for sufficiently low levels of capital stock but once the capital stock reaches a critical threshold, $\tilde{K}$, the second effect again dominates. Thus except for sufficiently low levels of capital, the variability of the growth rate is everywhere decreasing in the income (or capital) level of the economy.

17.6.4. Efficiency. The previous subsection completely characterized the stochastic equilibrium of the economy. Is this equilibrium Pareto efficient? Since all agents are price takers, it may be conjectured that the answer to this question must be yes. In this subsection I show that this is not the case. Though at first surprising, this result will turn out to be intuitive and interesting. First, it results from an economically meaningful *pecuniary externality.* Second, it makes sense from the viewpoint of the theory of general equilibrium; though all agents are price takers, this is not an Arrow-Debreu economy because the set of traded commodities is determined endogenously by a zero profit condition. To illustrate these issues in the most transparent way I ignore any potential source of intertemporal inefficiency (which, we know from Chapter 9, may arise in overlapping generations economies). For this reason, the analysis of efficiency takes a particular level of savings $s(t)$, or equivalently the current level of the capital stock $K(t)$, as given and looks at whether the way in which savings are allocated across different sectors of the economy is (constrained) efficient. We do this, by considering the social planner's problem. This problem can be written as:

$$\max_{n(t),X(t),[I(j,t)]_{0\le j\le n(t)}} \int_0^{n(t)} \log(qX(t)+QI(j,t))dj+(1-n(t))\log(qX(t)) \tag{17.56}$$

subject to

$$X(t)+\int_0^{n(t)} I(j,t)\,dj\le s(t)\,.$$

More specifically, the social planner chooses the set of sectors that are active, which is denoted by $[0,n(t)]$, the amount that will be invested in the safe sector $X(t)$ and the allocation of funds among the other sectors denoted by $[I(j,t)]_{0\le j\le n(t)}$. In principle, the social planner could have chosen the set of active sectors not to be an interval of the form $[0,n(t)]$, but the same argument as in Exercise 17.24 ensures that there is no loss of generality in imposing this form. The constraint makes sure that the sum of investments in the safe and the risky sectors is less than the amount of savings available to the planner. The main difference between this program and the maximization problem of the representative household (17.40) is that the social planner also chooses $n(t)$, while the representative household took the set of available assets as given. The social planner's allocation (and thus the Pareto optimal allocation) is given by the solution to this maximization problem. The next proposition characterizes the solution.

Proposition 17.12. *Let $n^*[K(t)]$ be given by (17.50), and $s(t)$ and $K(t)$ denote current level of savings and capital stock available to the social planner. Then, the unique solution to the maximization problem in (17.56) is as follows:*

- *For all $s(t)<D$, the set of active sectors is given by $\left[0,n^S[K(t)]\right]$, where $n^S[K(t)]>n^*[K(t)]$. The amount of investment in the safe sector is given by $X^S(t)$, where $X^S(t)<X^*(K(t))$. Finally, there exists a sector $j^*(t)\in\left(0,n^S[K(t)]\right)$ such that the portfolio of risky sectors for each household takes the form*

$$\begin{array}{ll} I^S(j,t)=M(j^*)>M(j) & \text{for } \; j<j^*(t) \\ I^S(j,t)=M(j) & \text{for } j\in\left[j^*(t),n^S[K(t)]\right] \\ I^S(j,t)=0 & \text{for } \; j>n^S(K(t)) \end{array}. \tag{17.57}$$

- *For all $s(t)\ge D$, $n^S[K(t)]=n^*[K(t)]=1$ and $I^S(j,t)=s(t)$ for all $j\in[0,1]$.*

Proof. See Exercise 17.29. □

This proposition implies that, when the economy has not achieved full diversification, the social planner will open more sectors than the decentralized equilibrium. She will finance these additional sectors by deviating from the *balanced portfolio*, which was always a feature of the equilibrium allocation. In other words, she will invest less in the sectors without the minimum size requirement. The Pareto optimal allocation of funds is shown in Figure 17.5.

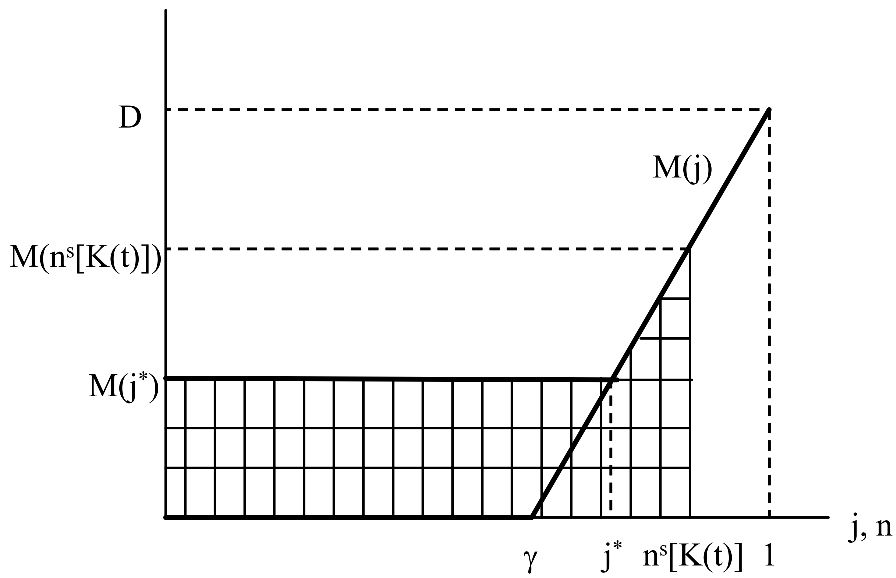

Figure 17.5. The efficient portfolio allocation.

The deviation from the balance portfolio implies that the social planner is implicitly *cross-subsidizing* the sectors with high a minimum size requirements at the expense of sectors with low or no minimum size requirements. This is because, starting with a balanced portfolio, opening a few more sectors always benefits consumers, who will be able to achieve better risk diversification. The only way the social planner can achieve this is by implicitly taxing sectors that have low or no minimum size requirements (so that they have high Lagrange multipliers and thus lower investments) and subsidizing the marginal sectors with high minimum size requirements.

Why does the decentralized equilibrium not achieve the same allocation? There are two complementary ways of providing the intuition for this. The first is that a marginal dollar of investment by an household in a high minimum size requirements sector creates a *pecuniary externality*, because this investment makes it possible for the sector to be active and thus provide better risk diversification possibilities to all the other agents. However, each household, taking equilibrium prices as given, ignores this pecuniary externality and tends to underinvest in marginal sectors with high minimum size requirements. Thus the source of inefficiency is that each household ignores its impact on others' diversification opportunities. The second intuition for this result is related. Because households take the set of prices as given and in equilibrium $P(j,t) = 1$ for all active sectors, they will always hold a balanced portfolio. However, the Pareto optimal allocation involves cross-subsidization across sectors in a non-balanced portfolio. Market prices do not induce the households to hold the right portfolio.

At this point, the reader may wonder why the First Welfare Theorem does not apply. In particular, all households are price takers. The reason why the First Welfare Theorem does not apply that the decentralized equilibrium here does not correspond to an Arrow-Debreu equilibrium. In particular, this is an equilibrium for an economy with endogenously incomplete markets, where the set of active markets is determined by zero profit (free entry) condition. All commodities that are traded in equilibrium are priced competitively but there is no "competitive pricing" for commodities that are not traded. Instead, in an Arrow-Debreu equilibrium, all commodities, even those that are not traded in equilibrium, are priced and in fact a potential commodity would not be traded in equilibrium only if its price were equal to numeral a zero and at zero prices generated excess supply. In this sense, the equilibrium characterized here is not an Arrow-Debreu equilibriumand in fact such an equilibrium does not exist in this economy because of the nonconvexity in the production set. Instead, the equilibrium concept used here is a more natural competitive equilibrium notion, which requires that all commodities that are traded in equilibrium are priced competitively and then determines the set of traded commodities by a free entry condition. Some additional discussion of this equilibrium concept is provided in the References and Literature section below

17.6.5. Inefficiency with Alternative Market Structures. Would the market failure in portfolio choices be overcome if some financial institution could coordinate households' investment decisions? Imagine that rather than all agents acting in isolation and ignoring their impact on each others' decisions, funds are intermediated through a financial coalition-intermediary. This intermediary can collect all the savings and offer to each saver a *complex* security (as different from a Basic Arrow Security) that pays $QI^S(j,t)+qX^S(t)$ in each state j, where $I^S(j,t)$ and $X^s(t)$ are as in the optimal portfolio. Holding this security would make each consumer better off compared to the equilibrium.

Although from this discussion it may appear that the inefficiency we identified may not be robust to the formation of more complex financial institutions, we will show that this is not the case. The remarkable result is that unless some rather strong assumptions are made about the set of contracts that a financial intermediary can offer, equilibrium allocations resulting from competition among intermediaries will be identical to the equilibrium allocation in Proposition 17.8. A full analysis of this issue is beyond the scope of the current book, but a brief discussion gives the flavor. Let us model more complex financial intermediaries as "intermediary-coalitions," that is, as a set of households who join their savings together and invest in a particular portfolio intermediate sectors. Such coalitions may be organized by a specific household, and if it is profitable for other households to join the coalition, the organizer of the coalition can charge a premium (or a joining fee) thus making profits. We assume that there is free entry into financial intermediation or coalition-building, so that any household can attempt to exploit profit opportunities if there are any. Let us also impose some structure on how the timing of financial intermediation works and also how individual households can participate into different coalitions. In particular, let us adopt the following assumptions.

(1) Coalitions maximize a weighted utility of their members at all points in time. In particular, a coalition cannot commit to a path of action that will be against the interests of its members in the continuation game.
(2) Coalitions cannot exclude other agents (or coalitions) from investing in a particular project.

Acemoglu and Zilibotti (1997) prove the following result.

Proposition 17.13. *The set of equilibria of the financial intermediation game described above is always non-empty and all equilibria have exactly the same structure as those characterized in subsection 17.6.2 and Proposition 17.8.*

I will not provide a proof of this proposition, since a formal statement of the proposition and the proof require additional notation. But the intuition is straightforward: as shown in Proposition 17.12, the Pareto optimal allocation involves a non-balanced portfolio and cross-subsidization across different sectors. This implies that the shadow price of investing in some sectors should be higher than in others, even though the cost of investing in each sector is equal to 1 (terms of date t final goods). These differences in shadow prices will then support a non-balanced portfolio. Recall also that it is the sectors with no minimum size requirement or low minimum size requirements that are being implicitly taxed in this allocation. This kind of cross-subsidization is difficult to sustain in equilibrium because each household would deviate towards slightly reducing its investments in coalitions/intermediaries that engage in cross-subsidization and undertake investments on the side to move its portfolio towards a balanced one (by investing in no minimum size or low minimum size sectors). At the end, only allocations without cross-subsidization can survive as equilibria, and those are identical to equilibria characterized in subsection 17.6.2.

The most important implication of this result is that even with unrestricted financial intermediaries or coalitions, the inefficiency resulting from endogenously incomplete markets cannot be prevented. The key economic force is that each household creates a positive pecuniary externality by holding a non-balanced portfolio but in a decentralized equilibrium each household wishes to and can easily move towards a balanced portfolio, undermining efforts to sustain the efficient allocation.

17.7. Taking Stock

This section presented a number of different models of stochastic growth. My selection of topics was geared towards achieving two objectives. First, I introduced a number of workhorse models of macroeconomics, such as the neoclassical growth model under uncertainty and the basic Bewley model. These models not only useful for the analysis of economic growth but also have a wide range of applications in the macroeconomics literature.

Second, the model in Section 17.6 demonstrated how stochastic models can significantly enrich the analysis of economic growth and economic development. In particular, this model showed how a simple extension of our standard models can generate an equilibrium path in which economies spend a long time with low productivity and suffer frequent crises. They take off into sustained and steady growth once they receive a sequence of favorable realizations. The takeoff process not only reduces volatility and increases growth but is also associated with better management of risk and greater financial development. Though stylistic, this model provides a good approximation to the economic development process that much of Western Europe underwent over the past 700 years or so. It also emphasizes the possibility that luck may have played an important role in the timing of takeoff and perhaps even in determining which countries were early industrializers. Therefore, this model provides an attractive formalization of the luck hypothesis discussed in Chapter 4. Nevertheless, underlying the equilibrium in this model is a set of market institutions that enable households to working competitive markets and invest freely. Thus my interpretation would be that the current model shows how random elements and luck can matter for the timing of takeoff among countries that satisfies some prerequisites for takeoff. This could account for some of the current-day cross-country income differences and may provide important insights about the beginning of the process of sustained growth. However, institutional factors—whether those prerequisites are satisfied—are more important for understanding why some parts of the world did not takeoff during the 19th century and have not yet embarked on a path of sustained and steady growth. These are topics that will be discussed in the rest of the book.

It is also worth noting that the model in Section 17.6 introduce a number of important ideas related to incomplete markets. The Bewley model presented in Section 17.4 is a prototypical incomplete markets model and as most incomplete markets models in the literature, it takes the set of markets that are open as given. In contrast, the model in Section 17.6 is a model with endogenously incomplete markets. The analysis showed that the fact that the set of markets that are open (the set of sectors that are active) is determined in equilibrium with a free entry condition can lead to a novel type of Pareto inefficiency due to pecuniary externalities (even though all households take prices is given). Although this type of Pareto inefficiency is different from those we have encountered so far, there are some important parallels between the fact that an insufficient number of markets are open in this model and too few intermediate goods being produced in the baseline endogenous technological change model of Chapter 13.

17.8. References and Literature

The neoclassical growth model under uncertainty, presented in Section 17.1, was first analyzed by Brock and Mirman (1972). Because the analysis of the social planner's problem is considerably easier than the study of equilibrium growth under uncertainty, most analyses in the literature look at the social planner's problem and then appeal to the Second Welfare Theorem. Stokey, Lucas and Prescott (1989) provide an example of this approach. An analysis of the full stochastic dynamics of this model requires a more detailed discussion of the general theory of Markov processes. Space restrictions preclude me from presenting these tools. The necessary material can be found in Stokey, Lucas and Prescott (1989, Chapters 8, 11, 12 and 13) or the reader can look at Futia (1982) for a more compact and excellent treatment. More advanced and complete treatments are presented in Ethier and Kurtz (1985) or Gikhman and Skorohod (1974). The tools in Stokey, Lucas and Prescott (1989) are sufficient to prove that the optimal path of capital-labor ratio in the neoclassical growth model under uncertainty converges to a unique invariant distribution and they can also be used to prove the existence of a stationary equilibrium in the Bewley economy.

The first systematic analysis of competitive equilibrium under uncertainty is provided in Lucas and Prescott (1971) and Mehra and Prescott (1979). Sargent and Ljungqvist (2004, Chapter 12) provides a good textbook treatment. The material in Section 17.2 is similar to that in Sargent and Ljungqvist but is more detailed and provides a few additional results.

The Real Business Cycle literature is enormous and the treatment in Section 17.3 only scratches the surface. The classic papers in this literature are Kydland and Prescott (1982) and Long and Plosser (1983). Sargent and Ljungqvist (2004) again provides a good introduction. The collection of papers in Cooley and Prescott (1995) is an excellent starting point, emphasizing the achievements of the RBC literature and providing a range of tools for theoretical as well as quantitative analysis using recursive competitive models. Blanchard and Fischer (1989) discusses the critiques of the RBC approach. The interested reader is also referred to the exchange between Edward Prescott and Lawrence Summers (Prescott, 1986, and Summers, 1986) and to the review of the more recent literature in King and Rebelo (1999).

Section 17.4 presents the incomplete markets model first introduced by Truman Bewley (1977, 1980, 1986). This model has now become one of the workhorse models of macroeconomics and has been used for analysis of business cycle dynamics, income distribution, optimal fiscal policy, monetary policy, and asset pricing. A more modern treatment is provided in Aiyagari (1994), though the published version of the paper does not contain any of the mathematical analysis. The reader is referred to Bewley (1977, 1980) and to the working paper version, Aiyagari (1993), for more details on some of the propositions stated in Section 17.4 as well as a proof of existence of a stationary equilibrium, which I did not provide in the text. Krusell and Smith (1998, 1999, 2005), among others, have used this model for business cycle analysis and have also provided new quantitative tools for the study of incomplete market economies.

Section 17.5 is a simple stochastic extension of the baseline overlapping generations model. I am not aware of any similar treatment, though none of the material in this section is new or difficult.

Section 17.6 builds on Acemoglu and Zilibotti (1997) and more details on some of the results stated in that section are provided in Acemoglu and Zilibotti (1997). Evidence on the relationship between economic development and volatility is provided in Acemoglu and Zilibotti (1997), Imbs and Wacziarg (2003), and Koren and Tenreyro (2007). Ramey and Ramey (1994) also provide related evidence, though they emphasize the effect of volatility on growth using cross-country regression analysis. As noted above, the concept of decentralized equilibrium used in this model is not Arrow-Debreu. Instead it imposes price-taking behavior in all open markets and determines the set of open markets via a free entry condition. This type of equilibrium concept is commonly used in general equilibrium theory, for example, Hart (1980), Makowski (1980), and Allen and Gale (1991). Koren and Tenreyro (2007) present a generalization of the Acemoglu and Zilibotti (1997) model. Acemoglu and Zilibotti (1997) also contain an analysis of international capital flows in a similar framework and this analysis is extended in Martin and Rey (2002).

17.9. Exercises

Exercise 17.1. Proposition 17.2 shows that $k(t+1)$ is increasing in $k(t)$ and $z(t)$. Provide sufficient conditions such that $c(t)$ is also increasing in these variables.

Exercise 17.2. Consider the neoclassical growth model under uncertainty analyzed in Section 17.1 and assume that $z(t)$ is realized after $c(t)$ and $k(t+1)$ are chosen.

(1) Show that if $z(t)$ is distributed independently across periods, the choice of capital stock and consumption in this economy is *identical* to that in a neoclassical growth model under certainty with a modified production function. Explain this result.
(2) Now suppose that $z(t)$ is not distributed independently across periods, Establish the equivalent of Proposition 17.1 for this case. How does the behavior in this economy differ from the canonical version of the neoclassical growth model under uncertainty in Section 17.1.

Exercise 17.3. Consider the same production structure as in Sections 17.1 and 17.2 but assume that irrespective of the level of the capital stock and the realization of the stochastic variable, each household saves a constant fraction s of its income. Characterize the stochastic laws of motion of this economy. How does behavior in this economy differ from that in the canonical neoclassical growth model under uncertainty.

Exercise 17.4. Consider the neoclassical growth model under uncertainty studied in Section 17.1.

(1) Provide conditions under which $\pi(k, z)$ is strictly increasing in both of its arguments.
(2) Show that when this is the case, the capital-labor ratio can never converge to a constant unless z has a degenerate distribution (i.e., it always takes the same value).

Exercise 17.5. Consider Example 17.1.

(1) Prove that equation (17.9) cannot be satisfied for any $B_0 \neq 0$.
(2) Conjecture that the value function for this example takes the form $V(k, z) = B_2 + B_3 \log k + B_4 \log z$. Verify this guess and compute the parameters B_2, B_3 and B_3.

Exercise 17.6. Show that the policy function in Example 17.1 $\pi(k, z) = \beta\alpha z k^{\alpha}$ applies when z follows a general Markov process rather than a Markov chain. [Hint: instead of the summation, replace the expectations sign with an appropriately defined (Lebesgue) integral and cancel terms under the integral sign].

Exercise 17.7. (1) Consider the economy analyzed in Example 17.1 with $0 < z_1 < z_N < \infty$. Characterize the limiting invariant distribution of the capital-labor ratio and show that the stochastic correspondence of the capital stock can be represented by Figure 17.1 in Section 17.5. Use this figure to show that the capital-labor ratio, k, will always grow when it is sufficiently small and always decline when it is sufficiently large.
(2) Next consider the special case where z takes two values z_h and z_l, with each value persisting with probability $q > 1/2$ and switching to the other value with probability $1 - q$. Show that as $q \to 1$, the behavior of the capital-labor ratio converges to the equilibrium behavior of the same object in the neoclassical growth model under certainty.

Exercise 17.8. Consider the economy studied in Example 17.1 but suppose that $\delta < 1$. Show that in this case there does not exist a closed-form expression for the policy function $\pi(k, z)$.

Exercise 17.9. Consider an extended version of the neoclassical growth model under uncertainty such that the instantaneous utility function of the representative household is $u(c, b)$, where b is a random variable following a Markov chain.

(1) Setup and analyze the optimal growth problem in this economy. Show that the optimal consumption sequence satisfies a modified stochastic Euler equation.
(2) Prove that Theorem 5.7 can be applied to this economy and the optimum growth path can be decentralized as a competitive growth path.

Exercise 17.10. Write the maximization problem of the social planner explicitly as a sequence problem, with output, capital and labor following different histories interpreted as a different Arrow-Debreu commodity. Using this formulation, show that the conditions of Theorem 5.7 are satisfied, so that the optimal growth path can be decentralized as an equilibrium growth path.

Exercise 17.11. Explain why in subsection 16.5.1 in the previous chapter, the Lagrange multiplier $\tilde{\lambda}\left[y^t\right]$ was conditioned on the entire history of labor income realizations, while in the formulation of the competitive equilibrium with a full set of Arrow-Debreu commodities (contingent claims) in Section 17.2, there is a single multiplier λ associated with the lifetime budget constraint.

Exercise 17.12. Consider the model of competitive equilibrium in Section 17.2. Repeat the analysis of the competitive equilibrium of the neoclassical growth model under uncertainty by assuming that instead of a price for buying and selling capital goods in each state (at the price sequence in terms of date 0 final good given by $R_0\left[z^t\right]$) there is a market for renting capital goods. Let the rental price of capital goods in terms of date 0 final good be $r_0\left[z^t\right]$ when the sequence of stochastic variables is z^t. Characterize the competitive equilibrium and show that it is equivalent to that obtained in Section 17.2. Explain why the two formulations give identical results.

Exercise 17.13. Prove Proposition 17.3.

Exercise 17.14. Characterize the competitive equilibrium path of the neoclassical growth model under uncertainty analyzed in Section 17.2 using sequential trading, but the sequence problem formulation rather than the recursive formulation.

Exercise 17.15. Show that Theorems 16.1-16.7 can be applied to $V(a,z)$ defined in (17.19) and establish that $V(a,z)$ is continuous, strictly increasing in both of its arguments, concave and differentiable in a.

Exercise 17.16. Derive equation (17.21).

Exercise 17.17. Prove Proposition 17.4.

Exercise 17.18. Consider the RBC model presented in Section 17.3 and suppose that the production function takes the form $F(K, zAL)$, with both z and A corresponding to labor-augmenting technological productivity terms. Suppose that z follows a Markov chain and $A(t+1) = (1+g)A(t)$ is an exogenous and deterministic productivity growth process. Setup the social planner's problem in this case. [Hint: use a transformation of variables to make the recursive equation stationary]. What restrictions do we need to impose on

$U(C, L)$ such that the optimal growth path corresponds to a "balanced growth path," where labor supply does not (with probability 1) go to zero or infinity?

Exercise 17.19. In Example 17.2, suppose that the utility function of the representative household is $u(C, L) = \log C + h(L)$, where $h(\cdot)$ is a continuous, decreasing and concave function. Show that the equilibrium level of labor supply is constant and independent of the level of capital stock and the realization of the productivity shock.

Exercise 17.20. Prove Proposition 17.5.

Exercise 17.21. Prove Proposition 17.6.

Exercise 17.22. What would happen if, instead of the logarithmic preferences (17.37), the utility function of the representative household in Section 17.6 took the more general form $u(c_1(t)) + \mathbb{E}_t u(c_2(t+1))$? Could the growth rate on an economy be higher in this case when the level of diversification is limited? [Hint: first show that full diversification always achieves higher growth; then consider extremely risk-averse preferences and construct an example in which lower diversification can encourage sufficiently more savings to increase growth].

Exercise 17.23. In the model of Section 17.6, prove that the maximization problem of the representative household involves for any $j, j' \in J(t)$, $I^*(j, t) = I^*(j', t)$.

Exercise 17.24. In the model of Section 17.6, prove that if an intermediate sector $j^* \in J(t)$, then all sectors $j \leq j^*$ are also in $J(t)$.

Exercise 17.25. In the model of Section 17.6, prove that the condition $Q \geq (2 - \gamma) q$ is sufficient to ensure that there is a unique intersection between the curves for (17.36) and (17.49) in Figure 17.2.

Exercise 17.26. Prove Proposition 17.8. In particular, show that (i) if the equilibrium $n < n^*[K]$, then there exists a profitable deviation for a financial intermediary to offer securities based on a previously-unavailable sector and make positive profits; and (ii) if $n > n^*[K]$, feasibility is violated.

Exercise 17.27. (1) Prove Proposition 17.10.

(2) Suppose that condition (17.54) is not satisfied. Does the stochastic process $\{K(t)\}_{t=0}^{\infty}$ converge? Does it converge to a point?

Exercise 17.28. Prove Proposition 17.11.

Exercise 17.29. Prove Proposition 17.12. [Hint: setup the Lagrangian for the social planner and show that when all sectors can not be active the social planner will not choose a balanced portfolio.]

Exercise 17.30. * Consider the following two-period economy similar to the environment described in Section 17.6. There are I financial intermediaries who compete a la Bertrand without using any resources. They invest funds on behalf of consumers in any of the projects of this economy. There is a continuum of measure 1 of projects each denoted by j. Asset j

requires a minimum size investment $M(j)$ and without loss of generality rank the projects in ascending order of minimum size.

There is continuum of consumers with measure normalized to 1, each with the utility function $u(c) + \mathbb{E}v(c')$, where c is consumption today, c' is consumption tomorrow, so that $\mathbb{E}v(c')$ denotes expected utility from tomorrow's consumption. Each consumer has total resources equal to w and decides how much to consume and how much to save and then how to allocate his savings. Assume that $u(\cdot)$ and $v(\cdot)$ are strictly concave and increasing. Funds today are turned into consumption tomorrow by financial intermediaries. Alternatively, funds can also be invested in a safe linear technology with rate of return q. Let the investment in asset j be $K(j)$, then if $K(j) \geq M(j)$, then asset j has probability π of paying out $Qk(j)$ such that $\pi Q > q$ (thus the safe technology is less productive). On the other hand if $K(j) < M(j)$, the pay-out is equal to zero.

(1) Denote the "share price" of \$1 invested in project j, which pays out \$$Q$ with probability π and zero otherwise, by $p(j)$. Show that financial competition ensures that if $K(j)$, $K(j') > 0$, then $p(j) = p(j') = 1$.

(2) Now assume that the returns of each project is *independent* from the return of all other projects and show that $K(j) = K(j')$—that is, the probability that asset j pays out Q conditional on asset j' having paid out Q (or 0) is π for all j and j'.

(3) Characterize the decentralized equilibrium of this economy.

(4) Show that when some projects are inactive, the decentralized equilibrium is constrained Pareto inefficient.

(5) Characterize the efficient allocation.

(6) Can you establish the inefficiency of the decentralized equilibrium without independence?

(7) Informally discuss what happens if $M(j)$ is not a minimum size requirement but a fixed cost (such that average costs are falling). [Hint: there are two cases to distinguish; (1) linear prices; (2) price discrimination].

Part 6

Technology Diffusion, Trade and Interdependences

One of the most major shortcomings of the models presented so far is that each country is treated as an "island" on to itself, not interacting with the rest of the countries in the world. This is problematic for at least two reasons. The first is related to the technological interdependences across countries and the second to international trade (in commodities and in assets). In this part of the book, we will investigate the implications of technological and trade interdependences on the process of economic growth.

The models presented so far treat technology either as exogenous or as endogenously generated within the boundaries of the economy in question. We have already seen how allowing for endogeneity of technology provides new and important insights about the process of growth. But should we think of the potential technology differences between the United States and Nigeria as resulting from lower R&D in Nigeria? The answer to this question is most probably no. To start with, Nigeria, like most less-developed or developing countries, imports many of its technologies from the rest of the world. This suggests that a framework in which *frontier* technologies in the world are produced in the United States or other advanced economies and then copied or adopted by other "follower" countries provides a better approximation to reality. Therefore, to understand technology differences between advanced and developing economies, we should not only, or not even primarily, focus on differential rates of endogenous technology generation in these economies, but on their decisions concerning *technology adoption* and *efficient technology use*.

While the exogenous growth models of Chapters 2 and 8 have this feature, they too have important shortcomings. First, technology is entirely exogenous, so interesting economic decisions only concern investment in physical capital. There is a conceptually and empirically compelling sense in which technology is different from physical capital (and also from human capital), so we would like to understand sources of differences in technology arising endogenously across countries. Thus the recognition that technology adoption from the world frontier matters is not the same as accepting that the Solow or the neoclassical growth model are the best vehicle for studying cross-country income differences. Second, while the emphasis on technology adoption makes the process of growth resemble the exogenous growth models of Chapters 2 and 8, technological advances at the world level are unlikely to be "manna from heaven". Instead, economic growth at the world level either results from the interaction of the adoption and R&D decisions of all countries or perhaps from the innovations by frontier economies. This implies that models in which the growth rate at the world level is endogenous and interacts (and coexists) with technology adoption may provide a better approximation to reality and a better framework for the analysis of the mechanics of economic growth. In addition to technology adoption, other interactions across countries, such as international trade, may also play the same role of allowing for endogenous growth at the world level together with growth in each specific country that depends on technological and other developments at the world level.

In Chapter 18, we will start with models of technology adoption and investigate the factors affecting the speed and nature of technology adoption. We will also place special emphasis on whether or not technologies that are available from the world technology frontier are *appropriate* for the needs of less-developed countries. Recall also that "technology differences" not only reflect differences in techniques used in production, but also differences in the organization of production affecting the efficiency with which existing factors of production are utilized. A satisfactory theory of technology differences among countries must therefore pay attention to barriers to technology adoption and to potential inefficiencies in the organization of production, leading to apparent technology differences across countries. In Chapter 18 I will also provide a simple model of inefficient technology adoption resulting from contracting problems among firms.

The second major element missing from our analysis so far, international trade and international capital flows, will be addressed in Chapter 19. International trade in commodities and assets link the economic fortunes of the countries in the world as well. For example, economies with low capital-labor ratios may be able to borrow internationally, which would naturally change equilibrium dynamics. Similarly and perhaps more importantly, less productive countries that export certain goods to the world economy will be linked with other economies because of changes in relative prices—i.e., because of changes in their "terms of trade". This type of terms of trade effects may also work towards creating a framework in which, while the world economy grows endogenously, the growth rates of each country is linked to those of others through trading relationships. Finally, we will see that the process of international trade and technology adoption are intimately linked, so that models of a collection of economies trading with each other will allow us to study the interaction between technology diffusion and the "international product cycle".

Throughout the rest of the book, including this part, I will be somewhat less comprehensive than in the previous chapters. In particular, to economize on space I will be more selective in the range of models covered, focusing on the models which I believe provide the main insights in an economical fashion. I will leave some of the alternative models that also relate to the economic issues under consideration to the discussion of the literature at the end or to exercises. In addition, I will make somewhat greater use of simplifying assumptions and I will leave the proofs of results that are similar to those we have encountered so far as well as the relaxation of some of the simplifying assumptions to exercises.

CHAPTER 18

Diffusion of Technology

In many ways, the problem of innovation ought to be harder to model than the problem of technology adoption. Nevertheless, the literature on economic growth and development has made more progress on models of innovation, such as those we discussed in Chapters 13-15, than on models of technology diffusion. This is in part because the process of technology adoption involves many challenging features. First, even within a single country, we observe considerable differences in the technologies used by different firms in the same narrowly-defined industry. Second and relatedly, it is difficult to explain how in the globalized world which we live in some countries may fail to import and use technologies that would significantly increase their productivity. In this chapter, we begin the study of these questions. Since potential barriers to technology adoption are intimately linked to the analysis of the political economy of growth, we will return to some of these themes in Part 8 of the book. For now the emphasis will be on how technological interdependencies change the mechanics of economic growth and can thus enrich our understanding of the potential sources of cross-country income differences and economic growth over time.

I will first provide a brief overview of some of the empirical patterns pertaining to technology adoption and diffusion within countries and industries, and how this appears to be important for within-industry productivity differences. I will then turn to a benchmark model of world equilibrium with technology diffusion, which will provide a reduced-form model for analyzing the slow diffusion of technological know-how across countries. I will then enrich this model by incorporating investments in R&D and technology adoption. Next, I will discuss issues of appropriate technology, and finally, I will turn to the impact of contractual imperfections on technology adoption decisions. Throughout this chapter, the only interaction among the countries in the world will be through technological exchange, and there will be no international trade in assets or in commodities.

18.1. Productivity Differences and Technology

Let us first start with a brief overview of productivity and technology differences within countries. This overview will help us place the cross-country differences in productivity and technology into perspective. The most important lesson from the within-country studies is that productivity and technology differences are ubiquitous even across firms within narrow sectors in the same country.

18.1.1. Productivity and Technology Differences within Narrow Sectors. A large literature uses longitudinal micro-data (often for the manufacturing sector) to study labor and total factor productivity differences across plants within narrow sectors (for example three-digit or four-digit manufacturing sectors). For our focus, the most important pattern that emerges from these studies is that, even within a narrow sector of the US economy, there is a significant amount of productivity differences across plants, with an approximately two or three-fold difference between the top and the bottom of the distribution (see, for example the survey in Bartelsman and Doms, 2000, for a summary of various studies and estimates). In addition, these productivity differences appear to be highly persistent (e.g., Bailey, Hulten and Campbell, 1992).

There is little consensus on what the causes of these differences are. Many studies find a correlation between plant productivity and plant or firm size, various measures of technology (in particular IT technology), capital intensity, the skill level of the workforce and management practices (e.g., Davis and Haltiwanger, 1991, Doms, Dunn and Troske, 1997, Black and Lynch, 2004). Nevertheless, since all of these features are choice variables for firms, these correlations cannot be taken to be causal. Thus to a large extent the determinants of productivity differences across plants are still unknown. In this light, it should not appear as a surprise that there is no consensus on the the determinants of cross-country differences in productivity.

Nevertheless, technology differences appear to be an important factor, at least as an approximate cause, for productivity differences. For example, Doms, Dunn and Troske (1997) and Haltiwanger, Lane and Spletzer (1999) document significant technology differences across plants within narrow sectors. Interestingly, as emphasized by Doms, Dunn and Troske (1997) and Caselli and Coleman (2001), a key determinant of the technology-level or the technology adoption decision seems to be the skill level of the workforce of the plant (often proxied by the share of non-production workers), though adoption of new technology does not typically lead to a significant change in employment structure. These results suggest that, consistent with some of the models discussed in Chapters 10 and 15, differences in the availability of skills and skilled workers might be an important determinant of technology adoption (and development) decisions. We will return to the role of skills in productivity and technology differences in the cross-country context below.

The distribution of productivity differences across firms appears to be related to the entry of new and more productive plants (and the exit of less productive plants). For example, consistent with the basic Schumpeterian models of economic growth discussed in Chapter

14, Bartelsman and Doms (2000) and Foster, Haltiwanger and Krizan (2000) document that entry of new plants plays has an important contribution to industry productivity growth. Nevertheless, entry and exit appear to account for only about 25% of average TFP growth, with the remaining productivity improvements are accounted for by continuing plants. This suggests that models in which firms continually invest in technology and productivity (for example such as the model of step-by-step innovation in Section 14.4 in Chapter 14) may be important for understanding the productivity differences across firms and plants and also for the study of cross-country productivity differences.

18.1.2. Technology Diffusion. A key implication of the sectoral studies is that, despite our presumption that technology and know-how are freely available and can be adopted easily, there are considerable technology and productivity differences among firms operating under similar circumstances. Nevertheless, cross-sectional distributions of productivity and technology are not stationary. In particular, new and more productive technologies, once they arrive on the scene, diffuse and over time are adopted by more firms and plants. The literature on technology diffusion studies this process of adoption of new technologies. As one might expect, there are parallels between the issue of technology diffusion across countries and slow technology diffusion across firms. It is therefore important to briefly overview the main findings all the technology diffusion literature.

The classic paper in this area is Griliches's (1957) study of the adoption of hybrid corn in the US. Griliches showed that the more productive hybrid corn diffused only slowly in the US agriculture and that this diffusion was affected by the local economic conditions of different areas. Consistent with the theoretical models presented so far, his study found evidence that the likelihood of adoption was related to the contribution of the hybrid corn in a particular area, the market size and the skill level of the area. The importance of these factors has been found in other studies as well (see, for example, Mansfield, 1998). Another important result of Griliches's study was to uncover the famous S-shape of diffusion, whereby a particular technology first spreads slowly and then once it reaches a critical level of adoption, it starts spreading much more rapidly. Finally, once a large fraction of the target population adopts the technology, the rate of adoption again declines. The overall pattern thus approximates an S curve or a logistic function. Jovanovic and Lach (1989), among others, show how this type of diffusion process can emerge as an equilibrium of an industry model with knowledge spillovers.

The important lesson for our focus here is that productivity and technology differences are not only present across countries, but also within countries. Moreover, even within countries better technologies do not immediately get adopted by all firms. In light of these patterns, the presence of significant productivity and technology differences across countries should not be entirely surprising. Nevertheless, the causes of within-country and cross-country productivity and technology differences might be different, and despite the presence of within-country differences, the significant cross-country differences do pose a puzzle that requires investigation. For example, within-country productivity differences might be due to differences in managerial (entrepreneurial) ability or related to the success of the match between the manager and the technology (or the product). These types of explanations would be unlikely to account for why almost all firms in many less developed countries are much less productive than the typical firms in the United States or other advanced economies. Motivated by the evidence briefly surveyed here, I will discuss both models in which technology diffuses slowly across countries and also models in which productivity differences may remain even when instantaneous technology diffusion and adoption are possible.

18.2. A Benchmark Model of Technology Diffusion

18.2.1. A Model of Exogenous Growth. In the spirit of providing the main insights with the simplest possible models, let us return to the Solow growth model of Chapter 2. In particular, suppose that the world economy consists of J countries, indexed $j = 1, ..., J$, each with access to an aggregate production function for producing the unique final good of the world economy,

$$Y_j(t) = F(K_j(t), A_j(t) L_j(t)),$$

where $Y_j(t)$ is the output of this unique final good in country j at time t, and $K_j(t)$ and $L_j(t)$ are the capital stock and labor supply. Finally, $A_j(t)$ is the technology of this economy, which is both country-specific and time-varying. In line with the result in Theorem 2.7 in Chapter 2, we have already imposed that technological change takes a purely labor-augmenting (Harrod-neutral) form. The aggregate production function F is assumed to satisfy the standard neoclassical assumptions, that is, Assumptions 1 and 2 from Chapter 2. In particular, recall that these assumptions imply that F exhibits constant returns to scale. Throughout this chapter and the next, whenever we study a world economy consisting of J countries, we assume that J is large enough so that each country is "small" relative to the rest of the world and thus ignores its effect on world aggregates.[1]

[1] This can be thought of in two ways. Either we can think of J as a large finite number, or consider the limit where $J \to \infty$.

Alternatively, we could have assumed that there is a continuum rather than a countable number of countries. None of the results in this and the next chapter depend on whether the number of countries are taken to be a continuum or a countable set, and throughout we work with a countable, in fact, finite, number of countries to simplify the exposition.

Using our usual approach, we can write income per capita as

$$\begin{aligned} y_j(t) &\equiv \frac{Y_j(t)}{L_j(t)} \\ &= A_j(t) F\left(\frac{K_j(t)}{A_j(t) L_j(t)}, 1\right) \\ &\equiv A_j(t) f(k_j(t)), \end{aligned}$$

where

$$k_j(t) \equiv \frac{K_j(t)}{A_j(t) L_j(t)}$$

is the effective capital-labor ratio of country j at time t.

We assume that time is continuous, that there is population growth at the constant rate $n_j \geq 0$ in country j, and that there is an exogenous saving rate equal to $s_j \in (0,1)$ in country j and a depreciation rate of $\delta \geq 0$ for capital, so that the law of motion of capital for each country is given by

$$\dot{k}_j(t) = s_j f(k_j(t)) - (n_j + g_j(t) + \delta) k_j(t), \tag{18.1}$$

where

$$g_j(t) \equiv \frac{\dot{A}_j(t)}{A_j(t)} \tag{18.2}$$

is the (endogenously-determined) growth rate of technology of country j at time t (see Exercise 18.1). We take $k_j(0) > 0$ and $A_j(0) > 0$ as exogenously given initial conditions.

To start with, technology diffusion is modeled in a reduced-form way. Let us assume that the world's technology frontier, denoted by $A(t)$, grows exogenously at the constant rate

$$g \equiv \frac{\dot{A}(t)}{A(t)} > 0,$$

with an initial condition $A(0) > 0$. We refer to $A(t)$ as the world technology or sometimes as the "world technology frontier". It encapsulates the maximal knowledge that any country can have, so that $A_j(t) \leq A(t)$ for all j and t. Moreover, each country's technology progresses as a result of *absorbing* the world's technological knowledge. In particular, let us posit the following law of motion for each country's technology:

$$\dot{A}_j(t) = \sigma_j (A(t) - A_j(t)) + \lambda_j A_j(t), \tag{18.3}$$

where $\sigma_j \in (0,\infty)$ and $\lambda_j \in [0, g)$ for each $j = 1, ..., J$ (see Exercise 18.2). Equation (18.3) implies that each country absorbs world technology at some exogenous constant rate σ_j. We refer to this parameter as the *technology absorption* rate. In practice, absorption corresponds both to straightforward adoption of existing technologies and to *adaptation* of existing blueprints to the conditions prevailing in a specific country, so that they can be used with the other technologies and practices in place. This parameter will vary across countries because of differences in their human capital or other investments (see below) and also because of institutional or policy barriers affecting technology adoption. This parameter multiplies the difference $A(t) - A_j(t)$, since it is this difference that remains to be absorbed by the country in question—if $A(t) = A_j(t)$, there is nothing to absorb from the world technology frontier. Though natural, this formulation has important economic consequences. In particular, it implies that countries that are relatively "backward" in the sense of having a low $A_j(t)$ compared to the frontier, will tend to grow faster, because they have more technology to absorb or or more room for *catch-up*. This potential advantage for relatively backward economies will play an important role in ensuring a stable world income distribution across countries. It also formalizes an idea of going back to Gerschenkron's (1962) essay *Economic Backwardness in Historical Perspective*. Gerschenkron argued that rapid catch-up by relatively backward countries was important for understanding cross-country growth patterns. He also suggested that the organization of production in the process of catching up is (or should) be different than the organization of production appropriate for frontier economies. We will return to this theme in Chapter 20.

Equation (18.3) also implies that technological progress can happen "locally" as well, that is building upon the knowledge stock of country j, $A_j(t)$. The parameter λ_j captures the speed at which this happens. This equation therefore contains the two major forms of technological progress that a particular country can experience; absorption from the world technology frontier and local technological advances. Its functional form is adopted for simplicity.

Notice that (18.3) already sidesteps one of the major issues raised at the beginning of this chapter: it posits that despite the level of globalization the world has reached and the relatively free-flow of information among individuals across the globe, the process of technology transfer between countries is a slow one. The assumption that $\sigma_j < \infty$ imposes this feature. In particular, since $\sigma_j < \infty$, $A_j(t) < A(t)$ will imply that $A_j(t + \Delta t) < A(t + \Delta t)$, at least for $\Delta t > 0$ and sufficiently small. Consequently, countries that have access to only a subset of the production techniques (blueprints) available in the world will not immediately acquire all of the knowledge that they do not currently have access to.

To proceed with the analysis of this model, let us define

$$a_j(t) \equiv \frac{A_j(t)}{A(t)}$$

as an inverse measure of the proportional technology gap between country j in the world or alternatively as an inverse measure of country j's *distance to the frontier* (distance to the world technology frontier), we can then write the above equation as (see Exercise 18.3):

$$\dot{a}_j(t) = \sigma_j - (\sigma_j + g - \lambda_j)\, a_j(t)\,. \tag{18.4}$$

Clearly, the initial conditions $A(0) > 0$ and $A_j(0) > 0$ give a unique initial condition for the differential equation for a_j, $a_j(0) \equiv A_j(0)/A(0) > 0$.

Given the description of the environment above, the dynamics of the world income per capita levels and technology are determined by $2J$ differential equations. For each j, we have one of (18.1) and one of (18.4). These equations characterize the state-state distribution of technology and income per capita in the world economy and its transitional dynamics. What makes the analysis of this world equilibrium relatively straightforward is the *block recursiveness* of the system of differential equations governing the behavior of income per capita and technology across countries. The law of motion of (18.4) for country j only depends on $a_j(t)$, so it can be solved without reference to the law of motion of $k_j(t)$ and to the law of motion of $\left\{k_{j'}(t), a_{j'}(t)\right\}_{j' \neq j}$. Once (18.4) is solved, then (18.1) becomes a first-order nonautonomous differential equation in a single variable. The fact that it is nonautonomous is a consequence of the fact that it has $g_j(t)$ on the right-hand side, which can be determined as

$$g_j(t) = \frac{\dot{a}_j(t)}{a_j(t)} + g.$$

Once we solve for the law of motion of $a_j(t)$, this is simply a function of time, making (18.1) a simple nonautonomous differential equation.

Let us start the analysis with the steady-state world equilibrium. A *world equilibrium* is defined as an allocation $\left\{[k_j(t), a_j(t)]_{t \geq 0}\right\}_{j=1}^{J}$ such that (18.1) and (18.4) are satisfied for each $j = 1, ..., J$ and for all t, starting with the initial conditions $\{k_j(0), a_j(0)\}_{j=1}^{J}$. A *steady-state world equilibrium* is then defined as a steady-state of this equilibrium process, i.e., an equilibrium with $\dot{k}_j(t) = \dot{a}_j(t) = 0$ for each $j = 1, ..., J$. The "steady-state equilibria" studied in this chapter will exhibit constant growth, so I could have alternatively referred to them as *balanced growth path equilibria*. Throughout I will use the term steady-state equilibrium for consistency.

Now imposing these steady-state conditions, we obtain the following straightforward proposition, which states that there exists a unique and globally stable steady-state equilibrium.

Proposition 18.1. *In the above-described model, there exists a unique steady-state world equilibrium in which income per capita in all countries grows at the same rate $g > 0$. Moreover, for each $j = 1, ..., J$, we have*

$$a_j^* = \frac{\sigma_j}{\sigma_j + g - \lambda_j}, \tag{18.5}$$

and k_j^* is uniquely determined by

$$s_j \frac{f\left(k_j^*\right)}{k_j^*} = n_j + g + \delta.$$

The steady-state world equilibrium $\left\{k_j^*, a_j^*\right\}_{j=1}^J$ *is globally stable in the sense that starting with any strictly positive initial values* $\{k_j(0), a_j(0)\}_{j=1}^J$*, the equilibrium path* $\{k_j(t), a_j(t)\}_{j=1}^J$ *converges to* $\left\{k_j^*, a_j^*\right\}_{j=1}^J$.

Proof. (Sketch) First solve (18.1) and (18.4) for each $j = 1, ..., J$ imposing the steady-state condition that $\dot{k}_j(t) = \dot{a}_j(t) = 0$. This yields a unique solution, establishing the uniqueness of the steady-state equilibrium. Then standard arguments show that the steady state a_j^* of the differential equation for $a_j(t)$ is globally stable. Next using this result, the global stability of the steady state of the differential equation for $k_j(t)$ follows straightforwardly. Exercise 18.4 asks you to complete the details of this proof. □

A number of features about this world equilibrium are noteworthy. First, we have a unique steady-state world equilibrium that is globally stable. This enables us to perform simple comparative static and comparative dynamic exercises (see Exercise 18.5). Second and most importantly, despite differences in saving rates and technology absorption rates across countries, income per capita in all economies grows at the same rate equal to the growth rate of the world technology frontier, g. Why is this? The technology adoption equation, (18.3), provides the answer to this question; the rate of technology diffusion (absorption) is higher when the gap between the world technology frontier in the technology level of a particular country is greater. Thus there is a force pulling backward economies towards the technology frontier, and in steady state this force is powerful enough to ensure that all countries grow at the same rate.

Does this imply that all countries will converge to the same level of income per capita? The answer is clearly no. Differences in saving rates and absorption rates translate into *level differences* (instead of growth rate differences) across countries. For example, a society with a low level of σ_j will initially grow less than others, until it is sufficiently behind the world technology frontier. At this point, it will also grow at the world rate, g. This discussion illustrates that it is precisely the endogenous technology gap between a country and the world frontier that ensures growth at the rate g for all countries. Thus societies that are unsuccessful in absorbing world technologies, those that impose barriers slowing technology diffusion (i.e., those with low σ_j) and those that are not sufficiently innovative in developing their own local technologies (i.e., those with low λ_j) will be poorer. Moreover, as in the baseline Solow model, those with low saving rates will also be poorer. These results are summarized in the following proposition.

Proposition 18.2. *Let steady-state income per capita level of country* j *be* $y_j^*(t) = \exp(gt)\, y_j^*$. *Then* y_j^* *is increasing in* σ_j, λ_j *and* s_j *and decreasing in* n_j *and* δ. *It does not depend on* $\sigma_{j'}, \lambda_{j'}$, $s_{j'}$ *and* $n_{j'}$ *for any* $j' \neq j$.

Proof. See Exercise 18.7. □

A particularly convenient—but also restrictive—feature of the equilibrium studied here is that even though there is technology diffusion and interdependence in this world equilibrium, there is no interaction among countries. Each country's steady-state income per capita (and in fact path of income per capita) only depends on the behavior of the world technology frontier and its own parameters. Later in this chapter, we will see models in which there is more interaction between the decisions of individual countries.

18.2.2. Consumer Optimization. It is straightforward to incorporate consumer optimization into this benchmark model of technology transfer. In particular, let us now suppose that each country admits a representative household with preferences at time $t = 0$ given by

$$U_j = \int_0^{\infty} \exp\left(-\left(\rho - n_j\right) t\right) \left[\frac{\tilde{c}_j\left(t\right)^{1-\theta} - 1}{1-\theta}\right] dt, \tag{18.6}$$

where $\tilde{c}_j(t) \equiv C_j(t)/L_j(t)$ is per capita consumption in country j at time t and we have imposed that all countries have the same time discount rate, ρ. This latter feature is to simplify the discussion in the text, and Exercise 18.9 generalizes the results in this subsection to the world economy with different discount rates. This is an important generalization, since it highlights that a stable world income distribution does not depend on equal discount rates or asymptotically equal saving rates across countries.

As in the neoclassical growth model, the flow resource constraint facing the representative household can be written as

$$\dot{k}_j(t) = f\left(k_j(t)\right) - c_j(t) - \left(n_j + g_j(t) + \delta\right) k_j(t),$$

where $c_j(t) \equiv \tilde{c}_j(t)/A_j(t) \equiv C_j(t)/A_j(t)\,L_j(t)$ is consumption normalized by effective units of labor. This equation now replaces (18.1) as the law of motion of effective capital-labor ratio of country j.

The world equilibrium and the steady-state world equilibrium are defined in a similar fashion, except that instead of a constant saving rate consumption sequences must now maximize the utility of the representative household in each country subject to their resource constraint. An analysis similar to that in Chapter 8 leads to the following proposition:

Proposition 18.3. *Consider the above-described model with consumer optimization with preferences given by (18.6) and suppose that $\rho - n_j > (1-\theta) g$. Then, there exists a unique steady-state world equilibrium where for each $j = 1, ..., J$, a_j^* is given by (18.5) and k_j^* is uniquely determined by*

$$f'\left(k_j^*\right) = \rho + \delta + \theta g,$$

and consumption per capita in each country grows at the rate $g > 0$.

Moreover, the steady-state world equilibrium is globally saddle-path stable in the sense that starting with any strictly positive initial values $\{k_j(0), a_j(0)\}_{j=1}^J$*, initial consumption to effective labor ratios are* $\{c_j(0)\}_{j=1}^J$ *and the equilibrium path* $\{k_j(t), a_j(t), c_j(t)\}_{j=1}^J$ *converges to* $\left\{k_j^*, a_j^*, c_j^*\right\}_{j=1}^J$*, where* c_j^* *is the steady-state consumption to effective labor ratio in economy* j*.*

Proof. (Sketch) We can first show that a_j^* can be determined from the differential equation in (18.4) without reference to any other variables and satisfies (18.5). The consumer Euler equations and the analysis of capital accumulation are the same analysis as in the baseline neoclassical growth model, taking into account that in steady state $g_j(t) = g$. To complete the proof of the proposition, we need to show the stability of a_j^*, and then taking into account the behavior of $g_j(t)$, we must establish the saddle path stability of k_j^* using the same type of analysis as in Chapter 8—which is slightly more complicated here because the differential equation for capital accumulation is not autonomous. You are asked to complete these details in Exercise 18.8. □

This proposition shows that all of the qualitative results of the benchmark model of technology diffusion apply irrespective of whether we assume constant saving rates or dynamic consumer maximization (as long as we ensure that the growth rate is not so high as to violate the transversality condition). Naturally, an equilibrium now corresponds not only to sequences of $\{k_j(t), a_j(t)\}$ but also includes the time path of consumption per unit of effective labor, $c_j(t)$. Consequently, the appropriate notion of stability is saddle-path stability, which the equilibrium in Proposition 18.3 satisfies.

18.2.3. The Role of Human Capital in Technology Diffusion. The model presented above is in part inspired by a classic short paper by Richard Nelson and Edmund Phelps (1966). The Nelson-Phelps model focused on the role of human capital in technology adoption and is well known for having proposed a new role of human capital, different from those emphasized by Becker and Mincer. Recall that Becker and Mincer emphasized how human capital increases the productivity of the labor hours supplied by an individual. While this approach allows the effect of human capital to be different in different tasks, in most applications it is presumed that greater human capital translates into higher productivity in all or most tasks, with the set of productive tasks typically taken as given.

In contrast, Nelson and Phelps and Ted Schultz, who was at the same time writing on the role of human capital in technology adoption in agriculture, argued that the main role of human capital was not to increase productivity in existing tasks. Instead, human capital, they argued, was most important in facilitating the adoption of new technologies. In particular, Schultz viewed the agricultural world (or perhaps the less developed economies more generally) to be in perpetual "disequilibrium" and argued that the main role of human capital was to enable individuals to deal with and adapt to situations of disequilibrium. Today, we would recognize what Schultz dubbed "disequilibrium" as an equilibrium of a dynamical system that is far from steady state (see Chapter 20 for a similar perspective). Thus his argument can be viewed as emphasizing the role of human capital in environments where certain key state variables, such as technology, are undergoing important changes. A similar argument was advanced by Nelson and Phelps's famous short paper.

In terms of the model described above, the simplest way of capturing this argument is to posit that the parameter σ_j is a function of the human capital of the workforce. The greater is the human capital of the workforce, the higher is the absorption capacity of the economy. If so, high human capital societies will be richer because, as shown in Proposition 18.2, economies with higher σ_j have higher steady-state levels of income.

While this modification leaves the mathematical exposition of the model unchanged, the implications for how we view growth experiences of societies with different levels of human capital are potentially quite different than in the Becker-Mincer approach (or at the very least, than in the simplest version of the Becker-Mincer approach). The latter approach suggests that we can approximate the role of human capital in economic development by carefully accounting for its role in the aggregate production function. This, in turn, can be done by estimating individual returns to schooling and returns to other dimensions of human capital in the labor market. The Nelson-Phelps-Schultz view, on the other hand, suggests that the main role of human capital will be during periods of technological change and in the process of technology adoption. Thus it is the process of technological diffusion and adoption that makes human capital particularly valuable.

What is unclear in this view, however, is whether the contribution of human capital to national income through this particular role will be reflected in the wages of individual workers. For example, in the context of technology adoption decisions in agriculture, studied by Schultz, even though there is an important distinction between the role of human capital in facilitating technology absorption and its role in directly increasing productivity in existing tasks, both types of contributions will be reflected in an individual's earnings—an individual with higher human capital, who successfully adopts new technologies, will become richer. If, on the other hand, the parameter σ_j in the above model is a function of human capital, this could be the result of each individual firm's adoption decisions as a function of its own employees' human capital or it might be working at the some higher level of aggregation than the firm, thus corresponding to *externalities* (recall Chapter 10). Therefore, while the Nelson-Phelps-Schultz view of human capital is conceptually different from the Becker-Mincer approach to human capital, whether this has major implications for empirically assessing the contribution of human capital to income differences across countries and over time depends on whether the benefits created by human capital are internalized by firms and workers or whether they take the form of externalities. Consequently, as in the Becker-Mincer approach, if there are significant external effects, many of the empirical strategies discussed in Chapter 3 will understate the role of human capital. This discussion, however, emphasizes that were this to happen, it would be a consequence of human capital externalities, not a direct implication of the particular channel via which human capital affects productivity and growth. Therefore, if, as suggested by the discussion in Chapter 3, human capital externalities, except through global R&D effects, are limited, the Nelson-Phelps-Schultz view of human capital will not significantly change the conclusions about the contribution of human capital to differences in income differences across countries and over time.

18.2.4. Barriers to Technology Adoption. As discussed in Chapter 8, one of the main criticisms against the neoclassical growth model has been its inability to generate quantitatively large cross-country income per capita differences. Most economists view this as related to the fact that the basic neoclassical growth model does not provide an explanation for "technology differences". The model in this section presents a reduced-form model of technology differences across countries, thus enables us to enrich the neoclassical growth model and the Solow model to incorporate technology differences. Nevertheless, such a theory will be useful only to the extent that the key parameters such as σ_j and λ_j can be mapped to reality. The previous subsection discussed ideas linking the parameter σ_j to human capital. An alternative, emphasized by Parente and Prescott (1994), is to link σ_j to barriers to technology adoption. Parente and Prescott construct a variant of the neoclassical growth model in which investments affect technology absorption, and countries differ in terms of the "barriers" that they place on the path of firms in this process. In terms of the reduced-form model here, the Parente-Prescott mechanism can be captured by interpreting σ_j as a function of property rights institutions or other institutional or policy features.

This perspective is useful as it gives us a concrete way of thinking of the reasons why σ_j may vary across countries. Nevertheless, it is still unsatisfactory in two important respects. First, exactly how these institutions affect technology adoption is left as a black box. Second and more importantly, why some societies choose to create barriers against technology adoption while others do not is left unexplained. The models that combine technology diffusion with endogenous technology decisions, which will be presented in the next section, make some progress on the first point. In fact, Parente and Prescott constructed a model in which firms undertook investments to acquire technologies from the world technology frontier. Nevertheless, their model is closer to the neoclassical growth model than to the endogenous technology models presented in Part 4 of the book, because it does not feature investments in the creation or adoption of new technologies that can be identified with R&D decisions. Since the endogenous technological change models we have seen above are more widely used and offer richer insights about the nature of technology, I will introduce endogenous technology adoption decisions in the context of these models. The question of why some societies block technology adoption will be the topic of Part 8 below.

18.3. Technology Diffusion and Endogenous Growth

In the previous section, technology diffusion took place "exogenously," in the sense that firms did not engage in R&D or investment type activities in order to improve their technologies. In this section, we introduce these types of purposeful activities directed at improving technology. The material in this section therefore complements the models of technology diffusion of the previous section in the same way that endogenous technological change models complemented (and advanced upon) the neoclassical framework with exogenous technology. The section is separated into two parts. In the first, the world growth rate will be taken as exogenous, while it will be endogenized in the second part.

18.3.1. Exogenous World Growth Rate. To keep the exposition as brief as possible, I will use the baseline endogenous technological change model with expanding machine variety and lab equipment specification as in Section 13.1 of Chapter 13 and I will frequently refer to the analysis there. Clearly, different versions of the endogenous technological change models could be used for the same purposes.

The aggregate production function of economy $j = 1, ..., J$ at time t is

$$Y_j(t) = \frac{1}{1-\beta} \left[\int_0^{N_j(t)} x_j(v,t)^{1-\beta} dv \right] L_j{}^{\beta}, \tag{18.7}$$

where L_j is the aggregate labor input, which is assumed to be constant over time, $N_j(t)$ denotes the different number of varieties of machines available to country j at time t, and $x_j(v,t)$ is the total amount of machine type v used at time t. We continue to assume that x's depreciate fully after use. As in Chapter 13, each variety in economy j is owned by a technology monopolist, which will sell machines embodying this technology at the profit maximizing (rental) price $\chi_j(v,t)$. This monopolist can produce each unit of the machine at a cost of $\psi \equiv 1-\beta$ units on the final good, where this normalization is again introduced to simplify the expressions.

Since there is no international trade, firms in country j can only use technologies supplied by technology monopolists in their country. This assumption introduces the potential differences in the knowledge stock available to different countries.

Each country admits a representative household with the same preferences as in (18.6), except that there is no population growth, i.e., $n_j = 0$ for all j. New varieties are again produced by investment, and thus the resource constraint for each country at each point in time is

$$C_j(t) + X_j(t) + \zeta_j Z_j(t) \leq Y_j(t), \tag{18.8}$$

where $X_j(t)$ is investment or spending on inputs at time t and $Z_j(t)$ is expenditure on technology adoption at time t, which may take the form of R&D or other expenditures, such as the purchase or rental of machines embodying new technologies. The parameter ζ_j is introduced as a potential source of differences in the cost of technology adoption across countries, which may result from institutional barriers against innovation as emphasized by Parente and Prescott (1994), from subsidies to R&D and to technology, or from other tax policies. As discussed in Section 8.9 in Chapter 8, many authors identify this parameter with tax distortions on investment-type activities and often proxy it with the relative price of investment to consumption goods. In the next chapter, we will see when this might be valid.

The main difference from the environment in Chapter 13 is in the *innovation possibilities frontier*,

$$\dot{N}_j(t) = \eta_j \left(\frac{N(t)}{N_j(t)}\right)^{\phi} Z_j(t), \tag{18.9}$$

where $\eta_j > 0$ for all j, and $\phi > 0$ and is common to all economies. This form of the innovation possibilities frontier captures the same basic idea as (18.3) in the previous section, but what matters is not the absolute gap in technology, but the proportional gap. This functional form is again adopted for simplicity. We assume that each economy starts with some initial technology stock $N_j(0) > 0$. Finally, as noted above, for now we assume that the world technology frontier of varieties expands at an exogenous rate $g > 0$, i.e.,

$$\dot{N}(t) = gN(t). \tag{18.10}$$

The analysis in Chapter 13 implies that the flow profits of a technology monopolist at time t in economy j is given by

$$\pi_j(t) = \beta L_j.$$

Suppose a steady-state (balanced growth path) equilibrium exists in which the interest rate is constant at some level $r_j^* > 0$. Then the net present discounted value of a new machine is

$$V_j^* = \frac{\beta L_j}{r_j^*}.$$

If the steady state involves the same rate of growth in each country, then $N_j(t)$ will also grow at the rate g, so that $N_j(t)/N(t)$ will remain constant, say at some level ν_j^*. In that case, an additional unit of technology spending will create benefits equal to $\eta_j\left(\nu_j^*\right)^{-\phi} V_j^*$

counterbalanced against the cost of ζ_j. Free-entry (with positive activity) then requires

$$\nu_j^* = \left(\frac{\eta_j \beta L_j}{\zeta_j r^*}\right)^{1/\phi}, \tag{18.11}$$

where I have also used the fact that given the preferences (18.6), equal growth rate across countries implies that the interest rate will be the same in all countries (and in fact it will be equal to $r^* = \rho + \theta g$).

Since a higher ν_j implies that country j is technologically more advanced and thus richer than others, equation (18.11) shows that societies with better innovation possibilities frontiers, as captured by the parameter η_j, and those with lower cost of R&D, corresponding to lower ζ_j, will be more advanced and richer. This equation also incorporates a scale effect as in the standard endogenous technological change models, so a country with a greater labor force will also be richer. This is for the same reason as a greater labor force leads to faster growth in the baseline endogenous technological change model: a greater labor force creates more demand for machines, making R&D more profitable.

This analysis leads to the following proposition:

Proposition 18.4. *Consider the model with endogenous technology adoption described in this section. Suppose that $\rho > (1-\theta) g$. Then there exists a unique steady-state world equilibrium in which relative technology levels are given by (18.11) and all countries grow at the same rate $g > 0$.*

Moreover, this steady-state equilibrium is globally saddle-path stable, in the sense that starting with any strictly positive vector of initial conditions $N(0)$ and $(N_1(0), ..., N_J(0))$, the equilibrium path of $(N_1(t), ..., N_J(t))$ converges to $(\nu_1^* N(t), ..., \nu_J^* N(t))$.

Proof. (Sketch) First show that the specified steady-state equilibrium is the only steady state equilibrium in which all countries grow at the same rate. Then consider the value function of technology monopolist in each country as in Chapter 13 and show that the number of varieties in each countries must asymptotically grow at the rate g. Exercise 18.11 asks you to complete this proof. □

This result and the preceding analysis therefore show that endogenizing investments in technology adoption leads to an equilibrium pattern similar to that we saw in the previous section. The main difference is that we can now pinpoint the factors that affect the rates of technology adoption and relate them to the profit incentives of firms. An explicit model of technology decisions also allows us to investigate how differences in the cost of investing in technology might affect cross-country differences in technology and income (see Exercise 18.12).

18.3.2. Endogenous Growth in the World. The model in the previous subsection was simplified by the fact that the world growth rate was exogenous. A more satisfactory model would derive the world growth rate from the technology adoption and R&D activities of each country. Such models are typically more involved, because the degree of interaction among countries in the world equilibrium is now considerably greater. In addition, a certain amount of care needs to be taken so that the world economy grows at a constant endogenous rate, while there are still forces that ensure relatively similar growth rates across countries. Naturally, one may also wish to construct models in which countries grow at permanently different long run rates (see, for example, Exercise 13.8 in Chapter 13). The evidence we have seen in Chapter 1 suggests that such long-run growth differences are present when we look at the past 200 or 500 years, but there are more limited sustained growth rates differences over the past 60 years or so (implying only small changes in the postwar world income distribution). Thus whether one wants to have long-run growth rate differences across countries is a modeling choice—it partly depends on whether one thinks of a model with a long transition leading to the large income differences, or wishes to approximate the past 200 or 500 years as corresponding to "steady-state behavior". Since such growth rates differences emerge straightforwardly in many models (including all of the endogenous technology models we have seen so far, see again Exercise 13.8), in this subsection we focus on forces that will keep countries growing at similar rates in the presence of endogenous technological change at the world level.

The main difference from the model in the previous subsection is that we now replace the world growth equation, (18.10), which specified exogenous world growth at the rate g, with an equation that links the improvements in the world technology to technological improvements in each country. In particular, we assume the simplest way of aggregating the technologies of different countries, which is by taking their arithmetic average:

$$N(t) = \frac{1}{J} \sum_{j=1}^{J} N_j(t). \tag{18.12}$$

With this new equation, $N(t)$ no longer corresponds to the "world technology frontier". Instead, it represents average technology in the world, and as long as there are some differences across countries, it will naturally be the case that $N_j(t) > N(t)$ for at least some j. Nevertheless, having the world technology an average of the technology of each country is a natural generalization of the ideas presented so far in this chapter. One disadvantage of this formulation is that it implies that the contribution of each country to the world technology is the same. Exercise 18.18 discusses alternative ways of aggregating individual country technologies into a world technology term and shows that the qualitative results here do not depend on the specification in (18.12). Besides equation (18.12), we assume that all the other equations from the previous subsection continue to hold.

The main result of this section is that the pattern of cross-country growth will be similar to that in the previous subsection, but now the growth rate of the world economy, g, will be endogenous, resulting from the investments in technologies made by firms in each country. In particular, suppose that there exists a steady-state world equilibrium in which each country grows at the rate g. Then, (18.12) implies that the world technology index, $N(t)$, will also grow at the same rate g. Now, as in the previous subsection, the net present discounted value of a new machine in country j is

$$\frac{\beta L_j}{r^*},$$

and the no-arbitrage condition in R&D investments implies that, for given g, each country j's relative technology, ν_j^*, should satisfy (18.11). However, now dividing both sides of equation (18.12) by $N(t)$ implies that the steady-state world equilibrium must satisfy:

$$\frac{1}{J}\sum_{j=1}^{J}\nu_j^* = 1$$

$$\frac{1}{J}\sum_{j=1}^{J}\left(\frac{\eta_j \beta L_j}{\zeta_j(\rho+\theta g)}\right)^{1/\phi} = 1, \tag{18.13}$$

where the second line uses the definition of ν_j^* from (18.11) and substitutes for the common interest rate r^* as a function of the world growth rate. The only unknown in equation (18.13) is g. Moreover, the left-hand side is clearly strictly decreasing in g, so this equation can be satisfied for at most one value of g, say g^*. A well-behaved world equilibrium would require the growth rates to be positive and not so high as to violate the transversality condition. The following condition is necessary and sufficient for the world growth rate to be positive:

$$\frac{1}{J}\sum_{j=1}^{J}\left(\frac{\eta_j \beta L_j}{\zeta_j \rho}\right)^{1/\phi} > 1. \tag{18.14}$$

Moreover, by usual arguments, when this condition is satisfied, there will exist a unique $g^* > 0$ that will satisfy (18.13) (if this condition were violated, (18.13) would not hold, and we would have $g = 0$ as the world growth rate). Therefore, the following proposition follows:

Proposition 18.5. *Suppose that (18.14) holds and that the solution g^* to (18.13) satisfies $\rho > (1-\theta)g^*$. Then there exists a unique steady-state world equilibrium in which growth at the world level is given by g^* and all countries grow at this common rate. This growth rate is endogenous and is determined by the technologies and policies of each country. In particular, a higher η_j or L_j or a lower ζ_j for any country $j = 1, ..., J$ increases the world growth rate.*

Proof. See Exercise 18.15. □

A number of features about this equilibrium are noteworthy. First, taking the world growth rate as given, the structure of the equilibrium is very similar to that in Proposition 18.4. Thus the fact that all countries grow at the same rate and that differences in the innovation possibilities frontier, η_j, the size the labor force, L_j, and the extent of potential distortions in technology investments, ζ_j, translate into level differences across countries has exactly the same intuition as in that proposition. What is more interesting is that essentially the same model as in the previous subsection now gives us an "endogenous" growth rate for the world economy. In particular, while growth for each country appears "exogenous" in the sense that, each country accumulates towards a world-determined growth rate, the growth rate of the world economy is endogenous and results from the investments of the firms in each country. As such, the current model provides a more satisfactory framework for the analysis of the process of world growth than both the purely exogenous growth models and the purely endogenous growth models. In the current model, technological progress and economic growth are the outcome of investments by all countries in the world, but there are sufficiently powerful forces in the world economy, here working through technological spillovers that pull relatively backward countries towards the world average, ensuring equal long-run growth rates for all countries in the long run. Naturally, equal growth rates are still consistent with quite *large level differences* across countries (see Exercise 18.12).

Proposition 18.5 used a number of simplifying assumptions. First, each country was assumed to have the same discount rate. This was only for simplicity, and Exercise 18.16 considers the case in which countries differ according to their discount rates. Second, the proposition only describes the steady-state equilibrium. Transitional dynamics are now more complicated, since the "block recursiveness" of the dynamical system is lost. The differential equations describing the equilibrium path for all countries need to be analyzed together. Nevertheless, local stability of the steady-state world equilibrium can be established, and this is analyzed in Exercise 18.15.

18.4. Appropriate and Inappropriate Technologies and Productivity Differences

The models presented so far in this chapter explicitly introduced a slow process of technology diffusion from the world stock of knowledge to the set of techniques used in production in each country. This was motivated either by some process of costly (and slow) technology absorption or because of barriers to technology adoption. However, as noted at the beginning of the chapter, in the highly globalized world we live in, where information technology and information flows make a wide range of blueprints easily accessible to most individuals and firms around the world, we should perhaps expect even faster technology transfer across countries. Why does rapid diffusion of ideas not remove all, or at least most, cross-country technology differences? Leaving the discussion of institutional or policy barriers preventing technology diffusion to later, in this section we focus on how "technology" differences and income gaps can remain substantial even with free flow of ideas.

A first important idea is that productivity differences may remain even if all differences in "techniques" disappear, because production is organized differently and the extent of inefficiency in production may vary across countries. A model along these lines will be discussed later in this chapter. Another important idea, which we now discuss, is that technologies of the world technology frontier may be *inappropriate* to the needs of specific countries, so that importing the most advanced frontier technologies may not guarantee the same level of productivity for all countries. At some level, this idea is both simple and attractive. Clearly, technologies and skills consist of bundles of complementary attributes and these bundles vary across countries, so that there is no guarantee that a new technology that works well given the skills and competences in the United States or Switzerland will also do so in Nigeria or Turkey. Nevertheless, without specifying these attributes that make some technologies work well in certain nations and not in others, this story will have little explanatory power. In this section, we will discuss three versions of this story that may have some theoretical and empirical appeal. First, we will discuss how differences in exogenous (e.g., geographic) conditions may make the same set of technologies differentially productive in different areas. Second, we will discuss how differences in capital intensity across countries may change the appropriateness of different types of technologies. Finally, we will spend most of this section on the implications of differences in skill intensity across countries for the appropriateness of frontier technologies to developing economies. In this context, we will show how the degree of appropriateness or inappropriateness of technologies may arise endogenously in the world equilibrium and also introduce a model of economic growth where labor has to be allocated across different sectors, which is of independent interest.

18.4.1. Inappropriate Technologies. The idea of inappropriate technologies can be best illustrated by an example on health innovations. Suppose that productivity in country j at time t, $A_j(t)$, is a function of whether there are effective cures against certain diseases affecting their populations. Suppose that there are two different diseases, heart attack and malaria. Countries $j = 1, ..., J'$ are affected by malaria and not by heart attacks, while $j = J' + 1, ..., J$ are affected by heart attacks and are unaffected by malaria. If the disease affecting country j has no cure, then productivity in that country given by $A_j(t) = \underline{A}$, while when a cure against this disease is introduced, we have $A_j(t) = \overline{A}$. Now imagine that a new cure against heart attacks is discovered and becomes freely available to all countries. Consequently, the productivity in countries $j = J' + 1, ..., J$ increases from $\underline{A}$ to $\overline{A}$, but productivity in countries $j = 1, ..., J'$ remains at $\underline{A}$. This simple example thus illustrates how technologies of the world frontier may be "inappropriate" to the needs of some of the countries (in this case, the J' countries affected by malaria). In fact, in this extreme case, a technological advance that is freely available to all countries in the world increases productivity in a subset of the countries and creates cross-country income differences.

Is there any reason to expect that issues of the sort might be important? The answer is both yes and no. Over 90% of the world R&D is carried out in OECD economies. There is therefore natural reasons to expect that new technologies should be optimized for the conditions in OECD countries or should explicitly deal with the problems that these countries are facing. This suggests that an analysis of the implications of appropriate technology is a promising area. Nevertheless, other than the issue of disease prevention, there are not many obvious *fixed* country characteristics that will create this type of "inappropriateness". Instead, the issue of appropriate technology is much more likely to be important in the context of whether new technologies increasing productivity via process and product innovations will function well at different *factor intensities.* The next two subsections focus on whether technologies developed in advanced economies can be productively used at different capital-labor and skilled-unskilled labor ratios than those for which they have been *designed.*

18.4.2. Capital-Labor Ratios and Inappropriate Technologies. A classic paper by Atkinson and Stiglitz (1969) entitled "A New View of Technical Change" argued that a useful way of modeling technological change is to view it as shifting isoquants (increasing productivity) at a given capital-labor ratio. For example, a firm that is using a specific machine, say a particular type of tractor, with a single worker, may discover a way to increase the productivity of the worker. This innovation can be used by any other firm employing the same tractor with a single worker. But it would be much less valuable to firms using oxen or less advanced tractors, or even to firms using more advanced tractors. Thus technological changes are localized for specific capital-labor ratios and when used with different capital labor ratios, they do not bring the same benefits. The implications of this observation for cross-country income differences can be quite major. If new technologies are developed for high capital-intensive production processes in OECD countries, they may be of little use to labor-abundant less-developed economies, where most production units will be functioning at lower capital-labor ratios than those in the OECD. This point is developed in the context of a Solow-type growth model by Basu and Weil (1998). I provide a simple version of their argument here.

Suppose that the production technology for all countries in the world is

$$Y = A\left(k \mid k'\right) K^{1-\alpha} L^{\alpha},$$

so that output per worker becomes

$$y \equiv \frac{Y}{L} = A\left(k \mid k'\right) k^{1-\alpha},$$

where $k = K/L$ is the capital-labor ratio, and $A(k \mid k')$ is the (total factor) productivity of technology designed to be used with capital-labor ratio k' when used instead with capital-labor ratio k. I have suppressed the time and country indices to simplify notation.

For example, suppose that

$$A\left(k \mid k'\right) = A \min\left\{1, \left(\frac{k}{k'}\right)^{\gamma}\right\}$$

for some $\gamma \in (0,1)$. That is, when a technology designed for the capital labor ratio k' is used with a lower capital-labor ratio, there is a loss in efficiency.

Now suppose that new technologies are developed in richer economies, which have greater capital-labor ratios. Then productivity in a less developed country with the capital-labor ratio $k < k'$ will be

$$y = A\left(k \mid k'\right) k^{1-\alpha} = Ak^{1-\alpha+\gamma}\left(k'\right)^{-\gamma}. \tag{18.15}$$

An immediate implication of equation (18.15) is that less-developed countries will be less productive even when they are producing with the same techniques. Moreover this productivity disadvantage will be larger when the gap in the capital intensity of production between these countries and in the technologically advanced economies is greater. Depending on the value of the parameter γ, the implication of this type of inappropriateness might be important for understanding cross-country income differences. With the same arguments as in Chapters 2 and 3, we may want to think of $\alpha \approx 2/3$. Then an economy with an eight times higher capital-labor ratio then another would only be twice as rich, when both countries have access to the same technology and there is no issue of inappropriate technologies. But if $\gamma = 2/3$ and the county with the higher capital-labor ratio is the frontier one setting the level of k' in terms of the function $A(k \mid k')$, the implied difference would be eightfold rather than the twofold difference implied by the model that overlooked the issue of appropriate technology. Thus inappropriateness of technologies have the potential to increase the implied cross-country income differences, even when all countries have access to the same technologies. Exercise 18.20 provides more details on this model.

18.4.3. Endogenous Technological Change and Appropriate Technology. The Atkinson-Stiglitz and Basu-Weil approach discussed in the previous subsection emphasizes differences in capital intensity between rich and poor economies. The evidence discussed in Section 18.1 suggests that differences in human capital may be particularly important in the adoption of technology. Moreover, the past 30 years have witnessed the introduction of a range of skill-biased technologies both in developed economies and in many developing countries (see Autor, Katz and Krueger, 1998, Acemoglu, 2002b, for general surveys, Berman, Bound and Machin, 1998, for evidence across OECD countries, and Berman and Machin, 2000, for evidence on skill-biased technological change in developing economies). Given this evidence, we may expect a mismatch between the skill requirements of frontier technologies and the available skills of the workers in less-developed countries to be potentially more important than differences in capital intensity. In this subsection, I outline the model introduced in Acemoglu and Zilibotti (2001), which emphasizes the implications of the mismatch between technologies developed in advanced economies and the skills of the work force of the less-developed countries. Furthermore, this will enable us to use the ideas related to directed technical change developed in Chapter 15 and also provide us with a tractable multi-sector growth model.

The world economy consists of two groups of countries, the North and the South, and as in Chapter 15, two types of workers, skilled and unskilled. There are two differences between the North and the South. First, all R&D and new innovations take place in the North (so that the North approximates the OECD or the US and some of the other advanced economies). Instead, the South simply copies technologies developed in the North. Because of lack of intellectual property rights in the South, we will presume that the main market of new technologies will be Northern firms. Second, the North is more skill-abundant than the South, in particular,

$$H^n/L^n > H^s/L^s,$$

where H^j denotes the number of skilled workers in country j and L^j denotes the number of unskilled workers. We will use $j = n$ or s to denote the North or the South, and assume that there are many Northern and many Southern countries. There is no population growth. Throughout, all countries have access to the same set of technologies, so there will be no issue of slow technology diffusion. All differences in productivity will arise from the potential mismatch between technology and skills.

On the preference side, all economies are assumed to admit a representative household with the standard preferences, e.g., (18.6) above with $n_j = 0$ for all countries, since there is no population growth. The final good in each country is produced as a Cobb-Douglas aggregate of a continuum 1 of intermediate goods, that is,

$$Y_j(t) = \exp\left[\int_0^1 \ln y_j(i,t)di\right] \tag{18.16}$$

where $Y_j(t)$ is the amount of final good in country j at time t, while $y_j(i,t)$ is the output of intermediate i. As usual, total output is spent on consumption, $C_j(t)$, intermediate expenditures, $X_j(t)$, and also in the North, there will be R&D expenditures equal to $Z_j(t)$. The South will not undertake R&D, but can adopt technologies developed in the North.

Let us assume that the technology for producing intermediate i in country j at time t is given as follows:

$$(18.17)\qquad \begin{aligned} y_j(i,t) &= \frac{1}{1-\beta}\left[\int_0^{N_L(t)} x_{L,j}(i,\nu,t)^{1-\beta}d\nu\right][(1-i)l_j(i,t)]^{\beta} \\ &+\frac{1}{1-\beta}\left[\int_0^{N_H(t)} x_{H,j}(i,\nu,t)^{1-\beta}d\nu\right][i\omega h_j(i,t)]^{\beta}. \end{aligned}$$

A number of features about this intermediate production function is worth noting. First each intermediate can be produced using two alternative technologies, one using skilled workers, the other one using unskilled labor. Here $l_j(i,t)$ is the number of unskilled workers working in intermediate i in country j at time t. $h_j(i,t)$ is defined similarly. Second, skilled and unskilled workers have different productivities in different industries—incorporating a pattern of cross-industry comparative advantage. In particular, the presence of the terms $1-i$ and i in the production function (18.17) implies that skilled workers are relatively more productive in higher indexed intermediates, while unskilled workers have comparative advantage and lower indexed intermediates. Third, skilled workers also have an absolute advantage, captured by the parameter ω, which is assumed to be greater than 1. Fourth, as in the standard models with machine varieties, $x_{L,j}(i,\nu)$ denotes the quantity of machines of type ν used with unskilled workers, and $x_{H,j}(i,\nu)$ is defined similarly. This part of the production function is parallel to those used in Chapter 15. The number of machine varieties available to be used with skilled and unskilled workers differ and are equal to $N_L(t)$ and $N_H(t)$. The important point here is that these quantities are not indexed by j, since all technologies are available to all countries, that is, we are ignoring the issue of slow diffusion and focusing on differences arising purely from inappropriateness of technology. Finally, as usual, the term $1/(1-\beta)$ is introduced as a convenient normalization.

We assume that the final good sectors and the labor markets are competitive. Again as in Chapters 13 and 15, a technology monopolist can produce these machines at marginal cost ψ and supplies the quantities of machines. Let the prices of these machines be denoted by $p^x_{L,j}(\nu,t)$ and $p^x_{H,j}(\nu,t)$ for the two sectors in country j for machine of type ν at time t. Note that these prices do not depend on i, since the machines are not sector-specific. Instead, they are skill-specific. As in Chapters 13 and 15, profit maximization by the final good producers leads to the following demands for machines:

$$\begin{aligned} x_{L,j}(i,\nu,t) &= \left[p_j(i,t)\left((1-i)l_j(i,t)\right)^{\beta}/p^x_{L,j}(\nu,t)\right]^{1/\beta}, \\ x_{H,j}(i,\nu,t) &= \left[p_j(i,t)\left(i\omega h_j(i,t)\right)^{\beta}/p^x_{L,j}(\nu,t)\right]^{1/\beta}, \end{aligned}$$

where $p_j(i,t)$ is the relative price of intermediate i in country j at time t in terms of the final good (which is set as the numeraire in each country). The technology monopolist in the

North will be the firm that invents the new type of machine, so here the analysis is identical to that in Chapters 13 and 15.

What about in the South? To keep the treatment of Northern and Southern economies symmetric, we assume that in each Southern economy a "technology" firm adopts the new technology invented in the North (at no cost) and acts as the monopolist supplier of that machine for the producers in its own country. Moreover, we assume that the marginal cost of producing machines for this firm is the same as the inventor in the North, equal to ψ.

As we have seen a number of times before, the isoelastic demand for machines imply that the profit-maximizing price for the technology monopolists will be a constant markup over marginal cost, and we normalize the cost to $\psi \equiv 1-\beta$. The symmetry between the North and the South we have introduced above implies that the price of machines and thus the demand for machines will take the same form in all countries. In particular, we obtain output in sector i in any country j as

$$y_j(i,t) = \frac{1}{1-\beta} p_j(i,t)^{(1-\beta)/\beta} \left[N_L(t)(1-i) l_j(i,t) + N_H(t) i \omega h_j(i,t) \right]. \tag{18.18}$$

For each economy, $N_L(t)$ and $N_H(t)$ are the state variables. Given these state variables the equilibrium is straightforward to characterize. In particular, the following proposition characterizes the structure of equilibrium in each country.

PROPOSITION 18.6. *In any country j, given the world technologies $N_L(t)$ and $N_H(t)$, there will exist a threshold $I_j(t) \in [0,1]$ such that skilled workers will be employed only in sectors $i > I_j(t)$, that is, for all $i < I_j(t)$, $h_j(i,t) = 0$, and for all $i > I_j(t)$, $l_j(i,t) = 0$.*

Moreover, prices and labor allocations across sectors will be such that: for all $i < I_j(t)$, $p_j(i,t) = P_{L,j}(t)(1-i)^{-\beta}$ and $l_j(i,t) = L_j/I_j(t)$, while for all $i > I_j(t)$, $p_j(i,t) = P_{H,j}(t) i^{-\beta}$ and $h_j(i,t) = H_j/(1-I_j(t))$ where the positive numbers $P_{L,j}(t)$ and $P_{H,j}(t)$ can be interpreted as the price indices for labor-intensive and skill-intensive intermediates.

PROOF. See Exercise 18.21. □

With Proposition 18.6, the characterization of equilibrium given the level a world technologies $N_L(t)$ and $N_H(t)$ is straightforward. In particular, the technology for the final goods sector in (18.16) implies that the price indices in country j at time t must satisfy

$$\frac{P_{H,j}(t)}{P_{L,j}(t)} = \left(\frac{N_H(t)\omega H_j/(1-I_j(t))}{N_L(t) L_j/I_j(t)} \right)^{-\beta}. \tag{18.19}$$

Moreover, the threshold sector $I_j(t)$ in country j at time t is indifferent between using skilled and unskilled workers (and technologies) for production, thus $P_{L,j}(t)(1-I_j(t))^{-\beta} = P_{H,j}(t) I_j(t)^{-\beta}$. Combining this with (18.19), we obtain

$$\frac{P_{H,j}(t)}{P_{L,j}(t)} = \left(\frac{N_H(t)}{N_L(t)} \frac{\omega H_j}{L_j} \right)^{-\beta/2}, \tag{18.20}$$

and the equilibrium threshold $I_j(t)$ is uniquely pinned down by

$$\frac{I_j(t)}{1-I_j(t)} = \left(\frac{N_H(t)}{N_L(t)}\frac{\omega H_j}{L_j}\right)^{-1/2}. \tag{18.21}$$

Combining these two equations, we can also derive the level of total output in economy j as

$$Y_j(t) = \exp(-\beta)\left[(N_L(t)L_j)^{1/2} + (N_H(t)\omega H_j)^{1/2}\right]^2, \tag{18.22}$$

and the skill premium as

$$\frac{w_{H,j}(t)}{w_{L,j}(t)} = \omega\left(\frac{N_H(t)}{N_L(t)}\right)^{1/2}\left(\frac{\omega H_j}{L_j}\right)^{-1/2} \tag{18.23}$$

(see Exercise 18.22). An interesting feature of this characterization, apparent from equation (18.22) is that the multi-sector model in this section leads to an equilibrium allocation so that the level of output is identical to that given a constant elasticity of substitution production function within elasticity of substitution equal to 2. In fact, this phenomenon is more general and by changing the pattern of comparative advantage of skilled and unskilled workers in different sectors, one can obtain models with aggregate production functions of any elasticities of substitution.

The characterization of the equilibrium above already shows that the type of technologies, $N_L(t)$ and $N_H(t)$, will impact economies with different factor proportions differently. For example, consider the extreme case in which $H^s = 0$, so that there are no skilled workers in the south. Then an increase in $N_H(t)$ will increase productivity in the North, but will have no effect in the South. Naturally, when there are skilled and unskilled workers in both the North and the South, the implications of the changes in these two technologies will not be as extreme, but the general principle will continue to apply: an increase in $N_H(t)$ relative to $N_L(t)$ will benefit the skill-abundant North more than the skill-scarce South. But conversely, an increase in $N_L(t)$ will tend to benefit Southern economies relatively more. Thus the question becomes whether the world technology will be appropriate to the needs of the North or the South. Here the features that new technologies are developed in the North and that there are no intellectual property rights for Northern R&D in the South become important. In particular, these features imply that new technologies will be developed—*designed*—for the needs of the North.

To communicate the main ideas related to the emergence of technologies that are inappropriate to factor proportions in the South, let us adopt the simplest version of the directed technical change model from Chapter 15 (in particular, Section 15.3 with the lab equipment specification) and suppose that

$$\dot{N}_L(t) = \eta Z_L(t) \text{ and } \dot{N}_H(t) = \eta Z_H(t), \tag{18.24}$$

which is the same as the innovation possibilities frontier in Section 15.3, except that η_L and η_H have been set equal to each other for simplification. The analysis there, combined with the fact that the relevant market sizes are given by H^n and L^n (because research firms can only sell their technologies to Northern firms) implies that the steady-state (balanced growth) equilibrium must take the following form:

Proposition 18.7. *With the lab equipment specification of directed technical change as in (18.24) and no intellectual property rights in the South, the unique steady-state equilibrium involves Northern relative prices*

$$\frac{P_H^n}{P_L^n} = \left(\frac{\omega H^n}{L^n}\right)^{-\beta}$$

and world relative technology ratio

$$\frac{N_H^*}{N_L^*} = \frac{\omega H^n}{L^n}. \tag{18.25}$$

Moreover, in the North the threshold sector satisfies

$$\frac{1 - I^{n*}}{I^{n*}} = \frac{\omega H^n}{L^n}$$

and the skill premium is

$$\frac{w_H^{n*}}{w_L^{n*}} = \omega.$$

This steady-state equilibrium is globally saddle path stable.

Proof. (Sketch) Equation (18.18) immediately implies that, given $N_L(t)$ and $N_H(t)$ and the prices of skilled and unskilled workers, relative profitability on employing skilled workers is strictly increasing in $i \in [0,1]$. This implies that there must exist a threshold $I_j(t)$ as specified in the proposition. The Cobb-Douglas specification in (18.16) implies an allocation of labor across intermediates, and the corresponding relationship between the prices of intermediates using skilled labor and those using unskilled labor, so that expenditures on different intermediates are equalized. You are asked to complete the details of this argument, derive the expression for the threshold and the skill premium, and also establish the stability of the equilibrium in Exercise 18.23. □

To understand the implications of directed technical change for equilibrium relative technologies N_L and N_H, let us next introduce three simple concepts. The first is *net output* in country j defined as

$$NY_j \equiv Y_j - X_j,$$

that is, output minus the spending on intermediates. The second and the third are *income per capita* and *income per effective unit of labor* in different countries, defined as

$$y_j \equiv \frac{Y_j}{L_j + H_j} \text{ and } y_j^{eff} \equiv \frac{Y_j}{L_j + \omega H_j}.$$

All of these quantities are functions of labor supplies and of relative technologies, in particular of N_H/N_L. These dependences are suppressed to simplify notation.

The next result shows that the steady-state technologies N_L^* and N_H^* are indeed "appropriate" for the conditions (factor proportions) in the North, and that this creates endogenous income differences between the North and the South.

Proposition 18.8. *Consider the above-described model. Then:*

(1) *The steady-state equilibrium technology ratio N_H^*/N_L^* is such that, given a constant level of for given $N_H + N_L$, it achieves the unique maximum of net output in the North, NY^n, as a function of relative technology N_H/N_L.*

(2) *At the steady-state equilibrium technology ratio N_H^*/N_L^*, we have $y_n > y_s$ and $y_n^{eff} > y_s^{eff}$.*

Proof. See Exercise 18.24. □

This proposition establishes two important results. First, the steady-state equilibrium technology is indeed appropriate for the needs of the North. This is intuitive, since research firms are innovating targeting the Northern markets (in particular the relative supply of skills in the North). Moreover, the statement that there is a unique maximum of NY_n (given the total amount of "technology" $N_H + N_L$) also implies that net output in the South, NY_s, given by a similar expression, will *not* be maximized by N_H^*/N_L^*. This is the essence of the second result contained in this proposition: because technologies are developed in the North (in practice, corresponding loosely to the OECD) and are designed for the needs (factor proportions) of Northern economies, they are inappropriate for the needs of the South. As a result, income per capita and income per effective units of labor in the North will be higher than in the South. Thus the process of directed technical change, combined with import of frontier technologies to less-developed economies, creates an advantage for the more advanced economies and acts as a force towards greater cross-country inequality. Therefore, the issue of potential mismatch between the technologies of the world frontier and the skills of less-developed countries creates a force towards large income per capita differences among these countries. Acemoglu and Zilibotti (2001) show that this source of cross-country income differences can be quite substantial in practice. Therefore, inappropriateness of technologies of the world to the needs of the less-developed countries, especially the potential mismatch between technology and skill, can create significant income differences.

18.5. Contracting Institutions and Technology Adoption

An important determinant of differences in technology and technology adoption are institutional differences across societies. We have already noted how the parameter σ_j in the model of Section 18.2 can be interpreted as varying across countries because of differences in policies and institutions erecting barriers against technology adoption. Naturally, an approach that links σ_j to such "technology barriers" is rather reduced-form and is most useful in providing a perspective in discussions. To make further progress, we need more micro-founded models of why there are barriers to technology adoptions and how these barriers affect technology choices. The reasons why certain groups may want to erect barriers against the introduction of new technologies will be discussed in detail in Part 8 below. In Part 7, we will discuss other factors affecting the efficiency of the organization of production, which can also be loosely related to "technology choices". However, before turning to these models, it is useful to show how differences in the ability to write contracts between firms and their suppliers (or firms and their workers) may have first-order effect on technology adoption decisions. I will now briefly discuss a model of endogenous technology adoption, which again builds on the framework developed in Chapter 13. The purpose of this model is to illustrate how contractual difficulties can lead to important technological differences across countries and to emphasize the other side of the issue of technology adoption, i.e., how the conditions in the adopting country affect the use of these technologies by firms. The model I will present is a slight simplification of that by Acemoglu, Antras and Helpman (2007). The main focus is how differences in contracting institutions across countries will affect relationships between producers and suppliers and thus change the profitability of technology adoption. I will also use this model to illustrate how analysis of contracting problems (in this instance between firms) can be easily incorporated into the types of models we have studied so far.

18.5.1. Description of the Environment. For simplicity, consider a static world and focus on a single country. There exists a continuum of final goods $q(z)$, with $z \in [0, M]$, where M represents the number (measure) of final goods (I use M here, since N will denote technology choice). All consumers have identical preferences,

$$u = \left(\int_0^M q(\nu)^\beta \, d\nu \right)^{1/\beta} - \psi e, \quad 0 < \beta < 1, \tag{18.26}$$

where e is the total effort exerted by this individual, with ψ representing the cost of effort in terms of real consumption. The parameter $\beta \in (0,1)$ determines the elasticity of demand and implies that the elasticity of substitution between final goods, $1/(1-\beta)$, is greater than 1. These preferences imply the demand function

$$q(\nu) = \left[\frac{p(\nu)}{p^I} \right]^{-1/(1-\beta)} \frac{A}{p^I},$$

for each producer $\nu \in [0, M]$, where $p(\nu)$ is the price of good ν, A is the aggregate spending level, and

$$p^I \equiv \left[\int_0^M p(\nu)^{-\beta/(1-\beta)} \, d\nu \right]^{-(1-\beta)/\beta}$$

is the ideal price index, which is taken as the numeraire, i.e., $p^I = 1$. This implies that each final good producer will face a demand function of the form $q = Ap^{-1/(1-\beta)}$, where q denotes quantity and p denotes price, and I have dropped the conditioning on z, since I will focus on the decisions of a single firm. The resulting revenue function for the firm can therefore be written as

$$R = A^{1-\beta} q^{\beta}. \tag{18.27}$$

Production depends on the technology choice of the firm, which is denoted by $N \in \mathbb{R}_+$. More advanced technologies involve a greater range of intermediate goods (inputs), supplied by different suppliers. The transactions between the producer and the suppliers will necessitate contracting relationships. For each $j \in [0, N]$, let $X(j)$ be the quantity of intermediate input j. The production function of the representative firm takes the standard CES form

$$q = N^{\kappa+1-1/\alpha} \left[\int_0^N X(j)^{\alpha} \, dj \right]^{1/\alpha}, \tag{18.28}$$

where we assume that $0 < \alpha < 1$, so that the elasticity of substitution between inputs, $\varepsilon \equiv 1/(1-\alpha)$, is always greater than one. In addition, we assume $\kappa > 0$. The standard specification of the CES (Dixit-Stiglitz) aggregator would not involve the term $N^{\kappa+1-1/\alpha}$ (i.e., it would be implicitly setting $\kappa = 1/\alpha - 1$). In that case, as we have seen in Section 12.4 in Chapter 12, when $X(j) = X$, total output would be given by $q = N^{1/\alpha} X$, and both the elasticity of substitution between inputs and the elasticity of output to changes in technology, N, would be governed by the same parameter, α. By introducing the term $N^{\kappa+1-1/\alpha}$ in front of the integral, we are separating these two elasticities.

There is a large number of profit-maximizing suppliers that can produce the necessary intermediate goods. We assume that each supplier has the same outside option $w_0 > 0$. For now, let us take w_0 as given and also assume that each intermediate input needs to be produced by a different supplier with whom the firm needs to contract (see Exercise 18.31 on endogenizing this outside option). A supplier assigned to the production of an intermediate input needs to undertake relationship-specific investments in a unit measure of (symmetric) activities. The marginal cost of investment for each activity is ψ as specified in (18.26). The production function of intermediate inputs is Cobb-Douglas and symmetric in the activities:

$$X(j) = \exp \left[\int_0^1 \ln x(i,j) \, di \right], \tag{18.29}$$

where $x(i,j)$ is the level of investment in activity i performed by the supplier of input j. This formulation will allow a tractable parameterization of contractual incompleteness, whereby a subset of the investments necessary for production will be nonverifiable and thus noncontractible. Finally, we assume that adopting a technology N involves costs $\Gamma(N)$, and impose the following two restrictions on $\Gamma(N)$:

(i) For all $N > 0$, $\Gamma(N)$ is twice continuously differentiable, with $\Gamma'(N) > 0$ and $\Gamma''(N) > 0$.

(ii) For all $N > 0$, $N\Gamma''(N) / [\Gamma'(N) + w_0] > [\beta(\kappa+1) - 1] / (1-\beta)$.

These restrictions are standard. In particular, they introduce enough convexity to ensure interior solutions.

The relationship between the producer and its suppliers requires contracts to ensure that the suppliers deliver the required inputs. Let the payment to supplier j consist of two parts: an ex ante payment $\tau(j) \in \mathbb{R}$ before the investment levels $x(i,j)$ take place, and a payment $s(j)$ after the investments. Then, the payoff to supplier j, also taking account of her outside option, is

$$\pi_x(j) = \max\left\{\tau(j) + s(j) - \int_0^1 \psi x(i,j)\, di, w_0\right\}. \tag{18.30}$$

Similarly, the payoff to the firm is

$$\pi = R - \int_0^N [\tau(j) + s(j)]\, dj - \Gamma(N), \tag{18.31}$$

where R is revenue and the other two terms on the right-hand side represent costs. Substituting (18.28) and (18.29) into (18.27), revenue can be expressed as

$$R = A^{1-\beta} N^{\beta(\kappa+1-1/\alpha)} \left[\int_0^N \left(\exp\left(\int_0^1 \ln x(i,j)\, di\right)\right)^{\alpha} dj\right]^{\beta/\alpha}. \tag{18.32}$$

18.5.2. Equilibrium under Complete Contracts. As a benchmark, consider the "idealized" case of complete contracts, where the firm has full control over all investments and pays each supplier her outside option. Conceptually, complete contracts correspond to the case in which markets are complete, and intermediates of different qualities can be bought and sold in a quasi-competitive fashion. While this is a good approximation for many commodities, complete contracts (or the corresponding complete markets) may not always capture the essence of the interaction between firms and their suppliers, especially when contracting institutions are somewhat imperfect, so that using courts or other legal sanctions against firms that breach their contractual agreements might be costly.

To prepare for our treatment below of technology adoption under incomplete contracts, consider a game where the firm chooses a technology level N and makes a contract offer $\left[\{x(i,j)\}_{i\in[0,1]}, \{s(j), \tau(j)\}\right]$ for every input $j \in [0,N]$. If a supplier accepts this contract for input j, she is obliged to supply $\{x(i,j)\}_{i\in[0,1]}$ as stipulated in the contract in exchange for the payments $\{s(j), \tau(j)\}$. A *subgame perfect equilibrium* of this game is a strategy combination for the firm and the suppliers such that suppliers maximize (18.30) and the firm maximizes (18.31). An equilibrium can be alternatively represented as a solution to the following maximization problem:

$$\max_{N, \{x(i,j)\}_{i,j}, \{s(j),\tau(j)\}_j} R - \int_0^N [\tau(j) + s(j)]\, dj - \Gamma(N) \tag{18.33}$$

subject to (18.32) and the suppliers' *participation constraint,*

$$s(j) + \tau(j) - \psi \int_0^1 x(i,j)\,di \geq w_0 \text{ for all } j \in [0,N]. \tag{18.34}$$

Since the firm has no reason to provide rents to the suppliers, it chooses payments $s(j)$ and $\tau(j)$ that satisfy (18.34) with equality. Moreover, with complete contracts, $\tau(j)$ and $s(j)$ are perfect substitutes, so only the sum $s(j)+\tau(j)$ matters and is determined in equilibrium—this will not be the case when contracts are incomplete.

Moreover, since the firm's objective function, (18.33), is (jointly) concave in the investment levels $x(i,j)$ and these investments are all equally costly, the firm chooses the same investment level x for all activities in all intermediate inputs. Now, substituting for (18.34) in (18.33), we obtain the following simpler unconstrained maximization problem for the firm:

$$\max_{N,x} A^{1-\beta} N^{\beta(\kappa+1)} x^{\beta} - \psi N x - \Gamma(N) - w_0 N. \tag{18.35}$$

The first-order conditions of this problem imply:

$$(N^*)^{\frac{\beta(\kappa+1)-1}{1-\beta}} A\kappa\beta^{1/(1-\beta)}\psi^{-\beta/(1-\beta)} = \Gamma'(N^*) + w_0, \tag{18.36}$$

$$x^* = \frac{\Gamma'(N^*) + w_0}{\kappa\psi}. \tag{18.37}$$

Equations (18.36) and (18.37) can be solved recursively. The restrictions on the function Γ above ensure that equation (18.36) has a unique solution for N^*, which, together with (18.37), yields a unique solution for x^*.

When all the investment levels are identical and equal to x, output is $q = N^{\kappa+1}x$. Since a total of $NX = Nx$ inputs are used in the production process, a natural measure of productivity is output divided by total input use, $P = N^{\kappa}$. In the case of complete contracts this productivity level is

$$P^* = (N^*)^{\kappa}, \tag{18.38}$$

which is increasing in the level of technology. Summarizing this analysis, we have:

Proposition 18.9. *Consider the above described model, take A as given and suppose that there are complete contracts. Then there exists a unique equilibrium with technology and investment levels $N^* > 0$ and $x^* > 0$ given by (18.36) and (18.37). Furthermore, this equilibrium satisfies:*

$$\frac{\partial N^*}{\partial A} > 0,\ \frac{\partial x^*}{\partial A} \geq 0,\ \frac{\partial N^*}{\partial \alpha} = \frac{\partial x^*}{\partial \alpha} = 0.$$

Proof. See Exercise 18.27. □

In the case of complete contracts, the size of the market, which corresponds to A and from the viewpoint of the individual firm is exogenous, has a positive effect on investments by suppliers of intermediate inputs and productivity, because a greater market size makes both suppliers' and the producer's investments more productive. The other noteworthy implication of this proposition is that under complete contracts, the level of technology and thus productivity do not depend on the elasticity of substitution between intermediate inputs, $1/(1-\alpha)$.

18.5.3. Equilibrium under Incomplete Contracts. We now consider the same environment under incomplete contracts. We model the imperfection of the contracting institutions by assuming that there exists a $\mu \in [0,1]$ such that, for every intermediate input j, investments in activities $0 \leq i \leq \mu$ are observable and verifiable and therefore contractible, while investments in activities $\mu < i \leq 1$ are not contractible. Consequently, a contract stipulates investment levels $x(i,j)$ for the μ contractible activities, but does not specify the investment levels in the remaining $1-\mu$ noncontractible activities. Instead, suppliers choose their investments in noncontractible activities in anticipation of the ex post distribution of revenue, and may decide to withhold their services in these activities from the firm. In economies with weak contracting institutions, we will have a low μ, thus only a small set of tasks are contractible, whereas more developed contracting institutions will correspond to high levels of μ.

The ex post distribution of revenues in activities that are not ex ante contractible will be determined by multilateral bargaining between the firm and its suppliers. The exact bargaining protocol will determine investment incentives of suppliers and the profitability of investment for the firm. Below we will adopt the *Shapley value* as a natural solution concept for this multilateral bargaining game. First, consider the timing of events:

- The firm adopts a technology N and offers a contract $[\{x_c(i,j)\}_{i=0}^{\mu}, \tau(j)]$ for every intermediate input $j \in [0,N]$, where $x_c(i,j)$ is an investment level in a contractible activity and $\tau(j)$ is an upfront payment to supplier j. The payment $\tau(j)$ can be positive or negative.
- Potential suppliers decide whether to apply for the contracts. Then the firm chooses N suppliers, one for each intermediate input j.
- All suppliers $j \in [0,N]$ simultaneously choose investment levels $x(i,j)$ for all $i \in [0,1]$. In the contractible activities $i \in [0,\mu]$ the suppliers will invest $x(i,j) = x_c(i,j)$.
- The suppliers and the firm bargain over the division of revenue, and at this stage, suppliers can withhold their services in noncontractible activities.
- Output is produced and sold, and the revenue R is distributed according to the bargaining agreement.

We will characterize a *symmetric subgame perfect equilibrium* (SSPE) of this game, where bargaining outcomes in all subgames are determined by Shapley values.

Behavior along the SSPE can be described by a tuple $\left\{\tilde{N}, \tilde{x}_c, \tilde{x}_n, \tilde{\tau}\right\}$ in which $\tilde{N}$ represents the level of technology, $\tilde{x}_c$ the investment in contractible activities, $\tilde{x}_n$ the investment in noncontractible activities, and $\tilde{\tau}$ the upfront payment to every supplier. That is, for every $j \in \left[0, \tilde{N}\right]$ the upfront payment is $\tau(j) = \tilde{\tau}$, and the investment levels are $x(i,j) = \tilde{x}_c$ for $i \in [0, \mu]$ and $x(i,j) = \tilde{x}_n$ for $i \in (\mu, 1]$. With a slight abuse of terminology, we will denote the SSPE by $\left\{\tilde{N}, \tilde{x}_c, \tilde{x}_n\right\}$.

As is typically the case in extensive form complete information games, the SSPE can be characterized by backward induction. First, consider the penultimate stage of the game, with N as the level of technology, x_c as the level of investment in contractible activities. Suppose also that each supplier other than j has chosen a level of investment in noncontractible activities equal to $x_n(-j)$ (these are all the same, because we are constructing a symmetric equilibrium), while the investment level in every noncontractible activity by supplier j is $x_n(j)$. Given these investments, the suppliers and the firm will engage in multilateral bargaining. Denote the return to supplier j resulting from this bargaining by $\bar{s}_x[N, x_c, x_n(-j), x_n(j)]$. The optimal investment by supplier j then implies that $x_n(j)$ must be chosen to maximize $\bar{s}_x[N, x_c, x_n(-j), x_n(j)]$ minus the cost of investment in noncontractible activities, $(1-\mu)\psi x_n(j)$. In a symmetric equilibrium, $x_n(j) = x_n(-j)$, or in other words, x_n needs to be a fixed-point given by:

$$x_n \in \arg\max_{x_n(j)} \bar{s}_x[N, x_c, x_n, x_n(j)] - (1-\mu)\psi x_n(j). \tag{18.39}$$

Equation (18.39) can be thought of as an "incentive compatibility constraint," with the additional symmetry requirement. While this equation is written with "$\in$" to allow for the fact that there may be more than one maximizers of the expression on the right-hand side, the structure of the current model ensures that there will be a unique maximizer, thus "$\in$" can be replaced with the equal sign, "$=$".

In a symmetric equilibrium with technology N, with investment in contractible activities given by x_c and with investment in noncontractible activities equal to x_n, the revenue of the firm is given by $R = A^{1-\beta}\left(N^{\kappa+1}x_c^{\mu}x_n^{1-\mu}\right)^{\beta}$. Moreover, let $s_x(N, x_c, x_n) = \bar{s}_x(N, x_c, x_n, x_n)$, then the Shapley value of the firm is obtained as a residual:

$$s_q(N, x_c, x_n) = A^{1-\beta}\left(N^{\kappa+1}x_c^{\mu}x_n^{1-\mu}\right)^{\beta} - Ns_x(N, x_c, x_n).$$

Now consider the stage in which the firm chooses N suppliers from a pool of applicants. If suppliers expect to receive less than their outside option, w_0, this pool is empty. Therefore, for production to take place, the final-good producer has to offer a contract that satisfies the participation constraint of suppliers under incomplete contracts, i.e.,

$$\bar{s}_x(N, x_c, x_n, x_n) + \tau \geq \mu\psi x_c + (1-\mu)\psi x_n + w_0 \quad \text{for } x_n \text{ that satisfies (18.39).} \tag{18.40}$$

In other words, given N and (x_c, τ), each supplier $j \in [0, N]$ should expect her Shapley value plus the upfront payment to cover the cost of investment in contractible and noncontractible activities and the value of her outside option.

The maximization problem of the firm can then be written as:

$$\max_{N, x_c, x_n, \tau} s_q(N, x_c, x_n) - N\tau - \Gamma(N)$$

subject to (18.39) and (18.40).

With no restrictions on τ, the participation constraint (18.40) will be satisfied with equality; otherwise the firm could reduce τ without violating (18.40) and increase its profits. We can therefore solve τ from this constraint, substitute the solution into the firm's objective function and obtain the simpler maximization problem:

$$\max_{N, x_c, x_n} s_q(N, x_c, x_n) + N\left[\bar{s}_x(N, x_c, x_n, x_n) - \mu\psi x_c - (1-\mu)\psi x_n\right] - \Gamma(N) - w_0 N, \tag{18.41}$$

subject to (18.39).

The SSPE $\left\{\tilde{N}, \tilde{x}_c, \tilde{x}_n\right\}$ solves this problem, and the corresponding upfront payment satisfies

$$\tilde{\tau} = \mu\psi\tilde{x}_c + (1-\mu)\psi\tilde{x}_n + w_0 - \bar{s}_x\left(\tilde{N}, \tilde{x}_c, \tilde{x}_n, \tilde{x}_n\right). \tag{18.42}$$

The key issue in the presence of incomplete contracts is that the payments from the firm to its suppliers will be determined ex post through bargaining rather than through contractual arrangements. As noted above, different bargaining protocols between suppliers and the producer will lead to somewhat different results. In the current context, the most natural choice appears to be the Shapley value, since it provides a plausible and tractable division rule for multilateral bargaining problems. The derivation of this formula is not essential for the results here, thus it is included for completeness at the end of this section. The next proposition provides the form of this bargaining solution.

Proposition 18.10. *Suppose that supplier j invests $x_n(j)$ in her noncontractible activities, all the other suppliers invest $x_n(-j)$ in their noncontractible activities, every supplier invests x_c in her contractible activities, and the level of technology is N. Then the Shapley value of supplier j is*

$$\bar{s}_x\left[N, x_c, x_n(-j), x_n(j)\right] = (1-\gamma) A^{1-\beta}\left[\frac{x_n(j)}{x_n(-j)}\right]^{(1-\mu)\alpha} x_c^{\beta\mu} x_n(-j)^{\beta(1-\mu)} N^{\beta(\kappa+1)-1}, \tag{18.43}$$

where

$$\gamma \equiv \frac{\alpha}{\alpha+\beta}. \tag{18.44}$$

Proof. See subsection 18.5.4. □

A number of features of (18.43) are worth noting. First, the derived parameter $\gamma \equiv \alpha/(\alpha+\beta)$ represents the bargaining power of the firm; it is increasing in α and decreasing in β. A higher elasticity of substitution between intermediate inputs, i.e., a higher α, raises the firm's bargaining power, because it makes every supplier less essential in production and therefore raises the share of revenue appropriated by the firm. In contrast, a higher elasticity of demand for the final good, i.e., higher β, reduces the firm's bargaining power, because, for any coalition, it reduces the marginal contribution of the firm to the coalition's payoff as a fraction of revenue.

Second, in equilibrium, all suppliers invest equally in all the noncontractible activities, i.e., $x_n(j) = x_n(-j) = x_n$, and so

$$\text{(18.45)} \quad \begin{aligned} s_x(N, x_c, x_n) &= \bar{s}_x(N, x_c, x_n, x_n) = (1-\gamma) A^{1-\beta} x_c^{\beta\mu} x_n^{\beta(1-\mu)} N^{\beta(\kappa+1)-1} \\ &= (1-\gamma)\frac{R}{N}, \end{aligned}$$

where $R = A^{1-\beta} x_c^{\beta\mu} x_n^{\beta(1-\mu)} N^{\beta(\kappa+1)}$ is the total revenue of the firm. Thus, the joint Shapley value of the suppliers, $Ns_x(N, x_c, x_n)$, equals the fraction $1-\gamma$ of the revenue, and the firm receives the remaining fraction γ, i.e.,

$$\text{(18.46)} \quad \begin{aligned} s_q(N, x_c, x_n) &= \gamma A^{1-\beta} x_c^{\beta\mu} x_n^{\beta(1-\mu)} N^{\beta(\kappa+1)} \\ &= \gamma R. \end{aligned}$$

This is a relatively simple rule for the division of revenue between the firm and its suppliers.

Finally, when α is smaller, $\bar{s}_x[N, x_c, x_n(-j), x_n(j)]$ is more concave with respect to $x_n(j)$, because greater complementarity between the intermediate inputs implies that a given change in the relative employment of two inputs has a larger impact on their relative marginal products. The impact of α on the concavity of $\bar{s}_x(\cdot)$ will play an important role in the following results. The parameter β, on the other hand, affects the concavity of revenue in output (see (18.27)), but has no effect on the concavity of $\bar{s}_x$, because with a continuum of suppliers, a single supplier has an infinitesimal effect on output.

To characterize a SSPE, we first derive the incentive compatibility constraint using (18.39) and (18.43):

$$x_n = \arg\max_{x_n(j)} \ (1-\gamma) A^{1-\beta} \left[\frac{x_n(j)}{x_n}\right]^{(1-\mu)\alpha} x_c^{\beta\mu} x_n^{\beta(1-\mu)} N^{\beta(\kappa+1)-1} - \psi(1-\mu) x_n(j).$$

Relative to the producer's first-best choice characterized above, we see two differences. First, the term $(1-\gamma)$ implies that the supplier is not the full residual claimant of the return from her investment in noncontractible activities and thus underinvests in these activities. Second, as discussed above, multilateral bargaining distorts the perceived concavity of the private return relative to the social return. Using the first-order condition of this problem and solving for the fixed point by substituting $x_n(j) = x_n$ yields a unique x_n:

$$x_n = \bar{x}_n(N, x_c) \equiv \left[\alpha(1-\gamma)\psi^{-1} x_c^{\beta\mu} A^{1-\beta} N^{\beta(\kappa+1)-1}\right]^{1/[1-\beta(1-\mu)]}. \tag{18.47}$$

This equation implies that investments in noncontractible activities are increasing in α. Mathematically, this follows from the fact that $\alpha(1-\gamma) = \alpha\beta/(\alpha+\beta)$ is increasing in α. The economics of this relationship is the outcome of two opposing forces. The share of the suppliers in revenue, $(1-\gamma)$, is decreasing in α, because greater substitution between the intermediate inputs reduces the suppliers' ex post bargaining power. But a greater level of α also reduces the concavity of $\bar{s}_x(\cdot)$ in x_n, increasing the marginal reward from investing further in noncontractible activities. Because the latter effect dominates, x_n is increasing in α.

Another interesting feature is that contractible and noncontractible activities are complements, and in particular, $\bar{x}_n(N, x_c)$ is increasing in x_c. Finally, the effect of N on x_n is ambiguous, since investment in noncontractible activities declines with the level of technology when $\beta(\kappa+1) < 1$ and increases with N when $\beta(\kappa+1) > 1$. This is because an increase in N has two opposite effects on a supplier's incentives to invest; a greater number of inputs increases the marginal product of investment due to the "love for variety" embodied in the technology, but at the same time, the bargaining share of a supplier, $(1-\gamma)/N$, declines with N. For large values of κ the former effect dominates, while for small values of κ the latter dominates.

Now, using (18.45), (18.46) and (18.47), the firm's optimization problem (18.41) can be expressed as the maximization of

$$A^{1-\beta}\left[x_c^{\mu}\bar{x}_n(N, x_c)^{1-\mu}\right]^{\beta} N^{\beta(\kappa+1)} - \psi N\mu x_c - \psi N(1-\mu)\bar{x}_n(N, x_c) - \Gamma(N) - w_0 N \tag{18.48}$$

with respect to N and x_c, where $\bar{x}_n(N, x_c)$ is defined in (18.47). Substituting (18.47) into (18.48) and differentiating with respect to N and x_c results in two first-order conditions, which yield a unique solution $\left(\tilde{N}, \tilde{x}_c\right)$ to (18.48):

$$\begin{aligned}\tilde{N}^{\frac{\beta(\kappa+1)-1}{1-\beta}} A\kappa\beta^{\frac{1}{1-\beta}}\psi^{-\frac{\beta}{1-\beta}} & \left[\frac{1-\alpha(1-\gamma)(1-\mu)}{1-\beta(1-\mu)}\right]^{\frac{1-\beta(1-\mu)}{1-\beta}} \left[\beta^{-1}\alpha(1-\gamma)\right]^{\frac{\beta(1-\mu)}{1-\beta}} \\ &= \Gamma'\left(\tilde{N}\right) + w_0,\end{aligned} \tag{18.49}$$

$$\tilde{x}_c = \frac{\Gamma'\left(\tilde{N}\right) + w_0}{\kappa\psi}. \tag{18.50}$$

As in the complete contracts case, these two conditions determine the equilibrium recursively. First, (18.49) gives $\tilde{N}$, and then given $\tilde{N}$, (18.50) yields $\tilde{x}_c$. Moreover, using (18.47), (18.49), and (18.50) gives the level of investment in noncontractible activities as

$$\tilde{x}_n = \frac{\alpha(1-\gamma)[1-\beta(1-\mu)]}{\beta[1-\alpha(1-\gamma)(1-\mu)]}\left(\frac{\Gamma'\left(\tilde{N}\right)+w_0}{\kappa\psi}\right). \tag{18.51}$$

Comparing (18.37) to (18.50), we see that for a given N the implied level of investment in contractible activities under incomplete contracts, $\tilde{x}_c$, is identical to the investment level in contractible activities under complete contracts, x^*. This highlights the fact that differences in investments in contractible activities between these economic environments only result from differences in technology adoption. In fact, comparing (18.36) with (18.49), we see that $\tilde{N}$ and N^* differ only because of the two bracketed terms on the left-hand side of (18.49). These represent the distortions created by bargaining between the firm and its suppliers. Intuitively, technology adoption is distorted because incomplete contracts reduce investments in noncontractible activities below the level of investment in contractible activities and this "underinvestment" reduces the profitability of technologies with high N. As $\mu \to 1$ (and contractual imperfections disappear), both of these bracketed terms on the left-hand side of (18.49) go to 1 and $\left(\tilde{N}, \tilde{x}_c\right) \to (N^*, x^*)$.

We next provide a number of comparative static results on the SSPE under incomplete contracts and compare the incomplete-contracts equilibrium to the equilibrium under complete contracts. The comparative static results are facilitated by the block-recursive structure of the equilibrium; any change in A, μ or α that increases the left-hand side of (18.49) also increase $\tilde{N}$, and the effect on $\tilde{x}_c$ and $\tilde{x}_n$ can then be obtained from (18.50) and (18.51). The main results are provided in the next proposition:

Proposition 18.11. *Consider the above described model with incomplete contracts and suppose that the restrictions on Γ hold. Then there exists a unique SSPE under incomplete contracts, $\left\{\tilde{N}, \tilde{x}_c, \tilde{x}_n\right\}$, characterized by (18.49), (18.50) and (18.51). Furthermore, $\left\{\tilde{N}, \tilde{x}_c, \tilde{x}_n\right\}$ satisfies $\tilde{N}, \tilde{x}_c, \tilde{x}_n > 0$,*

$$\tilde{x}_n < \tilde{x}_c,$$

$$\begin{aligned}
\frac{\partial \tilde{N}}{\partial A} &> 0,\ \frac{\partial \tilde{x}_c}{\partial A} \geq 0,\ \frac{\partial \tilde{x}_n}{\partial A} \geq 0,\\
\frac{\partial \tilde{N}}{\partial \mu} &> 0,\ \frac{\partial \tilde{x}_c}{\partial \mu} \geq 0,\ \frac{\partial (\tilde{x}_n/\tilde{x}_c)}{\partial \mu} > 0,\\
\frac{\partial \tilde{N}}{\partial \alpha} &> 0,\ \frac{\partial \tilde{x}_c}{\partial \alpha} \geq 0,\ \frac{\partial (\tilde{x}_n/\tilde{x}_c)}{\partial \alpha} > 0.
\end{aligned}$$

Proof. See Exercise 18.28. □

This proposition states that suppliers invest less in noncontractible activities than in contractible activities. In particular, we have that

$$\frac{\tilde{x}_n}{\tilde{x}_c} = \frac{\alpha(1-\gamma)[1-\beta(1-\mu)]}{\beta[1-\alpha(1-\gamma)(1-\mu)]} < 1, \tag{18.52}$$

which follows from equations (18.50) and (18.51) and from the fact that $\alpha(1-\gamma) = \alpha\beta/(\alpha+\beta) < \beta$ (recall (18.44)). This is intuitive: the producer firm is the full residual claimant of the return to investments in contractible activities and it dictates these investments in the contract. In contrast, investments in noncontractible activities are decided by the suppliers, who are not the full residual claimants of the returns generated by these investments (recall (18.45)) and thus underinvest in these activities.

In addition, the level of technology and investments in both contractible and noncontractible activities are increasing in the size of the market, in the fraction of contractible activities (quality of contracting institutions), and in the elasticity of substitution between intermediate inputs. The impact of the size of the market is intuitive; a greater A makes production more profitable and thus increases investments and equilibrium technology. Better contracting institutions, on the other hand, imply that a greater fraction of activities receive the higher investment level $\tilde{x}_c$ rather than $\tilde{x}_n < \tilde{x}_c$. This makes the choice of a more advanced technologies more profitable. A higher N, in turn, increases the profitability of further investments in $\tilde{x}_c$ and $\tilde{x}_n$. Better contracting institutions also close the (proportional) gap between $\tilde{x}_c$ and $\tilde{x}_n$ because with a higher fraction of contractible activities, the marginal return to investment in noncontractible activities is also higher.

A higher α, i.e., lower complementarity between intermediate inputs, also increases technology choices and investments. The reason is related to the discussion in the previous subsection where it was shown that a higher α reduces the share of each supplier but also makes $\bar{s}_x(\cdot)$ less concave. Because the latter effect dominates, a lower degree of complementarity increases supplier investments and makes the adoption of more advanced technologies more profitable.

One of the main implications of this analysis is that contractual frictions (here captured by the incomplete contracts equilibrium) lead to underinvestment in quality, discourage technology adoption and reduce productivity. This is summarized in the next proposition. Note that productivity under incomplete contracts is $\tilde{P} = \tilde{N}^{\kappa}$, while productivity on the complete contracts, P^*, is given in (18.38).

Proposition 18.12. *Let $\{\tilde{N}, \tilde{x}_c, \tilde{x}_n\}$ be the unique SSPE with incomplete contracts and let $\{N^*, x^*\}$ be the unique equilibrium with complete contracts. Then*

$$\tilde{N} < N^* \text{ and } \tilde{x}_n < \tilde{x}_c < x^*.$$

Proof. See Exercise 18.29. □

This proposition implies that since incomplete contracts lead to the choice of less advanced (lower N) technologies, they also reduce productivity and investments in contractible and noncontractible activities. Acemoglu, Antras and Helpman (2007) also show that the technology and income differences resulting from relatively modest differences in contracting institutions can be quite large. Therefore, the link between contracting institutions and technology adoption provides us with a theoretical mechanism that might generate significant technology differences across countries.

18.5.4. Appendix to Section 18.5: The Shapley Value and the Proof of Proposition 18.10 *. The concept of Shapley values, first proposed by Shapley (1953) has both intuitive and game theoretic appeal. In a bargaining game with a finite number of players, each player's Shapley value is the average of her contributions to all coalitions that consist of players ordered below her in all feasible permutations. More explicitly, in a game with $T+1$ players, let $g = \{g(0), g(1), ..., g(T)\}$ be a permutation of $0, 1, 2, ..., T$, where player 0 is the firm and players $1, 2, ..., T$ are the suppliers, and let $z_g^j = \{j' \mid g(j) > g(j')\}$ be the set of players ordered below j in the permutation g. We denote the set of all feasible permutations by G, the set of all subsets of $T+1$ players by S, and the value of a coalition consisting of a subset of the $T+1$ players by $v : S \to \mathbb{R}$. Then the Shapley value of player j is

$$s_j = \frac{1}{(T+1)!} \sum_{g \in G} \left[v\left(z_g^j \cup j\right) - v\left(z_g^j\right) \right].$$

We now derive the *asymptotic* Shapley value proposed by Aumann and Shapley (1974), which involves considering the limit of this expression as the number of players goes to infinity. Let there be T suppliers each one controlling a range $\xi = N/T$ of the continuum of intermediate inputs. Due to symmetry, all suppliers provide an amount x_c of contractible activities. As for the noncontractible activities, consider a situation in which a supplier j supplies an amount $x_n(j)$ per noncontractible activity, while the $T-1$ remaining suppliers supply the same amount $x_n(-j)$ (note that we are again appealing to symmetry).

To compute the Shapley value for this particular supplier j, we need to determine the marginal contribution of this supplier to a given coalition of agents. A coalition of n suppliers and the firm yields a sales revenue of

$$F_{IN}(n, N; \xi) = A^{1-\beta} N^{\beta(\kappa+1-1/\alpha)} x_c^{\beta\mu} \left[(n-1)\, \xi x_n(-j)^{(1-\mu)\alpha} + \xi x_n(j)^{(1-\mu)\alpha} \right]^{\beta/\alpha},$$

when the supplier j is in the coalition, and a sales revenue

$$F_{OUT}(n, N; \xi) = A^{1-\beta} N^{\beta(\kappa+1-1/\alpha)} x_c^{\beta\mu} \left[n \xi x_n(-j)^{(1-\mu)\alpha} \right]^{\beta/\alpha}$$

when supplier j is *not* in the coalition. Notice that even when $n < N$, the term $N^{\beta(\kappa+1-1/\alpha)}$ remains in front, because it represents a feature of the technology affecting output independent of the amount and quality of the inputs provided by the suppliers. On the other hand, productivity suffers because the term in square brackets is lower.

The Shapley value of player j is then

$$s_j = \frac{1}{(T+1)!}\sum_{g\in G}\left[v\left(z_g^j \cup j\right) - v\left(z_g^j\right)\right]. \tag{18.53}$$

The fraction of permutations in which $g(j) = i$ is $1/(T+1)$ for every i. If $g(j) = 0$ then $v\left(z_g^j \cup j\right) = v\left(z_g^j\right) = 0$, because in this event the firm is necessarily ordered *after* j. If $g(j) = 1$ then the firm is ordered before j with probability $1/T$ and after j with probability $1-1/T$. In the former case $v\left(z_g^j \cup j\right) = F_{IN}(1, N; \xi)$, while in the latter case $v\left(z_g^j \cup j\right) = 0$. Therefore the conditional expected value of $v\left(z_g^j \cup j\right)$, given $g(j) = 1$, is $\frac{1}{T}F_{IN}(1, N; \xi)$. By similar reasoning, the conditional expected value of $v\left(z_g^j\right)$ is $\frac{1}{T}F_{OUT}(0, N; \xi)$. Repeating the same argument for $g(j) = i$, $i > 1$, the conditional expected value of $v\left(z_g^j \cup j\right)$, given $g(j) = i$, is $\frac{i}{T}F_{IN}(i, N; \xi)$, and the conditional expected value of $v\left(z_g^j\right)$ is $\frac{i}{T}F_{OUT}(i-1, N; \xi)$. It then follows from (18.53) that

$$\begin{aligned} s_j &= \frac{1}{(T+1)T}\sum_{i=1}^{T} i\left[F_{IN}(i, N; \xi) - F_{OUT}(i-1, N; \xi)\right] \\ &= \frac{1}{(N+\xi)N}\sum_{i=1}^{T} i\xi\left[F_{IN}(i, N; \xi) - F_{OUT}(i-1, N; \xi)\right]\xi. \end{aligned}$$

Substituting for the expressions of F_{IN} and F_{OUT}, we obtain

$$\begin{aligned} s_j \;=\; & \frac{A^{1-\beta}N^{\beta(\kappa+1-1/\alpha)}x_c^{\beta\mu}}{(N+\xi)N}\sum_{i=1}^{T} i\xi\left\{i\xi x_n(-j)^{(1-\mu)\alpha} + \xi\left[x_n(j)^{(1-\mu)\alpha} - x_n(-j)^{(1-\mu)\alpha}\right]\right\}^{\beta/\alpha}\xi \\ & - \frac{A^{1-\beta}N^{\beta(\kappa+1-1/\alpha)}x_c^{\beta\mu}}{(N+\xi)N}\sum_{i=1}^{T} i\xi\left[i\xi x_n(-j)^{(1-\mu)\alpha} - \xi x_n(-j)^{(1-\mu)\alpha}\right]^{\beta/\alpha}\xi. \end{aligned}$$

The first-order Taylor expansion implies that

$$s_j = \frac{A^{1-\beta}N^{\beta(\kappa+1-1/\alpha)}x_c^{\beta\mu}(\beta/\alpha)\xi x_n(j)^{(1-\mu)\alpha}}{(N+\xi)N}\sum_{i=1}^{T}(i\xi)\left[i\xi x_n(-j)^{(1-\mu)\alpha}\right]^{(\beta-\alpha)/\alpha}\xi + o(\xi),$$

where $o(\xi)$ represents terms such that $\lim_{\xi\to 0} o(\xi)/\xi = 0$. Rearranging this expression and dividing by $o(\xi)$, we obtain

$$\frac{s_j}{\xi} = \frac{A^{1-\beta}N^{\beta(\kappa+1-1/\alpha)}(\beta/\alpha)\left[\frac{x_n(j)}{x_n(-j)}\right]^{(1-\mu)\alpha} x_c^{\beta\mu}x_n(-j)^{\beta(1-\mu)}}{(N+\xi)N}\sum_{i=1}^{T}(i\xi)^{\beta/\alpha}\xi + \frac{o(\xi)}{\xi}.$$

Now taking the limit as $T \to \infty$, which is also equivalent to the limit $\xi = N/T \to 0$, we obtain $\lim_{\xi \to 0} o(\xi)/\xi = 0$, so that we are left with the Riemann integral:

$$\lim_{T\to\infty}\left(\frac{s_j}{\xi}\right) = \frac{A^{1-\beta}N^{\beta(\kappa+1-1/\alpha)}\left(\beta/\alpha\right)\left[\frac{x_n(j)}{x_n(-j)}\right]^{(1-\mu)\alpha} x_c^{\beta\mu}x_n\left(-j\right)^{\beta(1-\mu)}}{N^2}\int_0^N z^{\beta/\alpha}dz.$$

Solving this integral delivers

$$\lim_{T\to\infty}\left(s_j/\xi\right) = \left(1-\gamma\right)A^{1-\beta}\left[\frac{x_n\left(j\right)}{x_n\left(-j\right)}\right]^{(1-\mu)\alpha} x_c^{\beta\mu}x_n\left(-j\right)^{\beta(1-\mu)}N^{\beta(\kappa+1)-1},$$

with $\gamma \equiv \alpha/\left(\alpha+\beta\right)$. This corresponds to equation (18.43) and completes the proof of the proposition.

18.6. Taking Stock

In this chapter, we have discussed models of technology differences across societies. While the baseline endogenous growth models, such as those studied in Part 4, are useful in understanding the incentives of research firms to create new technologies and can generate different rates of technological change across different economies, two factors suggest that a somewhat different perspective is necessary for understanding technology differences across nations. First, technology and productivity differences do not only exist across nations, but are ubiquitous within countries. Even within narrowly-defined sectors, there are substantial productivity differences across firms and only a small portion of these differences can be attributed to differences in capital intensity of production. This within-country pattern suggests that technology adoption and use decisions of firms are complex and new technologies only diffuse slowly across firms. This pattern gives us some clues about potential sources of productivity and technology differences across nations and suggests that a somewhat slow process of technology diffusion across countries may not be unreasonable. Second, while the United States or Japan can be thought of as creating their own technologies via the process of research and development, most countries in the world are technology importers rather than technological leaders. This is not to deny that some firms in these societies do engage in R&D nor to imply that a number of important technologies, most notably those related to the Green Revolution, have been invented in developing countries. These exceptions notwithstanding, adoption of existing frontier technologies appears more important for most firms in developing countries than the creation of entirely new technologies. This perspective also suggests that a detailed analysis of technology diffusion and technology adoption decisions is necessary for obtaining a good understanding of productivity and technology differences across countries.

A number of important lessons have emerged from our study in this chapter.

(1) We can make considerable progress in understanding technology and productivity differences across nations by positing a slow process of technology transfer across countries. Namely, in light of the within-country evidence, which suggests that even within narrowly-defined sectors in the same country different technologies can survive side-by-side for long periods of time, it seems reasonable to assume that technologically backward economies will only slowly catch up to those at the frontier. Such an approach enables us to have a tractable model of technology differences across countries. An important element of models of technology diffusion is that they create a built-in advantage for countries (or firms) that are relatively behind; since there is a larger gap for them to close, it is relatively easier for them to close it. This catch-up advantage for backward economies ensures that models of slow technology diffusion will lead to differences in income levels, not necessarily in growth rates. In other words, the canonical model of technology diffusion implies that countries that create barriers against technology diffusion or those that are slow in adopting new technologies for other reasons will be poor, but they will eventually converge to the growth rate of the frontier economies. Thus a study of technology diffusion enables us to develop a model of world income distribution, whereby the position of each country in the world income distribution is determined by their ability to absorb new technologies from the world frontier. This machinery is also useful in enabling us to build a framework in which, while each country may act as a neoclassical exogenous growth economy, importing its technology from the world frontier, the entire world behaves as an endogenous growth economy, with its growth rate determined by the investment in R&D decisions of all the firms in the world. This class of models becomes useful when we wish to think of the joint process of world growth and world income distribution across countries. This class of models also emphasizes that much is being lost in terms of insights when we focus our attention on the baseline neoclassical growth model in which each country is treated as an "island onto itself," not interacting with others in the world. Technological interdependences across countries implies that we should often consider the world equilibrium, not simply the equilibrium of each country on its own.

(2) While slow diffusion of existing technologies across countries is reasonable, in the globalized world we live in today, it is becoming increasingly easier for firms to adopt technologies that have already been tried and implemented in other parts of the world. Once we allow a relatively rapid diffusion of technologies, does there remain any reason for technology or productivity differences across countries (beyond differences in physical and human capital)? The second part of the chapter has

argued that the answer to this question is also yes and is related to the "appropriateness" of technologies. A given technology will not have the same impact on the productivity of all economies, because it may be a better match to the conditions or to the factor proportions of some countries than of others. Part of this chapter was devoted to explaining how the issue of appropriate technologies can play a role in different contexts. In our current age of pervasive skill-biased technologies, a particularly important channel of appropriateness is the potential match between technologies developed at the world frontier and the skills of the adopting country's workforce. A potential technology-skill mismatch can lead to large endogenous productivity differences. If the types of technologies developed at the world frontier were random, the possibility of the technology-skill mismatch creating a significant gap between rich and poor nations would be a mere possibility, no more. However, there are reasons to suspect that technology-skill mismatch may be more important, because of the organization of the world technology market. Two features are important here. First, the majority of frontier technologies are developed in a few rich countries. Second, the lack of effective intellectual property rights enforcement implies that technology firms in rich countries target the needs of their own domestic market. This creates a powerful force towards new technologies that are appropriate to ("designed for") the needs of the rich countries, and thus are typically inappropriate to the factor proportions of developing nations. In particular, new technologies will be "too skill-biased" to be effectively used in developing countries. This source of inappropriateness of technologies can create a large endogenous technology and income gap among nations.

(3) Productivity differences do not stem simply from differences in the use of different techniques of production, but also because production is organized differently around the world. A key reason for such differences is institutions and policies in place in different parts of the world. The last part of the chapter showed how contracting institutions, affecting what types of contracts firms can write with their suppliers, can have an important effect on their technology adoption decisions and thus on cross-country differences on productivity. Contracting institutions are only one of many potential organizational differences across countries that might impact upon equilibrium productivity. My purpose in presenting these ideas in this chapter is to emphasize the importance of endogenous productivity differences resulting from differences in the organization of production. We will see more on this when we turn to the relationship between the process of economic growth and the process of economic development in Part 7 of the book.

18.7. References and Literature

The large literature documenting productivity and technology differences across firms and the patterns of technology diffusion were discussed in Section 18.1 and the relevant references can be found there. The simple model of technology diffusion presented in Section 18.2 is inspired by Gerschenkron (1962) essay and Nelson and Phelps's (1966) classic paper, though I am not aware of a paper that presents a simple general equilibrium treatment similar to that in Section 18.2. Ideas similar to those of Nelson and Phelps were also developed independently by Schultz (1967), who went further than Nelson and Phelps in showing how these ideas could be applied in a variety of different settings, especially in the context of technology adoption in agriculture. The Nelson-Phelps approach has been important in a number of recent papers. Benhabib and Spiegel (1994) reinterpret and modify Barro-style growth regressions in light of Nelson-Phelps's view of human capital. Aghion and Howitt (1997) also provide a similar reinterpretation of growth regressions. Caselli (1999), Greenwood and Yorukoglu (1997), Galor and Moav (2001) and Aghion, Howitt and Violente (2001) provide models inspired by the Nelson-Phelps-Schultz view of human capital and applied them to understanding the recent increase in the returns to skills and the United States and other OECD economies. Acemoglu (2002b) contains a critique of these explanations of the rise in wage inequality.

The model in Section 18.3 is inspired by Howitt (2000), but is different in a number of important respects. First, Howitt uses a model of Schumpeterian growth rather than the baseline expanding input variety model used here. This difference is not important, and the choice here was motivated to simplify the exposition. Second, Howitt uses a model without scale affects. Since our interest here is not with scale effects, the added complication necessary to remove scale effects was deemed unnecessary. Finally, there are more widespread technological externalities in Howitt's model. Thus in many ways, the model in Section 18.3 is a much simplified version of Howitt's model, but it involves all the necessary ingredients for a benchmark model of endogenous growth at the world level.

The ideas of appropriate technology discussed in Section 18.4 have a long pedigree. Many development economists in the 1960s realized the importance of the issues of appropriate technology. The classic work here is Stewart (1977), though similar ideas were also discussed in Salter (1966) and David (1974). A classic treatment was provided Atkinson and Stiglitz (1969), who suggested a simple and powerful formalization of how technological change can be localized and thus not transfer from one productive units to another (or from one country to another). Atkinson and Stiglitz's idea is incorporated into a growth model by Basu and Weil (1998), which was the basis of one of the models in Section 18.4. The last part of this section draws on Acemoglu and Zilibotti (2001), who develop a model of appropriate technologies due to skill differences across countries and combine it with directed technical change to show how there will be a bias towards technologies inappropriate to the needs of poorer nations. Acemoglu and Zilibotti also provide evidence that these effects could be quantitatively large and patterns of sectoral differences are consistent with the importance of this type of technology-skill mismatch. Acemoglu (2002b) shows that technology-skill mismatch applies in a more general model of directed technical change than the one in Acemoglu and Zilibotti (2001) discussed here (see Exercise 18.26).

Finally, the model presented in Section 18.5 draws upon Acemoglu, Antras and Helpman (2007). A number of other models also generate endogenous productivity or technology differences across countries as a result of differences in the organization of production. Some of these will be discussed in Chapter 21.

18.8. Exercises

EXERCISE 18.1. Derive equation (18.1).

EXERCISE 18.2. Show that if the restriction that $\lambda_j \in [0, g)$ in Section 18.2 is relaxed, the requirement that $A_j(t) \leq A(t)$ can be violated.

EXERCISE 18.3. Derive equation (18.4).

EXERCISE 18.4. Complete the proof of Proposition 18.1.

EXERCISE 18.5. Derive the effect of an increase in λ_j on the law of motion of $a_j(t)$ and $k_j(t)$. How does this differ from the effect of an increase in σ_j? Explain why these two parameters have different effects on technology and capital stock dynamics.

EXERCISE 18.6. In the model of Section 18.2, show that if $g = 0$, then all countries converge to the same level of technology. Explain carefully why $g > 0$ leads to steady-state technology differences, while these differences disappear when $g = 0$.

EXERCISE 18.7.
(1) Set up the world equilibrium problem in subsection 18.2.2 as one in which the Second Welfare Theorem holds within each country. Under this assumption, carefully define an equilibrium path. Explain the significance of this assumption.
(2) Now set up the world equilibrium problem without appealing to the Second Welfare Theorem. Explain why the mathematical problem is identical to that in part 1 of this exercise.
(3) Prove Proposition 18.2.

EXERCISE 18.8.
(1) Why is the condition $\rho - n_j > (1 - \theta) g$ necessary in Proposition 18.3?
(2) Complete the proof of Proposition 18.3.

Exercise 18.9. In the model of Section 18.2 with consumer optimization, suppose that preferences in country j are given by

$$U_j = \int_0^\infty \exp\left(-\left(\rho_j - n_j\right)t\right)\left[\left(c_j\left(t\right)^{1-\theta} - 1\right)/\left(1-\theta\right)\right]dt,$$

where ρ_j differs across countries.

(1) Show that a unique steady-state world equilibrium still exists and all countries grow at the rate g.
(2) Provide an intuition for why countries grow at the same rate despite different rates of discounting.
(3) Show that this steady-state equilibrium is globally saddle-path stable.

Exercise 18.10. * Consider the model of Section 18.2 with F corresponding to the production function of an individual firm j (with a slight abuse of notation) and (18.3) corresponding to the law of motion of the technology of the firm, with $\sigma_j = \sigma\left(h_j\right)$, where h_j is the average human capital of the workers of firm j and σ is a strictly increasing and differentiable function. To simplify the discussion, suppose that each firm employs a single worker (which is without loss of any generality given constant returns to scale).

(1) Derive the wage of the worker of human capital h_j. [Hint: this consists of the workers value of marginal product in production plus the increase in the productivity of the firm because of the improvement in the firm's technology due to the higher human capital of the worker].
(2) Show that an increase in g (at any point t) will increase worker wages. Derive the implications of changes in g on the returns to human capital. Contrast an increase in the returns to human capital driven by an increase in g with those discussed in Chapter 15.

Exercise 18.11. Complete the proof of Proposition 18.4.

Exercise 18.12. Consider the model in subsection 18.3.1 and suppose that all countries have the same labor force size $L_j = 1$ and the same $\eta_j = \eta$, and only differ in terms of their ζ_j's. Imagine that the range of ζ_j's is the same as used in the quantitative evaluation of the neoclassical growth model in Chapter 8.

(1) Evaluate the impact of these differences in ζ_j's on cross-country technology and income differences for different values of ϕ.
(2) What value of ϕ is necessary so that a fourfold difference in ζ_j's translates into a thirtyfold difference in income per capita?
(3) How would you interpret the economic significance of such a value of ϕ? Would this be a satisfactory model of cross-country technology and income differences? If yes, explain why it is more attractive than the neoclassical model and other alternatives

we have seen so far. If not, suggest what important features are missing and how they might be introduced.

Exercise 18.13. * Consider the model in subsection 18.3.1. Suppose that preferences are given by $U_j = \int_0^\infty \exp\left(-\rho_j t\right)\left[\left(c_j\left(t\right)^{1-\theta} - 1\right)/\left(1-\theta\right)\right]dt$, where ρ_j differs across countries. Show that an equivalent of Proposition 18.4, with a unique globally saddle-path stable world equilibrium where all countries grow at the same rate, applies.

Exercise 18.14. Show that (18.14) is necessary and sufficient for a positive world growth rate in the model of subsection 18.3.2. Write down the conditions that characterize the world equilibrium when this condition is not satisfied.

Exercise 18.15. Prove Proposition 18.5.

Exercise 18.16. * Analyze the local dynamics of the steady-state world equilibrium in Proposition 18.5.

Exercise 18.17. * Consider Proposition 18.5 with the discount rates, the ρ_j's differing across countries. Prove that a unique steady-state world equilibrium, with all countries growing at the same rate, still exists.

Exercise 18.18. In the model of subsection 18.3.2, replace equation (18.12) with

$$N\left(t\right) = G\left(N_1\left(t\right), ..., N_J\left(t\right)\right),$$

where G is homogeneous of degree 1.

(1) Generalize the results in Proposition 18.5 to this case and derive an equation that determines the world growth rate implicitly.
(2) Derive an explicit equation for the world growth rate for the specific case in which $N\left(t\right) = \max_j N_j\left(t\right)$.

Exercise 18.19. In the model of subsection 18.3.2, there is a strong scale effect.

(1) Show that if population grows at some constant rate $n_j > 0$ in each country, there will not exist a steady-state equilibrium.
(2) Construct a variation of this model along the lines of the semi-endogenous growth models of Section 13.3 in Chapter 13, where this strong scale effect is removed. [Hint: modify equation (18.9), so that $\dot{N}_j\left(t\right) = \eta_j N\left(t\right)^{\phi} N_j\left(t\right)^{-\tilde{\phi}} Z_j\left(t\right)$, where $\tilde{\phi} > \phi$].
(3) Provide a full characterization of the steady-state world equilibrium in this case.

Exercise 18.20. Consider the model in subsection 18.4.2. Suppose that the world consists of two countries with constant and equal populations, and constant savings rates $s_1 > s_2$. Suppose that the production function in each country is given by (18.15) with k' corresponding to the highest capital-labor ratio in any country experienced until then. There is no technological progress and both countries start with the same capital-labor ratio.

(1) Characterize the steady-state world equilibrium (that is, the steady-state capital-labor ratios in both countries).
(2) Characterize the output per capita dynamics in the two economies. How does an increase in γ affect these dynamics?
(3) Show that the implied income per capita differences (in steady state) between the two countries are increasing in γ. Interpret this result.
(4) Do you think this model provides a good/plausible mechanism for generating large income differences across countries? Substantiate your answer with theoretical or empirical arguments.

Exercise 18.21. Complete the proof of Proposition 18.6. In particular, explicitly derive the expression for the threshold and the skill premium.

Exercise 18.22. Derive the equilibrium expressions (18.20)-(18.23).

Exercise 18.23. Prove Proposition 18.7. [Hint: in steady state the profits from owning a skill-complementary and unskilled labor-complementary machine must be equal].

Exercise 18.24. Prove Proposition 18.8.

Exercise 18.25. Consider the model of appropriate technology in subsection 18.4.3.

(1) Suppose that now research firms can sell their machines to all producers in the world, including those in the South and can charge the same markup. Derive the steady-state equilibrium under these conditions.
(2) Comparing your answer in part 1 to the analysis in the text, derive the implications of intellectual property rights enforcement in the South on equilibrium technologies? What are the implications on income per capita differences between the North and the South?
(3) In view of your answer to 1 and 2 above, could it be the case that Southern economies prefer lack of intellectual property rights enforcement to full intellectual property rights enforcement? [Hint: distinguish between a world in which there is a single Southern country versus one in which there are many].

Exercise 18.26. * Instead of the multi-sector model in subsection 18.4.3, suppose that output is given by an aggregate production function of the form
$Y(t) = \left[\gamma Y_L(t)^{\frac{\varepsilon-1}{\varepsilon}} + (1-\gamma) Y_H(t)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}$ as in Chapter 15, with Y_L and Y_H being produced exactly is in that chapter. Assume as in the model of subsection 18.4.3 that new technologies are developed in the North for the Northern market only.

(1) Characterize the steady-state (balanced growth path) equilibrium of this economy. [Hint: use exactly the same analysis as in Chapter 15 and subsection 18.4.3].
(2) Show that if $\sigma \equiv \varepsilon - (\varepsilon-1)(1-\beta)$ is equal to 2, the results are identical to those in subsection 18.4.3.

(3) Derive the equivalents of Proposition 18.8.

(4) Do the implications of inappropriate technologies become more or less important when σ increases?

Exercise 18.27. Prove Proposition 18.9.

Exercise 18.28. Prove Proposition 18.11.

Exercise 18.29. Prove Proposition 18.12.

Exercise 18.30. * Consider the model of Section 18.5. Suppose that there is a total population of L. Assume that each individual can work as a supplier for one of the M products, or he can work in the process of technology adoption. For this reason, suppose that the cost of technology adoption is given by $\Gamma(N) \equiv w\Gamma_0(N)$, where w is the wage rate, corresponding to the outside option of each supplier.

(1) Characterize the general equilibrium of the economy by endogenizing A for a given number of products M. In particular, show that in equilibrium the following market clearing condition must be satisfied: $M\Gamma_0(N^*) = L$, where N^* is the equilibrium technology choice (number of suppliers).

(2) What is the effect of an increase in μ on N^*? Explain the result.

(3) Now suppose that the M products differ according to their elasticity of substitution, in particular, each product has a different α, with the distribution of α's across products given by a distribution function $G(\alpha)$ with support within the interval $[0,1]$. Let $N^*(\alpha)$ be the equilibrium technology choice (number of suppliers) for a product with parameter α. Show that the market clearing condition now takes the form: $M\int_0^1 \Gamma_0(N^*(\alpha))\,dG(\alpha) = L$.

(4) What is the effect of an increase in μ on the equilibrium in this case?

(5) How would you endogenize Q in this model? What types of insights would this generate?

Exercise 18.31. Consider the model of Section 18.5. What types of organizational forms might emerge when contracting institutions are imperfect (i.e., μ is very low)? In particular, discuss how vertical integration and repeated interactions between suppliers and producers might change the results discussed in that section. How would you model each of these?

CHAPTER 19

Trade and Growth

The previous chapter discussed how technological linkages across countries and technology adoption decisions lead to a pattern of interdependent growth across countries. In this chapter, we study world equilibria when countries can trade financial assets or commodities. We start with growth in economies that can borrow and lend internationally, and discuss how this affects cross-country income differences and growth dynamics. We then turn to the growth implications of international trade in commodities.

Our first task is to construct models of world equilibria, which feature both international trade in commodities (or intermediate goods) and economic growth. The exact interactions between trade and growth depend on the nature of trade that countries engage in. I will try to provide an overview of these different interactions. I start with a model in which trade is of the Heckscher-Ohlin type, that is, it originates only because of differences in factor abundance across countries, and growth is driven by capital accumulation. Then I will turn to a model of Ricardian type, where trade is driven by technological comparative advantage. The main difference between these two approaches concerns whether the prices of the goods that a country supplies to the world are affected by its own production and accumulation decisions. These models will shed new light on the patterns of interdependences across countries, for example, showing that growth in one country cannot be analyzed in isolation from the growth experiences of other nations in the world.

Our second task is to turn to a central question of the literature on trade and growth: whether international trade encourages economic growth. We will see that the answer to this question also depends on exactly how trade is modeled, as well as on what the source of economic growth is (in particular learning by doing versus innovation). Throughout, the emphasis will be on the importance of considering the world equilibrium rather than the equilibrium of a closed economy in isolation.

19.1. Growth and Financial Capital Flows

In a globalized economy, if the rates of return to capital differ across countries, we would expect capital to flow towards areas where its rate of return is higher. This simple observation has a number of important implications for growth theory. First, it implies a very different pattern of economic growth in a financially integrated world. Our first task in this section is to illustrate the implications of international capital flows for economic growth and show how they significantly change transitional dynamics in the basic neoclassical growth model. Our second task is to highlight what new lessons can be derived from the analysis of economic growth in the presence of international capital flows. In particular, we will see that the presence of international capital flows raises a number of puzzles, most notably, the one emphasized by Lucas (1990): “Why Does Capital Not Flow from Rich to Poor Countries?”. We will see in the next section that this simple question helps us think about a range of important issues in economic growth and economic development. While a model of free flow of capital around the world is a good starting point, the existing evidence is not entirely consistent with such free flows. In particular, free flows of capital lead to a pattern of growth that appears counterfactual. Moreover, a large literature in international finance, starting with Feldstein and Horioka (1980), points out that there is much less *net* flows of capital from countries with high saving rates towards those with lower saving rates than a theory of frictionless international capital markets would suggest. In the next section we will briefly discuss why capital flows across countries may be hampered and what the implications of this are for cross-country growth dynamics.

19.1.1. A World Equilibrium with Free Financial Flows. Consider a world economy consisting of of J countries, indexed $j = 1, ..., J$, each with access to an aggregate production function for producing a unique final good:

$$Y_j(t) = F(K_j(t), A_j(t) L_j(t)),$$

where $Y_j(t)$ is the output of this unique final good in country j at time t, and $K_j(t)$ and $L_j(t)$ are the capital stock and labor supply, $A_j(t)$ is again the country-specific Harrod-neutral technology term. As in the previous chapter, we assume that each country is “small” and ignores its effects on world aggregates. Throughout the section we assume that technological change occurs at a constant rate across countries, though there may be level differences in technology, that is,

$$A_j(t) = A_j \exp(gt),$$

where g is the common growth rate of technology in the world.

We assume that each country admits a representative household with the standard preferences at time $t = 0$ given by

$$U_j = \int_0^\infty \exp(-(\rho - n)t) \left[\frac{\tilde{c}_j(t)^{1-\theta} - 1}{1-\theta}\right] dt, \tag{19.1}$$

where $\tilde{c}_j(t)$ is per capita consumption in country j at time t and we have imposed that all countries have the same time discount rate, ρ, and the same population growth rate n. Moreover, we assume that all countries start with the same population at time $t = 0$, which, without loss of any generality, is normalized to 1, i.e., $L_j(0) = 1$ for all $j = 1, ..., J$, so that

$$L_j(t) = L(t) = \exp(nt),$$

for all j. In addition, we assume that Assumption 4 from Chapter 8 is satisfied, i.e., $\rho - n > (1-\theta)g$.

The key feature of this economy is the presence of international borrowing and lending. Consistent with the permanent income hypothesis for individual consumption decisions, borrowing and lending will allow a smoother consumption profile for households (in particular for the representative household) in each country. But since the desire for a smoother consumption profile was one of the main reasons why the capital stock did not adjust immediately to its steady-state (or balanced growth path) value, the opportunities for international financial transactions will influence the dynamics of capital accumulation and growth.

More specifically, let $B_j(t) \in \mathbb{R}$ denote the net borrowing of country j from the world at time t. Let $r(t)$ denote the world interest rate. Free capital flows imply that this interest rate is independent of which country is borrowing and whether a country is borrowing or lending to others. Moreover, consistent with our assumption that each country is small relative to the world, all countries are price takers in the international financial markets, so they can borrow or lend as much as they like at this interest rate. Consequently, the flow resource constraint facing the representative household in each country will be somewhat different from that in subsection 18.2.2 and can be written as

$$\dot{k}_j(t) = f(k_j(t)) - c_j(t) + b_j(t) - (n+g+\delta)k_j(t), \tag{19.2}$$

where, as usual, $k_j(t) \equiv K_j(t)/A_j(t)L_j(t)$ is the effective capital-labor ratio in country j at time t, $c_j(t) \equiv C_j(t)/A_j(t)L_j(t)$ is the ratio of consumption to effective labor, and

$$y_j(t) \equiv \frac{Y_j(t)}{L_j(t)} \equiv A_j(t)f(k_j(t))$$

is income per capita, while

$$b_j(t) \equiv \frac{B_j(t)}{A_j(t)L_j(t)}$$

denotes the net borrowing normalized by effective labor. The most important feature of equation (19.2) is that, in contrast to all other resource equations we have encountered so far, it does not require domestic consumption and investment to be equal to domestic production. Instead, there are potential transfers of resources from the rest of the world, $B_j(t)$, which can be used for consumption or investment. Conversely, the country may be transferring resources to the rest of the world, so that it consumes and invests less than its production. Naturally, once we allow for international borrowing and lending, we must ensure that each country, thus each representative household, satisfies an international budget constraint. For this purpose, let $\mathcal{A}_j(t)$ denote the international asset position of country j at time t. If $\mathcal{A}_j(t)$

is positive, the country is a net lender and has positive claims on output produced in other countries, while if it is negative, the country is a net borrower. The flow international budget constraint for country j at time t can then be written as:

$$\dot{\mathcal{A}}_j(t) = r(t)\mathcal{A}_j(t) - B_j(t), \tag{19.3}$$

which simply states that the country earns the world interest rate, $r(t)$, on its existing asset position $\mathcal{A}(t)$ (or accumulates further debt if the latter is negative) and in addition receives transfers $B(t)$ from the rest of the world (or makes transfers to the rest of the world when $B(t)$ is negative). If transfers from the rest of the world exceed the interest earned on current assets, the asset position of the country deteriorates, that is, $\dot{\mathcal{A}}_j(t) < 0$. The no-Ponzi game condition we encountered in Chapter 8 now applies to the international asset position of a country, and requires

$$\lim_{t\to\infty} \mathcal{A}_j(t)\exp\left(-\int_0^t r(s)\,ds\right) = 0$$

for each $j = 1, ..., J$. In writing this equation, we have incorporated that each country faces the world interest rate, $r(t)$, at all points in time. The intuition for this expression is the same as the no-Ponzi game condition, (8.11) in Chapter 8.

As with the other variables, it is convenient to express the net asset position of the country in terms of effective labor units, so let us define

$$\mathsf{a}_j(t) \equiv \frac{\mathcal{A}_j(t)}{A_j(t)L_j(t)},$$

which implies that (19.3) can be rewritten as

$$\dot{\mathsf{a}}_j(t) = (r(t) - g - n)\,\mathsf{a}_j(t) - b_j(t) \tag{19.4}$$

and the no-Ponzi game condition becomes

$$\lim_{t\to\infty} \mathsf{a}_j(t)\exp\left(-\int_0^t (r(s) - g - n_j)\,ds\right) = 0. \tag{19.5}$$

Naturally, the amount of borrowing and lending in the world has to balance out. This implies the world capital market clearing condition

$$\sum_{j=1}^{J} B_j(t) = 0$$

must hold at all times t. Now dividing and multiplying each term by $A_j(t)L_j(t)$, and recalling that $A_j(t) = A_j\exp(gt)$ and $L_j(t) = L(t)$ for all j, the world capital market clearing condition can be written as:

$$\sum_{j=1}^{J} A_j b_j(t) = 0 \tag{19.6}$$

for all $t \geq 0$.

With access to international capital markets, the problem of the representative household in each country can be written as maximizing (19.1) subject to (19.2), (19.4) and (19.5).

A world equilibrium is now defined as a sequence of normalized consumption levels, capital stocks and asset positions for each country, that is, $\left\{[k_j(t), c_j(t), \mathsf{a}_j(t)]_{t\geq 0}\right\}_{j=1}^{J}$ and a time path of world interest rates, $[r(t)]_{t\geq 0}$, such that each country's allocation maximizes the utility of the representative household in each country, and the world financial market clears, that is, (19.6) is satisfied. A steady-state world equilibrium is defined as a world equilibrium in which $k_j(t)$ and $c_j(t)$ are constant and output in each country grows at a constant rate. As in previous chapters, we could alternatively refer to this allocation as a balanced growth path rather than a steady-state equilibrium.

While the equilibrium of this world economy with free financial flows is quite straightforward to characterize, it is useful to present a number of simple intermediate results to emphasize a number of important economic ideas. The first one is the following proposition:

Proposition 19.1. *In the world equilibrium of the economy with free flows of capital, we have that*

$$k_j(t) = k(t) = f'^{-1}(r(t) + \delta) \text{ for all } j = 1, ..., J,$$

where $f'^{-1}(\cdot)$ is the inverse function of $f'(\cdot)$ and $r(t)$ is the world interest rate.

Proof. See Exercise 19.1. □

This result is very intuitive. With free flows of capital, each firm in each country will stop renting capital only when its marginal product is equal to the opportunity cost, which is given by the world rental rate (the world interest rate plus the depreciation rate). Consequently, effective capital-labor ratios are equalized across countries. Note, however, that this does *not* imply equalization of capital-labor ratios. To the extent that two countries j and j' have different levels of productivity, $A_j(t)$ and $A_{j'}(t) \neq A_j(t)$, their capital-labor ratios are not, and *should not*, be equalized. This is an important point to which we will return below.

The next proposition focuses on the steady-state world equilibrium.

Proposition 19.2. *Suppose that Assumption 4 is satisfied. Then in the world economy with free flows of capital, there exists a unique steady-state world equilibrium in which output, capital and consumption per capita in all countries grow at the rate g and effective capital-labor ratios are given by*

$$k_j^* = k^* = f'^{-1}(\rho + \delta + \theta g) \text{ for all } j = 1, ..., J.$$

Moreover, in the steady-state equilibrium, we have that

$$\lim_{t\to\infty} \dot{\mathcal{A}}_j(t) = 0 \text{ for all } j = 1, ..., J.$$

Proof. See Exercise 19.2. □

At some level this result is very intuitive: with free capital flows, we have an *integrated* world economy. This integrated world economy has a unique steady-state equilibrium similar to that in the standard neoclassical growth model. This steady-state equilibrium not only determines the effective capital-labor ratio and its growth rate, but also the distribution of the available capital across different countries in the world economy. Even though this proposition is intuitive, its proof requires some care, to ensure that no country runs a Ponzi scheme and that this implies the *normalized* asset position of each country (and each household within each country), i.e., $\mathsf{a}_j(t)$ for each j, must asymptote to a constant. This last feature is no longer the case when the model is extended so that countries differ according to their discount rates (see Exercise 19.2).

Let us next consider the transitional dynamics of the world economy. The analysis of transitional dynamics is simplified by the fact that the world behaves as an integrated economy rather than an independent collection of economies (see Exercise 19.2). Consequently, the following result is straightforward:

Proposition 19.3. *In the world equilibrium of the economy with free flows of capital, there exists a unique equilibrium path* $\left\{[k_j(t), c_j(t), \mathsf{a}_j(t)]_{t\geq 0}\right\}_{j=1}^{J}$ *that converges to the steady-state world equilibrium. Along this equilibrium path* $k_j(t)/k_{j'}(t) = 1$ *and* $c_j(t)/c_{j'}(t) =$ *constant for any two countries* j *and* j'.

Proof. See Exercise 19.3. □

Intuitively, the integrated world economy acts as if it has a single neoclassical aggregate production function, thus the characterization of the dynamic equilibrium path and of transitional dynamics from Chapter 8 applies. In addition, Proposition 19.1 implies that $k_j(t)/k_{j'}(t)$ is constant and the standard Euler equations imply that $c_j(t)/c_{j'}(t)$ is constant. Therefore, both production and consumption in each economy grow in tandem.

The following is an important corollary to Proposition 19.3:

Corollary 19.1. *Consider the world economy with free flows of capital. Suppose that at time* t*, a fraction* λ *of the capital stock of country* j *is destroyed. Then capital flows immediately to this country (i.e.,* $\dot{\mathsf{a}}_j(t) \to -\infty$*) to ensure that* $k_j(t')/k_{j'}(t') = 1$ *for all* $t' \geq t$ *and for all* $j' \neq j$.

Proof. This is a direct implication of Propositions 19.1 and 19.3. The latter implies that there exists a unique globally stable equilibrium, while the former implies that at any point in the equilibrium we must have $k_j(t)/k_{j'}(t) = 1$. This is only possible by an immediate inflow of capital into country j. □

This result implies that in the world economy with free flows of capital, there are only transitional dynamics for the aggregate world economy, but no transitional dynamics separately for each country (in particular, $k_j(t)/k_{j'}(t) = 1$ for all t and any j and j'). This is intuitive, since international capital flows will ensure that each country has the same effective capital-labor ratio, thus dynamics resulting from slow capital accumulation are removed. This corollary therefore implies that any theory emphasizing the role of transitional dynamics in explaining the evolution of cross-country income differences must implicitly limit the extent or the speed of international capital flows. The evidence on this point is mixed. While the amount of gross capital flows in the world economy is large, the Feldstein-Horioka puzzle still remains a puzzle—as we will see below, countries that save more also tend to invest more rather than lending this money internationally. One reason for this might be the potential risk of sovereign default by countries that borrow significant amounts from the world financial markets. Exercise 19.4 investigates this issue.

Although the implications of this corollary for cross-country patterns of divergence can be debated, its implications for cross-regional convergence are clear; cross-regional patterns of convergence cannot be related to slow capital accumulation as in the baseline neoclassical growth model. Exercise 19.5 asks you to apply this corollary to investigations of income convergence across U.S. regions and states.

19.2. Why Doesn't Capital Flow from Rich to Poor Countries?

The model studied in the previous section provides us with a framework to answer the question posed above and in the title of this section. In the basic Solow and neoclassical growth models, a key source of cross-country income differences is capital-labor ratios. For example, if we consider a world economy in which all countries have access to the same technology and there are no human capital differences, the *only* reason why one country would be richer than another is differences in capital-labor ratios. But if two countries with the same production possibilities set differ in terms of their capital-labor ratios, then the rate of return to capital will be lower in the richer economy and there will be incentives for capital to flow from rich to poor countries. We now discuss reasons why capital may not flow from societies with higher capital-labor ratios to those with greater capital scarcity.

19.2.1. Capital Flows under Perfect International Capital Markets. One potential answer to the question posed above is provided by the analysis in the previous section. With perfect international capital markets, capital flows will equalize effective capital-labor ratios. But this does not imply equalization of capital-labor ratios. This result, which follows directly from the analysis in the previous section, is stated in the next proposition. Note that this result does *not* give a complete answer to our question, since it takes productivity differences across countries as given. Nevertheless, it explains how, given these productivity differences, there is no compelling reason to expect capital to flow from rich to poor countries.

Proposition 19.4. *Consider a world economy with identical neoclassical preferences across countries and free flows of capital. Suppose that countries differ according to their productivities, the A_j's. Then there exists a unique steady state equilibrium in which capital-labor ratios differ across countries (in particular, effective capital-labor ratios, the k_j's, are equalized), and there are no capital flows across countries.*

Proof. See Exercise 19.7. □

This proposition states that there is no reason to expect capital flows when countries differ according to their productivities. The more productive countries will have higher capital-labor ratios. To the extent that two countries j and j' have different levels of productivity, $A_j(t)$ and $A_{j'}(t) > A_j(t)$, their capital-labor ratios should not be equalized, instead, country j' should have a higher capital-labor ratio than j. Consequently, capital need not flow from rich to poor countries, because rich countries are more "productive". This is in fact similar to the explanation suggested in Lucas (1990), except that Lucas also linked differences in A_j's to differences in human capital and in particular to human capital externalities. Instead, Proposition 19.4 emphasizes that any sources of differences in A_j's will generate this pattern.

The reader would be right to object at this point that this is only a "proximate" answer to the question, since it provides no reason for why productivity differs across countries. This objection is largely correct. Nevertheless, this proposition is still useful, since it suggests a range of explanations for the lack of capital flows from rich to poor countries that do not depend on the details of the world financial system, but instead focus on productivity differences across countries. We have already made some progress in understanding the potential sources of productivity differences across countries, and as we make more progress, we will start having better answers to the question of why capital does not flow from rich to poor countries (in fact, why it might sometimes flow from poor to rich countries).

19.2.2. Capital Flows under Imperfect International Financial Markets. It is also useful to note that there are other reasons, besides Proposition 19.4, why capital may not flow from poor to rich countries. In particular, it may be the case that the rate of return to capital is higher in poor countries, but financial market frictions or issues of sovereign risk may prevent such flows. For example, lenders might worry that a country that has a negative asset position might declare international bankruptcy and not repay its debts. Alternatively, domestic financial problems in developing countries (which will be discussed in Chapter 21) may prevent or slow down the flows of capital from rich to poor countries. For whatever reason, if the international financial markets are not perfect and capital cannot flow freely from rich to poor countries, we may expect large differences in the return to capital across countries.

Existing evidence on this topic is mixed. Three different types of evidence are relevant. First, a number of studies, including Trefler's (1993) important work discussed in Chapter 3 and recent work by Caselli and Feyrer (2007), suggest that differences in the return to capital across countries are relatively limited. These estimates are directly relevant to the question of whether there are significant differences in the returns to capital across countries, but they are computed under a variety of assumptions (in Trefler's case, they rely on data on factor contents of trade and make a variety of assumptions on the impact of trade on factor prices; Caselli and Feyrer, on the other hand, require comparable and accurate measures of quality-adjusted differences in capital stocks across countries).

Second and somewhat in contrast to the aggregate results, a number of papers exploiting microdata, for example, summarized in Banerjee and Duflo (2005), suggest that the rate of return for additional investment in some firms in less-developed countries could be as high as 100%. Nonetheless, this evidence, even if taken at face value, does not suggest that there will be strong incentives for capital to flow from rich to poor countries, since it may be generated by within-country credit market imperfections. In particular, it may be that the rate of return is very high for a range of credit-rationed firms, but various incentive problems make it impossible for domestic or foreign financial institutions to lend to these firms on profitable terms. If these developing economies were to receive an infusion of additional foreign capital, the rate of return would not be given by the rate of return to credit-rationed firms, but by the rate of return to unconstrained firms, which is presumably much lower. Consequently, the incentives for capital to flow from rich to poor countries may be quite weak as suggested by Proposition 19.4.

Finally, directly related to the issue of the flow of capital across countries is the evidence related to the Feldstein-Horioka puzzle. In an influential paper, Martin Feldstein and Charles Horioka (1980) pointed out a striking fact: differences in savings and investment rates across countries are highly correlated. In particular, Feldstein and Horioka used various different samples to run a regression of the form:

$$\Delta\left(\frac{I_j(t)}{Y_j(t)}\right) = \alpha_0 + \alpha_1 \Delta\left(\frac{S_j(t)}{Y_j(t)}\right),$$

where $\Delta(I_j(t)/Y_j(t))$ is the change in the investment to GDP ratio of country j between some prior date and date t, and $\Delta(S_j(t)/Y_j(t))$ is the change in the savings to GDP ratio. Imagine that savings to GDP ratio varies across countries and over time because of "shocks" to the saving rate or other reasons. In a world with free capital flows, we would expect these changes in savings to have no effect on investment, thus we should estimate a coefficient of $\alpha_1 \approx 0$. In contrast, Feldstein and Horioka estimated a coefficient close to 1 (around 0.9) for OECD economies. Similar results have been found for other samples of countries, though other studies, most notably Taylor (1994), argue that including additional controls removes the puzzle. Feldstein and Horioka and much of the literature that has followed them has interpreted the positive correlation between investment and savings as evidence against free capital flows. Naturally, in practice there are a number of econometric issues one needs to worry about before one can reach a precise conclusion. For example, Exercise 19.6 shows how correlation between investment and savings can arise without imperfections in international financial markets, when the major difference across countries is in investment opportunities. Nevertheless, the Feldstein-Horioka puzzle suggests that issues of sovereign risk might be important in practice and may create barriers to the free flow of capital across countries. Models incorporating endogenous sources of sovereign risk together with the process of economic growth could be an interesting area for future research.

19.3. Economic Growth in a Heckscher-Ohlin World

We have so far focused on the growth implications of trade in financial assets, which enables countries to change the time profile of their consumption. Perhaps more important is international trade in commodities, which allows countries to exploit their comparative advantages (resulting from technology or differences in factor proportions). We now turn to a simple model of growth in a world consisting of countries that trade in commodities. This model builds on work by Ventura (1997), who constructed a tractable model of world equilibrium based on the Heckscher-Ohlin model of trade.

The Heckscher-Ohlin model is the benchmark model of international trade. It posits that countries have access to the same (or similar) technologies, and the main source of trade is differences in factor proportions—that some countries have more capital relative to labor than others or more human capital than others, etc. Clearly, an analysis of such an economy necessitates the specification of models in which there are multiple commodities used either in consumption or used as intermediates in the production of a final good. For the sake of concreteness, we will pursue the second alternative as in the models in Chapter 15, though this is without any loss of generality.

In particular, we assume that each country has access to an aggregate production function of the following form:

$$Y_j(t) = F\left(X_j^K(t), X_j^L(t)\right), \tag{19.7}$$

where $Y_j(t)$ is final output in country j at time t, F denotes a constant returns to scale production function, with the usual characteristics (in particular, satisfying Assumptions 1 and 2), except that it is defined over two intermediate inputs rather than labor and capital. Notice that Assumption 2 also incorporates the Inada conditions, which will play an important role in the analysis below. These intermediate inputs, $X_j^L(t)$ and $X_j^K(t)$ are respectively labor and capital intensive. We use the letter X to denote these inputs, since they refer to the amounts of these inputs *used in production* rather than the amount of inputs *produced* in country j. In the presence of international trade these two quantities will typically differ. We assume throughout that the production of the final good is competitive.

The theory of international trade is a well-developed and rich area of economics, and provides useful and fairly general theorems about the structure of production and trade. Here our purpose is not to review these results, but to illustrate the implications of Heckscher-Ohlin type international trade for economic growth, thus we assume the simplest possible setting in which the two intermediate inputs are each produced by one factor. In particular, we assume

$$Y_j^L(t) = A_j L_j(t) \tag{19.8}$$

and

$$Y_j^K(t) = K_j(t), \tag{19.9}$$

where the use of Y instead of X here emphasizes that these quantities refer to the local production, not the use, of these intermediates. Also, as usual, $L_j(t)$ is total labor input in country j at time t, supplied inelastically, and $K_j(t)$ is the total capital stock of the country. One feature about these intermediate production functions is worth noting: we have allowed productivity differences across countries in the production of the labor-intensive good, but not in the production of the capital-intensive good. This is the same assumption as the one adopted in Ventura (1997). Exercise 19.9 shows the implications of allowing differences in the productivity of the capital-intensive sector as well. For now, it suffices to note that this assumption makes it possible to derive a well-behaved world equilibrium, and is in the spirit of allowing only labor-augmenting technological progress in the basic neoclassical model. Moreover, this assumption is not entirely unreasonable, since we may think of differences in A_j's as reflecting differences in the human capital embodied in labor. Notice also that we have not introduced any technological progress. This is again to simplify the exposition, and Exercise 19.10 extends the model in this section to incorporate labor-augmenting technological progress.

Throughout the rest of this chapter, we assume that there is free international trade in commodities—in intermediate goods. This is an extreme assumption, since trading internationally involves costs, and many analyses of international trade incorporate the physical costs of transportation and tariffs. The main insights for economic growth do not depend on whether or not there are such costs, so I will simplify the analysis by assuming costless international trade. The most important implication of this assumption is that the prices of traded commodities, here the intermediate goods, are the same in all countries and are equal to their "world prices". Then the world supply and demand for these commodities will determine these prices. In particular, we denote the world price of the labor-intensive intermediate at time t by $p^L(t)$ and the price of the capital-intensive intermediate by $p^K(t)$. Both of these prices are in terms of the final good in the world market, which is taken as numeraire, with price normalized to 1.[1]

Given the production technologies in (19.8) and (19.9), this immediately implies that factor prices, the wage rate and the rental rate of capital in country j at time t are given by

$$\begin{aligned} w_j(t) &= A_j p^L(t) \\ R_j(t) &= p^K(t). \end{aligned}$$

These two equations summarize the most important economic insights of the model studied here. Factor prices shape the incentives to accumulate capital in the neoclassical growth model and are typically determined by the capital-labor ratio (recall Chapter 8). The specific structure we have here, in contrast, implies that these factor prices are determined by world prices. In particular, since capital is used only in the production of the capital-intensive intermediate and there is free trade in intermediates, the rental rate of capital in each country is given by the world price of the capital-intensive intermediate. A similar reasoning applies to the wage rate, with the only difference that, because of cross-country differences in the productivity of labor, wage rates are not equalized; instead it is the effective wage rates, i.e., $w_j(t)/A_j$'s, that are equalized. We follow Trefler (1993) in referring to this pattern as *conditional factor price equalization* across countries, meaning that, once we take into account intrinsic productivity differences of factors, there is equalization of factor prices across countries. Conditional factor price equalization is weaker than the celebrated factor price equalization of international trade theory, which would require $w_j(t)$'s to be equalized across countries. Instead we have that $w_j(t)/A_j$'s are equalized.

In this model, equalization of factor prices (or conditional factor prices) is an immediate consequence of free trade in goods, since each factor is only used in the production of a single traded intermediate. Nevertheless, factor price equalization results are considerably more general than the specialized structure here might suggest. In particular, factor price equalization or conditional factor price equalization results apply in general international trade models without trading frictions under fairly weak assumptions. Intuitively, trading commodities is a way of trading factors; if there is sufficient trade in commodities—especially sufficient trade in commodities with different factor intensities—then countries that are more

[1] In this model, there is no loss of generality in assuming that the price of the final good is normalized to 1 in each country even if there is no trade in final good. This is because all goods are traded and there are no differences in costs of living (purchasing power parity) across countries. This will no longer be the case in the model we study in the next section. Throughout, we take no position on whether there is trade in the final good, but, as specified below, we do not allow international lending and borrowing.

abundant in one factor will sell enough of the goods embedding that factor to equalize factor prices across countries. In the jargon of international trade theory, with free trade of commodities, there will exist *a cone of diversification*, such that when factor proportions of different countries are within this cone, there will be (conditional) factor price equalization. Our extreme assumption that labor is used in the production of the labor-intensive intermediate and capital is used in the production of the capital-intensive intermediate is useful as it ensures that the cone of diversification is large enough to include any possible configuration of the distribution of capital and labor stocks across countries.

The reader may also wonder why conditional factor price equalization is important. Its main importance for us is that when there is conditional factor price equalization, factor prices in each country are *entirely independent* of its capital stock and labor (provided that the country in question is "small" relative to the rest of the world; recall footnote 1 in the previous chapter). Thus each country will be taking intermediate prices, and consequently factor prices, as given when it makes its allocation and accumulation decisions. In fact, the distinguishing feature of the model analyzed in this section is this independence of factor prices from accumulation decisions, which is, in turn, a direct implication of a world of Heckscher-Ohlin trade.

We also assume that as in previous chapters, capital depreciates at an exponential rate δ in each country, so that the interest rate is

$$\begin{aligned} r_j(t) &= R_j(t) - \delta \\ &= p^K(t) - \delta. \end{aligned} \tag{19.10}$$

We next specify the resource constraints facing the economy. While there is free international trade in commodities, we assume that there is no international trade in assets. Thus we will be abstracting from the issues of international lending and borrowing discussed in the previous two sections. This will enable us to isolate the effects of international trade in the simplest possible way. Lack of international lending and borrowing implies that at every date, each country must run a *balanced international trade*. In terms of the variables introduced so far, this implies the following trade balance equation:

$$p^K(t)\left[X_j^K(t) - Y_j^K(t)\right] + p^L(t)\left[X_j^L(t) - Y_j^L(t)\right] = 0, \tag{19.11}$$

for all j and all t. This equation is intuitive; it requires that for each country (at each date) the value of their net sales of the capital-intensive good should be made up by their net purchases of the labor-intensive good. For example, if $X_j^K(t) - Y_j^K(t) < 0$, so that the country is a net supplier of the capital-intensive good (i.e., it uses less of the capital-intensive good in its final good sector than it produces), then it must be a net purchaser of the labor-intensive good, i.e., $X_j^L(t) - Y_j^L(t) > 0$.

In addition to this trade balance equation, there is the usual resource constraint affecting each country at each time, which we can write as

$$\dot{K}_j(t) = F\left(X_j^K(t), X_j^L(t)\right) - C_j(t) - \delta K_j(t), \tag{19.12}$$

for all j and t. In addition, world market clearing requires

$$\sum_{j=1}^{J} X_j^L(t) = \sum_{j=1}^{J} Y_j^L(t) \text{ and } \sum_{j=1}^{J} X_j^K(t) = \sum_{j=1}^{J} Y_j^K(t) \text{ for all } t. \tag{19.13}$$

The important feature in this equation is that both the consumption good and the capital good are produced with the same technology—one unit of the final good can be transformed into one unit of consumption good or one unit of capital or the investment good. In the next section, we will see how different factor intensities of consumption and capital goods can be allowed in models of international trade and growth. But for now, the simpler setup with the consumption and investment goods having the same factor intensities is sufficient for our purposes.

Finally, on the preference side, we assume that each country admits a representative household with standard preferences

$$U_j = \int_0^\infty \exp\left(-\left(\rho - n\right)t\right)\left[\frac{c_j(t)^{1-\theta} - 1}{1-\theta}\right]dt, \tag{19.14}$$

where $c_j(t) \equiv C_j(t)/L_j(t)$ is per capita consumption in country j at time t and we have imposed that all countries have the same time discount rate, ρ, and also the same rate of population growth. Throughout this section, without loss of any generality, we assume that all the decisions within each country is made by the representative household of that country, and we assume that $\rho > n$ to ensure positive discounting and finite lifetime utilities (see Chapter 8, in particular, Assumption $4'$).

With a reasoning similar to that in Chapter 8, a key object is the ratio of "capital-like" intermediates relative to "labor-like" intermediates in production. For this reason, we define

$$x_j(t) \equiv \frac{X_j^K(t)}{X_j^L(t)},$$

so that

$$\begin{aligned} Y_j(t) &= F\left(X_j^K(t), X_j^L(t)\right) \\ &= X_j^L(t)\, F\left(\frac{X_j^K(t)}{X_j^L(t)}, 1\right) \\ &\equiv X_j^L(t)\, f\left(x_j(t)\right), \end{aligned} \tag{19.15}$$

where the third line defines the function $f(\cdot)$ in the usual way exploiting the constant returns to scale nature of the function F. We refer to $x_j(t)$ as the capital intermediate intensity of country j. We also define $k_j(t) \equiv K_j(t)/L_j(t)$ as the capital labor ratio in country j at time t.

A *world equilibrium* can be expressed as a sequence of consumption, capital accumulation and capital intermediate intensity decision for each country and world prices, i.e., $\left[\{c_j(t), k_j(t), x_j(t)\}_{j=1}^J, p^K(t), p^L(t)\right]_{t\geq 0}$ such that $[c_j(t), k_j(t), x_j(t)]_{t\geq 0}$ maximizes the utility of the representative household in country j subject to (19.11) and (19.12) given the sequence the world prices $\left[p^K(t), p^L(t)\right]_{t\geq 0}$, and world prices are such that world markets clear, i.e., the equations in (19.13) hold. A *steady-state world equilibrium* is defined similarly as an equilibrium in which all of these quantities are constant.

Let us start with a straightforward result about the allocation of production around the world:

PROPOSITION 19.5. *Consider the above-described model. In any world equilibrium we have that*

$$x_j(t) = x_{j'}(t) = \frac{\sum_{j=1}^J k_j(t)}{\sum_{j=1}^J A_j} \text{ for any } j \text{ and } j' \text{ and any } t.$$

PROOF. Given world prices at time t, the representative household in each country maximizes $F\left(X_j^L(t), X_j^K(t)\right)$ subject to (19.11). Denoting the derivatives of this function by F_L and F_K, this implies

$$\frac{F_K\left(X_j^K(t), X_j^L(t)\right)}{F_L\left(X_j^K(t), X_j^L(t)\right)} = \frac{p^K(t)}{p^L(t)} \text{ for any } j \text{ and any } t.$$

Using the definition in (19.15) and the linear homogeneity of F, this can be written as

$$\frac{f'(x_j(t))}{f(x_j(t)) - x_j(t) f'(x_j(t))} = \frac{p^K(t)}{p^L(t)} \text{ for any } j \text{ and any } t,$$

where the left-hand side is strictly decreasing in $x_j(t)$, thus defines a unique $x_j(t)$ given the world price ratio. Since $x_j(t)$'s are equal across countries, they must all be equal to the ratio of capital-intensive intermediates to labor-intensive intermediates in the world, i.e.,

$$x_j(t) = \frac{\sum_{j=1}^J K_j(t)}{\sum_{j=1}^J A_j L_j(t)}.$$

Using the fact that $k_j(t) = K_j(t)/L_j(t) = K_j(t)/L(t)$ completes the proof of the proposition. □

This proposition implies that irrespective of differences and capital-labor ratios across countries, the ratio of capital-intensive to labor-intensive intermediates in production will be equalized across countries.

The equalization of the use of the ratio of capital-intensive to labor-intensive intermediates in the production of the final good enables us to aggregate the production and capital stocks of different countries to obtain the behavior of world aggregates. In particular, let $c(t)$ be the average consumption per capita in the world and $k(t)$ be the average capital-labor ratio in the world, given by

$$c(t) \equiv \frac{1}{J}\sum_{j=1}^{J} c_j(t) \text{ and } k(t) \equiv \frac{1}{J}\sum_{j=1}^{J} k_j(t).$$

The next proposition shows that world aggregates follow laws of motion very similar to that of the standard neoclassical closed economy.

PROPOSITION 19.6. *Consider the above-described model. Then in any world equilibrium, the world averages follow the laws of motion given by*

$$\begin{aligned}\frac{\dot{c}(t)}{c(t)} &= \frac{1}{\theta}\left(f'\left(\frac{k(t)}{A}\right)-\delta-\rho\right)\\ \dot{k}(t) &= Af\left(\frac{k(t)}{A}\right)-c(t)-(n+\delta)k(t),\end{aligned}$$

where $r(t) = p^K(t)$ is the world interest rate at time t and

$$A = \frac{1}{J}\sum_{j=1}^{J} A_j$$

is average labor productivity.

PROOF. Using (19.11), (19.12) and Proposition 19.5, the law of motion of the capital stock of country j can be written as

$$\dot{K}_j(t) = p^K(t)K_j(t) + p^L(t)A_jL(t) - C_j(t) - \delta K_j(t).$$

Now define $K(t) \equiv \frac{1}{J}\sum_{j=1}^{J} K_j(t)$, sum over $j = 1, ..., J$, and use the definitions of $p^K(t)$ and $p^L(t)$, Proposition 19.5 and the linear homogeneity of F (together with Theorem 2.1) to obtain

$$\sum_{j=1}^{J}\dot{K}_j(t) = F\left(\sum_{j=1}^{J}K_j(t), \sum_{j=1}^{J}A_jL(t)\right) - \sum_{j=1}^{J}C_j(t) - \delta\sum_{j=1}^{J}K_j(t).$$

Dividing both sides by $JL(t)$ and using Theorem 2.1 once more, we have

$$\frac{\dot{K}(t)}{L(t)} = Af\left(\frac{K(t)}{AL(t)}\right) - c(t) - \delta\frac{K(t)}{L(t)}.$$

Using the definition of $k(t)$ gives the second differential equation.

To obtain the differential equation for $c(t)$, aggregate the Euler equation for the representative household in each county, $\dot{c}_j(t)/c_j(t) = (r(t)-\rho)/\theta$, for each j. This completes the proof of the proposition. □

The result in this proposition is not surprising. With (conditional) factor price equalization, the world behaves as an integrated closed economy, and thus obeys the two key differential equations of the neoclassical model. Now using the previous two propositions, we can characterize the form of the steady-state world equilibrium.

Proposition 19.7. *Consider the above-described model. There exists a unique steady-state equilibrium whereby*

$$f'\left(x_j^*\right) = f'\left(\frac{k^*}{A}\right) = \rho + \delta \text{ for all } j, \tag{19.16}$$

where

$$x_j^* = x^* = \frac{\sum_{j=1}^J K_j\left(t\right)}{L\left(t\right)\sum_{j=1}^J A_j} \text{ and } k^* = \frac{\sum_{j=1}^J K_j\left(t\right)}{JL\left(t\right)}. \tag{19.17}$$

Moreover

$$p^{K*} = \rho + \delta. \tag{19.18}$$

Proof. The proof follows from Proposition 19.6. The Inada conditions in Assumption 2 rule out sustained growth. Therefore, world average consumption must remain constant in steady state, and the interest rate must satisfy $r^* = p^{K*} - \delta = \rho$. Propositions 19.5 and 19.6 then yield (19.16) and (19.17). □

Proposition 19.7 shows that the steady-state world equilibrium takes a very simple form, with the ratio of capital-intensive to labor-intensive intermediates pinned down purely by the aggregate production function F (or its transform, f) and by the ratio of total capital to total labor in the world. The reason why steady-state production structure is determined by world supplies of capital and labor is simple: in the presence of (conditional) factor price equalization, the world economy is *effectively integrated*. We have already seen in the previous two sections how capital flows can make the world become integrated. The analysis in this section shows that Heckscher-Ohlin trade also leads to the same result (as long as it guarantees conditional factor price equalization).

While the structure of the steady-state equilibrium is rather straightforward, transitional dynamics in this world economy are somewhat more involved. In fact, the behavior of individual economies can be quite rich and complicated. Nevertheless, the fact that world averages obey the equations of the neoclassical growth model ensures that the steady-state world equilibrium is globally stable.

Proposition 19.8. *Consider the above-described economy. The steady-state equilibrium characterized in Proposition 19.7 is globally saddle-path stable.*

Proof. With the arguments in the proof of Proposition 19.6, we have that for any sequence of world prices $\left[p^L\left(t\right), p^K\left(t\right)\right]_{t\geq 0}$, the problem of the representative household in each country j at any time t satisfies the differential equations:

$$\frac{\dot{c}_j(t)}{c_j(t)} = \frac{1}{\theta}\left(p^K(t) - \delta - \rho\right)$$

$$\dot{k}_j(t) = \left[p^K(t) - (n+\delta)\right] k_j(t) + p^L(t) A_j - c_j(t).$$

Standard arguments from Chapter 8 applied to world averages in Proposition 19.6 imply that world averages converge to the unique world steady state equilibrium and $\left[p^K(t)\right]_{t\geq 0}$ converges to $\rho+\delta$. This immediately implies that the law of motion for the consumption and capital-labor ratio of each country also converges. With $p^{K*} = \rho+\delta$, the convergence is necessarily to the unique steady-state world equilibrium. □

The analysis so far showed that a world economy consisting of a collection of economies engaged in Heckscher-Ohlin trade generates a pattern of growth similar to that we have seen in Chapter 8, with each country converging to a unique steady state. There is one important difference, however. As in the model with international borrowing and lending in the previous section, the nature of the transitional dynamics is very different from the closed-economy neoclassical growth models. Here, despite the absence of international capital flows, the rate of return to capital is equalized across countries. Thus there are no transitional dynamics resulting from a country with a higher rate of return to capital accumulating capital faster than the rest. This model therefore also emphasizes the potential pitfalls of using the closed-economy growth model for the analysis of output and capital dynamics across countries and regions.

Nevertheless, the results on transitional dynamics are perhaps the less interesting implications of the current model. One of my main objectives in this chapter is to illustrate how the presence of international trade changes the conclusions of closed economy growth models. The current framework already points out how this can happen. Notice that while the world economy has a standard neoclassical technology satisfying Assumptions 1 and 2, each country faces an "AK" technology, since it can accumulate as much capital as it wishes without running into diminishing returns. In particular, for every additional unit of capital at time t, a country receives a return of $p^K(t)$, which is independent of its own capital stock. So how is it that the world does not generate endogenous growth? The answer is that while each country faces an AK technology, and thus can accumulate when the price of capital-intensive intermediates is high, accumulation by all countries drives down the price of capital-intensive intermediate goods to a level that is consistent with steady state. In other words, the price of capital-intensive intermediates will adjust to ensure the steady state equilibrium where capital, output and consumption per capita are constant (see the proof of Proposition 19.8). While this process describes the long-run dynamics, it also opens the door for a very different type of short run (or "medium run") dynamics, especially for countries that have different saving rates than others.

To illustrate this possibility in the simplest possible way, consider the following thought experiment. Let us start with the world economy in steady state and suppose that one of the countries experiences a decline in its discount rate from ρ to $\rho' < \rho$. What will happen? The answer is provided in the next proposition.

Proposition 19.9. *Consider the above-described model. Suppose J is arbitrarily large and the world starts in steady state at time $t = 0$, then the discount rate of country 1 declines to $\rho' < \rho$. After this change, there exists some $T > 0$ such that for all $t \in [0, T)$, country one grows at the rate*

$$g_1 = \frac{\dot{c}_1(t)}{c_1(t)} = \frac{1}{\theta}(\rho - \rho').$$

Proof. In steady state, Proposition 19.8 and equation (19.18) imply that $p^{K*} = \rho + \delta$. Since country 1 faces this price as the return on capital and has a lower discount factor ρ', we are in the AK world of Chapter 11, Section 11.1, and the cost and growth rate follows from the analysis there. □

Essentially, given conditional factor-price equalization, each country faces an AK technology, thus can accumulate capital and grow without running into diminishing returns. The price of capital-intensive intermediates and thus the rate of return to capital is pinned down by the discount rate of other countries in the world, so that country 1, with its lower discount rate, will have an incentive to save faster than the rest of the world and can achieve positive growth of income per capita (while the rest of the world has constant income per capita).

Therefore, the model of economic growth with Heckscher-Ohlin trade can easily rationalize bouts of rapid growth ("growth miracles") by the countries that change their policies or their savings rates (or discount rates). Ventura (1997) suggests this model is a potential explanation for why, starting in the 1970s, East Asian tigers may have grown rapidly without running into diminishing returns. Since in the 1970s and 1980s East Asian economies were indeed more open to international trade than many other developing economies and have accumulated capital rapidly (e.g., Young, 1992, 1995, Vogel, 2006), this explanation is quite plausible. It shows how international trade can temporarily prevent the diminishing returns to capital that would set in because of rapid accumulation and enable sustained growth at higher rates.

Nevertheless, such behavior cannot go on forever. This follows from Assumption 2 above, which implies that world output cannot grow in the long run. So how is Proposition 19.9 consistent with this? The answer is that this proposition describes behavior in the "medium run". This is the reason why the statement of the proposition is for $t \in [0, T)$. At some point, country 1 will become so large relative to the rest of the world that it will essentially own almost all of the capital of the world. At that point or in fact even before this point is reached, country 1 can no longer be considered a "small" country; it will have a major impact, and it will recognize this impact, on the relative price of the capital-intensive intermediate. Consequently, the rate of return on capital will eventually fall so that accumulation by this country comes to an end. Naturally, an alternative path of adjustment could take place if, at some future date, the discount rate of country 1 increases back to ρ, so that the world economy again settles into a steady state.

The important lesson from this discussion is that while the current model can generate growth miracles, these can only apply in the "medium run". The fact that growth miracles can happen only in the medium run highlights another important feature of the current model. Exercise 19.8 shows that the current model does *not* admit a steady-state equilibrium (or even a well defined distribution of world income) when discount rates differ across countries. In other words, the well behaved world equilibrium in the world income distribution that emerges from this model relies on the knife-edge case in which all countries have the same discount rate (and also the same productivity of the capital-intensive intermediates, see Exercise 19.9). This feature is not only a shortcoming of the current model, but more generally a shortcoming of all Heckscher-Ohlin approach to trade and growth . In the traditional Heckscher-Ohlin model there is no comparative advantage coming from technology, so that each country is either small and takes world prices as given, or becomes sufficiently large to influence world prices for all commodities. This seems an unappealing feature on both empirical and intuitive grounds; while it is plausible that countries take prices of the goods that they import as given, they often influence the world prices of at least some of the goods that they export (such as copper for Chile, Microsoft windows for the United States or Lamborghinis for Italy). In the next section, we will see that models with more Ricardian features avoid these unappealing implications and provide a richer and more tractable framework for the analysis of the interaction between international trade and economic growth.

19.4. Trade, Specialization and the World Income Distribution

In this section, I will present a model of the world economy in which countries trade intermediate goods, but trade will have Ricardian features. In particular, each country will specialize in the production of a subset of the available goods in the world economy and will therefore affect the prices of the goods that it supplies to the rest of the world. Put differently, each country's terms of trade will be endogenous and will depend on the rate at which it accumulates capital. We will see that such a model is more flexible than the one discussed in the previous section, since it can allow for differences in discount rates (and saving rates) and also enables us to perform a richer set of comparative static results.

The model economy presented here builds on Acemoglu and Ventura (2002). I will start with a simplified version of this model, which features physical capital as the only factor of production. I will then present the full model in which both physical capital and labor are used to produce consumption and investment goods.

In addition to the nature of trade (Heckscher-Ohlin versus Ricardian), another major difference between the model in this section and the previous one will be that now, as in Section 18.3 in the previous chapter, the world economy will exhibit endogenous growth, with the growth rate determined by the investment decisions of all countries. Despite endogenous growth at the world level, international trade (without any technological spillovers) will create sufficient interactions to ensure a common long-run growth rate for all countries. Therefore, the current model will show how international trade, like technological spillovers, will create a powerful force limiting the extent to which divergence can occur across countries.

19.4.1. Basics. We consider a world economy consisting of a large number J of "small" countries, again indexed by $j = 1, ..., J$. There is a continuum of intermediate products indexed by $\nu \in [0, N]$, and two final products that are used for consumption and investment. There is free trade in intermediate goods and no trade in final products or assets. Lack of trade in consumption and investment goods enables us to focus on trade in intermediates, which is the main focus here. Lack of trade in assets again rules out international borrowing and lending.

Countries differ in their technology, savings and economic policies. For example, country j will be defined by its characteristics (μ_j, ρ_j, ζ_j), where μ is an indicator of how advanced the technology of the country is, ρ is its rate of time preference, and ζ is a measure the effect of policies and institutions on the incentives to invest. All of these characteristics potentially very across countries, but are constant over time. We take their distribution across countries as given. In addition, we assume that each country has a population normalized to 1 and there is no population growth.

All countries admit a representative household with utility function:

$$\int_0^\infty \exp\left(-\rho_j t\right) \ln C_j(t) dt\,, \tag{19.19}$$

where $C_j(t)$ is consumption of country j date t. Preferences are logarithmic and thus more specialized than the typical CRRA preferences we have used so far (for example, in terms of the preferences in (19.1), they involve $\theta \to 1$). Logarithmic preferences enable us to simplify the exposition without any substantive loss of generality. On the other hand, the preferences in (19.19) are significantly more flexible than those in the previous section because they allow the discount rates, the ρ_j's, to differ across countries. We also assume that country j starts with a capital stock of $K_j(0) > 0$ at time $t = 0$.

The budget constraint of the representative household in country j at time t is

$$p_j^I(t)\dot{K}_j(t) + p_j^C C_j(t) = Y_j(t) = r_j(t)K_j(t) + w_j(t), \tag{19.20}$$

where p_j^I and p_j^C are the prices of the investment and consumption goods in country j (in terms of the numeraire, which will be the ideal price index of traded intermediates; see below). Despite international trade in intermediates, because consumption and investment goods are not traded, their prices might differ across countries. As usual, $K_j(t)$ is the capital stock of country j at time t, $r_j(t)$ is the rental rate of capital, which may also differ across countries, and $w_j(t)$ is the wage rate. Notice that equation (19.20) imposes that there is no depreciation, which is adopted simply to reduce notation. The more important feature is that investment, $\dot{K}_j(t)$, is multiplied by $p_j^I(t)$, while consumption is multiplied by $p_j^C(t)$. This reflects the fact that investment and consumption goods will have different production technologies and thus their prices will differ. In this respect, the model in this section is closely related to that in Section 11.3 in Chapter 11. The second equality in (19.20) specifies that total output is equal to capital income plus labor income—$r_j(t)$ is the rental rate of capital, $K_j(t)$ is the total capital holdings in country j and $w_j(t)$ denotes total labor earnings, since population is normalized to 1.

As noted above, our focus here is with Ricardian models which feature specialization. We will introduce specialization in the simplest possible way. In particular, we will assume that the N intermediates available in the world economy are partitioned across the J countries, such that each intermediate can only be produced by one country. This assumption, which is often referred to as the *Armington* preferences or technology in the international trade literature, ensures that while each country is small in import markets, it will affect its own terms of trades by the amount of the goods it exports. Denoting the measure of goods produced by country j by μ_j, our assumption implies that

$$\sum_{j=1}^{J} \mu_j = N. \tag{19.21}$$

This equation immediately implies that a higher level of μ_j implies that country j has the technology to produce a larger variety of intermediates, so we interpret μ as an indicator of how advanced the technology of the country is. We assume throughout that all firms within each country have access to the technology to produce these intermediates, which ensures that all intermediates are produced competitively.

Moreover, let us assume that in each country the production technology of intermediates is such that one unit of capital produces one unit of any of the intermediates that the country is capable of producing and that there is free entry to the production of intermediates. This assumption immediately implies that the prices of all intermediates produced in country j at time t are given by

$$p_j(t) = r_j(t), \tag{19.22}$$

where recall that $r_j(t)$ is the rental rate of return in country j at time t.

19.4.2. The AK Model. Before presenting the full model, it is convenient to start with a simplified version, where capital is the only factor of production. Consequently, in terms of equation (19.20), we have $w_j(t) = 0$, and

$$Y_j(t) = r_j(t) K_j(t).$$

We assume that both consumption and investment goods are produced using domestic capital as well as a bundle of all the intermediate goods in the world (which are all traded freely). In particular, the production function for consumption goods in country j is:

$$C_j(t) = \chi K_j^C(t)^{1-\tau} \left(\int_0^N x_j^C(t,\nu)^{\frac{\varepsilon-1}{\varepsilon}} d\nu \right)^{\frac{\tau\varepsilon}{\varepsilon-1}}. \tag{19.23}$$

A number of features are worth noting. First, K_j^C denotes domestic capital used in the consumption goods sector and enters the production function with exponent $1-\tau$. Intuitively, this term corresponds to the services of the domestic capital stock used in the production of consumption goods. It represents the "non-traded" component of the production process, which depends on the services provided by non-traded goods using the capital available in the country. In particular, since there is no international trade in assets, it must be the domestic capital stock that is used in providing these non-traded services, and if a country has a relatively low capital stock, the relative price of capital will be high and less of it will be used in producing consumption goods (and investment goods; see below). Second, the term in parentheses represents the bundle of intermediates purchased from the world economy. In particular, $x_j^C(t,\nu)$ is the quantity of intermediate good ν purchased and used in the production of consumption goods in country j at time t. The expression implies that it is the CES aggregate of all the intermediates, with an elasticity of substitution ε, that matters in the production of consumption goods. Throughout we assume that

$$\varepsilon > 1,$$

which avoids the counterfactual and counterintuitive pattern of "immiserizing growth" (see Exercise 19.21). The use of constant elasticity of substitution aggregates is familiar by now and plays the same role here as in other models we have seen so far, and enables us to have tractable structure. The expression also makes it clear that there is a continuum N of intermediates (given by equation (19.21) above). Notice that this CES aggregator has an exponent τ, which ensures that the production function for consumption goods exhibits constant returns to scale. The parameter τ is not only the elasticity of the production function of consumption goods with respect to traded intermediates, but it will also be the share of trade in GDP for all countries in this world economy (see Exercise 19.13). Finally, χ is a constant introduced for normalization (see Exercise 19.11).

The production function for investment goods in country j is:

$$I_j(t) = \zeta_j^{-1} \chi K_j^I(t)^{1-\tau} \left(\int_0^N x_j^I(t,\nu)^{\frac{\varepsilon-1}{\varepsilon}} d\nu \right)^{\frac{\tau\varepsilon}{\varepsilon-1}}, \tag{19.24}$$

which is identical to that for consumption goods, except for the presence of the term ζ_j. This allows differential levels of productivity, due to technology or policy, in the production of investment goods across countries. The assumption that these differences are in the investment good sector rather than in the production of consumption goods is consistent with results on the relative prices of investment goods discussed previously, which suggested that in poorer economies investment goods are relatively more expensive. In terms of the production functions specified here, we may want to think of greater distortions as corresponding to higher levels of ζ_j, since we will see that higher ζ_j will reduce output and increase the relative price of investment goods. Because relative prices of investment and consumption goods are determined endogenously, the current model will enable us to explicitly link the policy parameter ζ_j to these prices rather than positing such a relationship. Moreover, in the full model with both capital and labor, we will see that the relative price all investment goods will depend on other factors, in particular, on technology and discount rates.

Market clearing for capital naturally requires

$$K_j^C(t) + K_j^I(t) + K_j^\mu(t) \leq K_j(t), \tag{19.25}$$

where $K_j^\mu(t)$ capital used in the production of intermediates and $K_j(t)$ is the total capital stock of country j at time t.

The reader can also see why we have referred to this model as the AK version; the production of both consumption and investment goods uses capital and intermediates that are directly produced from capital. Thus a doubling of the world capital stock will double the output of all intermediates and of consumption and investment goods.

While we can directly work with the production functions for consumption and investment goods, (19.23) and (19.24), as in many trade models, it is simpler to work with *unit cost functions*, which express the cost of producing one unit of consumption and investment goods in terms of the numeraire (which will be chosen as the ideal price index for intermediates, see equation (19.31) below). Exercise 19.11 shows that the production functions (19.23) and (19.24) are equivalent to the unit cost functions for consumption and production given by

$$B_j^C\left(r_j(t), [p(t,\nu)]_{\nu\in[0,N]}\right) = r_j(t)^{1-\tau} \left[\left(\int_0^N p(t,\nu)^{1-\varepsilon} d\nu \right)^{\frac{\tau}{1-\varepsilon}} \right], \tag{19.26}$$

$$B_j^I\left(r_j(t),[p(t,\nu)]_{\nu\in[0,N]}\right)=\zeta_j r_j(t)^{1-\tau}\left[\left(\int_0^N p(t,\nu)^{1-\varepsilon}d\nu\right)^{\frac{\tau}{1-\varepsilon}}\right], \tag{19.27}$$

where $p(t,\nu)$ is the price of the intermediate ν at time t and the constant χ in (19.23) and (19.24) is chosen appropriately (see Exercise 19.11). Notice that these prices are not indexed by j, since there is free trade in intermediates and thus all countries face the same intermediate prices. The specification using the unit cost functions simplifies the analysis.

A world equilibrium is defined in the usual fashion, as a sequence of prices, capital stock levels and consumption levels for each country, such that all markets clear and the representative household in each country maximizes his utility given the price sequences. Namely, an equilibrium is represented by

$$\left[\left\{p_j^C(t),p_j^I(t),r_j(t),K_j(t),C_j(t)\right\}_{j=1}^J,[p(t,\nu)]_{\nu\in[0,N]}\right]_{t\geq0}.$$

Notice that while the prices of consumption and investment goods and the return to capital are country specific, the prices of intermediates are not. A steady-state world equilibrium is also defined in the usual fashion, in particular, requiring that all prices are constant (as before, this "steady-state" equilibrium will involve balanced growth).

The characterization of the world equilibrium in this case is made relatively simple by the AK technology (and the logarithmic preferences). In particular, the maximization of the representative household, that is, the maximization of (19.19) subject to (19.20) for each j yields the following first-order conditions

$$\frac{r_j(t)+\dot{p}_j^I(t)}{p_j^I(t)}-\frac{\dot{p}_j^C(t)}{p_j^C(t)}=\rho_j+\frac{\dot{C}_j(t)}{C_j(t)} \tag{19.28}$$

for each j and t, and the transversality condition:

$$\lim_{t\to\infty}\exp\left(-\rho_j t\right)\frac{p_j^I(t)K_j(t)}{p_j^C(t)C_j(t)}=0, \tag{19.29}$$

for each j (see Exercise 19.12).

Equation (19.28) is the Euler equation. This equation might first appear slightly different from the standard Euler equations we have encountered throughout the book, but the reader will see that it is identical to the Euler equations implied by the two-sector model in Section 11.3 in Chapter 11 (recall, in particular, equation (11.31)). The difference from the standard Euler equations stems from the fact that we now have potentially different technologies for producing consumption and investment goods, thus individuals that delay consumption have to take into account the change in the relative price of consumption versus investment goods—which explains the presence of the term $\dot{p}_j^I(t)/p_j^I(t)-\dot{p}_j^C(t)/p_j^C(t)$. In this light, it is clear that this equation simply requires the (net) rate of return to capital to be equal to the rate of time preference plus the slope of the consumption path.

Equation (19.29) is the transversality condition. Integrating the budget constraint and using the Euler and transversality conditions, we obtain a particularly simple consumption function in this case:

$$p_j^C(t)\,C_j(t) = \rho_j p_j^I(t)\,K_j(t)\,, \tag{19.30}$$

which can be interpreted as individuals spending a fraction ρ_j of their wealth on consumption at every instant (recall that in this simplified model, there is no labor income and $p_j^I(t)\,K_j(t)$ is consumer wealth at current prices).

Our analysis so far has therefore characterized the prices of intermediates and the behavior of the consumption and capital stock for each country. We next need to determine the prices of consumption and investment goods and the relative prices of intermediates in the world economy. As a first step towards this, we define the numeraire for this world economy as the ideal price index for the basket of all the (traded) intermediates. Since the intermediates always appear in the CES form, the corresponding ideal price index is simply

$$\begin{aligned} 1 &= \left[\int_0^N p(t,\nu)^{1-\varepsilon}d\nu\right]^{\frac{1}{1-\varepsilon}} \\ &= \sum_{j=1}^J \mu_j p_j(t)^{1-\varepsilon}\,. \end{aligned} \tag{19.31}$$

Here the first term defines the ideal price index, while the second uses the fact that country j produces μ_j intermediates, and each of these intermediates have the same price $p_j(t) = r_j(t)$ as given by (19.22) above.

This choice of numeraire has another convenient implication. Our assumption that each country is small implies that each exports practically all of its production of intermediates and imports the ideal basket of intermediates from the world economy. Consequently, $p_j(t) = r_j(t)$ is not only the price of intermediates produced by country j, but also its *terms of trade*—defined as the price of the exports of a country divided by the price of its imports.

Next, using the price normalization in (19.31), (19.26) and (19.27) imply that the equilibrium prices of consumption and investment goods in country j at time t are given by

$$p_j^C(t) = r_j(t)^{1-\tau} \text{ and } p_j^I(t) = \zeta_j r_j(t)^{1-\tau}\,. \tag{19.32}$$

This completes the characterization of all the prices in terms of the rate of return to capital. To compute the rate of return to capital, we need to impose market clearing for capital in each country. In addition, we also have a trade balance equation for each country. However, by Walras' law, one of these equations is redundant. It turns out to be more convenient to use the trade balance equation, which can be written as

$$Y_j(t) = \mu_j r_j(t)^{1-\varepsilon}\,Y(t)\,, \tag{19.33}$$

where $Y(t) \equiv \sum_{j=1}^{J} Y_j(t)$ is total world income at time t. To see why this equation ensures balanced trade, note that each country spends a fraction τ of its income on intermediates, and since each country is small, this implies a fraction τ of its income being spent on imports. At the same time, the rest of the world spends a fraction $\tau\mu_j p_j(t)^{1-\varepsilon}$ of its total income on intermediates produced by country j (this follows because of the CES aggregator over intermediates combined with the observations that $p_j(t)$ is the relative price of each country j intermediate and there are μ_j of them). Noting that total world income is $Y(t)$ and that $p_j(t) = r_j(t)$, we obtain (19.33). Exercise 19.13 asks you to derive this equation from the capital market clearing equation, (19.25), thus verifying the use of the Walras' law.

The equations derived so far, in particular (19.22), (19.30), (19.32) and (19.33) together with the resource constraint, (19.20), characterize the world equilibrium fully.

Let us start by describing the state of the world economy, which can simply be represented by the distribution of capital stocks across the J economies (these are the only endogenous state variables). Their law of motion is obtained simply by combining (19.20), (19.30) and (19.32) on the one hand, and (19.20) and (19.33) on the other. In particular, for each j and t, the law of motion of the capital stock is described by the following pair of differential equations:

$$\frac{\dot{K}_j(t)}{K_j(t)} = \frac{r_j(t)^{\tau}}{\zeta_j} - \rho_j, \tag{19.34}$$

$$r_j(t)\,K_j(t) = \mu_j r_j(t)^{1-\varepsilon} \sum_{i=1}^{J} r_i(t)\,K_i(t)\,. \tag{19.35}$$

These two equations completely characterize the world equilibrium. Starting with a cross section of capital stocks at time t, $\{K_j(t)\}_{j=1}^{J}$, (19.35) gives the cross section of terms of trade and interest rates, $\{r_j(t)\}_{j=1}^{J}$. Given this cross section of interest rates, (19.34) describes exactly how the cross section of capital stocks will evolve.

The simplicity of these laws of motion are noteworthy. The first, (19.34), determines the evolution of the capital stock of each country simply as a function of their own parameters, ζ_j, the distortions on the investment good producing sector, and ρ_j, the discount rate, as well as the equilibrium rental rate. The second, (19.35), expresses the rental rate for each country as a function of the rental rates and capital stocks of other countries.

These two equations immediately establish the following important result:

Proposition 19.10. *There exists a unique steady-state world equilibrium where we have*

$$\frac{\dot{K}_j(t)}{K_j(t)} = \frac{\dot{Y}_j(t)}{Y_j(t)} = g^* \tag{19.36}$$

for $j = 1, ..., J$, and the world steady-state growth rate g^* is the unique solution to equation

$$\sum_{j=1}^{J} \mu_j \left[\zeta_j \left(\rho_j + g^*\right)\right]^{(1-\varepsilon)/\tau} = 1. \tag{19.37}$$

The steady-state rental rate of capital and the terms of trade in country j are given by

$$r_j^* = p_j^* = \left[\zeta_j \left(\rho_j + g^*\right)\right]^{1/\tau}. \tag{19.38}$$

This unique steady-state equilibrium is globally saddle-path stable.

Proof. (Sketch) By definition, a steady-state equilibrium must have constant prices, thus a constant r_j^*. This implies that in any state state, for each $j = 1, ..., J$, $\dot{K}_j(t)/K_j(t)$ must grow at some constant rate g_j. Suppose these rates are not equal for two countries j and j'. Taking the ratio of equation (19.35) for these two countries yields a contradiction, establishing that $\dot{K}_j(t)/K_j(t)$ is constant for all countries. Equation (19.33) then implies that all countries also grow at this common rate, say g^*. Given this common growth rate, (19.34) immediately implies (19.38). Substituting this back into (19.35) gives (19.37). Since these equations are all uniquely determined and (19.37) is strictly decreasing in g^*, thus has a unique solution, the steady-state world equilibrium is unique.

To establish global stability, it suffices to note that (19.35) implies that $r_j(t)$ is decreasing in $K_j(t)$. Thus whenever a country has a high capital stock relative to the world, it has a lower rate of return on capital, which from (19.34) slows down the process of capital accumulation in that country. This process ensures that the world economy, and all economies, move towards the unique steady-state world equilibrium. Exercise 19.14 asks you to provide a formal proof of stability. □

The results summarized in this proposition are quite remarkable. First, despite the high degree of interaction among the various economies, there exists a unique globally stable steady-state world equilibrium. Second, this equilibrium takes a relatively simple form. Third and most important, in this equilibrium all countries grow at the same rate g^*. This third feature is quite surprising, since each economy has access to a AK technology, thus without any international trade, each country would grow at a different rate (for example, those with lower ζ_j's or ρ_j's would have higher long-run growth rates). The process of international trade acts as a powerful force keeping countries together, ensuring that in the long run they will all grow at the same rate. In other words, international trade, together with specialization, leads to a stable world income distribution.

Why is this? The answer is related to the terms of trade effects encapsulated in equation (19.35). To understand the implications of this equation, considered the special case where all countries have the same technology parameter, i.e., $\mu_j = \mu$ for all j. Suppose also that a particular country, say country j, has lower ζ_j and ρ_j than the rest of the world. Then

(19.34) implies that this country will tend to accumulate more capital than others. But (19.35) makes it clear that this cannot go on forever and country j, by virtue of being richer than the world average, will also have a lower rate of return on capital. This lower rate of return will ultimately compensate the greater incentive to accumulate in country j, so that capital accumulation in this country converges to the same rate as in the rest of the world.

Intuitively, while each country is "small" relative to the world, it has market power in the goods that it supplies to the world. When it exports more of a particular good, the price of that good declines, so that world consumers should wish to consume the greater amount of this good that is being supplied in the world market. This implies that when a country accumulates faster than the rest of the world, and thus increases the supply of its exports relative to the supplies of other countries exports, it will face *worsening terms of trades.* This negative terms of trade effect will reduce the income of the country that is accumulating faster. However, more important than this level effect is the dynamic effects of the changes in terms of trades. Recall that equation (19.22) links the rate of return to capital to the terms of trade faced by the country. When a country experiences a worsening in its terms of trade, it also experiences a decline in the rate of return the capital and in the interest rate that the households face. This slows down its rate of capital accumulation, ensuring that in the steady state equilibrium all countries accumulate and grow at the same rate. Therefore, this model shows how pure trade linkages are sufficient to ensure that countries that would otherwise grow at different rates pull each other towards a common growth rate and the result is a stable world income distribution. Naturally, growth at a common rate does not imply that countries with different characteristics will have the same level of income. Exactly as in models of technological interdependences in the previous chapter, countries with better characteristics (higher μ_j and lower ζ_j and ρ_j) will grow at the same rate as the rest of the world, but will be *richer* than other countries. This is most clearly shown by the following equation, which summarizes the world income distribution.

Let $y_j^* \equiv Y_j(t)/Y(t)$ the the relative income of country j in steady state. Then equations (19.33) and (19.38) immediately imply that

$$y_j^* = \mu_j \left[\zeta_j \left(\rho_j + g^*\right)\right]^{(1-\varepsilon)/\tau}. \tag{19.39}$$

This equation shows that countries with better technology (high μ_j), lower distortions (low ζ_j) and lower discount rates (low ρ_j) will be relatively richer. Equation (19.39) also highlights that the elasticity of income with respect to ζ_j and ρ_j depends on the elasticity of substitution between the intermediates, ε, and the degree of openness, τ. When ε is high and τ is relatively low, small differences in ζ_j's and ρ_j's can lead to very large differences in income across countries. This observation is interesting for another reason; recall from Chapters 2 and 3 that the Solow growth model generates a similar equation linking the world income distribution to differences in savings rates and technology. In particular, recall that in a world with a Cobb-Douglas aggregate production function and no human capital differences, the Solow model implies that

$$y_j^* = A_j \left(\frac{s_j}{g^*}\right)^{\alpha/(1-\alpha)}, \tag{19.40}$$

where A_j is the relative labor-augmenting productivity of country j, s_j is its savings rate, g^* is again the world growth rate and α is the exponent of capital in the Cobb-Douglas production function, which is also equal to the share of capital in national income. Equation (19.39) shows that the implications of the world economy with trade are very similar, except that (1) the role of the labor-augmenting technologies is played by the technological capabilities of the country, which determine the range of goods in which it has a comparative advantage; (2) the role of the saving rate is played by the discount rate ρ_j and the policy parameter affecting the distortions on the production of investment goods, ζ_j; (3) instead of the share of capital in national income, the elasticity of substitution between intermediates and the degree of trade openness affects how spread out the world income distribution is. Exercise 19.15 develops these points further.

19.4.3. The General Model*. The model presented in the previous subsection has a number of striking implications. The most important is that despite the possibility of endogenous growth at the country level, world relative prices adjust in such a way as to keep the world income distribution stable. Consequently, differences in preferences and technology across countries translate into differences in income levels along a stable income distribution, rather than into differences in permanent growth rates. However, the reader may wonder how general this result is. The result was derived in the context of a collection of AK economies. In this subsection, I will show that the results generalize to an economy in which both capital and labor are used. To maintain the tractability of the model of the previous subsection, and in fact in order to obtain almost identical equations to those from the previous subsection, I will make use of the structure of production first used by Rebelo (1991), which we encountered in Section 11.3 in Chapter 11, where the production of investment goods only uses capital, while the production of consumption goods uses both capital and labor. While the exact mathematical derivations here depend on these specific assumptions, the general insights do not.

More specifically, preferences, demographics, the production functions for intermediates and the production function for investment goods are as same as in the previous subsection. The main difference is that the production function for consumption goods has now changed to

$$C_j(t) = \chi K_j^C(t)^{(1-\tau)(1-\gamma)} (L_j(t))^{(1-\tau)\gamma} \left(\int_0^N x_j^C(t,\nu)^{\frac{\varepsilon-1}{\varepsilon}} d\nu\right)^{\frac{\tau\varepsilon}{\varepsilon-1}}$$

for some $\gamma \in (0,1)$. Here χ is again a normalizing constant and where $L_j(t)$ is the total labor supply of country j at time t. All of this labor supply is used in the production of the consumption good, since neither the production of intermediates nor the production of the investment good use labor. The labor endowment in the economy is supplied inelastically by the representative household in the economy, and without loss of any generality, we normalize $L_j(t) = 1$. This implies that in terms of (19.20), $w_j(t)$ stands both for the wage rate per unit of labor and total labor income. The associated unit cost function for the consumption good is

$$B_j^C\left(w_j(t), r_j(t), [p(t,\nu)]_{\nu\in[0,N]}\right) = w_j(t)^{(1-\tau)(1-\gamma)} r_j(t)^{(1-\tau)\gamma}\left[\left(\int_0^N p(t,\nu)^{1-\varepsilon}d\nu\right)^{\frac{\tau}{1-\varepsilon}}\right].$$

Using the same price normalization, i.e., (19.31), we continue to have (19.22) for intermediate prices and

$$p_j^I(t) = \zeta_j r_j(t)^{1-\tau}$$

for the price of the investment good. The price of the consumption good is obtained, with a similar reasoning, as

$$p_j^C(t) = w_j(t)^{(1-\tau)(1-\gamma)} r_j(t)^{(1-\tau)\gamma}. \tag{19.41}$$

The maximization problem of the representative household in each country is essentially unchanged, except for the stream of labor income that the individual will receive. This maximization problem again leads to the necessary and sufficient conditions given by (19.28) and (19.29). Combining these two equations, we again obtain that consumption expenditure is given as the fraction of the lifetime wealth of the individual, which now consists of the value of capital plus the discounted value of future labor earnings (see Exercise 19.16):

$$p_j^C(t)C_j(t) = \rho_j\left(p_j^I(t)K_j(t) + \int_t^\infty \exp\left(-\int_t^z \frac{r_j(s)+\dot{p}_j^I(s)}{p_j^I(s)}ds\right)w(z)\,dz\right). \tag{19.42}$$

It is also straightforward to show that (19.33) still gives the necessary trade balance equation for each country.

The final condition we need to impose is market clearing for labor. Recall that labor demand comes only from the consumption goods sector, and given the Cobb-Douglas assumption, this demand is $(1-\gamma)(1-\tau)$ times consumption expenditure, $p_j^C C_j$, divided by the wage rate, w_j. So the market clearing condition for labor in country j at time t is:

$$1 = (1-\gamma)(1-\tau)\frac{p_j^C(t)C_j(t)}{w_j(t)}. \tag{19.43}$$

Finally, because (19.43) implies labor income, $w_j(t)$, is always proportional to consumption expenditure, the optimal consumption rule, (19.42), can be simplified to the following convenient equation:

$$p_j^C(t)\,C_j(t) = \frac{\rho_j}{1-(1-\gamma)(1-\tau)} p_j^I(t)\,K_j(t)\,. \tag{19.44}$$

In other words, households again consume a constant fraction of the value of the capital stock, but this fraction now depends not only on their discount rate, ρ_j, but also on the technology parameters, τ and γ. In light of this derivation, the following two propositions are straightforward:

Proposition 19.11. *In the general model with labor, the world equilibrium is characterized by (19.34) for each j and t, as well as two additional equations*

$$r_j(t)\,K_j(t) + w_j(t) = \mu_j r_j(t)^{1-\varepsilon} \sum_{i=1}^{J} \left[r_i(t)\,K_i(t) + w_i(t)\right], \text{ and} \tag{19.45}$$

$$\frac{w_j(t)}{r_j(t)\,K_j(t)+w_j(t)} = \frac{(1-\gamma)(1-\tau)\,\rho_j}{[\gamma+(1-\gamma)\,\tau]\,\zeta_j^{-1} r_j(t) + (1-\gamma)(1-\tau)\,\rho_j}. \tag{19.46}$$

Proof. See Exercise 19.17. □

The derivation and the intuition for this result follow the equilibrium characterization in the previous subsection. For a given cross section of capital stocks, equations (19.45) and (19.46) determine the cross section of rental rates and wage rates, and given the cross-sectional rental rates, (19.34) determines the evolution of the distribution of capital stocks in the world economy.

The next proposition shows that the structure of the world equilibrium is essentially identical to that in the previous subsection.

Proposition 19.12. *There exists a unique steady-state world equilibrium. In this equilibrium, capital stock and output in each country grows at the constant rate g^* as in (19.36) above, and the world steady-state growth rate g^* is the unique solution to (19.37). This unique steady-state equilibrium is globally stable.*

Proof. See Exercise 19.18. □

This proposition implies that the results regarding the stable income distribution continue to apply in this more general model. Moreover, equation (19.39) still gives the world income distribution in the steady-state world equilibrium.

The more general model does not simply replicate the results on the simpler AK model, however. One important implication of this more general model concerns the relative prices of investment and consumption goods. As discussed previously, the empirical evidence strongly suggests that the price of investments goods relative to consumption goods is greater in poor countries. Many models adopt a reduced-form approach to this empirical regularity and argue that it must be due to frictions affecting the investment sector in poor economies. However, only models that allow for trade and different production function for consumption and investment goods can be truly useful for understanding the sources of differences in these relative prices. The current model, which incorporates these features, naturally generates this pattern of relative prices. The equilibrium derivation above immediately implies the following relative price in each country:

$$\frac{p_j^I(t)}{p_j^C(t)} = \zeta_j \left(\frac{r_j(t)}{w_j(t)} \right)^{(1-\gamma)(1-\tau)},$$

so that the relative price of investment goods will be higher in countries that have high ζ_j and low wages. The first part of this result, that countries with high ζ_j's (high distortions on investment good sectors) have higher relative prices of investment goods, is consistent with the presumption in the literature. However, equation (19.46) above shows that countries with worse technology (low μ_j) and higher discount rates (high ρ_j) will also have lower wages and, via this channel, they will have higher relative prices of investment goods. Therefore, the current model not only provides us with a tractable framework for the analysis of international trade of economic growth, and how trade acts as a force stabilizing the world income distribution, but it also generates a cross section of the relative prices of investment and consumption goods that is consistent with the patterns we observe in the data. Furthermore, it highlights that the relative price of investment goods may vary across countries for reasons different from distortions on the investment sector, so that considerable care is necessary when using the observed variation in these relative prices in context of one-sector and/or closed-economy models as the previous literature has done.

In concluding this section, let us return to a comparison of the economic forces emphasized here with those of Section 19.3. Recall that in the model of the previous section, each country takes the world product and factor prices as given, and then accumulates without running into diminishing returns to scale. In contrast, the model in this section has emphasized how capital accumulation by a country will increase the world supply of goods in which it specializes, thus creating powerful terms of trade effects. These terms of trade effects are the reason why the long-run world income distribution is stable and the fast-growing countries tend to increase the growth rate of the rest of the world. Can the approaches in these two sections be reconciled? I believe the answer is yes. One way to reconcile these two approaches is to view them as applying at different stages of development and for different kinds of goods. Imagine, for example, a world in which some goods are "standardized" and can be produced in any country. When a country is producing these goods, it does not face terms of trade effects and can accumulate without running into diminishing returns to capital. As discussed in the previous section, this might be a good approximation to the situation experienced by the East Asian tigers in the 1970s and 80s, when they specialized in medium-tech goods (e.g., Vogel, 2006). However, as countries become richer they also produce and consume more specialized goods. These goods often come in differentiated varieties and thus a greater supply of any one of these goods will create terms of trade effects. Consequently, if a country is in the stage of development where it produces more of the specialized goods, further capital accumulation will run into diminishing returns because of terms of trade effects. An interesting research area is to construct models combining these two forces and determining when one becomes more important than the other. Whether a model that combines these forces will also generate new results (thus being more than the sum of its parts) is also an open question.

19.5. Trade, Technology Diffusion and the Product Cycle

The previous chapter highlighted the importance of technology diffusion in understanding cross-country income differences. But this was done in the context of a world consisting of a collection of closed economies. The presence of international trade enriches the process of technology diffusion, since it allows for the process of the "international product cycle," whereby technology diffusion goes hand-in-hand with certain products previously produced by technologically advanced economies migrating to less-developed nations. The idea of the international product cycle was first suggested by Vernon (1966). Here I present a simple model, originally developed by Krugman (1979), which provides a formalization of these ideas. The main use of the model presented here is that, thanks to its simplicity, it has many applications in the study of various different issues in macroeconomics, international trade and economic development.

19.5.1. The International Division of Labor. Consider the world economy consisting of two sets of economies, the North and the South. For the analysis in this section, it does not matter whether there is one Northern and one Southern country, or many countries within each group. There is free international trade, without any trading costs.

All individuals in all countries have the same CES preferences with love for variety defined over a consumption index. This consumption index for country $j \in \{n, s\}$ at time t is

$$C_j(t) = \left(\int_0^{N(t)} c_j(t, z)^{\frac{\varepsilon - 1}{\varepsilon}} dz \right)^{\frac{\varepsilon}{\varepsilon - 1}} \tag{19.47}$$

where $c_j(t, z)$ is the consumption of the zth good in country $j \in \{n, s\}$ at time t, $N(t)$ is the total number of goods in the world economy at time t that will be determined endogenously and traded freely, and $\varepsilon > 1$ is the elasticity of substitution between these goods. Naturally, without the free-trade assumption, the range of goods consumed by households in country j would not be $N(t)$, but a subset of these goods to which they have access to.

Each country admits a representative household with dynamic preferences defined over streams of consumption $C_j(t)$. For our purposes here, we do not need to specify what these dynamic preferences are, but for concreteness, the reader may want to assume that these are given by the standard CRRA preferences as in (19.1).

The key assumption of the model is that goods fall into two categories: new goods are just invented in the North and can only be produced there; old goods have been invented in the past and their production technology has been imitated by the South, so they can be produced both in the South and in the North.

The technology of production is simple: one worker produces one unit of any good to which the country in which he is located has access to.

Workers in the North have access to all goods, but workers in the South only have access to "old goods". It is important to emphasize that when producing old goods, Northern workers have no productive advantage. Their only advantage (and the only difference in technology) arises because they have access to a larger set of goods.

We assume that the total labor supply in the North is L^n at all times and the labor supply in the South is L^s. All labor is supplied inelastically.

An equilibrium is defined in the usual way as sequences of prices for all goods and allocation of labor across goods.

This description of the environment immediately implies that there can be two types of equilibria.

(1) *Equalization equilibrium*: in this type of equilibrium, there are sufficiently few new goods that both workers in the South and the North will produce some of the old goods. We will see below that in this type of equilibrium both new goods and old goods will command the same price, and incomes in the North and South will be the same. This justifies the label "equalization equilibrium".
(2) *Specialization equilibrium*: in this type of equilibrium the South specializes in the production of old goods, while the Northern producers specializes in the production of new goods.

Let us start by studying the international division of labor for a given set of new and old goods, $N^n(t)$ and $N^o(t)$, where naturally the total number of goods is $N(t) = N^n(t) + N^o(t)$. Since the North has access to all goods, while the South only has access to old goods, the ratio $N^n(t)/N^o(t)$ (or $N(t)/N^o(t)$) can be interpreted as an measure of the technology gap between the North and the South.

To start with let us suppose that the world is in a specialization equilibrium. Clearly, the prices of all new goods and the prices of all old goods will be equalized. Denote these two sets of prices by $p^n(t)$ and $p^o(t)$. Let the wage rate in the North be $w^n(t)$ and that in the

South $w^s(t)$. By its very nature, a specialization equilibrium implies that

$$\begin{aligned} p^n(t) &= w^n(t) \\ p^o(t) &= w^s(t). \end{aligned} \tag{19.48}$$

It must be the case that $w^n(t) \geq w^s(t)$, since otherwise Northern workers would prefer to produce the old goods. Thus a specialization equilibrium can exist only if when all old goods are produced in the South, the implied equilibrium wage rate in the South is lower than that in the North. To find out when this will be so is straightforward. The CES preferences specified in (19.47) imply that utility maximization requires the ratio of the consumption of new and old goods to satisfy

$$\frac{c^n(t)}{c^o(t)} = \left(\frac{p^n(t)}{p^o(t)}\right)^{-\varepsilon}. \tag{19.49}$$

Specialization implies that all of the labor force of the South is used to produce old goods, while all of the labor force of the North is employed in the production of new goods. Therefore

$$c^n(t) = \frac{L^n}{N^n(t)} \text{ and } c^o(t) = \frac{L^s}{N^o(t)}. \tag{19.50}$$

Combining the previous three equations, we obtain the following simple relationship between relative wages and labor supplies and technology:

$$\frac{w^n(t)}{w^s(t)} \equiv \omega(t) = \left(\frac{N^n(t)}{N^o(t)}\frac{L^s}{L^n}\right)^{1/\varepsilon}. \tag{19.51}$$

Notice that the right-hand side of (19.51) are all known quantities at time t. Thus they determine a unique relative wage between the North and the South. A specialization equilibrium will exist only if this ratio $\omega(t)$ is greater than or equal to 1. If it happens to be less than 1, then a specialization equilibrium does not exist; instead, the equilibrium will take the form of an equalization equilibrium. In this equalization equilibrium, wages in the North and the South are equalized, and some of the old goods are produced in the North. In particular, suppose that $\omega(t)$ as defined by (19.51) is strictly less than 1. Then, there exists a unique equilibrium, which takes the form of an equalization equilibrium, where new goods and old goods all command the same price, and are consumed in the same quantity. Therefore, we have

$$c^n(t) = \frac{\phi L^n}{N^n(t)} \text{ and } c^o(t) = \frac{L^s + (1-\phi) L^n}{N^o(t)},$$

where $\phi \in (0,1)$ is chosen such that $c^n(t) = c^o(t)$. We know that such a $\phi \in (0,1)$ exists, since $\omega(t) < 1$, which implies that $c^n(t) > c^o(t)$ at $\phi = 1$.

The characterization of the equilibrium is shown diagrammatically in Figure 19.1. This figure shows that there is a downward sloping relationship between the relative supply of labor in the North, L^n/L^s, and the earnings premium in the North, $\omega \equiv w^n/w^s$. It also shows that when $L^n/L^s = N^n(t)/N^o(t)$, the relationship becomes flat at $w^n/w^s = 1$, because in this case the relative supply of labor in the North is sufficiently large that we entered region of equalization equilibrium.

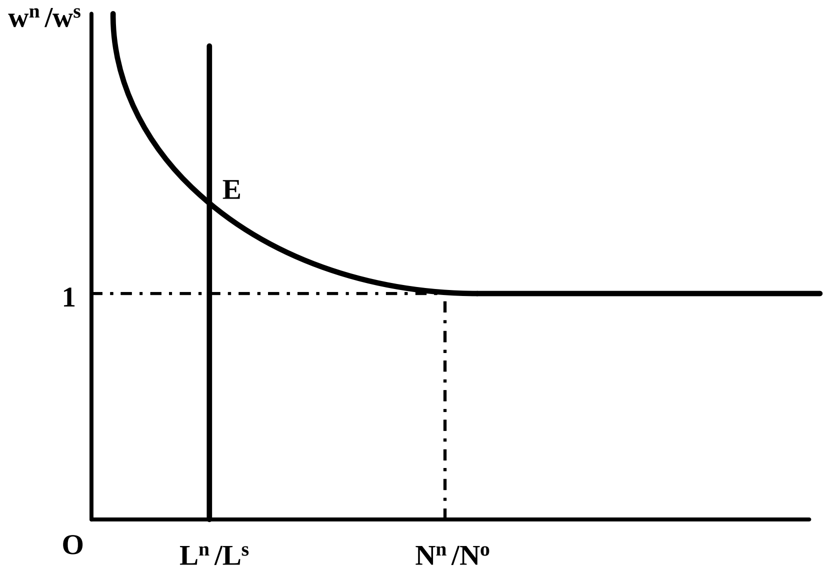

Figure 19.1. Determination on the relative wages in the North and the South in the basic product cycle model.

An interesting implication of this equilibrium is that even when there is a technology gap between the North and the South, Northern and Southern incomes may be equalized. There will only be an income gap between the North and the South when the technology gap is relatively large or when the labor supply in the South, L^s, is sufficiently large. This last feature is particularly interesting in the context of the current wave of globalization, which has involved the incorporation of India and China into the world economy as potential low-cost producers of "old" goods.

While we may think that the case with a sufficiently large technology gap and sufficiently large L^s, which leads to a positive income gap between the North and the South is more realistic, the possibility that such a gap may not exist is of theoretical interest and helps us understand the impact of the international division of labor on cross-country income differences. The possibility that incomes in the North and the South are equalized may appear surprising at first, but the intuition is straightforward. International trade ensures that the Southern consumers have access to goods that their country does not have the technology to produce. Consequently, despite the fact that the South is technologically behind the North, it may achieve the same consumption bundle and the same level of income. This discussion therefore suggests that international trade is a powerful force limiting the extent of cross-country income inequality (for example, resulting from technological differences). This is typically the case, but perhaps surprisingly, not always so. Exercise 19.26 goes through the implications of trade on cross-country income differences and shows that even in the context of the current model, it can sometimes lead to a larger gap of income between rich and poor countries.

19.5.2. Product Cycles and Technology Transfer. The characterization of the equilibrium in the previous subsection was for a given number of new and old goods. Our interest in this model originates because its relative simplicity enables us to endogenize the number of new and old goods, and generates a pattern of product cycle across countries. Here I will follow Krugman (1979) and endogenize the number of new and old goods using a model of exogenous technological change. Exercise 19.25 considers a version of this model with endogenous creation of new products.

In particular, let us suppose that new goods are created in the North according to the following simple differential equation

$$\dot{N}(t) = \eta N(t),$$

with some initial condition $N(0) > 0$ and innovation parameter $\eta > 0$. Goods invented in the North can be imitated by the South. As in the models of technology diffusion in the previous chapter, this process is assumed to be slow and follow the differential equation

$$\dot{N}^n(t) = \iota N^n(t),$$

where $\iota > 0$ is the imitation parameter, and this differential equation has a motivation similar to the technology diffusion equations in the previous chapter, and captures the idea that the South can only imitate from the set of goods that have not so far been imitated (of which there is a total of $N^n(t)$ at time t). Also, as specified above, $N(t) = N^n(t) + N^o(t)$. Combining these equations, we obtain a unique globally stable steady-state ratio of new to old goods given by

$$\frac{N^n(t)}{N^o(t)} = \frac{\eta}{\iota}. \tag{19.52}$$

This equation is intuitive: the ratio of new to old goods will be high when the rate of innovation in the North, η, is high relative to the rate of imitation from the South, ι. Combining this equation with (19.51), we obtain the equilibrium wage ratio between the North and the South as:

$$\frac{w^n(t)}{w^s(t)} = \max\left\{\left(\frac{\eta}{\iota}\frac{L^s}{L^n}\right)^{1/\varepsilon}, 1\right\}. \tag{19.53}$$

In this expression, when the max operator picks 1, then we are in the equalization equilibrium. Otherwise we are in the specialization equilibrium. Since the ratio $w^n(t)/w^s(t)$ also corresponds to the ratio of income between the North and the South, this equation also implies that a high rate of innovation by the North makes the South relatively poor (though not absolutely so), while a higher rate of imitation by the South makes the South relatively richer and the North relatively poorer (see Exercise 19.24). In view of the results from the previous chapter, these results are not surprising.

An important and interesting feature of this steady-state equilibrium is the product cycle. Let us focus on the specialization equilibrium. Then new goods are invented in the North and produced there by workers that receive relatively high wages (since in the specialization equilibrium, $w^n(t) > w^s(t)$). After a while, a given new good is imitated by the South, so its production shifts to the South, where labor costs are lower. Thus in this model we witness the international product cycle, starting with production at high labor costs in the North and then transitioning to a mode of "cheap production" in the South.

An important application of the product cycle model is to the implications of international protection of intellectual property rights (IPR). The rate of imitation ι can also be considered as an inverse measure of the international protection of IPR. Then as shown in Exercise 19.24, in this baseline model stronger international IPR protection will always increase the income gap between the North and the South. Interestingly, however, the exercise also shows that it does not always lead to a welfare improvement in the North.

19.6. Trade and Endogenous Technological Change

The effect of trade on growth has attracted much academic and policy attention. Most economists believe that trade promotes growth, and there is both micro and macro evidence consistent with this belief. A number of papers, for example, Dollar (1992) and Sachs and Warner (1995), find a positive correlation between openness to international trade and economic growth. While these studies are not entirely convincing, since they suffer from the typical difficulties of reaching causal conclusions from growth regressions (recall the discussion in Chapter 3), other papers have tried to overcome these difficulties by using instrumental-variables strategies. In this context, a well-known paper by Frankel and Romer (1999) exploits differences in the trade capacity of countries as given by the gravity equations of trade as a source of variation to estimate the effect of trade on long-run income differences. Gravity equations, which are widely used in the empirical trade literature, link the volume of trade between two countries to their geographic and economic characteristics and their interactions (such as size of country, GDP, distance, etc.). Frankel and Romer exploit the geographically-determined component of these gravity equations to construct a measure of "predicted trade" for each country and use this as an instrument for actual trade openness. Using this strategy, they show that greater trade is associated with higher income per capita (thus with greater long-run growth). In addition, recent microeconomic evidence by Bernard and Jensen (1997), Bernard, Eaton, Jensen and Kortum (2004) and others show that firms that engage in exporting are typically more productive, which might be partly due to "learning by exporting," though at least some part of this correlation is likely to be due to selection (Melitz, 2003). Similarly, firms in developing countries that import machinery from more advanced economies appear to be more productive (e.g., Goldberg and Pavnik, 2007). Nevertheless, a number of economists are skeptical of the growth effects of trade. Rodrik (1997) and Rodriguez and Rodrik (1999) argue that the empirical evidence that trade promotes growth is not entirely compelling. On the theoretical side, a number of authors, for example, Matsuyama (1992) and Young (1993), have presented models in which international trade can slow down growth in some countries.

In this and the next section, I investigate some of the simplest models that link trade to growth in order to investigate the potential impacts of international trade on economic growth. I start with a model illustrating how trade opening may change the pace of endogenous technological change. This model is inspired by Grossman and Helpman (1991b), who investigate many different interactions between international trade and endogenous technological change. Briefly, the model consists of two independent economies that can be approximated by the baseline endogenous technological change model with expanding input varieties as in Chapter 13. In fact, the model we will study is identical to the lab equipment specification in Section 13.1. The advantage of this model is that there are no knowledge spillovers, thus we do not have to make some potentially problematic assumption about knowledge spillovers occurring at the same time as trade opening.[2] We will look at these two economies first without any international trade and then with costless international trade. We will compare the equilibrium growth rates under these two scenarios. Naturally, a smoother transition, in which trade costs decline slowly, is more realistic in practice, but the sharp thought experiment of moving from autarky to full trade integration is sufficient for us to obtain the main insights concerning the effect of international trade on technological progress.

Given the analysis in Section 13.1 of Chapter 13, there is no need to repeat the analysis here. It suffices to say that we consider two economies, say 1 and 2, with identical technologies, identical preferences, and identical labor forces normalized to 1 (and no population growth). Preferences and technologies are identical to those specified in Section 13.1. Consequently, a slight variation on Proposition 13.1 in that section immediately implies the following result:

Proposition 19.13. *Suppose that condition*

$$\eta\beta > \rho \text{ and } 2(1-\theta)\eta\beta < \rho \tag{19.54}$$

[2] If, instead of the lab equipment specification, we were to use the specification with knowledge spillovers and the two countries produced different sets of inputs, it would be necessary to make additional modeling assumptions. For example, we would need to decide whether and how much the inputs produced in the foreign country increase the productivity of R&D in the home country before and after trade opening. Exercise 19.29 shows that assumptions concerning how the extent of knowledge spillovers change with trade opening influence the conclusions regarding the effects of trade on growth.

holds. Then in autarky there exists a unique equilibrium in which starting from any level of technology, both countries innovate and grow at the same rate

$$g^A = \frac{1}{\theta}(\eta\beta - \rho). \tag{19.55}$$

Proof. See Exercise 19.27. □

Next, we will analyze what happens when these two economies start trading. The exact implications of trade will depend on whether, before trade opening, the two countries were producing some of the same inputs or not (recall that there is a continuum of available inputs that can be produced). To the extent that they were producing some of the same inputs, the static gains from trade will be limited. If, on the other hand, the two countries were producing different inputs, there will be larger static gains. However, our interest here is with the dynamic effects of trade opening, that is, the effects of trade opening on economic growth. Once again, the analysis from Chapter 13 immediately leads to the following result:

Proposition 19.14. *Suppose that condition (19.54) holds. Then after trade opening, the world economy and both countries produce new technologies and grow at the rate*

$$g^T = \frac{1}{\theta}(2\eta\beta - \rho) > g^A,$$

where g^A *is the autarky growth rate given by (19.55).*

Proof. See Exercise 19.28. □

This proposition shows that opening to international trade encourages technological change and increases the growth rate of the world economy. The reason is simple. Trade enables each input producer to access a larger market, and this makes inventing new inputs more profitable. This greater profitability translates into a higher rate of innovation and more rapid growth.

The main effect captured in this simple model is reasonably robust. Grossman and Helpman (1991b) provide a number of extensions in richer models of international trade (for example with multiple factors). The economic force, a version of the market size effect, leading to the innovation gains from trade is also reasonably robust. Nevertheless, a number of caveats are necessary. First, as Exercise 19.29 shows, if the R&D sector competes with production, there will be powerful offsetting effects, because trade will also increase the demand for production workers. In this case, the qualitative result in this section, that trade opening increases the rate of technological progress, generally applies, but it is also possible to construct versions of this baseline model, where this effect is entirely offset. Exercise 19.29 also provides an example of this type of extreme offset, which should be borne in mind as a useful caveat. Second, Exercise 19.30 shows that if the full scale effect is removed and we focus on an economy with semi-endogenous growth (as the model studied in Section 13.3 in Chapter 13), trade opening will increase innovation temporarily, but not in the long run.

19.7. Learning-by-Doing, Trade and Growth

The previous section showed how international trade can increase economic growth in all countries in the world by encouraging faster technological progress. In addition to this effect of trade on growth working via technological change, the "static gains" from trade are well recognized and understood. By improving the allocation of resources in the world economy, these static gains can also encourage economic growth. Nevertheless, as mentioned in the previous section many commentators and some economists remain skeptical of the positive growth effects of international trade. A popular argument, often used to justify infant industry protection, is that the static gains from trade come at the cost of dynamic gains, because international trade induces some countries to specialize in industries with relatively low growth potential. In this section, I will outline a simple model with this feature. Richer models that also lead to similar conclusions have been presented by, among others, Matsuyama (1992), Young (1993) and Galor and Mountford (2006). There are also more subtle arguments for why trade may have negative effects on growth based on institutional differences across countries, which are discussed at the end of this chapter. My purpose here is to use the simplest model to illustrate the potential negative effects of trade—and also show when they may not apply. As in the models by Matsuyama and Young, the mechanism for potential dynamic losses from trade (for some countries) will be the presence of learning-by-doing externalities in some sectors.

In particular, consider a world economy consisting of two blocks of countries, the North and the South, and suppose that each block consists of many identical countries. The thought experiment is a move from autarky to full international trade integration between these two blocks. To simplify the exposition and to focus on the main ideas, let us assume that all countries are "almost identical". In particular, each country has a total labor force of 1, and labor can be used to produce one of two intermediate goods with the production functions

$$Y_j^1(t) = A_j(t) L_j^1(t) \text{ and } Y_j^2(t) = L_j^2(t),$$

with the labor market clearing condition

$$L_j^1(t) + L_j^2(t) \leq 1$$

for $j \in \{n, s\}$ denoting a Northern or Southern country. Moreover, let us assume that the total number of Northern and Southern countries are equal, and denote the total number of countries in the world by $2J$.

The final good is produced as a CES aggregate of these two intermediates. Once again distinguishing between the production of intermediates and their use in the final good sector, we write this as

$$Y_j(t) = \left[\gamma X_j^1(t)^{\frac{\varepsilon-1}{\varepsilon}} + (1-\gamma) X_j^2(t)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}},$$

where ε is the elasticity of substitution between the two intermediates. Throughout this section we assume that these two intermediates are gross substitutes, so that $\varepsilon > 1$. However, the case of $\varepsilon = 1$ (where the production function becomes Cobb-Douglas) is also of special-interest, so I will treat this case separately. Moreover, to simplify the algebra and the exposition below, I set $\gamma = 1/2$.

Learning-by-doing is modeled as follows:

$$\frac{\dot{A}_j(t)}{A_j(t)} = \eta L_j^1(t), \tag{19.56}$$

so that when more workers are employed in sector 1, the technology of sector 1 improves. There are no learning-by doing opportunities in sector 2. Thus one might think of sector 1 is manufacturing or some high-tech sectors, while sector 2 may correspond to agriculture or to low-tech sectors (though whether there are greater opportunities for learning-by-doing in manufacturing than in agriculture is quite debatable). As in Romer' (1986) model of growth through externalities, which was studied in Chapter 11, we assume that each producer ignores the positive externality that it creates on the future productivity of sector 1 by its production decisions today.

The only difference between the North and the South is a small "comparative advantage" for the North in the production of sector 1. In particular, we assume that

$$A_n(0) = 1 \text{ and } A_s(0) = 1 - \delta, \tag{19.57}$$

where δ is a small number.

Given this structure, the equilibrium both without international trade and with international trade are relatively straightforward to characterize.

The key in both cases is that the value of the marginal product of labor (the "wage rates") in the two sectors have to be equalized or only one of the two sectors will be active. Let us start with the closed economy, and suppose that both sectors have to be active at t. This implies that the marginal products have to be equalized in the two sector, thus

$$p_j^1(t) A_j(t) = p_j^2(t), \tag{19.58}$$

where $p_j^1(t)$ and $p_j^2(t)$ denote the prices of the two intermediates in country j in terms of the final good, and $A_j(t)$ is the level of productivity and sector 1 in country j. Notice that prices are indexed by j, since we are in the closed economy case. Profit-maximization by the final good producers immediately implies that

$$\begin{aligned} \frac{p_j^1(t)}{p_j^2(t)} &= \left(\frac{X_j^1(t)}{X_j^2(t)}\right)^{-\frac{1}{\varepsilon}} \\ &= \left(\frac{A_j(t) L_j^1(t)}{1 - L_j^1(t)}\right)^{-\frac{1}{\varepsilon}}, \end{aligned}$$

where $L_j^1(t)$ denotes the amount of labor allocated the sector 1 in country j at time t, and naturally, the amount of labor allocated to sector 2 is $L_j^2(t) = 1 - L_j^1(t)$. Combining this with (19.58), we obtain

$$L_j^1(t) = \frac{A_j(t)^{\varepsilon-1}}{1 + A_j(t)^{\varepsilon-1}}. \tag{19.59}$$

The evolution of the productivity of sector 1 is then given by (19.56).

Proposition 19.15. *Consider the above-described model and suppose that $\varepsilon > 1$. Then in the absence of international trade the equilibrium involves the allocation of labor given by (19.59) for all j and t. In particular, we have $L_j^1(t = 0) = 1/2$, and $L_j^1(t)$ monotonically converges to 1. The growth rate of each country $g_j(t)$ converges to $g^* = \eta$.*

If, on the other hand, $\varepsilon = 1$, then $L_j^1(t) = 1/2$ for all t, and the long-run growth rate of each country is $g^{**} = \eta/2$.

Proof. See Exercise 19.31. □

Next consider the same world economy with free international trade starting at time $t = 0$. For each intermediate good, there is now only a single world price, $p^1(t)$ for good 1 and $p^2(t)$ for good 2. With standard arguments, these prices satisfy

$$\begin{aligned} \frac{p^1(t)}{p^2(t)} &= \left(\frac{X_n^1(t) + X_s^1(t)}{X_s^2(t) + X_s^2(t)}\right)^{-\frac{1}{\varepsilon}} \\ &= \left(\frac{A_n(t) L_n^1(t) + A_s(t) L_s^1(t)}{2 - L_n^1(t) - L_s^1(t)}\right)^{-\frac{1}{\varepsilon}}, \end{aligned}$$

where the subscripts n and s denote Northern and Southern countries.

It is straightforward to verify that as a result of the slight comparative advantage introduced in equation (19.57), at $t = 0$, the marginal product of Northern workers in sector 1 is higher, and all of the labor force in the North will be employed in sector 1, and all of the labor force in the South will be employed in sector 2. Moreover, all of sector 1 production will be in Northern countries and all of sector 2 production will be in the South. In all subsequent periods, the productivity of Northern workers in sector 1 is even higher, while the productivity of Southern workers in sector 1 remains stagnant. Consequently, we obtain the following proposition:

Proposition 19.16. *Consider the above-described model. Then with free international trade, the equilibrium is as follows:* $L_n^1(t) = 1$ *and* $L_s^1(t) = 0$ *for all* t. *In this equilibrium, we have that*

$$\frac{\dot{A}_n(t)}{A_n(t)} = \eta \text{ and } \frac{\dot{A}_s(t)}{A_s(t)} = 0.$$

The world economy converges to a growth rate of $g^* = \eta$ *in the long run. Throughout, the ratio of income in the North and the South is given by*

$$\frac{Y_n(t)}{Y_s(t)} = A_n(t)^{\frac{\varepsilon-1}{\varepsilon}}.$$

Consequently, if $\varepsilon > 1$, *the North becomes progressively richer relative to the South, i.e.,* $\lim_{t\to\infty} Y_n(t)/Y_s(t) = \infty$. *If, on the other hand,* $\varepsilon = 1$, *the relative incomes of the North and the South remain constant, i.e.,* $Y_n(t)/Y_s(t) =$ *constant for all* t.

Proof. See Exercise 19.32. □

This proposition contains the main result about how international trade can harm certain countries when there are learning-by-doing externalities in some sectors. In particular, the South has a slight comparative disadvantage in sector 1. In the absence of trade, it devotes enough of its resources to that sector and achieves the same growth rate as the North. However, if there is free trade, the South specializes in sector 2 (because of its slight comparative disadvantage in sector 1) and fails to benefit from the learning-by-doing advantages emanating from production in sector 1. As a result, the South becomes progressively poorer relative to the North. This proposition therefore captures the main critique against international trade coming from models such as Young (1993) and proponents of the infant industry arguments.

However, the proposition also shows some of the shortcomings of these arguments. For example, if $\varepsilon = 1$ (or sufficiently close to 1), specialization in sector 2 does not hurt the South. The reason is closely related to the effects highlighted in Section 19.4: the increase in the productivity of sector 1 in the North creates a negative terms of trade effect against the North. This effect is always present, but when $\varepsilon = 1$ it becomes sufficiently powerful to prevent the impoverishment of the South despite the fact that they have specialized in the sector with the low growth potential. Another caveat is highlighted in Exercise 19.33: in the world economy described here, infant industry protection will not help the South. Even if there is no trade for some infant industry protection period of duration $T > 0$, the ultimate outcome will be the same as in Proposition 19.16.

So what are we to make of the results in this section and the general issue of the impact of trade on growth? An immediate answer is that the juxtaposition of the models of this and the previous section suggest that the effect of trade on growth must be an empirical one. Since there are models that highlight both the positive and the negative effects of trade on growth, the debate can be resolved only by empirical work. Having said that, the theoretical perspectives are still useful. A couple of issues are particularly worth noting. First, the effect of trade integration on the rate of endogenous technological progress may be limited because of the factors already discussed at the end of the previous section. For example, significant effects are possible only when trade opening does not increase wages in the final good sector competing for workers against the R&D sector (i.e., when the R&D sector does not compete for workers with the final good sector). Moreover, if the extreme scale effects are removed, trade opening creates a temporary boost in innovation, but does not necessarily change long-run growth rate. Nevertheless, the benefits of the greater market size for firms involved in innovation must be present according to any model of endogenous technological change. Taking all of these factors into account, we should expect some inducement to innovation from trade opening. Whether these effects are commensurate with or even greater than the static gains of international trade is much harder to ascertain. It may well be that the static gains from trade are more important than the subsequent innovation gains. On the other side of the tradeoff are the potential costs of trade in terms of inducing specialization of some economies in the wrong sectors. The model in this section illustrates this possibility. Nevertheless, I believe that the potential negative effects of trade on growth because of such "incorrect" specialization are much exaggerated. First, there is no strong evidence that learning-by-doing externalities are important in general and much more important in some sectors than in others (which is what is necessary for "incorrect" specialization). Second, even if this were the case, in most situations specialization is not perfect, thus some amount of learning-by-doing takes place in all economies. Third and most important, international flows of information, which often accompany trade opening but also exist independently, imply that improvements in productivity in some countries will affect productivity in others that were not initially specializing in those sectors (for example, Korea was initially an importer of cars, and is now a net exporter, its productivity in the automotive sector having increased with technology transfer). Finally, as the main result in this section showed, terms of trade effects ameliorate any negative impact of specialization in some countries. All in all, it seems that the theoretical case for worrying about the negative growth implications of trade is very weak.

19.8. Taking Stock

This section had three main objectives.

The first was to emphasize the shortcoming of using the closed-economy models for the analysis of the economic growth patterns across countries or regions. In this respect, we have shown how both international trade in assets (international borrowing and lending) and international trade in commodities change both the dynamics and potentially long-run implications of the closed-economy neoclassical growth models. For example, international capital flows remove transitional dynamics, because economies that are short of capital do not need to accumulate it slowly, but can borrow in international markets. Naturally, there are limits to how much international borrowing can take place. Countries are sovereign entities, thus it is relatively easy for them to declare bankruptcy once they have borrowed a lot. Consequently, the sovereign borrowing risk might place limits on the ability of countries to use international markets to smooth consumption and to increase their investments rapidly. Even in this case, some amount of international lending will take place and this will have an important effect on the equilibrium dynamics of output in the capital stock. The available evidence shows that the amount of gross capital flows are very large, though the Feldstein-Horioka puzzle, that fluctuations in investment are correlated with fluctuations in savings, shows that there are limits to net international capital flows. An investigation of why, despite the very large size of the gross capital flows, net international capital flows do not play a greater role in international consumption smoothing is an interesting area for future research. While there is some research on this topic in international finance, its implications for economic growth are important and need to be studied.

We have further seen that international trade in commodities also changes the implications of the neoclassical growth model. For example, in the model of economic growth with Heckscher-Ohlin trade in Section 19.3, trade in goods plays the same role as international lending and borrowing, and significantly changes cross-country output dynamics. Thus even in the absence of international lending and borrowing, the implications of approaches that model the entire world equilibrium are significantly different from those focusing on closed-economy dynamics. The model of economic growth with Ricardian trade in Section 19.4 also showed that output dynamics are very different in the presence of trade. In that model, there would be no convergence across countries without trade, but international trade, via the terms of trade effects it induces, creates a powerful force that links the real incomes of different countries. Consequently, the long-run equilibrium involves a stable world income distribution and the short-run dynamics are very different from the closed-economy model.

The second objective was to highlight how the nature of international trade interacts with the process of economic growth. Sections 19.3 and 19.4 focused on this issue. The model of economic growth with Heckscher-Ohlin trade showed how economic growth increases the effective elasticity of output with respect to capital for each country, because of (conditional) factor price equalization. This is useful in understanding how certain economies, such as East Asian tigers, can grow rapidly for extended periods relying on capital accumulation without running into diminishing returns. However, our analysis also showed that a pure Heckscher-Ohlin model may not be an appropriate framework for the analysis of the interactions across countries. In contrast, the model in Section 19.4 emphasized how Ricardian trade, based on technological comparative advantage, creates a new source of diminishing returns to accumulation for each country based on terms of trade effects. As a country accumulates more capital, it starts exporting more of the goods in which it specializes. The result is a worsening of its terms of trade, effectively reducing the rate of return to further capital accumulation. The analysis showed how this force leads to a stable world income distribution, whereby rapidly growing economies pull of the laggards together with them. How are we to reconcile the different implications of the models in Sections 19.3 and 19.4? One possibility is to imagine a world that is a mixture of the models of these two sections. It may be that some goods are "standardized" and can be produced in any country. When producing these goods, there are no terms of trade effects. So if a country can grow only by producing these goods, it can escape the standard diminishing returns to capital thanks to international trade. This might be a good approximation to the situation experienced by the East Asian tigers in the 1970s and 80s, when they specialized in medium-tech goods. However, as countries become richer they also produce and consume more specialized goods. These goods often come in differentiated varieties and thus a greater supply of any one of these goods will create terms of trade effects. Consequently, if a country is in the stage of development where it produces more of the specialized goods, further capital accumulation will run into diminishing returns through the mechanism highlighted in Section 19.4. Irrespective of how the forces emphasized in these two approaches are combined, they both show the importance of modeling the world equilibrium and also the importance of viewing the changes in the rate of return to capital in the context of the trading relations of an economy.

The third objective of this chapter was to investigate the effect of international trade on economic growth. Here, Sections 19.6 and 19.7 illustrated two different approaches, one emphasizing the beneficial effects of trade on growth, the other one the potential negative effects. Both classes of models are useful to have in one's arsenal in the analysis of world equilibrium and economic growth. The usefulness of these models notwithstanding, the impact of international trade of economic growth is ultimately an empirical question, though our theoretical analysis has already highlighted some important mechanisms and also suggested that the negative effects of trade on growth are unlikely to be important. Whether the positive effects of trade on technological progress are quantitatively significant remains an open question. It may well be that static gains of trade are more important than its dynamic gains. Nevertheless, any analysis of international trade must take its implications on economic growth and technological change into account.

19.9. References and Literature

This chapter covered a variety of models. Section 19.1 focused on the implications of international financial flows on economic growth. This topic is discussed in detail in Chapter 3 of Barro and Sala-i-Martin (2004), both with and without limits to financial flows. Obstfeld and Rogoff (1996) Chapters 1 and 2 provide a more detailed analysis of international borrowing and lending. Chapter 6 of Obstfeld and Rogoff provides an excellent introduction to the implications of imperfections in international capital markets. Work that models these imperfections explicitly includes Atkeson (1991), Bulow and Rogoff (1989a, 1989b), Kehoe and Perri (2002), Aguiar, Amador and Gopinath (2006). The Feldstein-Horioka puzzle, which was also discussed in Section 19.1, is still an active area of research. Obstfeld (1995) and Obstfeld and Taylor (2004) present surveys of much of the research on this topic. Taylor (1994), Baxter and Crucini (1993) and Kraay and Ventura (2002) propose potential resolutions for the Feldstein-Horioka puzzle.

Section 19.2 is motivated by Lucas's classic (1990) article. There is a large literature on why capital does not flow from rich to poor countries. Obstfeld and Taylor (2004) contain a survey of the work in this area. The work by Caselli and Feyrer (2007) discussed above provides a method for estimating cross-country differences in the marginal productive capital and argues that differences in the return to capital are limited. This work supports models that explain the lack of capital flows based on productivity differences, such as the model presented in Section 19.2. See also recent work by Alfaro, Kalemli-Ozcan and Volosovych (2005), which also emphasizes productivity differences and links these to institutional factors which we discussed in Chapter 4. Recent work by Chirinko and Mallick (2007) argues that the Caselli and Feyrer (2007) procedure may lead to misleading results because they do not incorporate adjustment costs in investment in their calculations and that once these costs are incorporated, returns to capital differ significantly across countries.

The rest of the chapter relies on some basic knowledge of international trade theory. Space restrictions preclude a detailed review. The reader is referred to a standard text, for example, Dixit and Norman (1990). Section 19.3 provides a slight generalization of the model in Ventura (1997) (it considers a general costs and returns to scale production function rather than CES production function is in Ventura, 1997), though it omits some of the more detailed characterization of transitional dynamics in the paper. A similar but less rich model was first analyzed by Stiglitz (1971). Stiglitz did not feature labor-augmenting productivity differences across nations and assumed exogenous saving rates. Other papers that combine Heckscher-Ohlin trade with models of economic growth include Atkeson and Kehoe (2000) and Cunat and Maffezoli (2001). Section 19.4 builds on Acemoglu and Ventura (2002). The importance of terms of trade effects are well recognized in the theory of international trade (see again

Dixit and Norman, 1990), but their growth implications had not previously been recognized. The model presented in Acemoglu and Ventura (2002) is a much simplified Ricardian model, exploiting the structure of preferences first introduced by Armington (1969), but in the production of the final good rather than in preferences. Richer Ricardian models typically build on the seminal article by Dornbusch, Fischer and Samuelson (1977), though this richer setup has not yet been integrated with growth models. Ventura (2005) provides a survey of international trade and economic growth, focusing on the models in Sections 19.3 and 19.4.

The model in Section 19.5 builds on Krugman's (1979) seminal article on product cycle. As noted in the text, Vernon (1966) was the first to formulate the problem of the international product cycle, emphasizing the economic forces modeled in Krugman (1979) and in Section 19.5 here. Grossman and Helpman (1991b) provide richer models of the product cycle with endogenous technology, similar to the economy discussed in Exercise 19.25. Antras (2006) provides a new perspective on the international product cycle that relies on the importance of incomplete contracts. Contractual problems between Northern producers and Southern subsidiaries constitute a barrier slowing down the transfer of goods to the South. Only after goods become sufficiently "standardized," the contracting problems become less severe and the transfer of production to the south takes place.

There is a large empirical literature on the impact of trade on growth. Many of the best-known papers in this literature were discussed at the beginning of Section 19.6. The rest of Section 19.6 builds on Romer and Rivera Batiz (1991) and Grossman and Helpman (1991b), but uses the formulation from Section 13.1 in Chapter 13. Grossman and Helpman (1991b) assume that R&D requires labor and introduce competition between the R&D sector and the final good sector. In this case, the nature of the knowledge spillovers becomes important for the implications of trade on the pace of endogenous technological progress. Romer and Rivera Batiz (1991) also discuss the implications of the form of the innovation possibilities frontier for the effects of trade on technological change. This point, which is developed in Exercise 19.29, also features in recent work by Atkeson and Burstein (2006). Grossman and Helpman (1991b) also present much richer models with multiple sectors and factor proportion differences across countries, leading to Heckscher-Ohlin type trade. Another potential effect of international trade on technological change would be by influencing the direction of technological change. This topic is analyzed in detail in Acemoglu (2003b), where I show that trade opening, with imperfect intellectual property rights, can make new technologies more skill-biased than before trade opening. Similar models are also analyzed in Thoenig and Verdier (2002) and Epifani and Gancia (2006).

Section 19.7 presents a model inspired by Young (2003) and Matsuyama (2002). Lucas (1988) and Galor and Mountford (2006) also present similar models, which feature interaction between specialization and learning-by-doing. Other models where international trade may be costly rely on differences in the amount of rents generated by different sectors because of imperfections in the labor market or institutional problems. Levchenko (2008) and Nunn (2007) present models in which trade leads to the transfer of rent-creating jobs from countries with weak institutions to those with better institutions and may be harmful to countries with weak institutions. Davis and Harrigan (2007) present a model in which trade leads to the reallocation of high-rent jobs to some countries and can be harmful to the economies that are losing these jobs.

19.10. Exercises

Exercise 19.1. Prove Proposition 19.1.

Exercise 19.2. Prove Proposition 19.2. [Hint: use (19.5) together with the fact that consumption and output grow at the same rate in each country to show that in the steady state it is optimal for each country (or each consumer in each country) to choose $\dot{\mathcal{A}}_j(t) \to 0$.] Consider the world economy with free flows of capital, but assume that each country has a different discount factor ρ_j.

(1) Prove that 19.1 still holds.
(2) Show that there does not exist a steady-state equilibrium with $\dot{\mathcal{A}}_j(t) = 0$ for all j. Explain the intuition for this result.
(3) Characterize the asymptotic equilibrium (i.e., the equilibrium path as $t \to \infty$). Suppose that $\rho_{j'} < \rho_j$ for all $j \neq j'$. Show that the share of the world capital that is used in country j' will tend to 1. What does this imply for the relationship between GDP and GNP across countries.
(4) Is the form of the asymptotic equilibrium in part 3 of this exercise realistic? If not, explain how you would modify the model to achieve a more realistic world equilibrium in the presence of free capital flows.

Exercise 19.3. This exercise asks you to prove Proposition 19.3.

(1) Show that $c_j(t)/c_{j'}(t)$ is constant for all j and j'.
(2) Show that given the result in Proposition 19.1, the integrated world equilibrium can be represented by a single aggregate production function. [Hint: use an argument similar to that leading to Proposition 19.6].
(3) Relate this result and Proposition 19.6 to Theorem 5.4 in Chapter 5. Explain why these "aggregation" results would not hold without free capital flows.
(4) Given the result in parts 1 and 2, apply an analysis similar to that for the global stability of the equilibrium path in the basic neoclassical growth model to establish the global stability of the equilibrium path here. Given global stability, prove the uniqueness of the equilibrium path.

Exercise 19.4. * Consider a world economy with international capital flows, but suppose that because of sovereign risk a country cannot borrow more than a fraction $\phi > 0$ of its capital stock. Consequently, in terms of the model in Section 19.1, we have the restriction that

$$b_j(t) \leq \phi k_j(t).$$

(1) Show that the steady-state equilibrium of the world economy is not affected by this constraint. Explain the intuition for this result carefully.
(2) Characterize the transitional dynamics of the world economy under this constraint. Show that Corollary 19.1 no longer holds.

Exercise 19.5. Barro and Sala-i-Martin (1994, 2004) use growth regressions to look at the patterns of convergence across U.S. regions and states. They find that there is a slow pattern of convergence across regions and states and they interpret this through the lenses of the neoclassical growth model. Explain why Corollary 19.1 implies that this interpretation is not appropriate. Suggest instead an alternative explanation for why convergence across regions and states might be slow. [Hint: should we expect technology or capital to flow more rapidly across regions?]

Exercise 19.6. Consider the the baseline AK model studied in Chapter 11, and suppose that countries have the same production technology, but differ according to their discount rates, the ρ_j's. Show that there will be persistent differences in saving and investment rates across countries that are correlated, even in the presence of free financial flows across countries. Provide a precise intuition for this result. Explain why this model could not account for the Feldstein-Horioka puzzle, which does not refer to the correlation between saving and investment in levels but in differences. Can you extend this model to account for the Feldstein-Horioka puzzle?

Exercise 19.7. Prove Proposition 19.4.

Exercise 19.8. Consider the model in Section 19.3 with different discount rates across countries. Prove that there does not exist a steady-state equilibrium.

Exercise 19.9. * Consider the model in Section 19.3, but assume that (19.9) is now modified to

$$Y_j^K(t) = B_j K_j(t),$$

where B_j's potentially differ across countries. Characterize the world equilibrium in this case.

Exercise 19.10. *

(1) Show that all the results in Section 19.3 continue to hold if the constant relative risk aversion preferences in (19.14) is now modified to an arbitrary strictly increasing, strictly concave utility function $u(c)$.

(2) Now let us go back to the preferences as in (19.14), but suppose that productivity of labor in each country is given by

$$A_j(t) = A_j \exp(gt).$$

Show that all of the results from the text continue to apply, and in particular, derive the equivalent of Proposition 19.4.

(3) Finally, let us suppose that F in (19.7) does not satisfy Assumption 2. How does this affect the analysis and the results?

Exercise 19.11. Derive the unit cost functions (19.26) and (19.27) from the production functions (19.23) and (19.24). Determine the value of the constant χ.

Exercise 19.12. Derive (19.28) and (19.29).

Exercise 19.13. Consider the model in Section 19.4.

(1) Derive the trade balance equation (19.33) from the capital market clearing equation, (19.25).

(2) Prove that the ratio of imports to GDP at each t is equal to τ.

Exercise 19.14. Provide a rigorous proof of the global stability of the steady-state world equilibrium in Proposition 19.10.

Exercise 19.15. (1) Derive (19.39) and (19.40).

(2) Explain why different parameters determine cross-country income dispersion in these two equations.

(3) Using reasonable parameter values show how the model with international trade can generate much larger differences in income per capita across countries resulting from small parameter differences.

Exercise 19.16. Derive equation (19.42).

Exercise 19.17. Prove Proposition 19.11.

Exercise 19.18. Prove Proposition 19.12.

Exercise 19.19. Consider the steady-state world equilibrium in the model of Section 19.4.

(1) Show that an increase in τ does not necessarily increase the steady-state world equilibrium growth rate g^* as given by (19.37). Provide an intuition for this result.

(2) Show that even when τ does not increase growth, it increases world welfare. [Hint: to simplify the answer to this part of the question, you can simply look at steady state welfare].

(3) Interpret this finding in light of the debate about the effect of trade on growth.

(4) Provide a sufficient condition for an increase in τ to increase the world growth rate and interpret this condition.

Exercise 19.20. * Consider the model of Section 19.4, except that instead of utility maximization by a representative household, assume that each country saves a constant fraction s_j

of its income. Show that terms of trade effects will be present in equilibrium, but the steady state will be "degenerate," with the relative prices of goods supplied by the highest saving country going to zero. Explain why exogenous savings versus dynamic utility maximization give different answers in this case.

Exercise 19.21. * Consider the model of Section 19.4, but assume that $\varepsilon < 1$. Characterize the equilibrium. Show that in this case countries that have lower discount rates will be relatively poor. Provide a precise intuition for this result. Explain why the assumption that $\varepsilon < 1$ may not be plausible.

Exercise 19.22. * Consider the baseline AK model in Section 19.4. Suppose that production and allocation decisions within each country is made by a "country-specific social planner" (who maximizes the utility of the representative consumer within the country).

(1) Show that the equilibrium in the text is no longer an equilibrium. Explain why.
(2) Characterize the equilibrium in this case and show that all of the qualitative results derived in the text apply. In particular, provide generalizations of Propositions 19.11 and 19.12.
(3) Show that world welfare is lower in this case than in the equilibrium in the text. Explain why.
(4) Do you find the equilibrium in this exercise or the one in the text more plausible? Justify your answer.

Exercise 19.23. * Consider the model with labor in Section 19.4. Suppose that countries can invest in order to create new varieties that they will be able to sell to the world. Suppose that if a particular firm creates such a variety, it can charge a markup equal to the monopoly price to all consumers in the world.

(1) Show that the optimal monopoly price for a firm in country j at time t is: $p_j(t) = (\varepsilon r_j(t)) / (\varepsilon - 1)$. Interpret this equation.
(2) Suppose that a new variety can be created by using $1/\eta$ units of labor. Show how this changes the labor market clearing condition and specify the free entry condition.
(3) Derive the world income distribution and show that it is stable, so that the same forces as in the model with exogenous distribution of products across countries apply in this model.
(4) What happens if new products can be produced using a combination of labor and capital?

Exercise 19.24. Show that in the model of Section 19.5 an increase in ι will always (weakly) close the relative income gap between the North and the South. Characterize the conditions under which an increase in ι will make the North worse-off (in terms of reducing its real income). Interpret these results.

Exercise 19.25. This exercise asks you to endogenize innovation decisions in the model of Section 19.5. Assume that new goods are created by technology firms in the North as in the model in Section 13.4 in Chapter 13, and these firms are monopolist suppliers until the good they have invented is copied by the South. The technology of production is the same as before, and assume that new goods can be produced by using final goods, with the technology $N(t) = \eta Z(t)$, where $Z(t)$ is final good spending. Imitation is still exogenous and takes place at the rate ι. Once a good is imitated, it can be produced competitively in the South.

(1) Show that for a good that is not copied by the South, the price will be
$$p(t, \nu) = \frac{\varepsilon}{\varepsilon - 1} w^n(t).$$
(2) Characterize the equilibrium for given levels of $N^n(t)$ and $N^o(t)$.
(3) Compute the net present value of a new product for a Northern firm. Why does it differ from the expression in Section 13.4?
(4) Impose the free entry condition and derive the equilibrium rate of technological change for the world economy. Compute the world growth rate.
(5) What is the effect of an increase in ι on the equilibrium? Can an increase in ι make the South worse-off? Explain the intuition for this result.

Exercise 19.26. Consider a variation of the product cycle model in Section 19.5. Suppose there is no international trade, so that, the number of goods produced and consumed in each country will differ.

(1) Show that wages and incomes in the North and the South at time t are
$$w^n(t) = N(t)^{\frac{1}{\varepsilon-1}} \text{ and } w^s(t) = N^o(t)^{\frac{1}{\varepsilon-1}}.$$
(2) Derive a condition for relative income differences to be smaller in this case than in the model with international trade. Provide a precise intuition for why international trade may increase relative income differences
(3) If trade increases the income differences between the North and the South, does it mean that it reduces welfare in the South? [Hint: if you wish, you can again use the steady-state welfare levels].

Exercise 19.27. Prove Proposition 19.13.

Exercise 19.28. Prove Proposition 19.14.

Exercise 19.29. Consider the model in Section 19.6, but assume that new products are created with the innovation possibilities frontier as in Section 13.2 in Chapter 13. Assume that before trade knowledge spillovers are created by the entire set of available inputs in the world economy, that is, the innovation possibilities frontier is similar to (13.24) in Section 13.2, except that
$$\dot{N}^j(t) = \eta N(t) L_R^j(t)$$

for country j, where $N(t) = N^1(t) + N^2(t)$ and $L_R^j(t)$ is the number workers working in R&D in country j. Consequently, trade opening does not change the structure of knowledge spillovers.

(1) Show that in this model, trade opening has no effect on the equilibrium growth rate. Provide a precise intuition for this result.
(2) Next assume that before trade opening the innovation possibly the frontier takes the form $\dot{N}^j(t) = \eta N^j(t) L_R^j(t)$. Show that in this case, trade opening leads to an increase in the equilibrium growth rate as in Proposition 19.14. Explain why the results are different.
(3) Which of the specifications in 1 and 2 is more plausible? In light of your answer to this question, how do you think trade opening should affect economic growth.

Exercise 19.30. Consider the model in Section 19.6, with two differences. First, population grows at the rate n in both countries. Second, the innovation possibilities frontier is given as

$$\dot{N}^j(t) = \eta N(t)^{-\phi} Z^j(t)$$

for country j, where $N(t) = N^1(t) + N^2(t)$. Show that trade opening leads to greater technological progress upon impact, but the long-run growth rate of each country remains unchanged.

Exercise 19.31. Prove Proposition 19.15.

Exercise 19.32. (1) Prove Proposition 19.16.
(2) Explain why when $\varepsilon = 1$, specialization in the sector without learning-by-doing does not have an adverse effect on the relative income of the South.
(3) What are the implications of trade opening on relative incomes if $\varepsilon < 1$?
(4) Characterize the equilibrium if all economies are closed until time $t = T$ and then open to international trade at time T. What are the implications of this result for infant industry protection.

Exercise 19.33. Consider the economy in Section 19.7, but assume that the South is bigger than the North.

(1) Show that in this case not all Southern workers will work in sector 2 and there will be some learning-by-doing in the South.
(2) How does this affect the long-run equilibrium? [Hint: show that the limiting value of L_s^1 is equal to 0].

Part 7

Economic Development and Economic Growth

In this part of the book, I discuss the relationship between economic development and economic growth. The first question that the reader will rightly ask is why there is (or there should be) a distinction between economic development and economic growth. This question is particularly apt because I have argued in Chapter 1 that societies that are rich—*developed*—today are those that have grown steadily over the past 200 years and those that are poor or *less-developed* are those that have not achieved this type of steady growth. This perspective suggests that economic development and economic growth are essentially the same thing and should be studied together. Nevertheless, there are two reasons, one good and one bad, for drawing a distinction between development and growth. The good reason is that even though economic development and growth are part of the same process, models of growth emphasize different aspects of this process than models of economic development. In particular, the models we have studied so far focus on either balanced growth or transitional dynamics leading to balanced growth. Even though we have emphasized these transitional dynamics in a number of contexts, our main interest has been to ensure that they take us towards a balanced growth path. Behavior along or near the balanced growth path of a neoclassical or endogenous growth economy provides a good approximation to the behavior of relatively developed societies. But many salient features of economic growth at lower incomes or at earlier stages of development are not easy to map to this "orderly" behavior of balanced growth. In fact, Simon Kuznets and other economists have documented that even in more developed economies, many aspects of the process of economic growth are far from the balanced benchmark implied by the standard neoclassical growth model.

Motivated by these patterns, in his classic book *Modern Economic Growth*, Simon Kuznets defines economic growth as follows:

> "We identify the economic growth of nations as a sustained increase in per capita or per worker product, most often accompanied by an increase in population and usually by sweeping structural changes. In modern times these were changes in the industrial structure within which product was turned out and resources employed—away from agriculture toward nonagricultural activities, the process of industrialization; in the distribution of population between the countryside and the cities, the process of urbanization; in the relative economic position of groups within the nation distinguished by employment status, attachment to various industries, level of per capita income, and the like; in the distribution of product by use—among household consumption, capital formation, and the government consumption, and within each of these major categories by further subdivisions; in the allocation of product by its origin within the nation's boundaries and elsewhere; and so on." Simon Kuznets (1966).

Although one might debate whether this is the most functional definition of economic growth, it does capture a range of important changes that accompany economic growth in most societies. And yet, the models of economic growth we have studied so far do not do justice to the complex process described by Kuznets. They provide a framework for explaining the sustained increase in income per capita or output per worker. But our models do not feature Kuznets's *sweeping structural changes*.

A complementary perspective to Kuznets's vision is provided by early development economists, such as Hirschman, Nurske and Rosenstein-Rodan, who emphasized the importance of potential *market failures* and *poverty traps* in the process of development. If such market failures and poverty traps are an important determinant of economic performance, then we may expect them to be more widespread in less-developed, poorer economies.[3] Thus one might expect Kuznets's structural change to be accompanied by a process that involves the organization of production becoming more efficient and the economy moving from the interior of the aggregate production possibilities set towards its frontier.

A useful theoretical perspective might therefore be to consider the early stages of economic development taking place in the midst of—or even via—this type of structural transformation, which includes both the structural changes emphasized by Kuznets and the process of less-developed economies approaching their efficiency frontier. We may then expect this structural transformation to ultimately bring the economy to the neighborhood of balanced growth, where our focus so far has been. If this perspective is indeed useful, we would like to develop unified models that explain both the structural changes at the early stages of development and the behavior approximated by balanced growth at the later stages.

Some of the models we have seen so far take steps in this direction. For example, the model of takeoff in Section 17.6 in Chapter 17 captures a specific type of transformation, from volatile, low-productivity growth into sustained, stable growth. In addition, many of the models in Chapter 18 emphasize the difference between frontier economies and technological followers. Nevertheless, we have not offered a framework that can do justice to Kuznets's vision and this is largely because the current growth literature is far from a satisfactory framework that can achieve this objective. In this light, the distinction between economic growth and economic development can be justified by arguing that, in the absence of a unified framework or perhaps precisely *before* we can develop a unified framework, we need to study the two aspects of the long-run growth process separately. Economic growth, according to this division of labor, focuses on balanced growth, the growth behavior of the world economy, and other aspects of the growth process approximating the behavior of relatively developed

[3] In fact, these theoretical perspectives may be the justification for referring to relatively poor economies as *underdeveloped* rather than as *developing*. In what follows, unless there is a special reason for using these terms, I stick with the less tainted adjectives "less-developed" or relatively poor.

economies. Economic development, on the other hand, becomes the study of structural transformations, and the efficiency implications of these transformations, at the early stages of development. Models of economic development would then focus on structural changes in the production and consumption, on urbanization, on the size and the composition of the population, on the occupational structure, and on changes in living and social arrangements. The study of economic development will then seek to understand when, why and how these processes take place and whether they contribute to a less-developed economy moving towards the frontier of its production possibilities set. Since, as emphasized by Kuznets, economic growth in relatively developed economies also incorporates important element of structural change, part of our analysis in the context of economic development will also shed light on the nature of economic growth in more advanced nations, for example, by helping us understand why and how relatively balanced growth can often go hand-in-hand with major changes in the sectoral composition of output and employment.

The second—the not-so-satisfactory—reason for the distinction between economic growth and economic development is that there are separate literatures on these two topics, with very different emphases and often different questions. The economic growth literature focuses on the theoretical and empirical questions we have so far addressed in this book. The economic development literature, on the other hand, focuses on empirical analyses of education, poverty, discrimination, women's economic and social status, child outcomes, health, lending relations and agriculture in less-developed economies. Much of this literature is non-theoretical. It documents how economic relationships work in less-developed economies or identifies specific market failures. This literature has provided us with numerous facts that are helpful in understanding the economic relations in less-developed economies and has sometimes acted as a conduit for micro reforms that have improved the lives of the citizens of these less-developed economies. But this literature does not ask questions about the aspects of the process of economic development I have emphasized here—that is, it does not pose the question of *why* some countries are less productive and poorer, and *how and why* these less-developed economies can undergo the process of structural transformation associated with, and necessary for, modern economic growth. This implies that even though the reason for drawing a distinction between economic growth and economic development might be literature-driven, it may nonetheless be useful. Moreover, based on this distinction one may attempt to bridge the gap that exists between the distinct development and growth literatures by combining the theoretical tools developed in this book with the wealth of evidence collected by the empirical development literature. Such a combination might ultimately lead to a more satisfactory framework for understanding the process of economic development (though unfortunately space restrictions preclude me from pursuing these issues in detail here).

These two reasons motivate my acceptance of the standard distinction between economic development and economic growth. Although I go along with this standard distinction, throughout I emphasize how it is exactly the same tools that are useful for understanding the process of economic development—the structural transformations emphasized by Kuznets, Hirschman, Nurske and Rosenstein-Rodan—as well as the more orderly process of economic growth. My hope is that this approach will engender both greater efforts to develop a unified theoretical framework useful for understanding the process of development and also theoretical approaches that can make contact with and benefit from the wealth of evidence collected by the empirical development literature.

I organize this part of the book into two chapters. The first, Chapter 20, will focus on models that take only a minimal departure from the balanced growth approaches we have seen so far, while still shedding some light on the structural changes emphasized by Kuznets. The models in this chapter can thus be viewed as extensions of the neoclassical growth models in Chapters 8 and 11 designed to shed light on various important empirical patterns. However, these models neither do full justice to the process of sweeping structural changes emphasized by Kuznets nor do they capture the complex aspects of the process of economic development associated with the move from the interior of the production possibilities set towards the frontier. The second, Chapter 21, will focus on a number of models that investigate various different aspects of this process, including financial development, the demographic transition, urbanization, and other social changes. Furthermore, they highlight the importance of potential market failures that may cause development traps. These models present a range of exciting questions and different modeling approaches, but at the expense of providing less unity. Each model makes a different set of assumptions and the profession is far from a unified framework for the analysis of the major structural transformations involved in the process of development. The purpose of Chapter 21 is not to provide such a unified framework but to introduce the reader to these interesting and important questions. It should also be noted that the division between the two chapters is not perfect. Some of the models of structural transformation studied in Chapter 21 can be seen as closely related to the structural change models in Chapter 20. Moreover, some topics, such as the beginning of industrialization, can be treated both as a process of structural change and also as an outcome of a society solving certain market failures. Thus, there is quite a bit of arbitrariness in the decision of whether a particular topic should be in Chapter 20 or Chapter 21.

CHAPTER 20

Structural Change and Economic Growth

In this chapter, I discuss various different approaches to the analysis of structural change. The next two sections focus on the shift of employment and production from agriculture to manufacturing, and then from manufacturing to services. This is a useful starting point both because changes in the composition of employment and production are an important part of the process of economic development and also because, as emphasized by Kuznets and others, similar changes are present even beneath the façade of balanced modern growth. Consequently, these two sections will focus on demand-side and supply-side reasons why we may expect structural change as an economy becomes richer but also emphasize how such structural changes can be reconciled with balanced growth. Section 20.3 turns to a related theme. As emphasized in Chapter 1, industrialization appears to be an important element underlying the takeoff that led to modern growth and thus to the large cross-country income differences we witness today. In this section, I present a simple model of industrialization, which again emphasizes the importance of structural change but also shows how pre-industrial agricultural productivity may be a key determinant of the process of industrialization and takeoff.

20.1. Non-Balanced Growth: The Demand Side

Figure 20.1 provides a summary of some of the major changes in the structure of production that the US economy has undergone over the past 150 years. It shows that the share of US employment in agriculture stood at around 90% of the labor force at the beginning of the 19th century, while only a very small fraction of the US labor force worked in manufacturing and services. By the second half of the 19th century, both manufacturing and services had expanded to over 20% of employment, accompanied by a steep decline in the share of agriculture. Over the past 150 years or so, the share of employment in agriculture has continued to decline and now stands at less than 5%, while over 70% of US workers now work in service industries. The share of manufacturing first increased when the share of agriculture started its decline, but has been on a downward trend over the past 40 years or so and now stands at just over 20%. When we look at consumption shares, the general trends are similar, though the share of consumption expenditures on agricultural products is still substantial because of changes in relative prices and relative productivities (and also partly because of imports of agricultural goods). The changes in the composition of employment in the British economy towards the end of the 18th century are also consistent with the US patterns shown in Figure 20.1 (see, for example, Mokyr, 1989). Consequently, similar patterns are present in all OECD economies. Some of the less-developed economies are still largely agricultural but the time trend is inexorably towards a smaller share of agriculture. Because of Kuznets's emphasis on structural change and seminal work on the topic, Kongsamut, Rebelo and Xie (2001) refer to these changes in the composition of employment and production as the *Kuznets facts*. They provide a tractable model to reconcile this type of structural change with the *Kaldor facts* we have emphasized so far, that is, the relative constancy of factor shares and the interest rate.

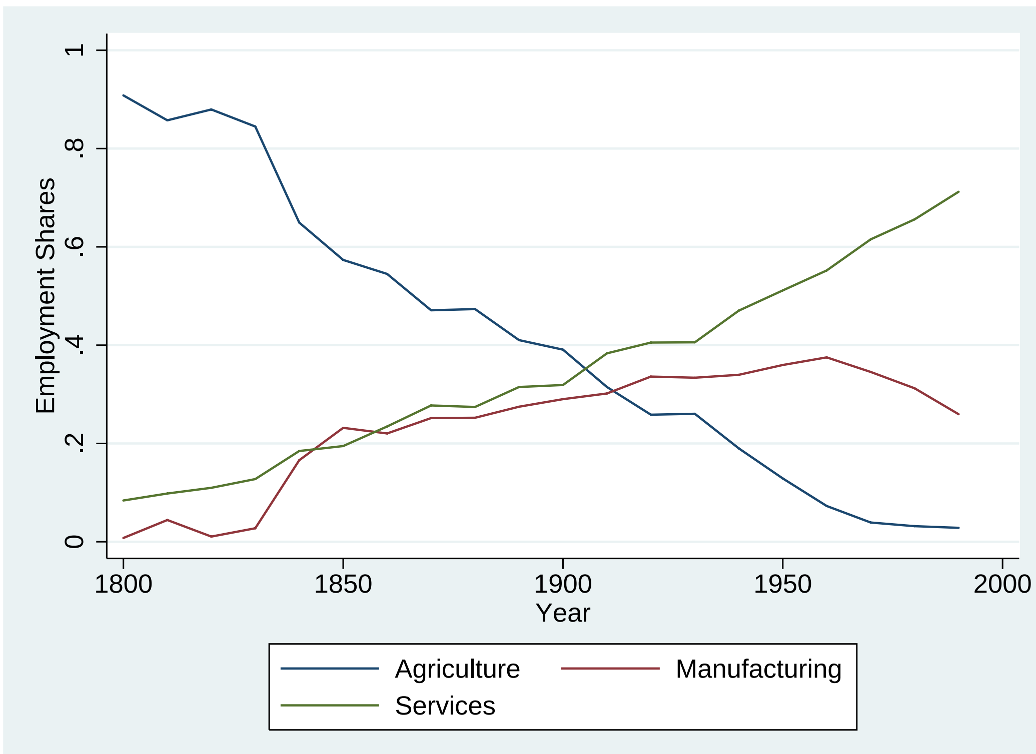

Figure 20.1. The share of US employment in agriculture, manufacturing and services, 1800-2000.

Figure 20.1 paints a picture of changes in sectoral employment that includes a significant *non-balanced* component. Consequently, models that depart from Kaldor facts over the early stages of the development process might be useful for understanding broader aspects of structural change. Kongsamut, Rebelo and Xie instead take a more modest departure from the baseline neoclassical growth model and propose a model that can account for a certain degree of non-balanced growth at the sectoral level, while still remaining consistent with the

Kaldor facts of aggregate balanced growth. Even though it is designed to match the Kaldor facts regardless of the stage of development, its tractability makes this model a useful starting point for our analysis. Moreover, as emphasized by Kuznets, once we look beneath the aggregate facts of balanced growth structural changes in the composition of employment and production are present even in relatively advanced economies. A model consistent with the Kaldor facts provides us with the simplest approach to these types of sectoral changes that appear to be ongoing even in relatively developed economies.

At the heart of Kongsamut, Rebelo and Xie's approach is the so-called *Engel's law*, which states that as a household's income increases, the fraction that it spends on food (agricultural products) declines. While calling this observation a law may exaggerate its status, this observation first made by the 19th-century German statistician Ernst Engel, appears to be a remarkably robust pattern in the data. Kongsamut, Rebelo and Xie extend Engel's law, by also positing that as a household becomes richer, it will desire not only to spend less on food, but will also wish to spend more on services. In particular, consider the following infinite-horizon economy. Population grows at the exogenous rate $n \geq 0$, so that total labor supply is

$$L(t) = \exp(nt) L(0). \tag{20.1}$$

The economy admits a representative household who supplies labor inelastically and has standard preferences given by

$$U(0) \equiv \int_0^\infty \exp(-(\rho - n)t) \frac{c(t)^{1-\theta} - 1}{1-\theta} dt, \tag{20.2}$$

with $\theta \geq 0$ and $c(t)$ denoting the consumption aggregate produced out of agricultural, manufacturing and service goods. I use the lower case letter here to emphasize that this is consumption per capita. Aggregate consumption itself consists of agricultural, manufacturing and services consumptions, with an aggregator of the form:

$$c(t) = \left(c^A(t) - \gamma^A\right)^{\eta^A} c^M(t)^{\eta^M} \left(c^S(t) + \gamma^S\right)^{\eta^S}, \tag{20.3}$$

where $c^A(t) \in [\gamma^A, \infty)$ denotes per capita agricultural consumption at time t, $c^M(t) \in \mathbb{R}_+$ is manufacturing consumption, and $c^S(t) \in \mathbb{R}_+$ is services consumption, while γ^A, γ^S, η^A, η^M and η^S are positive constants. This general functional form of the aggregator (preferences) is often referred to as the *Stone-Geary* preferences. It is a highly tractable way of introducing income elasticities that are different from one for different subcomponents of consumption, which will enable us to incorporate Engel's law. In particular, this aggregator implies that there is a minimum or *subsistence* level of agricultural (food) consumption equal to γ^A. The household must consume at least this much food to survive and in fact, consumption and utility are not defined when the household does not consume the minimum amount of food (recall (negative number)$^{1-\theta}$ is undefined for $\theta > 0$). After this level of food consumption has been achieved, the household starts to demand other items, in particular, manufactured goods (e.g., textiles and durables) and services (entertainment, retail, etc.). However, as we will see shortly, the presence of the γ^S term in the aggregator implies that the household will spend a positive amount on services only after certain levels of agricultural and manufacturing consumption have been reached.

We assume that the economy is closed, thus agricultural, manufacturing and services consumption must be met by domestic production. We follow Kongsamut, Rebelo and Xie and assume the following production functions for the agricultural, manufacturing and service goods:

$$\begin{aligned} Y^A(t) &= B^A F\left(K^A(t), X(t)L^A(t)\right), \\ Y^M(t) &= B^M F\left(K^M(t), X(t)L^M(t)\right), \\ Y^S(t) &= B^S F\left(K^S(t), X(t)L^S(t)\right), \end{aligned} \tag{20.4}$$

where $Y^j(t)$ for $j \in \{A, M, S\}$ denotes the output of agricultural, manufacturing and services at time t, $K^j(t)$ and $L^j(t)$ for $j \in \{A, M, S\}$ are the levels of capital and labor allocated to the agricultural, manufacturing and services sectors at time t, B^j for $j \in \{A, M, S\}$ is a Hicks-neutral productivity term for the three sectors and finally, $X(t)$ is a labor-augmenting (Harrod-neutral) productivity term affecting all sectors (I use the letter X instead of the standard A to distinguish this from the agricultural good). The function F satisfies the usual neoclassical assumptions, Assumptions 1 and 2, and thus in particular, exhibits constant returns to scale. Two other features in (20.4) are worth noting. First, the production function for all three sectors are identical. Second, the same labor-augmenting technology term affects all three sectors. Both of these features are clearly unrealistic but they are useful to isolate the demand-side sources of structural change and to contrast them with the supply-side factors that will be discussed in the next section. Furthermore, Exercises 20.7 below show that they can be relaxed to some degree. Throughout we take the initial population, $L(0) > 0$, and the initial capital stock, $K(0) > 0$, as given. Let us also assume that there is a constant rate of growth of the labor-augmenting technology term, i.e.,

$$\frac{\dot{X}(t)}{X(t)} = g \tag{20.5}$$

for all t, with initial condition $X(0) > 0$. To ensure that the transversality condition of the representative household holds, we impose the same assumption as in the basic neoclassical growth model of Chapter 8, Assumption 4 (which, recall, implies that $\rho - n > (1-\theta)g$).

Market clearing for labor and capital requires

$$K^A(t) + K^M(t) + K^S(t) = K(t), \tag{20.6}$$

and

$$L^A(t) + L^M(t) + L^S(t) = L(t), \tag{20.7}$$

where $K(t)$ and $L(t)$ are the total supplies of capital and labor at time t.

Another key assumption of the Kongsamut, Rebelo and Xie model builds on Rebelo (1991) and imposes that it is the manufacturing good that is used in the production of the investment good. Consequently, market clearing for the manufacturing good takes the form

$$\dot{K}(t) + c^M(t) L(t) = Y^M(t), \tag{20.8}$$

where, for simplicity, we have ignored capital depreciation (otherwise there would be an additional term $\delta K(t)$ on the left-hand side). This equation states that the total production of manufacturing goods is distributed between consumption of manufacturing goods and new capital stock, which will be used for production of agricultural, manufacturing and service goods in the future. Since the economy admits a representative household, equations (20.4)-(20.8) can also be taken to represent the representative household's budget constraint.

In addition, market clearing for the agricultural and service goods take the standard forms

$$c^A(t) L(t) = Y^A(t) \text{ and } c^S(t) L(t) = Y^S(t), \tag{20.9}$$

where the left-hand sides of both equations are multiplied by $L(t)$ to turn per capita consumption levels into total consumptions.

All market are competitive. Let us choose the price of the manufacturing good at each date as the numeraire, which leaves us with the prices of agricultural goods, $p^A(t)$, and of services, $p^S(t)$, and factor prices $w(t)$ and $r(t)$. The consumption aggregator (20.3) immediately implies that the prices of agricultural and service goods must satisfy

$$\frac{p^A(t)\left(c^A(t) - \gamma^A\right)}{\eta^A} = \frac{c^M(t)}{\eta^M}, \tag{20.10}$$

and

$$\frac{p^S(t)\left(c^S(t) + \gamma^S\right)}{\eta^S} = \frac{c^M(t)}{\eta^M}. \tag{20.11}$$

Competitive factor markets also imply

$$w(t) = \frac{\partial B^M F\left(K^M(t), X(t) L^M(t)\right)}{\partial L^M}, \tag{20.12}$$

and

$$r(t) = \frac{\partial B^M F\left(K^M(t), X(t) L^M(t)\right)}{\partial K^M}, \tag{20.13}$$

where I could have equivalently used the marginal products from other sectors, with identical results.

A competitive equilibrium is defined in the usual manner as sequences of sectoral factor demands $\left[K^A(t), K^M(t), K^S(t), L^A(t), L^M(t), L^S(t)\right]_{t=0}^{\infty}$ that maximize profits given the sequence of the total supplies of capital and labor $[K(t), L(t)]_{t=0}^{\infty}$ and the sequence of prices $\left[p^A(t), p^M(t), w(t), r(t)\right]_{t=0}^{\infty}$; price sequences $\left[p^A(t), p^M(t), w(t), r(t)\right]_{t=0}^{\infty}$ that satisfy (20.10)-(20.13) given $\left[K^A(t), K^M(t), K^S(t), L^A(t), L^M(t), L^S(t)\right]_{t=0}^{\infty}$; and sequences of consumption and capital $\left[c^A(t), c^M(t), c^S(t), K(t)\right]_{t=0}^{\infty}$ that maximize (20.2) subject to (20.4)-(20.8); and a sequence of labor supply $[L(t)]_{t=0}^{\infty}$ that satisfies (20.1). In addition, throughout, I assume that

$$B^A F\left(K^A(0), X(0) L^A(0)\right) > \gamma^A L(0), \tag{20.14}$$

so that the economy starts with enough capital and technological know-how to produce more than the minimum necessary amount of agricultural consumption

An equilibrium is straightforward to characterize in this economy. Because the production functions of the all three sectors are identical, the following result obtains immediately:

Proposition 20.1. *Suppose (20.14) holds. Then, in any equilibrium, the following conditions are satisfied:*

$$\frac{K^A(t)}{X(t) L^A(t)} = \frac{K^M(t)}{X(t) L^M(t)} = \frac{K^S(t)}{X(t) L^S(t)} = \frac{K(t)}{X(t) L(t)} \equiv k(t) \tag{20.15}$$

for all t, where the last equality defines $k(t)$ as the aggregate effective capital-labor ratio of the economy;

$$p^A(t) = \frac{B^M}{B^A} \tag{20.16}$$

for all t;

$$p^S(t) = \frac{B^M}{B^S} \tag{20.17}$$

for all t.

Proof. See Exercise 20.2. □

The results in this proposition are intuitive. First, the fact that the production functions are identical implies that the capital-labor ratios allocated to the three sectors must be equalized. Second, given (20.15), the equilibrium price relationships (20.16) and (20.17) follow from the fact that the marginal products of capital and labor have to be equalized in all three sectors.

Proposition 20.1 does not make use of the preference side. Next incorporating utility maximization on the side of the representative household, in particular, deriving the standard Euler equation for the representative consumer and then using equations (20.10)-(20.11), we obtain the following additional equilibrium conditions:

PROPOSITION 20.2. *Suppose (20.14) holds. Then, in any equilibrium, we have that*

$$\frac{\dot{c}^M(t)}{c^M(t)} = \frac{1}{\theta}(r(t) - \rho) \tag{20.18}$$

for all t and moreover, provided that Assumption 4 holds, the transversality condition of the representative household is satisfied. In addition, we have that for all t

$$\frac{p^A(t)\left(c^A(t) - \gamma^A\right)}{\eta^A} = \frac{c^M(t)}{\eta^M} \tag{20.19}$$

and

$$\frac{p^s(t)\left(c^S(t) + \gamma^S\right)}{\eta^S} = \frac{c^M(t)}{\eta^M}. \tag{20.20}$$

PROOF. See Exercise 20.3. □

In analogy to previous models we have seen so far, we may want to define a *balanced growth path* in this economy as an equilibrium path in which output and consumption of all three sectors grow at the same constant rate. The next proposition shows that such a balanced growth path does not exist.

PROPOSITION 20.3. *Suppose that either $\gamma^A > 0$ and/or $\gamma^S > 0$. Then a balanced growth path does not exist.*

PROOF. See Exercise 20.4. □

This result is not surprising. Since the preferences of the representative household incorporate Engel's law, the household would always like to change the composition of its consumption, and this will be reflected in a change in the composition of production. Instead of a balanced growth path, let us define a weaker notion of "balanced growth," which I will refer to as a *constant growth path* (CGP). A CGP requires that the rate of growth of aggregate consumption must be asymptotically constant.[1] Given the preferences in (20.2), the constant growth rate of consumption implies that the interest rate must also be constant asymptotically. In a CGP, output, consumption and employment in the three sectors may grow at different rates.

PROPOSITION 20.4. *Suppose (20.14) holds. Then, in the above-described economy a CGP exists if and only if*

$$\frac{\gamma^A}{B^A} = \frac{\gamma^S}{B^S}. \tag{20.21}$$

[1]Kongsamut, Rebelo and Xie, instead, define the concept of generalized balanced growth path, where the interest rate is constant. Clearly, given the CRRA preferences in (20.2), the two notions are equivalent.

In a CGP $k(t) = k^*$ for all t, and moreover we have the following evolution of consumption and employment in the three sectors

$$\frac{\dot{c}^A(t)}{c^A(t)} = g\frac{c^A(t) - \gamma^A}{c^A(t)},\ \frac{\dot{c}^M(t)}{c^M(t)} = g,\ \text{and}\ \frac{\dot{c}^S(t)}{c^S(t)} = g\frac{c^S(t) + \gamma^S}{c^S(t)}, \tag{20.22}$$

and

$$\frac{\dot{L}^A(t)}{L^A(t)} = n - g\frac{\gamma^A/L^A(t)}{B^A X(t) F(k^*, 1)},\ \frac{\dot{L}^M(t)}{L^M(t)} = n,\ \text{and}\ \frac{\dot{L}^S(t)}{L^S(t)} = n + g\frac{\gamma^S/L^S(t)}{B^S X(t) F(k^*, 1)}$$

for all t.

Moreover, in the CGP the share of national income accruing to capital is constant.

Proof. See Exercise 20.5. □

This model therefore delivers a tractable framework for the analysis of structural change that has potential relevance both for the experience of economies at the early stages of development and also for understanding the patterns of growth of relatively advanced countries. Engel's law (augmented with the highly income elastic demand for services) generates a demand-side force towards non-balanced growth. In particular, as their incomes grow, consumers wish to spend a greater fraction of their budget on services and a smaller fraction on food (agricultural goods). This makes an equilibrium with fully balanced growth impossible. Instead, different sectors grow at different rates and there is reallocation of labor and capital across sectors. Nevertheless, Proposition 20.4 shows that under condition (20.21) a constant growth path (CGP) exists and in this equilibrium, structural change takes place despite the fact that the interest rate and the share of capital in national income are constant. This model therefore delivers many of the features that are useful for thinking of the long haul of the process of development; in particular, the equilibrium path can be consistent with the Kaldor facts, and there is a continuous process of structural change, whereby the share of agriculture in production and employment declines over the development process and the share of services increases.

On the downside, a number of potential shortcomings of the current model are worth noting. First, one may argue that the process of structural change in this model falls short of the sweeping transformations discussed by Kuznets. It is straightforward to incorporate transitional dynamics into the model. Exercise 20.6 shows that if the effective capital-labor ratio starts out below its CGP value of k^* in Proposition 20.4, then there will be additional transitional dynamics in this model complementing the structural changes. Nevertheless, even these transitional dynamics probably fall short of the sweeping structural change emphasized by Kuznets. To some degree, whether or not this is so is a matter of taste and emphasis. The current model certainly does not incorporate the various different aspects of structural transformation which we will discuss in the next chapter, though it was also not meant to incorporate these transformations.

Second, the assumption that all three sectors have the same production function appears restrictive. Nevertheless, this assumption can be relaxed to some degree. Exercise 20.7 discusses how this can be done. Perhaps more important is the assumption that investments for all three sectors use only the manufacturing good. This assumption is similar in nature to the assumption that only capital is used to be produce capital (investment) goods in Rebelo's (1991) model, which we studied in Chapter 11. Exercise 20.10 shows that if this assumption is relaxed, it is no longer possible to reconcile the Kuznets and the Kaldor facts in the context of this model.

Third, the model presented here is designed to generate a constant share of employment in manufacturing. Although this pattern is broadly consistent with the US experience over the past 150 years, when we look at even earlier stages of development, almost all employment is in agriculture. This implies that early stages of structural change must also involve an increase in the share of employment in manufacturing. A number of models in the literature generate this pattern by also introducing land as an additional factor of production. Exercise 20.8 provides an example and Section 21.2 will further discuss models incorporating land as a major factor of production in the context of the study of population dynamics.

Finally, the condition necessary for a CGP, (20.21), is a rather "knife-edge" condition. We would not expect this condition to be satisfied naturally. Nevertheless, even when this condition is not satisfied, the behavior of the model may approximate the structural change we observe in practice and Exercise 20.9 illustrates this with an example in which sectoral production functions differ but are all of the Cobb-Douglas form.

20.2. Non-Balanced Growth: The Supply Side

The previous section showed how the process of structural change can be driven by a generalized form of Engel's law, that is, by the desires of the consumers to change the composition of their consumption as they become richer. An alternative approach to why growth may be non-balanced was first proposed by Baumol's (1967) seminal work. Baumol suggested that "uneven growth" (or what I am referring to here as non-balanced growth) will be a general feature of the growth process because different sectors will grow at different rates owing to different rates of technological progress (for example, technological progress might be faster in manufacturing than in agriculture or services). Although Baumol's original article derived this result only under a variety of additional assumptions, the general insight that there might be supply-side forces pushing the economy towards non-balanced growth is considerably more general. Here I review some ideas based on Acemoglu and Guerrieri (2006), who emphasize the supply-side causes of non-balanced growth. Ultimately, both the rich patterns of structural change during the early stages of development and those we witness in more advanced economies today require models that combine supply-side and demand-side factors. Nevertheless, isolating these factors in separate models is both more tractable and also conceptually more transparent. For this reason, in this section I focus on the supply side, abstracting from Engel's law throughout, and will only return to the combination of the supply-side and the demand-side factors in Exercise 20.17.

20.2.1. General Insights. At some level, Baumol's theory of non-balanced growth can be viewed as self-evident—if some sectors have higher rates of technological progress, there must be some non-balanced elements in equilibrium. My first purpose in this section is to show that there are more subtle and compelling reasons for supply-side non-balanced growth than those originally emphasized by Baumol. In particular, most growth models, like the Kongsamut, Rebelo and Xie model presented in the previous section, assume that production functions in different sectors are identical. In practice, however, industries differ considerably in terms of their capital intensity and also in terms of the intensity with which they use other factors (for example, compare the retail sector to durables manufacturing or transport). In short, different industries have different *factor proportions*. The main economic point I would like to emphasize in this section is that factor proportion differences across sectors combined with *capital deepening* will lead to non-balanced economic growth.

I will illustrate this point first using a simple but fairly general environment. This environment consists of two sectors each with a constant returns to scale production function and arbitrary preferences over the goods that are produced in these two sectors. Both sectors employ capital, K, and labor, L. To highlight that the exact nature of the accumulation process is not essential for the results, I take the sequence (process) of capital and labor supplies, $[K(t), L(t)]_{t=0}^{\infty}$, as given and assume that labor is supplied inelastically.

Preferences are defined over the final output or a consumption aggregator as in (20.3) in the previous section. Whether we use the specification with a consumption aggregator or a formulation with intermediates used competitively in the production of a final good makes no difference for any of the results. With this in mind, let final output be denoted by Y and assume that it is produced as an aggregate of the output of two sectors, Y_1 and Y_2,

$$Y(t) = F(Y_1(t), Y_2(t)).$$

Let us also assume that F satisfies Assumptions 1 and 2, so that, in particular, it exhibits constant returns to scale and is twice continuously differentiable. Sectoral production functions are given by

$$Y_1(t) = A_1(t) G_1(K_1(t), L_1(t)) \tag{20.23}$$

and

$$Y_2(t) = A_2(t) G_2(K_2(t), L_2(t)), \tag{20.24}$$

where $L_1(t)$, $L_2(t)$, $K_1(t)$ and $K_2(t)$ denote the amount of labor and capital employed in the two sectors, and the functions G_1 and G_2 are also assumed to satisfy the equivalents of Assumptions 1 and 2. The terms $A_1(t)$ and $A_2(t)$ are Hicks-neutral technology terms.

Market clearing for capital and labor implies that

$$\begin{aligned} K_1(t) + K_2(t) &= K(t), \\ L_1(t) + L_2(t) &= L(t), \end{aligned} \tag{20.25}$$

at each t. Without loss of any generality, I ignore capital depreciation.

Let us take the final good as the numeraire in every period and denote the prices of Y_1 and Y_2 by p_1 and p_2, and wage and rental rate of capital (interest rate) by w and r. Product and factor markets are competitive, so that product and factor prices satisfy

$$\frac{p_1(t)}{p_2(t)} = \frac{\partial F(Y_1(t), Y_2(t))/\partial Y_1}{\partial F(Y_1(t), Y_2(t))/\partial Y_2} \tag{20.26}$$

and

$$\begin{aligned} w(t) &= \frac{\partial A_1(t) G_1(K_1(t), L_1(t))}{\partial L_1} = \frac{\partial A_2(t) G_2(K_2(t), L_2(t))}{\partial L_2} \\ r(t) &= \frac{\partial A_1 G_1(K_1(t), L_1(t))}{\partial K_1} = \frac{\partial A_2 G_2(K_2(t), L_2(t))}{\partial K_2}. \end{aligned} \tag{20.27}$$

An equilibrium, given factor supply sequences, $[K(t), L(t)]_{t=0}^{\infty}$, is a sequence of product and factor prices, $[p_1(t), p_2(t), w(t), r(t)]_{t=0}^{\infty}$ and factor allocations, $[K_1(t), K_2(t), L_1(t), L_2(t)]_{t=0}^{\infty}$, such that (20.25), (20.26) and (20.27) are satisfied.

Let the shares of capital in the two sectors be defined as

$$\sigma_1(t) \equiv \frac{r(t) K_1(t)}{p_1(t) Y_1(t)} \text{ and } \sigma_2(t) \equiv \frac{r(t) K_2(t)}{p_2(t) Y_2(t)}. \tag{20.28}$$

There is *capital deepening* at time t if $\dot{K}(t)/K(t) > \dot{L}(t)/L(t)$. There are *factor proportion differences* at time t if $\sigma_1(t) \neq \sigma_2(t)$. And finally, technological progress is *balanced* at time t if $\dot{A}_1(t)/A_1(t) = \dot{A}_2(t)/A_2(t)$. Notice that factor proportion differences, that is, $\sigma_1(t) \neq \sigma_2(t)$, refers to the equilibrium factor proportions in the two sectors at time t. It does not necessarily mean that these will not be equal at some future date. The following proposition shows the supply side forces leading to structural change in the simplest possible way:

Proposition 20.5. *Suppose that at time t, there are factor proportion differences between the two sectors, technological progress is balanced, and there is capital deepening, then growth is not balanced, that is,* $\dot{Y}_1(t)/Y_1(t) \neq \dot{Y}_2(t)/Y_2(t)$.

Proof. First define the capital to labor ratio in the two sectors as

$$k_1(t) \equiv \frac{K_1(t)}{L_1(t)} \text{ and } k_2(t) \equiv \frac{K_2(t)}{L_2(t)},$$

and the "per capita production functions" (without the Hicks-neutral technology term) as

$$g_1(k_1(t)) \equiv \frac{G_1(K_1(t), L_1(t))}{L_1(t)} \text{ and } g_2(k_2(t)) \equiv \frac{G_2(K_2(t), L_2(t))}{L_2(t)}. \tag{20.29}$$

Since G_1 and G_2 are twice continuously differentiable by assumption, so are g_1 and g_2 and denote their first and second derivatives by g_1', g_2', g_1'' and g_2''.

Now, differentiating the production functions for the two sectors,

$$\frac{\dot{Y}_1(t)}{Y_1(t)} = \frac{\dot{A}_1(t)}{A_1(t)} + \sigma_1(t)\frac{\dot{K}_1(t)}{K_1(t)} + (1-\sigma_1(t))\frac{\dot{L}_1(t)}{L_1(t)}$$

and

$$\frac{\dot{Y}_2(t)}{Y_2(t)} = \frac{\dot{A}_2(t)}{A_2(t)} + \sigma_2(t)\frac{\dot{K}_2(t)}{K_2(t)} + (1-\sigma_2(t))\frac{\dot{L}_2(t)}{L_2(t)}.$$

To simplify the notation, I drop the time arguments for the remainder of this proof.

Suppose, to obtain a contradiction, that $\dot{Y}_1/Y_1 = \dot{Y}_2/Y_2$. Since F exhibits constant returns to scale, $\dot{Y}_1/Y_1 = \dot{Y}_2/Y_2$ together with (20.26) implies

$$\frac{\dot{p}_1}{p_1} = \frac{\dot{p}_2}{p_2} = 0. \tag{20.30}$$

Given the definition in (20.29), equation (20.27) gives the following conditions characterizing the equilibrium interest rate and wage:

$$\begin{aligned} r &= p_1 A_1 g_1'(k_1) \\ &= p_2 A_2 g_2'(k_2), \end{aligned} \tag{20.31}$$

and

$$\begin{aligned} w &= p_1 A_1 \left(g_1(k_1) - g_1'(k_1) k_1\right) \\ &= p_2 A_2 \left(g_2(k_2) - g_2'(k_2) k_2\right). \end{aligned} \tag{20.32}$$

Differentiating the interest rate condition, (20.31), with respect to time and using (20.30), we obtain:

$$\frac{\dot{A}_1}{A_1} + \varepsilon_{g_1'}\frac{\dot{k}_1}{k_1} = \frac{\dot{A}_2}{A_2} + \varepsilon_{g_2'}\frac{\dot{k}_2}{k_2}$$

where

$$\varepsilon_{g_1'} \equiv \frac{g_1''(k_1) k_1}{g_1'(k_1)} \text{ and } \varepsilon_{g_2'} \equiv \frac{g_2''(k_2) k_2}{g_2'(k_2)}.$$

Since $\dot{A}_1/A_1 = \dot{A}_2/A_2$,

$$\varepsilon_{g_1'}\frac{\dot{k}_1}{k_1} = \varepsilon_{g_2'}\frac{\dot{k}_2}{k_2}. \tag{20.33}$$

Differentiating the wage condition, (20.32), with respect to time, using (20.30) and some algebra gives:

$$\frac{\dot{A}_1}{A_1} - \frac{\sigma_1}{1-\sigma_1}\varepsilon_{g_1'}\frac{\dot{k}_1}{k_1} = \frac{\dot{A}_2}{A_2} - \frac{\sigma_2}{1-\sigma_2}\varepsilon_{g_2'}\frac{\dot{k}_2}{k_2}.$$

Since $\dot{A}_1/A_1 = \dot{A}_2/A_2$ and $\sigma_1 \neq \sigma_2$, this equation is inconsistent with (20.33), yielding a contradiction and proving the claim. □

The intuition for this result is straightforward. Suppose that there is capital deepening and that, for concreteness, sector 2 is more capital-intensive (i.e., $\sigma_1 < \sigma_2$). Now, if both capital and labor were allocated to the two sectors at constant proportions over time, the more capital-intensive sector, sector 2, would grow faster than sector 1. In equilibrium, the faster growth in sector 2 would change equilibrium prices, and the decline in the relative price of sector 2 would cause some of the labor and capital to be reallocated to sector 1. However, this reallocation *could not* entirely offset the greater increase in the output of sector 2, since, if it did, the relative price change that stimulated the reallocation would not take place. Consequently, equilibrium growth must be non-balanced.

Proposition 20.5 is related to the well-known Rybczynski's Theorem in international trade. Rybczynski's Theorem states that for an open economy within the "cone of diversification" (where factor prices do not depend on factor endowments), changes in factor endowments will be absorbed by changes in the sectoral output mix. Proposition 20.5 can be viewed both as a closed-economy analog and also as a generalization of Rybczynski's Theorem; it shows that changes in factor endowments (capital deepening) will be absorbed by faster growth in one sector than the other, even though relative prices of goods and factors will change in response to the change in factor endowments.

It is also straightforward to generalize Proposition 20.5 to an economy with $N \geq 2$ sectors. In particular, suppose that aggregate output is given by the constant returns to scale production function

$$Y = F\left(Y_1\left(t\right), Y_2\left(t\right), ..., Y_N\left(t\right)\right).$$

Defining $\sigma_j\left(t\right)$ as the capital share in sector $j = 1, ..., N$ as in (20.28), we have:

Proposition 20.6. *Suppose that at time t, there are factor proportion differences among the N sectors in the sense that there exists i and $j \leq N$ such that $\sigma_i\left(t\right) \neq \sigma_j\left(t\right)$, technological progress is balanced between i and j, i.e., $\dot{A}_i\left(t\right)/A_i\left(t\right) = \dot{A}_j\left(t\right)/A_j\left(t\right)$, and there is capital deepening, i.e., $\dot{K}\left(t\right)/K\left(t\right) > \dot{L}\left(t\right)/L\left(t\right)$, then growth is not balanced and $\dot{Y}_i\left(t\right)/Y_i\left(t\right) \neq \dot{Y}_j\left(t\right)/Y_j\left(t\right)$.*

Proof. See Exercise 20.11. □

20.2.2. Balanced Growth and Kuznets Facts. The previous subsection provided general insights about how supply-side factors can lead to non-balanced growth. To obtain a general result on the implications of capital deepening and factor proportion differences across sectors on non-balanced growth, Proposition 20.5 was stated for a given (arbitrary) sequence of capital and labor supplies, $\left[K\left(t\right), L\left(t\right)\right]_{t=0}^{\infty}$. However, without endogenizing the path of capital accumulation (and specifying the pattern of population growth) we cannot address whether a model relying on supply-side factors can also provide a useful framework for thinking about the Kaldor and the Kuznets facts.

For this purpose, I now specialize the environment of the previous subsection by incorporating specific preferences and production functions and then provide a full characterization of a simpler economy. The economy is again in infinite horizon and population grows at the exogenous rate $n > 0$ according to (20.1). Let us also assume that the economy admits a representative consumer, with standard preferences given by (20.2), who also supplies labor inelastically. Proposition 20.5 emphasized the importance of capital deepening, which will now result from exogenous technological progress.

Instead of a general production function for the final good as in the previous subsection, I now assume that the unique final good is produced with a constant elasticity of substitution aggregator:

$$Y(t) = \left[\gamma Y_1(t)^{\frac{\varepsilon-1}{\varepsilon}} + (1-\gamma) Y_2(t)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}, \tag{20.34}$$

where $\varepsilon \in [0, \infty)$ is the elasticity of substitution between the two intermediates and $\gamma \in (0,1)$ determines the relative importance of the two goods in aggregate production. Let us ignore capital depreciation again and also assume that the final good is distributed between consumption and investment, i.e.,

$$\dot{K}(t) + L(t)c(t) \leq Y(t), \tag{20.35}$$

where $c(t)$ is consumption per capita.

The two intermediates Y_1 and Y_2 are produced competitively with aggregate production functions

$$Y_1(t) = A_1(t) K_1(t)^{\alpha_1} L_1(t)^{1-\alpha_1} \text{ and } Y_2(t) = A_2(t) K_2(t)^{\alpha_2} L_2(t)^{1-\alpha_2}. \tag{20.36}$$

Throughout, I impose that

$$\alpha_1 < \alpha_2, \tag{20.37}$$

which implies that sector 1 is less capital-intensive than sector 2. This is without loss of any generality, since in the case in which $\alpha_1 = \alpha_2$, there are no supply-side effects and thus the issues I am concerned with in this section do not arise.

In (20.36) A_1 and A_2 correspond to Hicks-neutral technology term that grow at exogenous and potentially different rates given by

$$\frac{\dot{A}_1(t)}{A_1(t)} = a_1 > 0 \text{ and } \frac{\dot{A}_2(t)}{A_2(t)} = a_2 > 0. \tag{20.38}$$

Labor and capital market clearing again require that at each t,

$$L_1(t) + L_2(t) = L(t), \tag{20.39}$$

and

$$K_1(t) + K_2(t) = K(t). \tag{20.40}$$

Let us also denote the wage and the interest rate (the rental rate of capital) by $w(t)$ and $r(t)$ and the prices of the two intermediate goods by $p_1(t)$ and $p_2(t)$. We again normalize the price of the final good to 1 at each instant. An equilibrium is defined in the usual manner, as sequences of labor and capital allocations and prices, such that $[K_1(t), K_2(t), L_1(t), L_2(t)]_{t=0}^{\infty}$ maximize intermediate sector profits given the prices $[w(t),\ r(t),\ p_1(t), p_2(t)]_{t=0}^{\infty}$ and the aggregate capital and labor supplies $[K(t), L(t)]_{t=0}^{\infty}$; intermediate and factor markets clear at the prices $[w(t),\ r(t),\ p_1(t), p_2(t)]_{t=0}^{\infty}$; $[K(t),\ c(t)]_{t=0}^{\infty}$ maximize utility of the representative household given the prices $[w(t),\ r(t),\ p_1(t), p_2(t)]_{t=0}^{\infty}$; and population evolves according to (20.1).

It is useful to break the characterization of equilibrium into two pieces: *static* and *dynamic*. The static part takes the state variables of the economy, which are the capital stock, the labor supply and the technology, K, L, A_1 and A_2, as given and determines the allocation of capital and labor across sectors and the equilibrium factor and intermediate prices. The dynamic part of the equilibrium determines the evolution of the endogenous state variable, K (the dynamic behavior of L is given by (20.1) and those of A_1 and A_2 by (20.38)).

The choice of numeraire implies that at each instant

$$1 = \left[\gamma^{\varepsilon} p_1(t)^{1-\varepsilon} + (1-\gamma)^{\varepsilon} p_2(t)^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}},$$

and profit maximization of the final good sector implies

$$p_1(t) = \gamma \left(\frac{Y_1(t)}{Y(t)}\right)^{-\frac{1}{\varepsilon}} \quad \text{and} \quad p_2(t) = (1-\gamma)\left(\frac{Y_2(t)}{Y(t)}\right)^{-\frac{1}{\varepsilon}}. \tag{20.41}$$

Given this specification (and the fact that capital does not depreciate), the equilibrium allocation of resources will equate the marginal product of capital and labor into two sectors. The following equations give these equilibrium conditions and also expressions for the factor prices (see Exercise 20.12). The equilibrium conditions are

$$\gamma(1-\alpha_1)\left(\frac{Y(t)}{Y_1(t)}\right)^{\frac{1}{\varepsilon}} \frac{Y_1(t)}{L_1(t)} = (1-\gamma)(1-\alpha_2)\left(\frac{Y(t)}{Y_2(t)}\right)^{\frac{1}{\varepsilon}} \frac{Y_2(t)}{L_2(t)}, \tag{20.42}$$

and

$$\gamma\alpha_1\left(\frac{Y(t)}{Y_1(t)}\right)^{\frac{1}{\varepsilon}} \frac{Y_1(t)}{K_1(t)} = (1-\gamma)\alpha_2\left(\frac{Y(t)}{Y_2(t)}\right)^{\frac{1}{\varepsilon}} \frac{Y_2(t)}{K_2(t)}, \tag{20.43}$$

while the factor prices can be expressed as

$$w(t) = \gamma(1-\alpha_1)\left(\frac{Y(t)}{Y_1(t)}\right)^{\frac{1}{\varepsilon}} \frac{Y_1(t)}{L_1(t)}, \tag{20.44}$$

and

$$r(t) = \gamma\alpha_1\left(\frac{Y(t)}{Y_1(t)}\right)^{\frac{1}{\varepsilon}} \frac{Y_1(t)}{K_1(t)}. \tag{20.45}$$

The key to the characterization of the static equilibrium is to determine the fraction of capital and labor employed in the two sectors. Let us define $\kappa(t) \equiv K_1(t)/K(t)$ and $\lambda(t) \equiv L_1(t)/L(t)$. Combining equations (20.39), (20.40), (20.42), and (20.43), we obtain:

$$\kappa(t) = \left[1 + \frac{\alpha_2}{\alpha_1}\left(\frac{1-\gamma}{\gamma}\right)\left(\frac{Y_1(t)}{Y_2(t)}\right)^{\frac{1-\varepsilon}{\varepsilon}}\right]^{-1}, \tag{20.46}$$

and

$$\lambda(t) = \left[1 + \frac{\alpha_1}{\alpha_2}\left(\frac{1-\alpha_2}{1-\alpha_1}\right)\left(\frac{1-\kappa(t)}{\kappa(t)}\right)\right]^{-1}. \tag{20.47}$$

Equation (20.47) makes it clear that the share of labor in sector 1, λ, is monotonically increasing in the share of capital in sector 1, κ. This implies that in equilibrium capital and labor will be reallocated towards the same sector. The structure of the static equilibrium depends on how the allocation of capital and labor depends on the aggregate amount of capital and labor available in the economy. The following proposition answers this question.

Proposition 20.7. *In equilibrium,*

$$\frac{d\ln\kappa(t)}{d\ln K(t)} = -\frac{d\ln\kappa(t)}{d\ln L(t)} = \frac{(1-\varepsilon)(\alpha_2-\alpha_1)(1-\kappa(t))}{1+(1-\varepsilon)(\alpha_2-\alpha_1)(\kappa(t)-\lambda(t))} > 0 \text{ if and only if } (\alpha_2-\alpha_1)(1-\varepsilon) > 0. \tag{20.48}$$

$$\frac{d\ln\kappa(t)}{d\ln A_2(t)} = -\frac{d\ln\kappa(t)}{d\ln A_1(t)} = \frac{(1-\varepsilon)(1-\kappa(t))}{1+(1-\varepsilon)(\alpha_2-\alpha_1)(\kappa(t)-\lambda(t))} > 0 \text{ if and only if } \varepsilon < 1. \tag{20.49}$$

Proof. See Exercise 20.13. □

Equation (20.48) states that when the elasticity of substitution between sectors, ε, is less than 1, the fraction of capital allocated to the capital-intensive sector declines in the stock of capital (and conversely, when $\varepsilon > 1$, this fraction is increasing in the stock of capital). Intuitively, if K increases and κ remains constant, then the capital-intensive sector, sector 2, will grow by *more* than sector 1. Equilibrium prices given in (20.41) then imply that when $\varepsilon < 1$ the relative price of the capital-intensive sector will fall more than proportionately, inducing a greater fraction of capital to be allocated to the less capital-intensive sector 1. The intuition for the converse result when $\varepsilon > 1$ is similar.

Moreover, equation (20.49) implies that when the elasticity of substitution, ε, is less than one, an improvement in the technology of a sector causes the share of capital going to that sector to fall. The intuition is again the same: when $\varepsilon < 1$, increased production in a sector causes a more than proportional decline in its relative price, inducing a reallocation of capital away from it towards the other sector (again the converse results and intuition apply when $\varepsilon > 1$).

Combining (20.44) and (20.45), we also obtain relative factor prices as

$$\frac{w(t)}{r(t)} = \frac{1-\alpha_1}{\alpha_1}\left(\frac{\kappa(t)K(t)}{\lambda(t)L(t)}\right), \tag{20.50}$$

and the capital share in the economy as:

$$\sigma_K(t) \equiv \frac{r(t)K(t)}{Y(t)} = \gamma\alpha_1\left(\frac{Y_1(t)}{Y(t)}\right)^{\frac{\varepsilon-1}{\varepsilon}}\kappa(t)^{-1}. \tag{20.51}$$

Proposition 20.8. *In equilibrium,*

$$\frac{d\ln(w(t)/r(t))}{d\ln K(t)} = -\frac{d\ln(w(t)/r(t))}{d\ln L(t)} = \frac{1}{1+(1-\varepsilon)(\alpha_2-\alpha_1)(\kappa(t)-\lambda(t))} > 0. \tag{20.52}$$

$$\frac{d\ln(w(t)/r(t))}{d\ln A_2(t)} = -\frac{d\ln(w(t)/r(t))}{d\ln A_1(t)} = -\frac{(1-\varepsilon)(\kappa(t)-\lambda(t))}{1+(1-\varepsilon)(\alpha_2-\alpha_1)(\kappa(t)-\lambda(t))} < 0 \quad \text{if and only if} \quad (\alpha_2-\alpha_1)(1-\varepsilon) > 0. \tag{20.53}$$

$$\frac{d\ln\sigma_K(t)}{d\ln K(t)} < 0 \text{ if and only if } \varepsilon < 1. \tag{20.54}$$

$$\frac{d\ln\sigma_K(t)}{d\ln A_2(t)} = -\frac{d\ln\sigma_K(t)}{d\ln A_1(t)} < 0 \text{ if and only if } (\alpha_2-\alpha_1)(1-\varepsilon) > 0. \tag{20.55}$$

Proof. The results in (20.52) and (20.53) follow from differentiating equation (20.50) and Proposition 20.7. To prove the remaining claims, let me suppress time arguments and write:

$$\begin{aligned}\left(\frac{Y_1}{Y}\right)^{\frac{\varepsilon-1}{\varepsilon}} &= \left[\gamma+(1-\gamma)\left(\frac{Y_1}{Y_2}\right)^{\frac{1-\varepsilon}{\varepsilon}}\right]^{-1} \\ &= \gamma^{-1}\left(1+\frac{\alpha_1}{\alpha_2}\left(\frac{1}{\kappa}-1\right)\right)^{-1}\end{aligned}$$

Then, using the results of Proposition 20.7 and the definition of σ_K from (20.51), we have:

$$\frac{d\ln\sigma_K}{d\ln K} = -\Omega\frac{1-\sigma_K}{\sigma_K}\frac{\alpha_1}{\alpha_2}\frac{(1-\varepsilon)(\alpha_2-\alpha_1)(1-\kappa)/\kappa}{1+(1-\varepsilon)(\alpha_2-\alpha_1)(\kappa-\lambda)} \tag{20.56}$$

and

$$\frac{d\ln\sigma_K}{d\ln A_2} = -\frac{d\ln\sigma_K}{d\ln A_1} = \Omega\frac{1-\sigma_K}{\sigma_K}\frac{\alpha_1}{\alpha_2}\frac{(1-\varepsilon)(1-\kappa)/\kappa}{1+(1-\varepsilon)(\alpha_2-\alpha_1)(\kappa-\lambda)}, \tag{20.57}$$

where

$$\Omega \equiv \left[\left(1+\frac{\alpha_1}{\alpha_2}\left(\frac{1}{\kappa}-1\right)\right)^{-1} - \left(\frac{1-\alpha_1}{1-\alpha_2}+\frac{\alpha_1}{\alpha_2}\left(\frac{1}{\kappa}-1\right)\right)^{-1}\right].$$

Clearly, $\Omega > 0$ if and only if $\alpha_1 < \alpha_2$, which is satisfied in view of (20.37). Equations (20.56) and (20.57) then imply (20.54) and (20.55). □

The most important result in this proposition is (20.54), which links the equilibrium relationship between the capital share in national income and the capital stock to the elasticity of substitution. Since a negative relationship between the share of capital in national income and the capital stock is equivalent to capital and labor being gross complements in the aggregate, this result also implies that the elasticity of substitution between capital and labor is less than one if and only if ε is less than one. Recall from the discussion in Section 15.6 in Chapter 15 that a variety of different approaches suggest that the elasticity of substitution between capital and labor is less than one.

The intuition for Proposition 20.8 is informative about the workings of the model. Consistent with the discussion of Proposition 20.5 above, when $\varepsilon < 1$, an increase in the capital stock of the economy causes the output of the more capital-intensive sector, sector 2, to increase relative to the output in the less capital-intensive sector (despite the fact that the share of capital allocated to the less-capital intensive sector increases as shown in equation (20.48)). This then increases the production of the more capital-intensive sector and reduces the relative reward to capital (and the share of capital in national income). The converse result applies when $\varepsilon > 1$.

Recall also from Section 15.2 in Chapter 15 that when $\varepsilon < 1$, (20.55) in Proposition 20.8 implies that an increase in A_1 is "capital biased" and an increase in A_2 is "labor biased". The intuition for why an increase in the productivity of the sector that is intensive in capital is biased toward labor (and vice versa) is again similar: when the elasticity of substitution between the two sectors, ε, is less than one, an increase in the output of a sector (this time driven by a change in technology) decreases its price more than proportionately, thus reducing the relative compensation of the factor used more intensively in that sector. When $\varepsilon > 1$, we have the converse pattern, and an increase in A_2 is "capital biased," while an increase in A_1 is "labor biased"

We now turn to the characterization of the dynamic equilibrium path of this economy. We start with the Euler equation for consumers, which follows from the maximization of (20.2). The Euler equation for per capita consumption takes the familiar form:

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta}(r(t) - \rho). \tag{20.58}$$

Since the only asset of the representative household in this economy is capital, the transversality condition takes the standard form:

$$\lim_{t\to\infty} K(t) \exp\left(-\int_0^t r(\tau)\, d\tau\right) = 0, \tag{20.59}$$

which, together with the Euler equation (20.58) and the resource constraint (20.35), determines the dynamic behavior of consumption per capita and capital stock, c and K. Equations (20.1) and (20.38) give the behavior of L, A_1 and A_2.

A dynamic equilibrium is given by paths of wages, interest rates, labor and capital allocation decisions, w, r, λ and κ, satisfying (20.44), (20.42), (20.45), (20.43), (20.46) and (20.47), and of consumption per capita, c, capital stock, K, employment, L, and technology, A_1 and A_2, satisfying (20.1), (20.35), (20.38), (20.58), and (20.59).

Let us also introduce the following notation for growth rates of the key objects in this economy:

$$\frac{\dot{L}_s(t)}{L_s(t)} \equiv n_s(t), \frac{\dot{K}_s(t)}{K_s(t)} \equiv z_s(t), \frac{\dot{Y}_s(t)}{Y_s(t)} \equiv g_s(t) \text{ for } s=1,2 \text{ and } \frac{\dot{K}(t)}{K(t)} \equiv z(t), \frac{\dot{Y}(t)}{Y(t)} \equiv g(t),$$

Whenever they exist, we can also define the corresponding (limiting) asymptotic growth rates as follows:

$$n_s^* = \lim_{t\to\infty} n_s(t) \ , \ z_s^* = \lim_{t\to\infty} z_s(t) \text{ and } g_s^* = \lim_{t\to\infty} g_s(t),$$

for $s = 1, 2$. Similarly denote the asymptotic capital and labor allocation decisions by asterisks

$$\kappa^* = \lim_{t\to\infty} \kappa(t) \text{ and } \lambda^* = \lim_{t\to\infty} \lambda(t).$$

With this terminology, we can establish the following useful proposition.

PROPOSITION 20.9. (1) *If $\varepsilon < 1$, then $n_1(t) \gtreqless n_2(t) \Leftrightarrow z_1(t) \gtreqless z_2(t) \Leftrightarrow g_1(t) \lesseqgtr g_2(t)$.*

(2) *If $\varepsilon > 1$, then $n_1(t) \gtreqless n_2(t) \Leftrightarrow z_1(t) \gtreqless z_2(t) \Leftrightarrow g_1(t) \gtreqless g_2(t)$.*

PROOF. Omitting time arguments and differentiating (20.42) with respect to time, we obtain

$$\frac{1}{\varepsilon}g + \frac{\varepsilon-1}{\varepsilon}g_1 - n_1 = \frac{1}{\varepsilon}g + \frac{\varepsilon-1}{\varepsilon}g_2 - n_2, \tag{20.60}$$

which implies that $n_1 - n_2 = (\varepsilon - 1)(g_1 - g_2)/\varepsilon$ and establishes the first part of the proposition. Similarly differentiating (20.43) yields

$$\frac{1}{\varepsilon}g + \frac{\varepsilon-1}{\varepsilon}g_1 - z_1 = \frac{1}{\varepsilon}g + \frac{\varepsilon-1}{\varepsilon}g_2 - z_2 \tag{20.61}$$

and establishes the second part of the result. □

This proposition establishes the straightforward, but at first counter-intuitive, result that, when the elasticity of substitution between the two sectors is less than one, the equilibrium growth rate of the capital stock and labor force in the sector that is growing faster must be *less* than in the other sector. When the elasticity of substitution is greater than one, the converse result obtains. To see the intuition, note that terms of trade (relative prices) shift in favor of the more slowly growing sector. When the elasticity of substitution is less than one, this change in relative prices is more than proportional with the change in quantities and this encourages more of the factors to be allocated towards the more slowly growing sector.

PROPOSITION 20.10. *Suppose the asymptotic growth rates g_1^* and g_2^* exist. If $\varepsilon < 1$, then $g^* = \min\{g_1^*, g_2^*\}$. If $\varepsilon > 1$, then $g^* = \max\{g_1^*, g_2^*\}$.*

PROOF. Differentiating the production function for the final good (20.34) we obtain:

$$g(t) = \frac{\gamma Y_1(t)^{\frac{\varepsilon-1}{\varepsilon}} g_1(t) + (1-\gamma) Y_2(t)^{\frac{\varepsilon-1}{\varepsilon}} g_2(t)}{\gamma Y_1(t)^{\frac{\varepsilon-1}{\varepsilon}} + (1-\gamma) Y_2(t)^{\frac{\varepsilon-1}{\varepsilon}}}. \tag{20.62}$$

This equation, combined with $\varepsilon < 1$, implies that as $t \to \infty$, $g^* = \min\{g_1^*, g_2^*\}$. Similarly, combined with $\varepsilon > 1$, it implies that as $t \to \infty$, $g^* = \max\{g_1^*, g_2^*\}$. □

Consequently, when the elasticity of substitution is less than 1, the asymptotic growth rate of aggregate output will be determined by the sector that is growing more slowly, and the converse applies when $\varepsilon > 1$.

As in the previous section, we focus on a *constant growth path* (CGP), again defined as an equilibrium path where the asymptotic growth rate of consumption per capita exists and is constant, i.e.,

$$\lim_{t\to\infty} \frac{\dot{c}(t)}{c(t)} = g_c^*.$$

Let us also define the growth rate of total consumption as $\dot{C}(t)/C(t) \equiv g_C^* = g_c^* + n$, since it will be slightly more convenient to work with the growth rate of total consumption than the growth rate of consumption per capita. From the Euler equation (20.58), the fact that the growth rate of consumption or consumption per capita are asymptotically constant implies that the interest rate must also be asymptotically constant, that is, $\lim_{t\to\infty} \dot{r} = 0$.

To establish the existence of a CGP, we impose the following parameter restriction:

$$\rho - n \geq (1-\theta) \max\left\{\frac{a_1}{1-\alpha_1}, \frac{a_2}{1-\alpha_2}\right\}. \tag{20.63}$$

This assumption ensures that the transversality condition (20.59) holds. Terms of the form $a_1/(1-\alpha_1)$ or $a_2/(1-\alpha_2)$ appear naturally in equilibrium, since they capture the "augmented" rate of technological progress. In particular, recall that associated with the technological progress, there will also be endogenous capital deepening in each sector. The overall effect on labor productivity (and output growth) will depend on the rate of technological progress augmented with the rate of capital deepening. The terms $a_1/(1-\alpha_1)$ or $a_2/(1-\alpha_2)$ capture this, since a lower α_s corresponds to a greater share of capital in the sector $s = 1, 2$, and thus to a higher rate of augmented technological progress for a given rate of Hicks-neutral technological change. In this light, condition (20.63) can be understood as implying that the augmented rate of technological progress should be low enough to satisfy the transversality condition (20.59).

The next proposition will present the main result of this subsection and will characterize the relatively simple form of the CGP in the presence of non-balanced growth. However, rather than presenting the general case, it is useful to impose the following assumption
(20.64)
either (i) $a_1/(1-\alpha_1) < a_2/(1-\alpha_2)$ and $\varepsilon < 1$; or (ii) $a_1/(1-\alpha_1) > a_2/(1-\alpha_2)$ and $\varepsilon > 1$,

which will make it easier to state this result. In particular, this condition ensures that sector 1 is the *asymptotically dominant* sector, either because it has a slower rate of technological progress and $\varepsilon < 1$, or it has more rapid technological progress and $\varepsilon > 1$. Notice also that, for the reasons noted above, the appropriate comparison is not between a_1 and a_2, but between $a_1/(1-\alpha_1)$ and $a_2/(1-\alpha_2)$. Exercise 20.14 generalizes the results in this proposition to the case in which the converse of condition (20.64) holds.

Proposition 20.11. *Suppose that conditions (20.37), (20.63) and (20.64) hold. Then there exists a unique CGP such that*

$$g^* = g_C^* = g_1^* = z_1^* = n + g_c^* = n + \frac{1}{1-\alpha_1}a_1, \tag{20.65}$$

$$z_2^* = n - (1-\varepsilon)\,a_2 + (1 + (1-\varepsilon)(1-\alpha_2))\frac{a_1}{1-\alpha_1} < g^*, \tag{20.66}$$

$$g_2^* = n + \varepsilon a_2 + (1 - \varepsilon(1-\alpha_2))\frac{a_1}{1-\alpha_1} > g^*, \tag{20.67}$$

$$n_1^* = n \text{ and } n_2^* = n - (1-\varepsilon)(1-\alpha_2)\left(\frac{a_2}{1-\alpha_2} - \frac{a_1}{1-\alpha_1}\right) < n_1^*. \tag{20.68}$$

Proof. Suppose first that $g_2^* \geq g_1^* > 0$ and $\varepsilon > 1$. Then equations (20.46) and (20.47) imply that $\lambda^* = \kappa^* = 1$. In view of this, Proposition 20.10 implies $g^* = g_1^*$. This condition together with equations, (20.36) , (20.60) and (20.61), solves uniquely for n_1^*, n_2^*, z_1^*, z_2^*, g_1^* and g_2^* as given in equations (20.65), (20.66), (20.67) and (20.68). Note that this solution is consistent with $g_2^* > g_1^* > 0$, since conditions (20.37) and (20.63) imply that $g_2^* > g_1^*$ and $g_1^* > 0$. Finally, $C(t) \equiv c(t)L(t) \leq Y(t)$, (20.35) and (20.59) imply that the consumption growth rate, g_C^*, is equal to the growth rate of output, g^*. To see why, suppose that this last claim were not correct, then since $C(t)/Y(t) \to 0$ as $t \to \infty$, the resource constraint (20.35) would imply that asymptotically $\dot{K}(t) = Y(t)$. Integrating this we obtain $K(t) \to \int_0^t Y(s)\,ds$, and since Y is growing exponentially, this implies that the capital stock grows more than exponentially, thus violating the transversality condition (20.59).

Finally, we can verify that an equilibrium with z_1^*, z_2^*, m_1^*, m_2^*, g_1^* and g_2^* satisfies the transversality condition (20.59). Note that the transversality condition (20.59) will be satisfied if

$$\lim_{t\to\infty}\frac{\dot{K}(t)}{K(t)} < r^*, \tag{20.69}$$

where r^* is the constant asymptotic interest rate. Since from the Euler equation (20.58) $r^* = \theta g^* + \rho$, (20.69) will be satisfied when $g^*(1-\theta) < \rho$. Condition (20.63) ensures that this is the case with $g^* = n + a_1/(1-\alpha_1)$. The argument for the case in which $g_1^* \geq g_2^* > 0$ and $\varepsilon > 1$ is similar and is left to Exercise 20.14.

To complete the proof, we need to establish that in all CGPs $g_2^* \geq g_1^* > 0$ when $\varepsilon < 1$ ($g_1^* \geq g_2^* > 0$ when $\varepsilon > 1$ is again left to Exercise 20.14). We now separately derive a contradiction for two configurations, (1) $g_1^* \geq g_2^*$, or (2) $g_2^* \geq g_1^*$ but $g_1^* \leq 0$.

(1) Suppose $g_1^* \geq g_2^*$ and $\varepsilon < 1$. Then, following the same reasoning as above, the unique solution to the equilibrium conditions (20.36), (20.60) and (20.61), when $\varepsilon < 1$ is:

$$g^* = g_C^* = g_2^* = z_2^* = n + a_2/(1-\alpha_2),$$
$$z_1^* = n - (1-\varepsilon) a_1 + (1 + (1-\varepsilon)(1-\alpha_1)) \frac{a_1}{1-\alpha_1},$$
$$g_1^* = n + \varepsilon a_1 + (1 - \varepsilon(1-\alpha_1)) \frac{a_1}{1-\alpha_1} \tag{20.70}$$

and also similar expressions for n_1^* and n_2^*. Combining these equations implies that $g_1^* < g_2^*$, which contradicts the hypothesis $g_1^* \geq g_2^* > 0$. The argument for $\varepsilon > 1$ is analogous.

(2) Suppose $g_2^* \geq g_1^*$ and $\varepsilon < 1$, then the same steps as above imply that there is a unique solution to equilibrium conditions (20.36), (20.60) and (20.61), which are given by equations (20.65), (20.66), (20.67) and (20.68). But now (20.65) directly contradicts $g_1^* \leq 0$. Finally suppose $g_2^* \geq g_1^*$ and $\varepsilon > 1$, then the unique solution is given by the equations in subpart 1 above. But in this case, (20.70) directly contradicts the hypothesis that $g_1^* \leq 0$, completing the proof.

□

A number of implications of this proposition are worth emphasizing. First, as long as $a_1/(1-\alpha_1) \neq a_2/(1-\alpha_2)$, growth is non-balanced. The intuition for this result is the same as Proposition 20.5 in the previous subsection. Suppose, for concreteness, that $a_1/(1-\alpha_1) < a_2/(1-\alpha_2)$ (which would be the case, for example, if $a_1 \approx a_2$). Then, differential capital intensities in the two sectors combined with capital deepening in the economy (which itself results from technological progress) ensures faster growth in the more capital-intensive sector, sector 2. Intuitively, if capital were allocated proportionately to the two sectors, sector 2 would grow faster. Because of the changes in prices, capital and labor are reallocated in favor of the less capital-intensive sector, so that relative employment in sector 1 increases. However, crucially, this reallocation is not enough to fully offset the faster growth of real output in the more capital-intensive sector. This result also highlights that the assumption of balanced technological progress in Proposition 20.5 (which, in this context, corresponds to $a_1 = a_2$) was not necessary for the result there, but we simply needed to rule out the knife-edge case where the relative rates of technological progress between the two sectors were exactly in the right proportion to ensure balanced growth (in this context, $a_1/(1-\alpha_1) = a_2/(1-\alpha_2)$).

Second, the CGP growth rates are relatively simple, especially because we have restricted attention to the set of parameters that ensure that sector 1 is the asymptotically dominant sector (cf., condition (20.64)). If, in addition, we also have $\varepsilon < 1$, the model leads to the richest set of dynamics, whereby the more slowly growing sector determines the long-run growth rate of the economy, while the more rapidly growing sector continually sheds capital and labor, but does so at exactly the right rate to ensure that it still grows faster than the rest of the economy.

Third, in the limiting equilibrium the share of capital and labor allocated to one of the sector tends to one (e.g., when sector 1 is the asymptotically dominant sector, $\lambda^* = \kappa^* = 1$). Nevertheless, at all points in time both sectors produce positive amounts, so this limit point is never reached. In fact, at all times both sectors grow at rates *greater* than the rate of population growth in the economy. Moreover, when $\varepsilon < 1$, the sector that is shrinking grows *faster* than the rest of the economy at all point in time, even asymptotically. Therefore, the rate at which capital and labor are allocated away from this sector is determined in equilibrium to be *exactly* such that this sector still grows faster than the rest of the economy. This is the sense in which non-balanced growth is not a trivial outcome in this economy (with one of the sectors shutting down), but results from the positive but differential growth of the two sectors.

Finally, it can be verified that the capital share in national income and the interest rate are constant in the CGP. For example, when condition (20.64) holds, we have $\sigma_K^* = \alpha_1$. In contrast, when this condition does not hold $\sigma_K^* = \alpha_2$—in other words, the asymptotic capital share in national income will reflect the capital share of the dominant sector. It can also be verified that limiting interest rate is also constant (see Exercise 20.15), thus this model based on supply-side sources of non-balanced growth is also consistent with the Kaldor facts as well as the Kuznets facts. The analysis so far does not establish that the CGP is asymptotically stable. This is done in Exercise 20.16, which also provides an alternative proof of Proposition 20.11. Consequently, a model based on supply-side factors can also give useful insights about structural change. Naturally, to understand a sweeping long-run changes in the composition of output and employment, we need to combine the demand-side and the supply-side factors studied in the last two sections. Exercise 20.17 takes a first step in this direction.

20.3. Agricultural Productivity and Industrialization

Although the models presented in the last two sections have highlighted how demand-side and supply-side factors can lead to structural change (and also how structural change can be consistent with a constant balanced growth path and the Kaldor facts), they did not focus on the process of industrialization. Chapter 1 documented that the industrialization process, beginning at the end of the 18th century in Europe, lies at the root of modern economic growth and cross-country income differences. Thus a natural question is why industrialization started and then progressed rapidly in some countries while it did not in others. In view of the picture presented in Chapter 1, this question might hold important clues about the cross-country differences in income per capita today.

In this light, it would be useful to have a number of different approaches to this question and evaluate their pros and cons. Although this is part of my objective, I will not present these models all in one place. The first approach, based on the model of Acemoglu and Zilibotti (1997), was already presented as an application of stochastic growth models in Section 17.6 in Chapter 17. Although this theory focused on takeoff in general, the most relevant incident of takeoff in history is related to industrialization. Therefore, the theory in Section 17.6 can be interpreted as offering a potential explanation for the origins of industrialization based on whether the investments in different sectors undertaken by different societies turned out to be successful. In particular, societies that happen to have put a substantial fraction of their resources in sectors that turned out to be unlucky, or were ex post discovered not to be as productive, have been less successful than those that have invested in sectors and projects that were ex post more successful. The theory showed how success breeds success and a string of good outcomes can lead to a takeoff, whereby the society is able to diversify its sectoral and project-based risks successfully in a deeper financial markets and allocate its funds more productively towards high-return activities. In the next chapter, we will see another approach to the origins of industrialization based on the idea of *the big push* suggested by Rosenstein-Rodan. The model by Murphy, Shleifer and Vishny (1989) in Section 21.5 in the next chapter will formalize this notion and show how, in the presence of technologies with fixed costs and monopolistic competition, coordination failures might prevent industrialization. The attractive feature of this approach will be its close connection to the baseline endogenous growth models we have studied in Part 4. A potential shortcoming might be its reliance on multiple equilibria, without an explanation for why some societies manage to coordinate to the good equilibrium whereas others end up in the bad equilibrium.

Before turning to market failures in development, it is useful to consider another approach which will shed light on what factors might facilitate or even spur industrialization. A common argument in the economic history literature is that 18th-century England was particularly well-placed for industrialization because of its high agricultural productivity (e.g., Nurske, 1953, Rostow, 1960, Mokyr, 1989, or Overton, 2001). The basic idea is that societies with a high agricultural productivity can afford to shift part of their labor force to industrial activities. Some type of increasing returns coming from technology or demand is then invoked to argue that the ability to shift a critical fraction of the labor force to industry is an important element of the early industrial experience.

In this section, I present a model based on Matsuyama (1992), which formalizes this intuition and presents a number of comparative static results that are useful in thinking about the origins of industrialization. Matsuyama's model naturally complements the models we have already studied in this chapter, because it is, at some level, a model of structural change. It combines Engel's law and learning-by-doing externalities in the industrial sector. The model is not only a tractable framework for the analysis of the relationship between agricultural productivity and industrialization, but it also enables an insightful analysis of the impact of international trade on industrialization.

Consider the following infinite-horizon continuous time economy with a constant population normalized to 1. The preference side is modeled via a representative household with preferences given by

$$U(0) \equiv \int_0^\infty \exp(-\rho t)\,(c^A(t) - \gamma^A)^\eta c^M(t))dt, \tag{20.71}$$

which is similar to the preferences in (20.2). In particular, $c^A(t)$ denotes the consumption of the agricultural good and $c^M(t)$ is the consumption of the manufacturing good at time t. The parameter γ^A is again the minimum (subsistence) food requirement, ρ is the discount factor, and η designates the importance of agricultural goods versus manufacturing goods in the utility function. The representative household supplies labor inelastically. Let us also focus on the closed economy in the text, leaving some of the interesting extensions to open economy to Exercise 20.20.

Output in the two sectors is produced with the following production functions

$$Y^M(t) = X(t)\,F(L^M(t)) \tag{20.72}$$

and

$$Y^A(t) = B^A G(L^A(t)), \tag{20.73}$$

where as before Y^M and Y^A denote the total production of the manufacturing and the agricultural goods, and L^M and L^A denote the total labor employed in the two sectors. Both production functions F and G exhibit diminishing returns to labor. More formally, F and G are continuously differentiable and strictly concave. In particular, $F(0) = 0$, $F'(\cdot) > 0$, $F''(\cdot) < 0$, $G(0) = 0$, $G'(\cdot) > 0$, and $G''(\cdot) < 0$, where as usual F' and G' denotes first derivatives of these functions. Diminishing returns to labor might arise because they both use land or some other factor of production as well as labor. Nevertheless, it is simpler to assume diminishing returns rather than introduce another factor of production. The fact that there are diminishing returns implies that when labor is priced competitively, there will be equilibrium profits.

The key feature for this model of industrialization is that there is no technological progress in agriculture but the production function for the manufacturing good, (20.72), includes the term $X(t)$, which will allow for technological progress in manufacturing. Although there is no technological progress in agriculture, the productivity parameter B^A potentially differs across countries, reflecting either previous technological progress in terms of new agricultural methods or differences in land quality (even though here, for simplicity, we are focusing on a single country). Existing evidence shows that there are very large (perhaps too large) differences in labor productivity and TFP of agricultural activities among countries even today, thus allowing for potential productivity differences in agriculture is reasonable. Current research also shows that the image of the agricultural sector as a quasi-stagnant sector without technological progress is not accurate, and in fact, this sector experiences both substantial capital-labor substitution and major technological change (including the introduction of new varieties of seeds, mechanization, and organizational changes affecting productivity). Nevertheless, the current model provides a good starting point for our purposes.

Labor market clearing requires that

$$L^M(t) + L^A(t) \leq 1,$$

since total the labor supply is normalized to 1. Let $n(t)$ denote the fraction of labor employed in manufacturing as of time t. Since there will be full employment in this economy, $L^M(t) = n(t)$ and $L^A(t) = 1 - n(t)$.

The key assumption is that manufacturing productivity, $X(t)$, evolves over time as a result of learning-by-doing externalities as in Romer's (1986) model we studied in Chapter 11. In particular, suppose that the growth of the manufacturing technology, $X(t)$, is proportional to the amount of current production in manufacturing

$$\dot{X}(t) = \delta Y^M(t), \tag{20.74}$$

where $\delta > 0$ measures the extent of these learning-by-doing effects and we have an initial productivity level of $X(0) > 0$ at time $t = 0$ taken as given. As in the Romer model, learning-by-doing effects are external to individual firms. This type of external learning-by-doing effects are too reduced-form to generate insights about how productivity improvements take place in the industrial sector. Nevertheless, our analysis so far makes it clear that one can endogenize technology choices by introducing monopolistic competition and under the standard assumptions made in Part 4 above, this will generate a market size affect and lead to an equation similar to (20.74). Exercise 20.19 asks you to consider such a model.

In equilibrium, each firm will choose its labor demand in order to equate the value of the marginal product to the wage rate, $w(t)$. Let us choose the price of agricultural goods as the numeraire (i.e., normalize it to 1) and also assume that the equilibrium is interior with both sectors being active. Then, equilibrium labor demand equations in the two sectors will be given by

$$w(t) = B^A G'(1 - n(t)) \text{ and } w(t) = p(t) X(t) F'(n(t))$$

where $p(t)$ is the relative price of the manufactured good (in terms of the numeraire, the agricultural good). Market clearing then implies:

$$B^A G'(1 - n(t)) = p(t) X(t) F'(n(t)). \tag{20.75}$$

The presence of the term $\gamma^A > 0$ implies that as in Section 20.1, preferences are non-homothetic and that the income elasticity of demand for agricultural goods will be less than unity (while that for manufacturing goods will be greater than unity). As we have already seen, this is the simplest way of introducing Engel's law.

Let us also assume that aggregate productivity is high enough to meet the minimum agricultural consumption requirements of the entire population (which, here, is normalized to 1):

$$B^A G(1) > \gamma^A > 0. \tag{20.76}$$

If this inequality were violated, the economy's agricultural sector would not be productive enough to provide the subsistence level of food to all consumers.

Finally, the budget constraint of the representative household at each date t can be written as

$$c^A(t) + p(t) c^M(t) \leq w(t) + \pi(t)$$

where $\pi(t)$ is the profits per representative household, resulting from the diminishing returns in the production technologies.

An equilibrium in this economy is defined in the standard way as a sequence of consumption levels in the two sectors and allocations of labor between the two sectors at all dates, such that consumers maximize utility and firms maximize profits given prices, and goods and factor prices are such that all markets clear.

Maximization of (20.71) implies that for each household, and thus for the entire economy, we have

$$c^A(t) = \gamma^A + \eta p(t) c^M(t). \tag{20.77}$$

Since the economy is closed, production must equal consumption and thus

$$c^A(t) = Y^A(t) = B^A G(1 - n(t))$$

and

$$c^M(t) = Y^M(t) = X(t) F(n(t))$$

Now combining these equations with (20.75) and (20.77) yields

$$\phi(n(t)) = \frac{\gamma^A}{B^A}, \tag{20.78}$$

where

$$\phi(n) \equiv G(1 - n) - \eta G'(1 - n) F(n) / F'(n),$$

is a strictly decreasing function. Moreover, we have that $\phi(0) = G(1)$ and $\phi(1) < 0$. The ϕ function can be interpreted as the "excess demand" function for manufacturing over agriculture. An equilibrium has to satisfy (20.78). From Assumption (20.76) and the properties of the ϕ function, we can conclude that the equilibrium condition (20.78) has a unique interior solution in which

$$n(t) = n^* \in (0, 1).$$

Notice an important implication of this equation. Even though the current model is one of structural change like those in the previous two sections, it only generates changes in the composition of output—the fraction of the labor force working in agriculture remains constant at $1 - n^*$. This implies that, while the current model is useful for interpreting the origins of industrialization, it will not be sufficient to generate insights about why the composition of employment in different sectors of the economy has been changing over the past 150 or 200 years.

Next, using (20.78), the unique equilibrium allocation of labor between the two sectors satisfies

$$n^* = \phi^{-1}\left(\frac{\gamma^A}{B^A}\right). \tag{20.79}$$

Since ϕ is strictly decreasing, so is its inverse function ϕ^{-1} and thus the fraction of the labor force employed in manufacturing, n^*, is strictly increasing in B^A. This is the most important result of the current model and shows that a greater fraction of the labor force will be allocated to the manufacturing sector when agricultural productivity is higher. The reason for this result is intuitive: the Cobb-Douglas production function combined with homothetic preferences would imply a constant allocation of employment between the two sectors independent of their productivity. However, in the current model, preferences are non-homothetic preferences and a certain amount of food production is necessary first. When agricultural productivity, B^A, is high, a relatively small fraction of the labor force is sufficient to generate this minimal level of food production, and thus a greater fraction of the labor force can be employed in manufacturing.

This results, combined with learning-by-doing in manufacturing, cf. equation (20.74), is at the root of the relationship between agricultural productivity and industrialization. In particular, equation (20.74) implies that output in manufacturing grows at the constant rate, $\delta F(n^*)$, which is also positively related to B^A in view (20.79). Therefore, the current model generates a very simple representation of the often-hypothesized relationship between agricultural productivity and the origins of industrialization.

It is also useful to note that in the equilibrium of this model, because the shares of employment in manufacturing and agriculture are constant and there is no technological progress in the agricultural sector, agricultural output remains constant. All growth is generated by growth of manufacturing production. However, since manufacturing and agricultural goods are imperfect substitutes, the relative prices change, so expenditure on agricultural goods increases (see Exercise 20.18).

We can summarize these results as follows:

Proposition 20.12. *In the above-described model, the combination of learning-by-doing and Engel's law generates a unique equilibrium in which the share of employment of manufacturing is constant at* $n^* \equiv \phi^{-1}(\gamma^A/B^A)$*, and manufacturing output and consumption grow at the rate* $\delta F(n^*)$*, which is increasing in agricultural productivity* B^A*.*

We have so far characterized the equilibrium in a closed economy. A major result of this model is that higher agricultural productivity leads to faster industrial growth and thus to faster overall growth. The reason for this is intuitive: higher agricultural productivity enables the economy to allocate a larger fraction of its labor force to the knowledge-producing sector, which is manufacturing (where knowledge-production is introduced in a reduced-form manner as in Romer's (1986) model). Even though the presumption that most important knowledge-producing activities take place in the manufacturing sector is no longer generally accepted, this model provides a useful framework for the analysis of the origins of industrialization. An important advantage of the current model is its tractability. This enables us to adapt it easily to analyze other related questions, such as the impact of trade opening on industrialization. This is done in Exercise 20.20. The striking result in this case is that the implications of the closed and the open economies are very different. For example, that exercise shows that higher agricultural productivity, in the presence of international trade, can lead to delayed industrialization or even to deindustrialization, rather than being the source of rapid industrialization as in the closed economy. The reason for this is related to the forces we analyzed in Section 19.7 of Chapter 19; specialization according to comparative advantage may have negative long-run consequences in the presence of sector-specific externalities. However, as already discussed in that section, the evidence for large externalities of this sort are not very strong. Consequently, the model in this section and its implications regarding the role of international trade in the process of industrialization should be interpreted with some caution. Nevertheless, this model is an important tool in our arsenal of models of long-run economic development, especially because it illustrates in an elegant and tractable manner how agricultural productivity interacts with the process of industrialization.

20.4. Taking Stock

This chapter took a first step towards the analysis of structural changes involved in the process of economic development. Our first step has been relatively modest. The focus has been on the structural changes associated with the shifts in output and employment away from agriculture to manufacturing and to services and with the changes between sectors of different capital intensities. Section 20.1 focused on demand-side reasons for why growth can be non-balanced. In particular, it incorporated Engel's law into the basic neoclassical growth model so that households spend a smaller fraction of their budget on agricultural goods as they become richer. This framework is ideally suited for the analysis of the structural changes across broad sectors such as agriculture, manufacturing and services. Section 20.2, on the other hand, turned to supply-side reasons for non-balanced growth, which were first highlighted by Baumol's (1967) classic paper. However, instead of assuming exogenously-given different rates of technological progress across sectors, this section emphasized how sectoral differences in capital intensity can lead to non-balanced growth. Capital-intensive sectors tend to grow more rapidly as a result of an equi-proportionate increase in the capital-labor ratio. This feature, combined with capital deepening at the economy level, naturally leads to a pattern of non-balanced growth. This type of non-balanced growth may contribute to structural changes across agricultural, manufacturing and service sectors, but is more relevant when we look at sectors differentiated according to their capital intensity. A particular focus of both Sections 20.1 and 20.2 has been to reconcile non-balanced growth at the sectoral level with the pattern of relatively balanced growth at the aggregate. As already noted in Chapter 2, balanced growth need not be taken literally. It is at best an approximation to the growth behavior of advanced economies. Nevertheless, it seems to be a particularly accurate approximation to many features of the growth process, since interest rates and the share of capital income in GDP do appear to have been relatively constant over the past 100 years or more in most advanced economies. It is therefore important to understand how significant reallocation of resources at the sectoral and micro levels can coexist with the more "balanced" behavior at the aggregate. The models in Sections 20.1 and 20.2 suggested some clues about why this may be the case, but the answers provided here should be viewed as preliminary rather than definitive.

I also discussed a simple model of the origins of industrialization. This model showed how agricultural productivity might have an important effect on the timing of industrialization, but also demonstrated that the effect of agricultural productivity might depend on whether or not the economy is open to international trade. The origins of industrialization are important because, as discussed in Chapter 1, existing evidence suggests that the timing and nature of industrialization may have important implications for cross-country income differences we observe today, and thus the investigation of the economic development problem might necessitate an analysis of why some countries industrialized early while others were delayed or never started the process of industrialization.

Understanding the sources of the structural changes and how they can be reconciled with the broad patterns of balanced growth in the aggregate, as well as an analysis of the origins of industrialization, sheds light on both the process of economic growth and the process of economic development. In this sense, the models in this section enrich our understanding of economic growth considerably. And yet, this is only a modest step towards the investigation of the sweeping structural changes emphasized by Kuznets because we have not departed from the neoclassical approach to economic growth. In particular, Sections 20.1 and 20.2 used generalized versions of the basic neoclassical growth model of Chapter 8, and Section 20.3 used a variant of the Romer (1986) model from Chapter 11.

It should be emphasized again that the topics discussed in this chapter, though closely related to the basic neoclassical growth model, are areas of frontier research. We are far from a satisfactory framework for understanding the process of reallocation of capital and labor across sectors, how this changes at different stages of development, and how this remains consistent with relatively balanced aggregate growth and the Kaldor facts. I have therefore not attempted to provide a unified framework that combines the transition from agriculture to industrialization, the demand-side reasons for non-balanced growth and the supply-side forces (even though Exercise 20.17 provides some hints on how this may be achieved). The development of such unified models as well as richer models of non-balanced growth are areas for future research.

20.5. References and Literature

The early development literature contains many important works documenting the major structural changes taking place in the process of development. Kuznets (1957, 1973) and Chenery (1960) provide some of the best overviews of the broad evidence and the literature, though similar issues were discussed by even earlier development economists such as Rosenstein-Rodan (1943), Nurske (1953), and Rostow (1960). Figure 20.1, which uses data from The Historical Statistics of the United States, gives a summary of these broad changes.

The model of non-balanced growth based on Engel's law presented in Section 20.1 is based on Kongsamut, Rebelo and Xie (2001). Previous work that have analyzed similar models include Murphy, Shleifer and Vishny (1989), Echevarria (1997), Laitner (2000). More recent work building on Kongsamut, Rebelo and Xie (2001) includes Caselli and Coleman (2001) and Gollin, Parente and Rogerson (2002). Many of these models are considerably richer than the Kongsamut, Rebelo and Xie approach. For example, Murphy, Shleifer and Vishny (1989) incorporate monopolistic competition and analyzes the implications of income inequality for the demand for different types of goods. Echevarria (1997) and Laitner (2000) show how the initial phase of transitioning from agriculture to manufacturing will be associated with aggregate non-balanced growth. The distinguishing feature of these models is that land is also a factor of production and is more important for agriculture than for manufacturing. Exercise 20.8 provides an example of such a model. The recent literature also places greater emphasis on sources of agricultural productivity and emphasizes that differences in agricultural productivity across countries are often as large as or even larger than productivity differences in other sectors. Gollin, Parente and Rogerson (2002) is one of the first papers in this vein.

The works mentioned in the previous paragraph, like the model I presented in Section 20.1, appeal to Engel's law and model the resulting non-homothetic preferences by positing Stone-Geary preferences as in equation (20.3). A more flexible and richer approach is to allow for "hierarchies of needs" in consumption, whereby households consume different goods in a particular sequence (e.g., food need to be consumed before textiles, and textile need to be consumed before electronics, etc.). This approach is used in Stokey (1988), Matsuyama (2002), Foellmi and Zweimuller (2002), and Buera and Kaboski (2006) to generate richer models of structural change. Space restrictions precluded me from presenting these hierarchy of needs models, even though they are both insightful and elegant alternatives to the standard approach of using Stone-Geary preferences.

Section 20.2 builds on Acemoglu and Guerrieri (2006). The precursor to this work is Baumol (1967), which emphasized the importance of differential productivity growth on non-balanced growth. However, Baumol did not derive a pattern of non-balanced growth including reallocation of capital and labor across sectors, and assumed differential rates of productivity growth to be exogenous. Ngai and Pissarides (2006) and Zuleta and Young (2006) provide modern versions of Baumol's hypothesis. Instead, the approach in Section 20.2 emphasizes how the combination of different capital intensities and capital deepening in the aggregate can endogenously lead to this pattern.

The model in Section 20.3 is based on Matsuyama (1992) and is also closely related to the model I presented in Section 19.7 in Chapter 19. Excellent account of the role of agriculture in industrialization, especially in the British context, are provided in Mokyr (1993) and Overton (2001).

20.6. Exercises

Exercise 20.1.
(1) Show that the consumption aggregator in (20.3) leads to Engel's law.
(2) Suggest alternative consumption aggregators that will generate similar patterns.

Exercise 20.2. Prove Proposition 20.1.

Exercise 20.3.
(1) Set up the optimal control problem for a representative household in the model of Section 20.1.
(2) From the Euler equations and the transversality condition, verify part 1 of Proposition 20.2.
(3) Use equations (20.10)-(20.11) to derive parts 2 and 3 of the proposition.

Exercise 20.4.
(1) Prove Proposition 20.3.

(2) Show that even though a balanced growth path does not exist, an equilibrium path always exists.

EXERCISE 20.5. (1) Prove Proposition 20.4. In particular, show that if (20.21) is not satisfied, a CGP cannot exist, and that this condition is sufficient for a CGP to exist.

(2) Characterize the CGP effective capital-labor ratio, k^*.

EXERCISE 20.6. In the model of Section 20.1, show that as long as condition (20.21) is satisfied when the economy starts with an effective capital-labor ratio $K(0)/(X(0)L(0))$ different from k^*, the CGP is globally stable and the effective capital-labor ratio will monotonically converge to k^* as $t \to \infty$.

EXERCISE 20.7. * Consider a generalization of the model of Section 20.1, where the sectoral production functions are given by the following Cobb-Douglas forms

$$\begin{aligned} Y^A(t) &= K^A(t)^{\alpha^A}\left(B^A(t)L^A(t)\right)^{1-\alpha^A} \\ Y^M(t) &= K^M(t)^{\alpha^M}\left(B^M(t)L^M(t)\right)^{1-\alpha^M} \\ Y^S(t) &= K^S(t)^{\alpha^S}\left(B^S(t)L^S(t)\right)^{1-\alpha^S} \end{aligned}$$

and assume that $B^A(t)$, $B^M(t)$ and $B^S(t)$ grow respectively at the rates g^A, g^M and g^S.

(1) Derive the equivalent of Propositions 20.1 and 20.2.
(2) Show that as long as preferences are given by (20.3) and $\gamma^A > 0$ and/or $\gamma^S > 0$, balanced growth is impossible.
(3) Show that there exists a generalization of condition (20.21) such that this model will have a CGP as defined in Section 20.1. [Hint: the generalization includes two separate conditions that depend on technology growth rates as well as preference parameters].

EXERCISE 20.8. Consider a version of the model in Section 20.1, with only manufacturing and agricultural goods. The consumption aggregator is $c(t) = \left(c^A(t) - \gamma^A\right)^{\eta^A} c^M(t)^{\eta^M}$, with $\gamma^A > 0$. Assume that the production functions for agricultural and manufacturing goods take the form $Y^A(t) = X(t)\left(L^A(t)\right)^{\zeta}(Z)^{1-\zeta}$ and $Y^M(t) = X(t)L^M(t)$, where Z is land. There are no savings or capital.

(1) Characterize the competitive equilibrium in this economy.
(2) Show that this economy also exhibits structural change; in particular, show that the share of manufacturing sector grows over time.
(3) What happens to land rents along the equilibrium path?

EXERCISE 20.9. * In the model of Section 20.1, suppose that condition (20.21) is not satisfied. Assume that the production function F is Cobb-Douglas. Characterize the asymptotic growth path of the economy (i.e., the growth path of the economy as $t \to \infty$).

EXERCISE 20.10. Consider the model of Section 20.1 but assume that there exist a final good produced according to the technology $Y(t) = \left(Y^A(t) - \gamma^A\right)^{\eta^A} Y^M(t)^{\eta^M} \left(Y^S(t) + \gamma^S\right)^{\eta^S}$.

(1) Show that all the results in Section 20.1 hold without any change as long as capital goods are produced out of intermediate Y^M as implied by equation (20.8).
(2) Next assume that capital goods are produced out of the final good, so that the resource constraint becomes $\dot{K}(t) + c(t) L(t) = Y(t)$, where $c(t)$ is the per capita consumption of the final good. Show that in this model a CGP (as defined in Section 20.1) does not exist.

EXERCISE 20.11. Prove Proposition 20.6.

EXERCISE 20.12. Derive equations (20.42), (20.43), (20.44), and (20.45).

EXERCISE 20.13. Prove Proposition 20.7.

EXERCISE 20.14. (1) Complete the proof of Proposition 20.11 by considering the case in which $\varepsilon > 1$ and $g_1^* \geq g_2^* > 0$.
(2) State and prove the equivalent of Proposition 20.11, when the converse of condition (20.64) holds.

EXERCISE 20.15. Show that in the allocation in Proposition 20.11 the asymptotic interest rate is constant and derive a closed-form expression for this interest rate.

EXERCISE 20.16. * In this exercise, you are first asked to provide an alternative proof of Proposition 20.11 and then characterize the local transitional dynamics in the neighborhood of the constant growth path.

(1) Reexpress the equilibrium equations in terms of the following three variables $\varphi(t) \equiv c(t)/A_1(t)^{1/(1-\alpha_1)}$, $\chi(t) \equiv K(t)/\left(L(t) A_1(t)^{1/(1-\alpha_1)}\right)$ and $\kappa(t)$. In particular, show that the following three differential equations, together with the appropriate transversality condition and initial values $\chi(0)$ and $\kappa(0)$, characterize the dynamic equilibrium

$$
\begin{aligned}
\frac{\dot{\varphi}(t)}{\varphi(t)} &= \frac{1}{\theta}\left[\alpha_1 \gamma \eta(t)^{1/\varepsilon} \lambda(t)^{1-\alpha_1} \kappa(t)^{-(1-\alpha_1)} \chi(t)^{-(1-\alpha_1)} - \rho\right] - \frac{a_1}{1-\alpha_1}, \\
\frac{\dot{\chi}(t)}{\chi(t)} &= \lambda(t)^{1-\alpha_1} \kappa(t)^{\alpha_1} \chi(t)^{-(1-\alpha_1)} \eta(t) - \chi(t)^{-1} \varphi(t) - n - \frac{a_1}{1-\alpha_1}, \\
\frac{\dot{\kappa}(t)}{\kappa(t)} &= \frac{(1-\kappa(t))\left[(\alpha_2 - \alpha_1)\frac{\dot{\chi}(t)}{\chi(t)} + a_2 - \frac{1-\alpha_2}{1-\alpha_1} a_1\right]}{(1-\varepsilon)^{-1} + (\alpha_2 - \alpha_1)(\kappa(t) - \lambda(t))},
\end{aligned}
\tag{20.80}
$$

where $\kappa(t)$ and $\lambda(t)$ are given by (20.46) and (20.47), and

$$
\eta(t) \equiv \gamma^{\frac{\varepsilon}{\varepsilon-1}} \left[1 + \frac{\alpha_1}{\alpha_2}\left(\frac{1-\kappa(t)}{\kappa(t)}\right)\right]^{\frac{\varepsilon}{\varepsilon-1}}. \tag{20.81}
$$

[Hint: use the Euler equation of the representative consumer and the resource constraint of the economy, rearrange these to express the laws of motion of $\varphi(t)$ and

$\chi(t)$ in terms of $\kappa(t)$, $\lambda(t)$ and $\eta(t)$ as defined in (20.81), and then differentiate (20.46).]

(2) State the appropriate transversality condition.
(3) Show that if an allocation satisfies the three differential equations in (20.80) and the appropriate transversality condition, then it corresponds to an equilibrium path.
(4) Show that in a CGP equilibrium $\varphi(t)$ must be constant. Using this, show that the CGP requires that $\kappa(t) \to 1$ and that $\chi(t)$ be constant. From these observations, derive an alternative proof of Proposition 20.11.
(5) Now linearize these three equations around the CGP of Proposition 20.11 and show that the linearized system has two negative and one positive eigenvalues and using this fact conclude that the constant growth path is locally stable. [Hint: as part of this argument, explain why $\kappa(t)$ should be considered a state variable with $\kappa(0)$ taken as an initial value].

Exercise 20.17. Consider a model that combines the supply-side and the demand side features discussed in Sections 20.1 and 20.2. In particular, suppose that the consumption aggregator is given by $c(t) = \left(c^S(t) + \gamma^S\right)^{\eta^S} c^M(t)^{\eta^M}$, where c^S consumption of services and c^M denotes the consumption of manufacturing goods. Assume that the economy is closed and both services and manufacturing are produced by Cobb-Douglas technologies with the same Hicks-neutral rate of exogenous technological progress, but manufacturing is more capital-intensive. Characterize the equilibrium of this economy. Show that the relative price and the employment share of services will be increasing over time. Is it possible for the total consumption of manufacturing goods to increase faster than those of services?

Exercise 20.18. Consider the model of Section 20.3.

(1) Show that aggregate food (agricultural) consumption and production stay constant at

$$B^A G(1-\phi^{-1}(\gamma^A/B^A)) = \gamma^A + B^A \eta G'(1-\phi^{-1}(\gamma^A/B^A))\frac{F(\phi^{-1}(\gamma^A/B^A))}{F'(\phi^{-1}(\gamma^A/B^A))}.$$

(2) Show that this is increasing in B^A and provide the intuition for this result.
(3) Show that expenditure on agricultural goods increases at the same rate as aggregate output. [Hint: first characterize how $p(t)$ changes along the equilibrium path].

Exercise 20.19. * Consider the model of Section 20.3 and suppose that the production function for the manufacturing sector is given by

$$Y^M(t) = \frac{1}{1-\beta}\left[\int_0^{N(t)} x(\nu,t)^{1-\beta}\, d\nu\right] L^M(t)^\beta,$$

which is similar to the production functions in Part 4 of the book, with $N(t)$ denoting the range of machines (intermediates) and $x(\nu,t)$ corresponding to the amount of machine of type ν used by the manufacturing sector. Assume as in Part 4 that these machines are supplied by technology monopolists with perpetual patents and can be produced by using the manufacturing good only at constant marginal cost of $(1-\beta)$ units of the manufacturing good. Also assume the lab-equipment specification for creating new machines as in Section 15.7. Characterize the equilibrium of this economy and show that the qualitative features are the same as the model in the text.

Exercise 20.20. Consider an open economy version of the model of Section 20.3. In particular, suppose that the economy trades with the rest of the world taking product prices as given. The rest of the world is characterized by the same technology, except that it has an initial level of productivity in the manufacturing sector equal to $X^F(0)$ and an agricultural productivity given by A^F. Suppose that there are no spillovers in learning-by-doing, so that equation (20.74) applies to the "home" economy and the law of motion of manufacturing productivity in the rest of the world is given by $\dot{X}^F(t) = \delta Y^{M,F}(t)$, where $Y^{M,F}(t)$ is total foreign manufacturing production at time t.

(1) Show that comparative advantage in this economy is determined by the comparison of $X(0)/B^A$ versus $X^F(0)/A^F$. Interpret this.
(2) Suppose that $X(0)/B^A < X^F(0)/B^F$, so that the home economy has a comparative advantage in agricultural production. Show that the initial share of employment in manufacturing in the home economy, $n^*(0)$, must satisfy
$$\frac{X(0)F'(n^*(0))}{B^A G'(1-n^*(0))} = \frac{X^F(0)F'\left(n^{F*}(0)\right)}{B^F G'(1-n^{F*}(0))},$$
where $n^{F*}(0)$ is the share of manufacturing employment in the rest of the world. Show that $n^*(0)$ given by this equation is strictly less than n^* as given by (20.79).
(3) What happens to manufacturing employment in the home economy starting as in part 2 of this exercise? [Hint: derive an equivalent of the equation in part 2 for any t, differentiate this with respect to time and then use the laws of motion of X and X^F].
(4) Explain why agricultural productivity, which was conducive to faster industrialization in the closed economy, may lead to delayed industrialization or to deindustrialization in the open economy.
(5) Consider an economy specializing in agriculture as in the earlier parts of this exercise. Is welfare at time $t=0$ necessarily lower when this economy is open to trade than when it is closed to trade? Relate your answer to the analysis in Section 19.7 of Chapter 19.

CHAPTER 21

Structural Transformations and Market Failures in Development

Together with the process of economic development and the changes in the structure of production, there is also a transformation of the economy, which both involves major social changes and induces greater (and perhaps more "complex") coordination of economic activities. Loosely speaking, we can think of a society that is relatively developed as functioning along (or at any rate, nearer) the frontier of its production possibilities set, while a less-developed economy is in the interior of its "notional" production possibilities set. This may be caused by the fact that certain arrangements necessary for an economy to reach the frontier of its production possibility set require a large amount of capital or some specific technological advances (in which case, even though we may think of the society as functioning in the interior of its production possibility set, this may not be the outcome of market failure, thus the qualifier "notional" in the previous sentence). Alternatively, less developed economies may be in the interior of their production possibility set because these societies are subject to severe market failures. In this chapter, I will discuss both types of models.

I first focus on structural transformations and how these may be limited by amount of capital or technology available in a society. The main economic issues are most simply and effectively illustrated by a simple model in which economic development is accompanied with financial development, enabling better risk sharing and thus investment in higher productivity activities. This model, presented in Section 21.1, has certain clear similarities to the model we studied in Section 17.6 in Chapter 17, though it focuses on the sharing of idiosyncratic risks rather than diversification of aggregate risks. Section 21.2 is less explicitly about the economy moving from the interior of its production possibilities set towards the frontier. Nevertheless, it discusses a major aspect of the structural transformation of the economy during the development process—the demographic transition. In particular, this section discusses how population and fertility change over the process of development and emphasizes how these structural transformations may be linked to investments in human capital. This model provides a brief introduction to the rich literature on the demographic transition and its role in economic growth. Section 21.3 discusses another important structural change, the increase in the population living in urban areas via migration from the countryside to the cities. This model illustrates both how development is associated with a process of allocating workers to activities in which their marginal product is higher and also how a dual economy structure can emerge in equilibrium and slow down this reallocation process. All three of these sections are meant to give a flavor of vast literatures dealing with issues of financial development, the demographic transition, population growth, fertility, migration, urbanization, and other social changes taking place in the course of the development process. Some of these areas are at the forefront of current research in economic growth (and economic development!) but space restrictions preclude me from spending more than a few sections on these important topics.

In Section 21.4 I present a model that is complementary to those in Sections 21.1-21.3, where the stage of development is captured by the distance of an economy's technology to the world technology frontier. This model shows how the distance to the frontier measure of the stage of development influences how production is and should be organized. More specifically, it focuses on whether entrepreneurs that are unsuccessful will be immediately replaced by new entrepreneurs or whether the equilibrium will feature long-term relationships and the survival of experienced entrepreneurs even when they are not very productive early on. The simple framework presented in this section can be used for modeling a range of decisions related to the structure of production and the internal organization of firms over the process of development, which is another aspect of the sweeping structural changes suggested by Kuznets.

The models presented in Sections 21.1-21.4 focus on structural transformations. Part of the focus of these models is on the structural transformation that is an essential part of the process of economic development and the common theme is an investigation of the factors that help or hinder this structural transformation as well as the impact of this transformation on aggregate productivity. Some of these models feature market failures, though the focus is not on market failures per se.

A central question of economic development (and economic growth) is why so many societies have failed to take advantage of new and improve their technologies over the past 200 years. The perspective that emphasizes potential differences in efficiency and in the extent of market failures also suggests that some less-developed economies might be suffering disproportionately from market failures. In the extreme, some economists might interpret the implications of these models as stating that less-developed economies are "stuck" in a potential "development trap," that is, an equilibrium or a steady state where efficiency is low and market failures sustain this low efficiency equilibrium, though a different type of steady state (or equilibrium) with a higher level of income and/or a higher growth rate is also possible. The rest of this chapter turns to an investigation of some of the role of various different types of market failures in economic development and in particular, in causing poverty traps. Section 21.5 emphasizes the possibility of multiple equilibria due to aggregate demand externalities. In the model presented in this section, one of the multiple equilibria approximates a situation without industrialization and growth. Section 21.6 investigates the importance of income inequality for economic development and shows how the interaction of imperfect capital markets with income inequality can lead to multiple steady states, again with different levels of efficiency and productivity. I also use the models in this section to emphasize the difference between multiple equilibria and multiple steady states, and I provide a brief discussion of richer models of income inequality dynamics and their implications for economic development. Finally, Section 21.7 provides a reduced-form model that emphasizes some of the common themes in the approaches covered in this chapter. While each model in this chapter makes quite different assumptions, there are sufficiently many common elements that my hope is that an attempt to bring out the similarities, even if in a highly reduced-form way, will provide additional insights.

The topics covered in this chapter are part of a large and diverse literature. My purpose is not to do justice to this literature but to emphasize how certain major structural transformations take place as part of the process of economic development and also highlight the potential importance of market failures in this process. Given this objective and the large number of potential models, my choice of models is selective and my treatment will be more informal than the rest of the book. In addition, I will often make reduced-form assumptions in order to keep the exposition brief and simple, while communicating the main ideas.

21.1. Financial Development

An important aspect of the structural transformation brought about by economic development is a change in financial relations and deepening of financial markets. Section 17.6 in Chapter 17 already presented a model where economic growth goes hand-in-hand with financial deepening. However, the model in that section only focused on some specific aspects of the role of financial institutions. In general, financial development brings about a number of complementary changes in the economy. First, there is greater depth in the financial market, allowing better diversification of aggregate risks, a feature also emphasized in the model of Section 17.6. Second, one of the key roles of financial markets is to allow risk sharing and consumption smoothing for individuals. In line with this, financial development also allows better diversification of *idiosyncratic risks*. We have seen in Section 17.6 that better diversification of aggregate risks leads to a better allocation of funds across sectors/projects. Similarly, better sharing of idiosyncratic risks will lead to a better allocation of funds across individuals. Third, financial development might also reduce credit constraints on investors and thus also directly enable the transfer of funds to individuals with better investment opportunities. The second and the third channels not only affect the allocation of resources in the society but also the distribution of income, because diversification of idiosyncratic risks and relaxation of credit market constraints might lead to better income and risk sharing. On the other hand, as the possibility of such risk-sharing arrangements reduce consumption risk, individuals might also take riskier actions, potentially affecting the distribution of income. A complete analysis of the issues surrounding financial development and its interactions with economic growth are beyond the scope of this chapter. As already hinted, existing evidence suggests that financial development and economic development go hand-in-hand and many economists interpret this as, at least partly, reflecting the causal effect of financial development on economic growth. A full analysis of issues related to financial development must both study the relevant theoretical issues and also investigate the empirical relationship between finance and growth.

Here I will instead present a simple model of financial development, focusing on the diversification of idiosyncratic risks and complementing the analysis in Section 17.6. The model is inspired by the work of Townsend (1979) and Greenwood and Jovanovic (1990) and adopts some of the modeling features of the model of Acemoglu and Zilibotti (1997) in Section 17.6. It will illustrate how financial development takes place endogenously and interacts with economic growth, and will also provide some simple insights about the implications of financial development for income distribution. Given the similarity of the model to that in Section 17.6, my treatment here will be relatively quick and informal.

I consider an overlapping generations economy in which each individual lives for two periods and has preference given by

$$\mathbb{E}_t U(c(t), c(t+1)) = \log c(t) + \beta \mathbb{E}_t \log c(t+1), \tag{21.1}$$

where $c(t)$ denotes the consumption of the unique final good of the economy and $\mathbb{E}_t$ denotes the expectation operator given time t information. As we already seen Chapter 9 and also in Section 17.6, these preferences are very convenient since they ensure a constant savings rate.

There is no population growth and the total population of each generation is normalized to 1. Let us assume that each individual is born with some labor endowment l. The distribution of endowments across agents is given by the distribution function $G(l)$ over some support $\left[\underline{l}, \overline{l}\right]$ This distribution of labor endowments is constant over time with mean $L = 1$ and labor is supplied inelastically by all individuals in the first period of their lives. In the second period of their lives, individuals cannot supply labor and can only consume their capital income.

The aggregate production function of the economy is given by

$$Y(t) = K(t)^{\alpha} L(t)^{1-\alpha} = K(t)^{\alpha},$$

where $\alpha \in (0, 1)$ and the second equality uses the fact that total labor supply will be equal to 1 at each date. As in Section 17.6, the only risk is in transforming savings into capital, thus the lifecycle of an individual looks identical to that shown in Figure 17.3 in that section.

Moreover, we assume that agents can either save all of their labor earnings from the first period of their lives using a safe technology with rate of return q (in terms of capital at the next date) or invest all of their labor income in the risky technology with return $Q+\varepsilon$, where ε is a mean zero independently and identically distributed stochastic shock and

$$Q > q.$$

This implies that the risky technology is more productive. The assumption that individuals have to choose one of these two technologies rather than dividing their savings between the two is made for simplicity only (see Exercise 21.1).

Although the model looks very similar to that in Section 17.6, there is a crucial difference. Because ε is identically and independently distributed *across individuals*, if individuals could pool their resources, they would get rid of the idiosyncratic risk and enjoy the higher return Q. In particular, if a large number (a continuum) all individuals pooled their resources, they would guarantee an average return of Q. Let us assume that this is not possible because of a standard *informational problem*—the actual return of an individual's saving decision is not observed by other individuals unless some financial monitoring is undertaken. Let us assume that this type of financial monitoring has a cost $\xi > 0$ for each individual. This implies that by paying the cost of ξ, each individual can join the financial market (or in the language of Townsend, he can become part of a "financial coalition") and in this case, the actual return of his savings are all observed. Intuitively, this cost captures the fixed costs that individuals have to pay to be engaged in financial markets as well as the fixed cost associated with monitoring or being monitored. An immediate implication of this specification is that joining the financial markets is more attractive for richer individuals, since the fixed cost is less important for them. This feature is both plausible and also generates predictions consistent with microdata, where we observe richer individuals investing in more complex financial securities.

If the individual does not join the financial markets, then no other agent in the economy can observe the realization of the returns on his savings. In this case, no financial contract for sharing of idiosyncratic risks is possible, since such a contract would involve agents that have a high (realized) value of ε making transfers to those who are unlucky and have low realized values of ε. However, without monitoring, each agent will claim to have a low value of ε, thus receive rather than make ex post payments. The anticipation of this type of opportunistic behavior prevents any risk sharing in the absence of monitoring.

Let us also assume that ε has a distribution that places positive probability on $\varepsilon = -Q$. This implies that if an individual undertakes the risky investments, there is a positive probability that all his savings will be lost. This immediately implies that without some type of risk sharing, individuals would always choose the safe project. This observation significantly simplifies the analysis of the model. Now suppose that the economy starts with some initial capital stock of $K(0)$. This implies that an individual with labor endowment l_i will have labor earnings of

$$W_i(0) = w(0)\, l_i,$$

where

$$w(t) = (1-\alpha)\, K(t)^{\alpha} \tag{21.2}$$

is the competitive wage rate at time t. After labor incomes are realized, individuals first make their savings decisions and then choose which assets to invest in. The preferences in (21.1) imply that individuals will save a constant fraction

$$\frac{\beta}{1+\beta}$$

of their income regardless of their income level or the rate of return (in particular, independent of whether they are investing in the risky or the safe asset). In view of this, the value to not participating in the financial markets for individual i at time t is

$$V_i^N\left(W_i(t), R(t+1)\right) = \log\left(\frac{1}{1+\beta} W_i(t)\right) + \beta \log\left(\frac{\beta R(t+1)\, q}{1+\beta} W_i(t)\right),$$

which takes into account that the rate of return on capital in the second period of the life of the individual will be $R(t+1)$ and the individual will receive a gross return q on his savings of $\beta W_i(t) / (1+\beta)$. In contrast, when the individual decides to take part in financial markets (presuming that there are sufficiently many other individuals also taking part in financial markets to provide risk diversification, which here means a positive measure of individuals doing so), his value will be

$$V_i^F\left(W_i(t), R(t+1)\right) = \log\left(\frac{1}{1+\beta}\left(W_i(t) - \xi\right)\right) + \beta \log\left(\frac{\beta R(t+1)\, Q}{1+\beta}\left(W_i(t) - \xi\right)\right),$$

which takes into account that the individual will have to spend the amount ξ out of his labor income on the cost of joining the financial market, leaving him a net income of $W_i(t) - \xi$. He will then save a fraction $\beta/(1+\beta)$ of this, but in return, he will receive the higher gross return Q. The reason why the individual will necessarily receive Q, rather than a risky return, is because, conditional on joining the financial market, each individual is able to fully diversify his idiosyncratic risks and therefore receive the average return Q. The comparison of these two expressions immediately gives the threshold level

$$W^* \equiv \frac{\xi}{1-(q/Q)^{\beta/(1+\beta)}} > 0 \tag{21.3}$$

such that individuals with first-period earnings greater than W^* will join the financial market and those with less than W^* will not. A notable feature of this threshold W^* is that it is independent of the rate of return on capital in the second period of the lives of the individuals, R. This is an implication of log preferences in (21.1).

Now that we have determined the behavior of individuals concerning whether they will join the financial market, we can determine the evolution of the economy by studying the evolution of individual earnings. Individual earnings are determined by two factors: individual labor endowments and the capital stock at time t, which determines the wage per unit of labor, $w(t)$, as given in (21.2). In particular, suppose that at time t the wage is given by $w(t)$. Then the fraction of individuals who will join the financial market at time t, $g^F(t)$, is given by the fraction of individuals who have $l_i \geq W^*/w(t)$. Alternatively, using the fact that labor endowments have a distribution given by $G(\cdot)$, the fraction of individuals investing in financial markets is obtained as

$$g^F(t) \equiv 1 - G\left(\frac{W^*}{w(t)}\right) = 1 - G\left(\frac{W^*}{(1-\alpha)K(t)^\alpha}\right). \tag{21.4}$$

In view of this, the capital stock at time $t+1$ can be written as

$$K(t+1) = \frac{\beta}{1+\beta}\left[q\left(\int_{\underline{l}}^{\frac{W^*}{(1-\alpha)K(t)^\alpha}} l dG(l)\right)(1-\alpha)K(t)^\alpha + Q\int_{\frac{W^*}{(1-\alpha)K(t)^\alpha}}^{\bar{l}}((1-\alpha)K(t)^\alpha l - \xi)\, dG(l)\right], \tag{21.5}$$

which takes into account that all individuals with labor endowment less than $W^*/q(1-\alpha)K(t)^\alpha$ will choose the safe project and receive the gross return q on their savings, while those above this threshold will spend ξ on the fixed cost of monitoring and then receive the higher return Q. It can be verified that $K(t+1)$ is increasing in $K(t)$ and there will be growth in the capital stock (and thus output) of the economy provided that $K(t)$ is small enough (in particular, less than the "steady-state" level of capital when this is unique; see Exercise 21.2).

Now inspection of the accumulation equation (21.5) together with the threshold rule for joining the financial market leads to a number of interesting conclusions.

(1) As $K(t)$ increases, that is, as the economy develops, equation (21.4) implies that more individuals will join the financial market. Consequently, a greater level of capital will lead to more risk taking, but these risks will also be shared better. More importantly, economic development also induces a better composition of investment as a greater fraction of the individuals start using their savings more efficiently. Thus with a mechanism similar to the model in Section 17.6, economic development leads to endogenously higher productivity by improving the allocation of funds in the economy. Consequently, this model, like the one in Section 17.6, implies that economic development and financial development go hand-in-hand.

(2) However, there is also a distinct sense in which the economy here allows for a potential causal effect of financial development on economic growth. Imagine that

societies differ according to their ξs, which can be interpreted as a measure of the institutionally- or technologically-determined costs of monitoring or some cost of financial transactions that depend on the degree of investor protection. Societies with lower ξs will have a greater participation in financial markets and this will endogenously increase their productivity. Thus while the equilibrium behavior of financial and economic development are jointly determined, differences in financial development driven by exogenous institutional factors related to ξ will have a potential causal effect on economic growth.

(3) As noted above, at any given point in time it will be the richer agents—those with greater labor endowment—that will join the financial market. Therefore, initially, the financial market will help those who are already well-off to increase the rate of return on their savings. This can be thought of as the *unequalizing* effect of the financial market.

(4) The fact that participation in financial market increases with $K(t)$ also implies that as the economy grows, at least at the early stages of economic development, the unequalizing effect of financial intermediation will become stronger. Therefore, presuming that the economy starts with relatively few rich individuals, the first expansion of the financial market will increase the level of overall inequality in the economy as a greater fraction of the agents in the economy now enjoy the greater returns.

(5) As $K(t)$ increases even further, eventually the *equalizing* effect of the financial market will start operating. At this point, the fraction of the population joining the financial market and enjoying the greater returns is steadily increasing. If the steady-state level of capital stock K^* is such that $\underline{l} \geq W^* / (1-\alpha)(K^*)^\alpha$, then eventually all individuals will join the financial market and they will all receive the same rate of return on their savings.

The last two observations are interesting in part because the relationship between growth and inequality is a topic of great interest to development economics (one to which we will return later in this chapter). One of the most important ideas in this context is that of the *Kuznets curve*, based on Simon Kuznets's observations, which claims that growth first increases income inequality in the society and then leads to a decline in inequality. Whether or not the Kuznets curve is a good description of the relationship between growth and inequality is a topic of current debate. While many European societies seem to have gone through a phase of increasing and then decreasing inequality during the growth process over the 19th century, the evidence for the 20th century is more mixed. Nevertheless, the last two observations show that a model with endogenous financial development based on risk sharing among individuals can generate a pattern consistent with the Kuznets curve. Whether there is indeed a Kuznets curve in general and if so, whether the mechanism highlighted here plays an important role in generating this pattern are areas for future theoretical and empirical work.

21.2. Fertility, Mortality and the Demographic Transition

Chapter 1 highlighted the big questions related to growth of income per capita over time and its dispersion across countries today. Our focus so far has been on these per capita income differences. Equally striking differences exist in the level of population across countries and over time. Figure 21.1 uses data from Maddison (2002) and shows the levels and the evolution of population in different parts of the world over the past 2000 years. The figure is in log scale, so a linear curve indicates a constant rate of population growth. The figure shows that starting about 250 years ago there is a significant increase in the population growth rate in many areas of the world. This accelerated population growth continues in much of the world, but importantly, the rate of population growth slows down in Western Europe sometime in the 19th century (though not so in the Western Offshoots because of immigration). There is no similar slow down of population growth in less-developed parts of the world. On the contrary, in many less developed nations, the rate of population growth seems to have increased over the past 50 or so years. We have already discussed one of the reasons for this in Chapter 4—the spread of antibiotics, basic sanitation and other health-care measures around the world that reduced the very high mortality rates in many countries. However, equally notable is the *demographic transition* in Western Europe, which is the term coined for the decline in fertility sometime during the 19th century (more precisely at different points during this century for different countries). Understanding why population has grown slowly and then accelerated to reach a breakneck speed of growth over the past 150 years and why population growth rates differ across countries are major questions for economic development and economic growth. These questions are not only interesting because population levels are among the variables we would like to understand and explain, but also because one might sometimes wish to focus on differences in total income across societies rather than on income per capita differences. In this case, differences in population become a variable to focus on directly.

In this section, I discuss the most basic approaches to population dynamics and fertility. I first discuss a simple version of the famous Malthusian model and then use a variant of this model to investigate potential causes of the demographic transition. This is a vast and important area of research and one section can hardly do justice to the important empirical and theoretical issues. Thomas Malthus was one of the most brilliant and influential economists of the 19th century and is responsible for one of the first general equilibrium growth models. The next subsection will present a version of this model. The Malthusian model is responsible for earning the discipline of economics the name "the dismal science" because

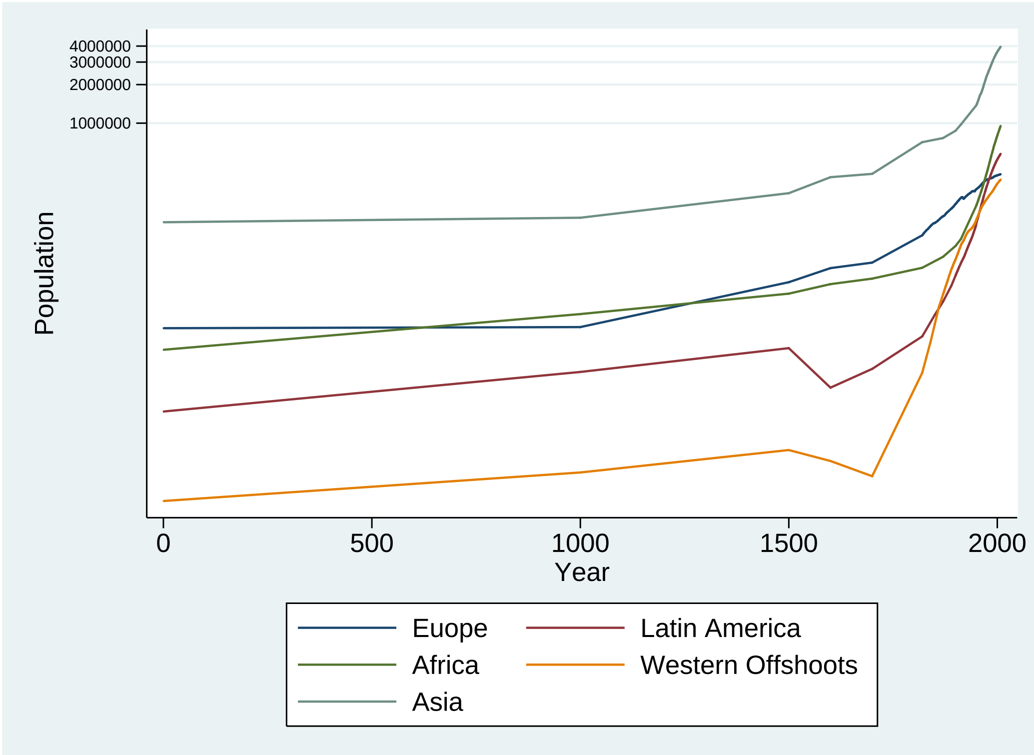

Figure 21.1. Total population in different parts of the world over the past 2000 years.

of its dire prediction that population will adjust up or down (by births or deaths) until all individuals are at the subsistence level of consumption. Nevertheless, this dire prediction is not the most important part of the Malthusian model. Instead, at the heart of this model is the *negative relationship* between population, which is itself endogenously determined, and income per capita. In this sense, it is closely related to the Solow model or the neoclassical growth model, augmented with a behavioral rule that determines the rate of population growth. It is this less extreme version of the Malthusian model that will be presented in the next subsection. I will then enrich this model by the important and influential idea due to Gary Becker that there is a tradeoff between the quantity and quality of children and that this tradeoff changes over the process of development. I will show how a simple model can capture the process via which parents start valuing the quality (human capital) of their offsprings more as the economy becomes richer and demands more human capital. This process will eventually lead to the demographic transition, with fewer but more skilled children. Since my objective here is to introduce the main ideas rather than give a full account of this active research area, my treatment will be informal.

21.2.1. A Simple Malthusian Model. Consider the following non-overlapping generations model. We start with a population of $L(0) > 0$ at time $t = 0$. Each individual living at time t supplies one unit of labor inelastically and has preferences given by

$$c(t)^{\beta}\left[y(t+1)(n(t+1)-1)-\frac{1}{2}\eta_0 n(t+1)^2\right], \tag{21.6}$$

where $c(t)$ denotes the consumption of the unique final good of the economy by the individual himself, $n(t+1)$ denotes the number of offsprings the individual has and $y(t+1)$ is the income of each offspring, and $\beta > 0$ and $\eta_0 > 0$. The last term in square brackets is the child-rearing costs and are assumed to be convex to reflect the fact that the costs of having more and more children will be higher (for example, because of time constraints of parents, though one can also make arguments for why the costs of child-rearing might exhibit increasing returns to scale over a certain range). Clearly, these preferences introduce a number of simplifying assumptions. First, each individual is allowed to have as many offsprings as it likes, which is unrealistic because it does not restrict the number of offsprings to an integer. The technology also does not incorporate possible specialization in child-rearing and market work within the family. Second, these preferences introduce the "warm glow" type altruism we encountered in Chapter 9, so that parents receive utility not from the utility of their offspring, but from some characteristic of their offsprings. Here it is a transform of the total income of all the offsprings that features in the utility function of the parent. Third, the costs of child-rearing are in terms of "utils" rather than forgone income and current consumption multiplies both the benefits and the costs of having additional children. This feature, which is motivated by balanced growth type reasoning, implies that the demand for children will be independent of current income (otherwise, growth will automatically lead to greater demand for children). All three of these assumptions are for simplicity and do not have important effects on the main insights of the model, though they naturally change many of the key expressions and some of the auxiliary implications. I have also written the number of offsprings that an individual has a time t as $n(t+1)$, since this will determine population at time $t+1$.

Each individual has one unit of labor and there are no savings. The production function for the unique good takes the form

$$Y(t) = Z^{\alpha} L(t)^{1-\alpha}, \tag{21.7}$$

where Z is the total amount of land available for production and $L(t)$ is total labor supply. There is no capital and land is introduced in order to create diminishing returns to labor, which is an important element of the Malthusian model. I normalize the total amount of land to $Z = 1$ without loss of any generality. A key question in models of this sort is what happens to the returns to land. The most satisfactory way of dealing with this problem would be to allocate the property rights to land among the individuals and let them bequeath this to their offsprings. This, however, introduces another layer of complication, and since my purpose here is to illustrate the basic ideas, I will follow the unsatisfactory assumption often made in the literature, that land is owned by another set of agents, whose behavior will not be analyzed here. The main important assumption made here is that those receiving land rents do not supply labor and/or their offsprings do not make an important contribution to overall population growth.

By definition, population at time $t+1$ is given as

$$L(t+1) = n(t+1)L(t), \tag{21.8}$$

which takes into account new births as well as the death of the parent.

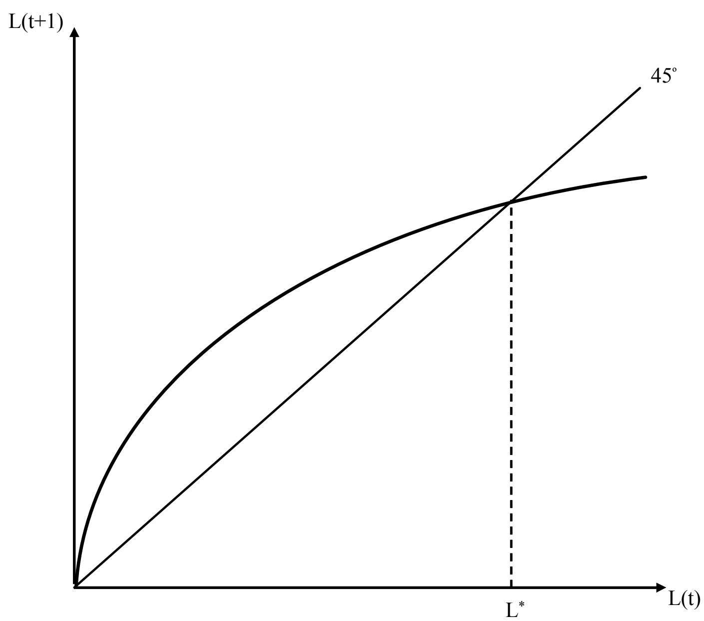

Figure 21.2. Population dynamics in this simple Malthusian model.

Labor markets are competitive, so the wage at time $t+1$ is given as

$$w(t+1) = (1-\alpha)L(t+1)^{-\alpha}. \tag{21.9}$$

Since there is no other source of income, this is also equal to the income of each individual living at time $t+1$, $y(t+1)$. Thus an individual with income $w(t)$ at time t will solve the problem of maximizing (21.6) subject to the constraint that $c(t) \leq w(t)$, together with $y(t+1) = (1-\alpha)L(t+1)^{-\alpha}$, which implies that the individual takes the population level in the next period as given (since he is infinitesimal). Let us focus on a symmetric equilibrium, which will naturally require the choice of $n(t+1)$ to be consistent with $L(t+1)$ according to equation (21.8). This maximization problem immediately gives $c(t) = w(t)$ and

$$n(t+1) = (1-\alpha) L(t+1)^{-\alpha} \eta_0^{-1}.$$

Now substituting for (21.8) and rearranging, we obtain

$$L(t+1) = (1-\alpha)^{\frac{1}{1+\alpha}} \eta_0^{-\frac{1}{1+\alpha}} L(t)^{\frac{1}{1+\alpha}}. \tag{21.10}$$

This equation implies that $L(t+1)$ is an increasing concave function of $L(t)$. In fact, the law of motion for population implied by (21.10) resembles the dynamics of capital-labor ratio in the Solow growth model (or the overlapping generations model) and is plotted in Figure 21.2. The figure makes it clear that starting with any $L(0) > 0$, there exists a unique globally stable state state L^* given by

$$L^* \equiv (1-\alpha)^{1/\alpha} \eta_0^{-1/\alpha}. \tag{21.11}$$

If the economy starts with $L(0) < L^*$, then population will slowly (and monotonically) adjust towards this steady-state level. Moreover, (21.9) shows that as population increases wages will fall. If in contrast, $L(0) > L^*$, then the society will experience a decline in population and rising real wages. It is straightforward to introduce shocks to population and show that in this case the economy will fluctuate around the state-state population level L^* (with an invariant distribution depending on the distribution of the shocks) and experience cycles reminiscent to the Malthusian cycles, with periods of increasing population and decreasing wages followed by periods of decreasing population and increasing wages (see Exercise 21.3). The main difference of this model from the simplest (or the crudest) version of the Malthusian model is that there is no biologically determined subsistence level of consumption. The level of consumption will tend to a constant given by

$$\begin{aligned} c^* &= (1-\alpha)(L^*)^{-\alpha} \\ &= \eta_0, \end{aligned}$$

though this is not determined biologically, but by preferences and technology.

21.2.2. The Demographic Transition. I now extend the basic Malthusian model of the previous subsection in two respects to study the demographic transition. First, I introduce a quality-quantity tradeoff along the lines of the ideas suggested by Gary Becker. Each parent can choose his offsprings to be unskilled or skilled. To make them skilled, the parent has to exert the additional effort for child-rearing denoted by $e(t) \in \{0, 1\}$. If he chooses not to do this, his offsprings will be unskilled.

The total population of unskilled individuals at time t is denoted by $U(t)$ and the total population of the skilled are denoted by $S(t)$, clearly with

$$L(t) = U(t) + S(t).$$

The second modification is that there are now two production technologies that can be used for producing the final good. The Malthusian (traditional) technology is still given by (21.7) and any worker can be employed with the Malthusian technology. The modern technology is given by

$$Y^M(t) = X(t)\, S(t)\,. \tag{21.12}$$

This equation implies that productivity in the modern technology is potentially time varying and also states that only skilled workers can be employed with this technology. It also imposes that all skilled workers will be employed with this technology. Naturally, this need not be true in general (there may be an excess supply of skilled workers). However, this will never be the case in equilibrium, since parents would not choose to exert the additional effort to endow their offsprings with skills if they would then work in the traditional sector. In the interest of keeping the exposition brief and simple (and with a slight abuse of notation), equation (21.12) already incorporates the fact that all skilled workers will be employed in the modern sector.

To incorporate the quality-quantity tradeoff, individual preferences are now modified from (21.6) to

$$c(t)^\beta \left[y(t+1)\,(n(t+1)-1) - \frac{1}{2}\left(\eta_0\,(1-e(t)) + \eta_1 X(t+1)\, e(t)\right) n(t+1)^2 \right]. \tag{21.13}$$

This formulation of the preferences states that if the individual decides invests in his offsprings' skills, instead of the fixed cost η_0 he has to pay a cost that is proportional to the amount of knowledge $X(t+1)$ that the offspring has to absorb to use the modern technology. I assume that $X(0)\,\eta_1 > \eta_0$, so that even at the initial level of the modern technology rearing a skilled child is more costly than an unskilled child.

Finally, I make the same external learning-by-doing assumption as in Romer (1986) or the model of industrialization in Section 20.3, and assume that

$$X(t+1) = \delta S(t)\,, \tag{21.14}$$

which implies that the improvement in the technology of the modern sector is a function of the number of skilled workers employed in this sector. This type of reduced form assumption is clearly unsatisfactory, but as noted above, one could get similar results with an endogenous technology model with the market size effect. Another important feature of this production function is that it does not use land. This assumption is consistent with the fact that most modern production processes make little use of land, instead relying on technology, physical capital and human capital. Equation (21.12) captures this in a simple form, though it does so without introducing physical capital.

The output of the traditional and the modern sector are perfect substitutes—they both produce the same final good. In view of the observation that all unskilled workers will work in the traditional sector and all skilled workers will work in the modern sector, we have wages of skilled and unskilled workers at time t as

$$w^U(t) = (1-\alpha) U(t)^{-\alpha}, \tag{21.15}$$

and

$$w^S(t) = X(t), \tag{21.16}$$

where (21.15) is identical to (21.9) in the previous subsection, except that it features only the unskilled workers instead of the entire labor force.

Let us next turn to the fertility and quality-quantity decisions of individuals. As before, each individual will consume all his income and his income level has no effect on his fertility and quality-quantity decisions. Thus we do not need to distinguish between high-skill and low-skill parents. Using this observation, let us simply look at the optimal number of offsprings that an individual will have when he chooses $e(t) = 0$. This is given by

$$\begin{aligned} n^U(t+1) &= w^U(t+1)\,\eta_0^{-1} \\ &= (1-\alpha) U(t+1)^{-\alpha}\,\eta_0^{-1}, \end{aligned} \tag{21.17}$$

where the second line uses (21.15). Instead, if the parent decides to exert effort $e(t) = 1$ and invest in the skills of his offsprings, then he will choose the number of offsprings equal to

$$\begin{aligned} n^S(t+1) &= w^S(t+1)\,X(t+1)^{-1}\,\eta_1^{-1} \\ &= \eta_1^{-1}. \end{aligned} \tag{21.18}$$

The comparison of equations (21.17) and (21.18) suggests that unless unskilled wages are very low, an individual who decides to provide additional skills to his offsprings will have fewer offsprings. This is because bringing up skilled children is more expensive. Thus the comparison of these two equations captures the quality-quantity tradeoff.

Now substituting these equations back into the utility function (21.13), we obtain the utility from the two strategies (normalized by consumption, which does not affect his decision) as

$$V^U(t) = \frac{1}{2}(1-\alpha)^2 U(t+1)^{-2\alpha}\,\eta_0^{-1}$$

and

$$V^S(t) = \frac{1}{2} X(t+1)\,\eta_1^{-1}.$$

Inspection of these two expressions shows that we can never have an equilibrium in which all offsprings are skilled, since otherwise V^U would become unboundly large. Therefore, in equilibrium we must have

$$V^U(t) \geq V^S(t). \tag{21.19}$$

This equilibrium condition implies that there are two possible configurations. First, $X(0)$ can be so low that (21.19) will hold as a strict inequality. In this case, all offsprings will be unskilled. The condition for this inequality to be strict is

$$X(0)\,\eta_1^{-1} < (1-\alpha)^2 L(1)^{-2\alpha}\,\eta_0^{-1},$$

which uses the fact that when there are no skilled workers there is no production in the modern sector and thus $X(1) = X(0)$. If this inequality satisfied, there would be no skilled children at date $t = 0$. However, as long as $L(1)$ is less than L^* as given in (21.11), population will grow. It is therefore possible that at some point (21.19) holds with equality. The condition for this never to happen is that

$$X(0)\,\eta_1^{-1} < (1-\alpha)^2 (L^*)^{-2\alpha}\,\eta_0^{-1}. \tag{21.20}$$

In this case, the law of motion of population is identical to that in the previous subsection and there is never any investment in skills. We can think of this is a pure Malthusian economy.

If, on the other hand, this condition is not satisfied, then at least at some point individuals will start investing in the skills of their offsprings and the modern sector will have skilled workers to employ. From then on (21.19) must hold as equality. In that case, let the fraction of parents having unskilled children at time t be denoted by $u(t+1)$. Then by definition

$$\begin{aligned} U(t+1) &= u(t+1)\left(n^U(t+1)-1\right)L(t) \\ &= u(t+1)^{1/(1+\alpha)}(1-\alpha)^{2/(1+\alpha)}\,\eta_0^{-1/(1+\alpha)}L(t)^{1/(1+\alpha)} \end{aligned} \tag{21.21}$$

and

$$\begin{aligned} S(t+1) &= (1-u(t+1))\left(n^S(t+1)-1\right)L(t) \\ &= (1-u(t+1))\,\eta_1^{-1}L(t). \end{aligned} \tag{21.22}$$

Moreover, to satisfy (21.19) as equality, we need $(1-\alpha)^2 U(t+1)^{-2\alpha}\,\eta_0^{-1} = X(t+1)\,\eta_1^{-1}$, or

$$X(t+1)\,\eta_1^{-1} = u(t+1)^{-2\alpha/(1+\alpha)}(1-\alpha)^{2(1-\alpha)/(1+\alpha)}\,\eta_0^{-(1-\alpha)/(1+\alpha)}L(t)^{-2\alpha/(1+\alpha)}. \tag{21.23}$$

Equilibrium dynamics are then determined by equations (21.21)-(21.23) together with (21.16). While the details of the behavior of this dynamical system are somewhat involved, the general picture is clear. If an economy starts with both a low level of $X(0)$ and a low level of $L(0)$, but does not satisfy condition (21.20), then the economy will start in the Malthusian regime, only making use of the traditional technology and not investing in skills. As population increases wages fall, and at that point parents start finding it beneficial to invest in the skills of their children and firms start using the modern technology. Those parents that invest in the skills of their children have fewer children than parents rearing unskilled offsprings. The rate of population growth and fertility are high at first, but as the modern technology improves and the demand for skills increases, a larger fraction of the parents start investing in the skills of their children and the rate of population growth declines. Ultimately, the rate of population growth approaches η_1^{-1}. Thus this model gives a very stylized representation of the demographic transition.

In the literature, there are richer models of the demographic transition. For example, there are many ways of introducing quality-quantity tradeoffs in the utility function of the parents, and what spurs a change in the quality-quantity tradeoff may be an increase in capital intensity of production, changes in the wages of workers, or even changes in the wages of women differentially affecting the desirability of market and home activities. Nevertheless, the general qualitative features are similar in that the quality-quantity tradeoff is often viewed as the major reason for the demographic transition. Despite this emphasis on the quality-quantity tradeoff, there is relatively little direct evidence that this tradeoff is important in general or in leading to the demographic transition. Other social scientists have suggested social norms, the large declines in mortality, or the reduced need for child labor as potential factors contributing to the demographic transition. As of yet, there is no general consensus on the causes of the demographic transition or on the role of the quality-quantity tradeoff in determining population dynamics. The study of population growth and demographic transition is an exciting and important area, and theoretical and empirical analyses of the factors affecting fertility decisions and how they interact with the allocation of workers across different tasks (sectors) remain important and interesting questions to be explored.

21.3. Migration, Urbanization and The Dual Economy

Another major structural transformation over the process of development relates to changes in social and living arrangements. For example, as an economy develops, more individuals move from rural areas to cities and also undergo the social changes associated with separation from a small community and becoming part of a larger, more anonymous environment. Other social changes might also be important. For instance, certain social scientists regard the replacement of "collective responsibility systems" by "individual responsibility systems" as an important social transformation. This is clearly related to changes in the living arrangements of individuals (villages versus cities, or extended versus nuclear families). It is also linked to whether different types of contracts are being enforced by social norms and community enforcement, or whether they are enforced by legal institutions. There may also be a similar shift in the importance of the market, as more activities are mediated via prices rather than taking place inside the home or an extended family or a community. This process of social change is both complex and interesting to study, though a detailed discussion of the literature and possible approaches to this complex set of issues falls beyond the scope of the current book. Nevertheless, a brief discussion of some of these social changes are useful to illustrate other, more diverse facets of structural change associated with economic development. I will illustrate the main ideas by focusing on the process of migration from rural areas and urbanization. Another reason to study migration and urbanization is that the reallocation of labor from rural to urban areas is closely related to the popular concept of *the dual economy*, which is an important theme of some of the older literature on development economics. According to this notion, less-developed economies consist of a modern sector and a traditional sector, but the connection between these two sectors is imperfect. The model of industrialization in the previous chapter (Section 20.3) featured a traditional and a modern sector, but these sectors traded their outputs and competed for labor in competitive markets. Dual economy approaches, instead, emphasize situations in which the traditional and the modern sectors function in parallel but with only limited interactions. Moreover, the traditional sector is often viewed as less efficient than the modern sector, thus the lack of interaction may also be a way of shielding the traditional economy from its more efficient competitor. A natural implication of this approach will then be to view the process of development as one in which the less efficient traditional sector is replaced by the more efficient modern sector. Lack of development may in turn correspond to an inability to secure such reallocation.

In this section I first present a model of migration that builds on the work by Arthur Lewis (1954). A less-developed economy is modeled as a dual economy, with the traditional sector associated with villages and the modern sector with the cities. The model enables us to study how and whether the reallocation of resources from the traditional sector to the modern sector will take place. I will then present a model inspired by Banerjee and Newman's (1998) article, as well as by Acemoglu and Zilibotti (1999), in which the traditional sector and the rural economy have a comparative advantage in community enforcement, even though in line with the other dual economy approaches, the modern economy (the city) enables the use of more efficient technologies. This model will also illustrate how certain aspects of the traditional sector can shield the less productive firms from more productive competitors and slow down the process of development. Finally, I will show how the import of technologies from more developed economies, along the lines of the models discussed in Section 18.4 of Chapter 18, will also naturally lead to a dual economy structure, as a consequence of the less-developed economy's efforts in using the more skill-intensive, modern technologies.

21.3.1. Surplus Labor and the Dual Economy. The main emphasis of Lewis's work was on the idea of *surplus labor*. Lewis argued that less-developed economies typically had surplus labor, that is, unemployed or underemployed labor, often in the villages. The dual economy can then be viewed as the juxtaposition of the modern sector where workers are gainfully and productively employed together with the traditional sector where they are underemployed. The general tendency of less-developed economies to have higher levels of unemployment (and lower levels of employment to population ratios) was one of the motivations for Lewis's model. A key feature of Lewis's model is the presence of some barriers preventing, or slowing down, the allocation of workers away from the traditional sector towards urban areas and the modern sector. I now present a reduced-form model that formalizes these notions.

Consider a continuous-time infinite-horizon economy that consists of two sectors or regions, which I will refer to as urban and rural. Total population is normalized to 1. At time $t = 0$, $L^U(0)$ individuals are in the urban area and $L^R(0) = 1 - L^U(0)$ are in the rural area. In the rural area, the only economic activity is agriculture and, for simplicity, we assume that the production function for agriculture is linear, thus total agricultural output is

$$Y^A(t) = B^A L^R(0),$$

where $B^A > 0$. In the urban area, the main economic activity is manufacturing. Manufacturing can only employ workers in the urban area. The production function therefore takes the form

$$Y^M(t) = F\left(K(t), L^U(t)\right),$$

where $K(t)$ is the capital stock, with initial condition $K(0)$. F is a standard neoclassical production function satisfying Assumptions 1 and 2. Let us also assume, for simplicity, that the manufacturing and the agricultural goods are perfect substitutes. Labor markets both in the rural and urban area are competitive. There is no technological change in either sector.

The key assumptions of this model will be twofold. First, the marginal product of labor, and thus the wage, in manufacturing will be higher than in agriculture. Second, because of barriers to mobility, there will only be slow migration of workers from rural to urban areas.

In particular, let us capture the dynamics in this model in a reduced-form way whereby capital accumulates only out of the savings of individuals in the urban area, thus we have

$$\dot{K}(t) = sF\left(K(t), L^U(t)\right) - \delta K(t), \tag{21.24}$$

where s is the exogenous saving rate and δ is the depreciation rate of capital. The important feature implied by this specification is that greater output in the modern sector leads to further accumulation of capital for the modern sector. An alternative, adopted in Section 20.3 of the previous chapter that will also be used in the next subsection, is to allow the size of the modern sector to directly influence its productivity growth, for example because of learning-by-doing externalities as in Romer (1986) or because of endogenous technological change depending on the market size commanded by this sector (e.g., Exercise 20.19). For the purposes of the model in this subsection, which of these alternatives is adopted has no major consequence.

Given competitive labor markets, the wage rates in the urban and rural areas at date t are given by

$$w^U(t) = \frac{\partial F\left(K(t), L^U(t)\right)}{\partial L} \text{ and } w^R(t) = B^A.$$

Let us assume that

$$\frac{\partial F(K(0), 1)}{\partial L} > B^A, \tag{21.25}$$

so that even if all workers are employed in the manufacturing sector at the initial capital stock, they will have a higher marginal product than working in agriculture.

Migration dynamics are assumed to take the following simple form:

$$\dot{L}^R(t) \begin{cases} = -\mu L^R(t) & \text{if } w^U(t) > w^R(t) \\ \in \left[0, -\mu L^R(t)\right] & \text{if } w^U(t) = w^R(t) \\ = 0 & \text{if } w^U(t) < w^R(t) \end{cases} \tag{21.26}$$

This equation implies that as long as wages in the urban sector are greater those in the rural sector, there is a constant rate of migration. The speed of migration does not depend on the wage gap, which is an assumption adopted only to simplify the exposition. We may want to think of μ as small, so that there are barriers to migration and even when there are substantial gains to migrating to the cities, migration will take place slowly. When there is no wage gain to migrating, there will be no migration.

Now (21.25) implies that at date $t = 0$, there will be migration from the rural areas towards the cities. Moreover, assuming that $K(0)/L^U(0)$ is below the steady-state capital-labor ratio, the wage will remain high and will continue to attract further workers. To analyze this process in slightly greater detail, let us define

$$k(0) \equiv \frac{K(0)}{L^U(0)}$$

as the capital-labor ratio in manufacturing. As usual, let us also define the per capita production function in manufacturing as $f(k(t))$. Clearly,

$$w^U(t) = f(k(t)) - k(t) f'(k(t)).$$

Combining (21.24) and (21.26), we obtain that, as long as $f(k(t)) - k(t) f'(k(t)) > B^A$, the dynamics of this capital-labor ratio will be given by

$$\dot{k}(t) = sf(k(t)) - (\delta + \mu\nu(t)) k(t), \tag{21.27}$$

where $\nu(t) \equiv L^R(t)/L^U(t)$ is the ratio of the rural to urban population. Notice that when urban wages are greater than rural wages, the rate of migration, μ, times the ratio $\nu(t)$, plays the role of the rate of population growth in the basic Solow model of Chapter 2. In contrast, when $f(k(t)) - k(t) f'(k(t)) \leq B^A$, there is no migration and $\dot{k}(t) = sf(k(t)) - \delta k(t)$. Let us focus on the former case. Define the level of capital-labor ratio $\bar{k}$ such that

$$f(\bar{k}) - \bar{k} f'(\bar{k}) = B^A, \tag{21.28}$$

where urban and rural wages are equalized. Once this level is reached, migration will stop, and therefore $\nu(t)$ will remain constant. After this level of capital-labor ratio is reached, equilibrium dynamics will again be given by $\dot{k}(t) = sf(k(t)) - \delta k(t)$. Therefore, the steady state must always involve

$$\frac{sf(\hat{k})}{\hat{k}} = \delta. \tag{21.29}$$

For the analysis of transitional dynamics, which are our primary interest here, there are several cases to study. Let us focus on the one that appears most relevant for the experiences of many less-developed economies (leaving the rest to Exercise 21.4). In particular, suppose that the following conditions hold:

(1) $k(0) < \hat{k}$, so that the economy starts with lower capital-labor ratio (in the urban sector) than the steady-state level. This assumption also implies that $sf(k(0)) - \delta k(0) > 0$.
(2) $k(0) > \bar{k}$, which implies that $f(k(0)) - k(0) f'(k(0)) > B^A$, that is, wages are initially higher in the urban sector than in the rural sector.
(3) $sf(k(0)) - (\delta + \mu\nu(0)) k(0) < 0$, given the distribution of population between urban and rural areas, the initial migration will lead to a decline in the capital-labor ratios.

In this case, the economy starts with rural to urban migration at date $t = 0$. Since initially $\nu(0)$ is high, this migration reduces the capital-labor ratio in the urban area (which evolve according to the differential equation (21.27)). There are then two possibilities. In the first, the capital-labor ratio never falls below $\bar{k}$, thus rural to urban migration takes place at the maximum possible rate, μ, forever. Nevertheless, the effect of this migration on the urban capital-labor ratio is reduced over time as $\nu(t)$ declines with migration. Since we know that $sf(k(0)) - \delta k(0) > 0$, at some point the urban capital-labor ratio will start increasing, and it will eventually converge to the unique steady-state level $\hat{k}$. This convergence can take a long time and notably, it is not monotonic. The capital-labor ratio, and urban wages, first fall and then increase. The second possibility is that the initial surge in rural to urban migration reduces the capital-labor ratio to $\bar{k}$ at some point, say at date t'. When this happens, wages remain constant at B^A in both sectors and the rate of migration $\dot{L}^R(t)/L^R(t)$ adjusts exactly so that capital-labor ratio remains at $\bar{k}$ for a while (recall that when urban and rural wages are equal, (21.26) admits any level of migration between zero and the maximum rate μ). In fact, the urban capital-labor ratio can remain at this level for an extended period of time. Ultimately, however, $\nu(t)$ will again decline sufficiently that the capital-labor ratio in the urban sector must start increasing. Once this happens, migration takes place at the maximal rate μ and the economy again slowly converge as to the capital-labor ratio $\hat{k}$ in the urban sector.

Therefore, this discussion illustrates how a simple model of migration can generate rich dynamics of population in rural and urban areas and wage differences between the modern and the traditional sectors.

In dynamics discussed above, especially in the first case, the economy has the flavor of a *dual economy*. Wages and the marginal product of labor are higher in the urban area than in the rural area. If, in addition, μ is low, the allocation of workers from the rural to the urban areas will be slow, despite the higher wages. Thus the pattern of dual economy may be pronounced and may persist for a long time. It is also notable that rural to urban migration increases total output in the economy, because it enables workers to be allocated to activities in which their marginal product is higher. This process of migration increasing the output level in the economy also happens slowly because of the relatively slow process of migration.

The above discussion implies that, for the parameter configuration refocused on, the dual economy structure not only affects the social outlook of the society, which remains rural and agricultural for an extended period of time (especially when μ is small), but also leads to lower output than the economy could have generated by allocating labor more rapidly to the manufacturing sector. One should be cautious in referring to this as a "market failure," however, since we did not specify the reason why migration is slow. Without providing a micro model for migration, it is difficult to conclude whether the migration decisions are socially optimal or not (in the same sense as without a micro-founded model of savings, we could not talk of whether there was the right amount of savings and capital accumulation in the basic Solow growth model).

The model presented in this subsection therefore gives us a first formalization of a dual economy structure, which many development economists view as a good representation of the workings of less-developed economies. While dual economy features indeed appear to be important in practice and the model presented here is indeed simple and tractable, there are various reasons for striving for more sophisticated models. First, the migration behavior in the current model is extremely reduced-form. This is important, since the migration behavior is at the heart of the model. The reduced-form formulation implies that we cannot ask questions about whether migration is optimal or suboptimal. Second, the model gives the flavor of too little migration, though in many less-developed economies many urban centers appear to be overpopulated. Thus it is useful to seek more insights on whether there will be too much or too little migration. Finally, the assumption that the manufacturing sector is more productive than agriculture is somewhat crude. While the dual economy structure suggests that one part of the economy will be more productive, it would be more satisfactory and insightful if there are some compensating differentials in the less productive sector. The model presented in the next subsection will rectify some of these shortcomings.

21.3.2. Community Enforcement, Migration and Development. I now present a model inspired by Banerjee and Newman (1998) and Acemoglu and Zilibotti (1999). Banerjee and Newman consider an economy where the traditional sector has low productivity but is less affected by informational asymmetries and thus individuals can engage in borrowing and lending with limited monitoring and incentive costs. In contrast, the modern sector is more productive but informational asymmetries create more severe credit market problems. Banerjee and Newman discuss how the process of development is associated with the reallocation of economic activity from the traditional to the modern sector and how this reallocation is slowed down by the informational advantage of the traditional sector. Acemoglu and Zilibotti (1999) view the development process as one of "information accumulation," and greater information enables individuals to write more sophisticated contracts and enter into more complex production relations. This process is then associated with changes in technology, changes in financial relations and social transformations, since greater availability on information and better contracts enable individuals to abandon less efficient and less information-dependent social and productive relationships.

The model in this subsection is simpler than both of these papers, but features a similar economic mechanism. Individuals who live in rural areas are subject to community enforcement. This means that they can enter into economic and social relationships without being unduly affected by moral hazard problems. When individuals move to cities, they can take part in more productive activities, but other enforcement systems are necessary to ensure compliance to social rules, contracts and norms. These systems will typically be associated with certain costs. As in the model of industrialization in the previous section, I will also assume that the modern sector is subject to learning-by-doing externalities. Thus the productivity advantage of the modern sector grows as more individuals migrate to cities and work in the modern sector. However, the community enforcement advantage of villages slows down this process and may even lead to a development trap. Since the mathematical structure of the model is similar to that in the last section, my treatment will be relatively brief.

The basic structure of the model is similar to that in the previous section. All labor markets are competitive and population is normalized to 1. There are three differences from the model in the previous subsection. First, migration between the rural and urban areas is costless. Thus at any point in time an individual can switch from one sector to another. Second, instead of capital accumulation, there is an externality in the manufacturing sector. In particular, suppose that output of the manufacturing sector is given by

$$Y^M(t) = X(t) F\left(L^U(t), Z\right),$$

where $X(t)$ denotes the productivity of the modern sector, which will be determined endogenously via learning-by-doing externalities. In addition, Z denotes another factor of production and fixed supply (so that there are diminishing returns to labor), and the production function F satisfies our standard assumptions, Assumptions 1 and 2. Moreover, let us assume that the technology in the manufacturing sector evolves according to the differential equation

$$\dot{X}(t) = \eta L^U(t) X(t)^\zeta,$$

where $\zeta \in (0,1)$. This equation builds in learning-by-doing externalities along the lines of Romer's (1986) paper and is also similar to the industrialization model of Section 20.3 in the previous chapter. The fact that $\zeta < 1$ implies that these externalities are less than what would be necessary to sustain endogenous growth.

Finally, let us also assume that rural areas have a comparative advantage in *community enforcement*. In particular, individuals engage in many social and economic activities, ranging from financial relations and employment to marriage and social relations. Many of these relationships in cities are anonymous and enforcement is through some type of monitoring by the law and relies on complex institutions. Such institutions often work imperfectly in most societies and particularly in less-developed economies. In contrast, rural areas house small number of individuals who are in long-term relationships. These long-term relationships enable the use of community enforcement in many activities. Thus with long-term relationships, individuals can pledge their reputation to borrow money to smooth consumption, to obtain information about which individual would be most appropriate for a particular job, or to ensure cooperation in other work or social relations. We represent these in a reduced-form way by assuming that an individual pays a flow cost of $\xi > 0$ due to imperfect monitoring and lack of community enforcement when he is in the urban area.

All individuals maximize their utility, and since savings do not matter for the key allocation decisions, I do not specify utility functions. The key observation is that all individuals would like to maximize the net present discounted value of their lifetime incomes. But since moving between urban and rural areas is costless, this implies that each individual should work in the sector that has the higher net wage at that time. This implies that in an interior equilibrium (where both the rural and the urban sectors are active), the following wage equalization condition must hold:

$$w^M(t) - \xi = w^A(t).$$

Competitive labor markets imply that

$$\begin{aligned} w^M(t) &= X(t) \frac{\partial F\left(L^U(t), Z\right)}{\partial L} \\ &\equiv X(t)\tilde{\phi}\left(L^U(t)\right), \end{aligned}$$

where the second line defines the function $\tilde{\phi}$, which is strictly decreasing in view of Assumption 1 on the production function F. Substituting from the above relationships, labor market clearing implies that

$$X(t)\,\tilde{\phi}\left(L^U(t)\right) = B^A + \xi,$$

or

$$\begin{aligned} L^U(t) &= \tilde{\phi}^{-1}\left(\frac{B^A+\xi}{X(t)}\right) \\ &\equiv \phi\left(\frac{X(t)}{B^A+\xi}\right), \end{aligned}$$

where again the second line defines the function ϕ, which is strictly increasing in view of the fact that $\tilde{\phi}$ (and thus $\tilde{\phi}^{-1}$) was strictly decreasing. Therefore, the evolution of this economy can be represented by the differential equation

$$\dot{X}(t) = \eta\phi\left(\frac{X(t)}{B^A+\xi}\right)X(t)^{\zeta}.$$

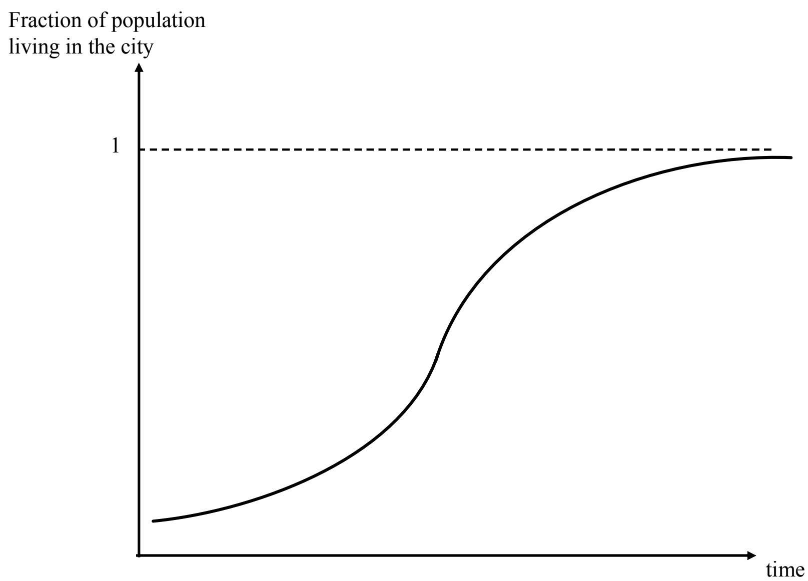

Figure 21.3. The dynamic behavior of the population in rural and urban areas.

A number of features about this law of motion are worth noting. First, the typical evolution of $X(t)$ will be given as in Figure 21.3, with an S-shaped pattern. This is because when $X(0)$ is low, we would expect that $\phi\left(X(t)/\left(B^A+\xi\right)\right)$ to be also low the early stages of development and thus manufacturing technology to progress only slowly. However, as $X(t)$ increases, $\phi\left(X(t)/\left(B^A+\xi\right)\right)$ will also increase, raising the rate of technological change in the manufacturing sector. Ultimately, however, $L^U(t)$ cannot exceed 1, so $\phi\left(X(t)/\left(B^A+\xi\right)\right)$ will tend to a constant, and thus the rate of growth of X will decline. Therefore, this reduced-form model generates an S-shaped pattern of technological change in the manufacturing sector, and associated with that the migration of workers from rural to urban areas will also follow an S-shaped pattern.

Second and more importantly, the process of technological change in the manufacturing sector and migration to the cities are slowed down by the comparative advantage of the rural areas in community enforcement. In particular, the greater is ξ, the slower is technological change and migration into urban areas. Since employment in the urban areas creates positive externalities, the community enforcement system in rural areas slows down the process of economic development in the economy as a whole. We may therefore conjecture that high levels of ξ, corresponding to greater community enforcement advantage of the traditional sector, will generally reduce growth and welfare in the economy. Contracting this, however, are the static gains created by the better community enforcement system in rural areas. A high level of ξ will increase the initial level of consumption in the economy. Consequently, there is a tradeoff between dynamic and static welfare implications of different levels of ξ and this tradeoff is investigated formally in Exercise 21.5.

Finally, this model also offers a formalization of some of the ideas related to the dual economy. In contrast to the model of subsection 21.3.1, there are no mobility barriers, thus workers in the villages and cities receive the same wage. However, the functioning of the economy and the structure of social relations are different in these two areas. While villages and the rural economy rely on community enforcement, the city uses the modern technology and impersonal institutional checks in order to enforce various economic and social arrangements. Consequently, the dual economy in this model exhibits itself as much in the social dimension as in the economic dimension.

21.3.3. Inappropriate Technologies and the Dual Economy. I now discuss how ideas related to the issue of appropriate technology discussed in Chapter 18 (Section 18.4) may provide promising clues about the nature of the dual economy patterns. Recall from Section 18.4 that less-developed economies often import their technologies from more advanced economies and that these technologies are typically designed for different factor proportions than those of the less-developed economy. For example, in Section 18.4, I emphasized the implications of a potential mismatch between the skills of the workforce of a less-developed economy and the skill requirements of modern technologies. However, the model in that section was designed such that the equilibrium (and the best option) for the less-developed economy was always to use the modern technology.

Consider a variant of a model in that section, where each technology is of the Leontief type, so that it requires a certain number of skilled and unskilled workers. For example, technology A_h will produce a total of $A_h L$ units of the unique final good, where L is the number of unskilled workers, but this technology requires a ratio of skilled to unskilled workers exactly equal to h (for example, the skilled workers will be the managers or the supervisors for the unskilled workers). Suppose A_h is increasing in h, so that more advanced technologies are more productive.

Now consider a less-developed economy that has access to all technologies A_h for $h \in \left[0, \bar{h}\right]$ for some $\bar{h} < \infty$. Suppose that the population of this economy consists of H skilled and L unskilled workers, such that $H/L < \bar{h}$. This inequality implies that not all workers can be employed with the most skill-intensive technology. What will be form of equilibrium be in this economy?

To answer this question, imagine that all markets are competitive, so that the allocation of workers to tasks will simply maximize output. Then the problem can be written as

$$\max_{[L(h)]_{h \in [0,\bar{h}]}} \int_0^{\bar{h}} A_h L\left(h\right) dh \tag{21.30}$$

subject to

$$\begin{aligned} \int_0^{\bar{h}} L\left(h\right) dh &= L, \text{ and} \\ \int_0^{\bar{h}} hL\left(h\right) dh &= H, \end{aligned}$$

where $L\left(h\right)$ is the number of unskilled workers assigned to work with technology A_h. The first-order conditions for this maximization problem can be written as

$$A_h \leq \lambda_L + h\lambda_H \text{ for all } h \in \left[0, \bar{h}\right], \tag{21.31}$$

where λ_L is the multiplier associated with the first constraint and λ_H is the multiplier associated with the second constraint. The first-order condition is written as an inequality, since not all technologies $h \in \left[0, \bar{h}\right]$ will be used, and those that are not active might satisfy this condition with a strict inequality.

Inspection of the first-order conditions implies that if $A_{\bar{h}}$ is sufficiently high and if $A_0 > 0$, the solution to this problem will have a very simple feature. All skilled workers will be employed at technology $\bar{h}$, and together with them, there will be $L\left(\bar{h}\right) = H/\bar{h}$ unskilled workers employed with this technology. The remaining $L - L\left(\bar{h}\right)$ workers will be employed with the technology $h = 0$ (see Exercise 21.6). This equilibrium will then have the feature of a dual economy. Two very different technologies will be used for production, one more advanced (modern), and the other corresponding to the least advanced technology that is feasible. This dual economy structure emerges because of a non-convexity—to maximize output, it is necessary to operate the most advanced technology, but this exhausts all of the available skilled workers, implying that unskilled workers have to be employed in technologies that do not require skilled inputs or supervision. This perspective therefore suggests that a dual economy structure might be the natural outcome of technology transfer, especially in situations where less-developed economies import their technologies from more advanced nations and these technologies are inappropriate to the needs of less-developed countries.

Models of dual economy based on this type of appropriate technology ideas have not been investigated in detail, though the literature on appropriate technology, which was discussed in Chapter 18, suggests that they may be important in practice. While this model focuses on the dual economy aspect in production, one can easily generalize this framework by assuming that the more advanced technology will be operated in urban areas and with contractual arrangements enforced by modern institutions, while the less advanced technology is operated in villages or rural areas. Thus models based on appropriate (or inappropriate) technology may be able to account for broader patterns related to dual economy, including rural to urban migration and changes in social arrangements.

21.4. Distance to the Frontier and Changes in the Organization of Production

In this section, I discuss how the structure of production changes over the process of development, and how this might be related both to changes in certain aspects of the internal organization of the firm and to a shift in the "growth strategy" of an economy. I will illustrate these ideas using a simple model based on Acemoglu, Aghion and Zilibotti (2006). Because of space restrictions, I only provide a sketch of the model, mainly focusing on the production side of the economy.

Consider a less-developed economy that is behind the world technology frontier. Throughout, I focus on the behavior of this economy, thus there is no need to use country indices. Time is discrete and the economies populated by two-period lived overlapping generations of individuals. Total population is normalized to 1. There is a unique final good, which is also taken as the numeraire. It is produced competitively using a continuum of intermediate inputs according to the standard Dixit-Stiglitz (constant elasticity of substitution) aggregator:

$$Y(t) = \int_0^1 A(\nu,t)^{1-\alpha} x(\nu,t)^{\alpha} \, d\nu, \tag{21.32}$$

where $A(\nu,t)$ is the productivity of the intermediate good in intermediate sector ν at time t, $x(\nu,t)$ is the amount of intermediate good ν used in the production of the final good at time t, and $\alpha \in (0,1)$.

Each intermediate good is produced by a monopolist $\nu \in [0,1]$ at a unit marginal cost in terms of the unique final good. The monopolist faces a competitive fringe of imitators that can copy its technology and also produce an identical intermediate good with productivity $A(\nu,t)$, but will do so more expensively. In particular, the competitive fringe can produce each intermediate good at the cost of $\chi > 1$ units of final good. The existence of this competitive fringe forces the monopolist to charge a *limit price*:

$$p(\nu, t) = \chi > 1. \tag{21.33}$$

Naturally, this limit price configuration will be an equilibrium when χ is not so high that the monopolist prefers to set a lower unconstrained monopoly price. The condition for this is simply

$$\chi \leq 1/\alpha,$$

which I impose throughout. Broadly, one can think of the parameter χ as capturing both technological factors and government regulations regarding competitive policy. A higher χ corresponds to a less competitive market. Given the demand implied by the final goods technology in (21.32) and the equilibrium limit price in (21.33), equilibrium monopoly profits are simply:

$$\pi(\nu, t) = \delta A(\nu, t), \tag{21.34}$$

where

$$\delta \equiv (\chi - 1)\, \chi^{-1/(1-\alpha)} \alpha^{1/(1-\alpha)}$$

is a measure of the extent of monopoly power. In particular it can be verified that δ is increasing in χ for all $\chi \leq 1/\alpha$.

In this model, the process of economic development will be driven not by capital accumulation—which was the force emphasized in some of the earlier models—but by technological progress, that is, by increases in $A(\nu, t)$. Let us assume that each monopolist $\nu \in [0, 1]$ can increase its $A(\nu, t)$ by two complementary processes: (i) imitation (adoption of existing technologies); and (ii) innovation (discovery of new technologies). The key economic trade-offs in the model arise from the fact that different economic arrangements (both in terms of the organization of firms and in terms of the growth strategy of the economy) will lead to different amounts of imitation and innovation.

To prepare for this point, let us define the average productivity of the economy in question at date t as:

$$A(t) \equiv \int_0^1 A(\nu, t)\, d\nu.$$

Let $\bar{A}(t)$ denote the productivity at the world technology frontier. The fact that this economy is behind the world technology frontier means that

$$A(t) \leq \bar{A}(t)$$

for all t. The world technology frontier progresses according to the difference equation

$$\bar{A}(t) = (1 + g)\, \bar{A}(t - 1), \tag{21.35}$$

where the growth rate of the world technology frontier is taken to be

$$g \equiv \underline{\eta} + \bar{\gamma} - 1, \tag{21.36}$$

where $\underline{\eta}$ and $\bar{\gamma}$ will be defined below.

We assume that the process of imitation and innovation leads to the following law of motion of each monopolist's productivity:

$$A(\nu,t) = \eta\bar{A}(t-1) + \gamma A(t-1) + \varepsilon(\nu,t), \tag{21.37}$$

where $\eta > 0$ and $\gamma > 0$, and $\varepsilon(\nu,t)$ is a random variable with zero mean, capturing differences in innovation performance across firms and sectors.

In equation (21.37), $\eta\bar{A}(t-1)$ stands for advances in productivity coming from *adoption* of technologies from the frontier (and thus depends on the productivity level of the frontier, $\bar{A}(t-1)$), while $\gamma A(t-1)$ stands for the component of productivity growth coming from innovation (building on the existing knowledge stock of the economy in question at time $t-1$, $A(t-1)$).

Let us also define

$$a(t) \equiv \frac{A(t)}{\bar{A}(t)}$$

as the (inverse) measure of the country's *distance to the technological frontier* at date t.

Now, we can integrate (21.37) over $\nu \in [0,1]$, use the fact that $\varepsilon(\nu,t)$ has mean zero, divide both sides by $\bar{A}(t)$ and use (21.35) to obtain a simple linear relationship between a country's distance to frontier $a(t)$ at date t and the distance to frontier $a(t-1)$ at date $t-1$ given by

$$a(t) = \frac{1}{1+g}(\eta + \gamma a(t-1)). \tag{21.38}$$

This equation is similar to the technological catch-up equation in Section 18.2 in Chapter 18. It shows how the dual process of imitation and innovation may lead to a process of convergence. In particular, as long as $\gamma > 1+g$, equation (21.38) implies that $a(t)$ will eventually converge to 1. Second, the equation also shows that the relative importances of imitation and innovation will depend on the distance to the frontier of the economy in question. In particular when $a(t)$ is large (meaning the country is close to the frontier), γ—thus innovation—matters more for growth. In contrast when $a(t)$ is small (meaning the country is farther from the frontier), η—thus imitation—is relatively more important.

To obtain further insights, let us now endogenize η and γ using a reduced-form approach. Following the analysis in Acemoglu, Aghion and Zilibotti (2006), I will model the parameters η and γ as functions of the investments undertaken by the entrepreneurs and the contractual arrangement between firms and entrepreneurs. The key idea is that there are two types of entrepreneurs: high-skill and low-skill. When an entrepreneur starts a business, his skill level is unknown, and is revealed over time through his subsequent performance. This implies that there are two types of "growth strategies" that are possible. The first one emphasizes *selection* of high-skill entrepreneurs and will replace any entrepreneur that is revealed to be low skill. This growth strategy will involve a high degree of turning (creative destruction) and a large number of young entrepreneurs (as older unsuccessful entrepreneurs are replaced by new young entrepreneurs). The second strategy maintains experienced entrepreneurs in place even when they are have low skills. This strategy therefore involves an organization of firms relying on "longer-term relationships" (here between entrepreneurs in the credit market), an emphasis on experience and cumulative earnings, and less creative destruction. While low-skill entrepreneurs will be less productive than high-skill entrepreneurs, there are potential reasons for why an experienced low-skill entrepreneur might be preferred to a new young entrepreneur. For example, this may be because entrepreneurial experience increases productivity, so that the low-skill experienced entrepreneur may be better at certain tasks than a high-skill inexperienced entrepreneur. Alternatively, Acemoglu, Aghion and Zilibotti (2006) show that in the presence of credit market imperfections, the retained earnings of an old entrepreneur may provide him with an advantage in the credit market (because he can leverage his existing earnings to raise more money). I denote the strategy based on selection by $R = 0$, while the strategy that maintains experienced entrepreneurs in place is denoted by $R = 1$.

The key reduced-form assumption here will be that experienced entrepreneurs (either because of the value of experience or because of their retained earnings) are better at increasing the productivity of their company when this involves the imitation of technologies from the world frontier (i.e., relatively "routine" tasks). High-skill entrepreneurs, on the other hand, are more innovative and generate higher growth due to innovation. Thus the tradeoff between $R = 1$ and $R = 0$ and the associated tradeoff between organizational forms boils down to the tradeoff between imitation of technologies from the world technology frontier versus innovation. For this reason, I will refer to the first one as *imitation-based growth strategy* and to the second one as the *innovation-based growth strategy.* Motivated by these considerations, let us assume that the equation for the law of motion of the distance to frontier, (21.38), takes the form

$$a(t) = \begin{cases} \frac{1}{1+g}\left(\bar{\eta} + \underline{\gamma} a(t-1)\right) & \text{if} \quad R(t) = 1 \\ \\ \frac{1}{1+g}\left(\underline{\eta} + \bar{\gamma} a(t-1)\right) & \text{if} \quad R(t) = 0 \end{cases} \tag{21.39}$$

as a function of the contractual/organizational decision at time t, $R(t) \in \{0, 1\}$. In this equation we assume that

$$\bar{\gamma} > \underline{\gamma} < 1 + g \text{ and } \bar{\eta} > \underline{\eta}. \tag{21.40}$$

The first part of this assumption follows immediately from the notion that high-skill entrepreneurs are better at innovation, while the second part, in particular, that $\bar{\gamma} > \underline{\gamma}$, builds in the feature that experienced entrepreneurs are better at imitation. When the imitation-based growth strategy is pursued, experienced entrepreneurs are not replaced, and consequently, there is greater transfer of technology from the world technology frontier. The final part of this assumption, that $\underline{\gamma} < 1 + g$ simply ensures that imitation-based growth will not lead to faster growth than the world technology frontier. Also in terms of (21.39), we can interpret the assumption (21.36) as stating that the world technology frontier advances due to innovation-based growth strategy, which is natural, since a country at the world technology frontier cannot imitate from others. This is what equation (21.40) introduces.

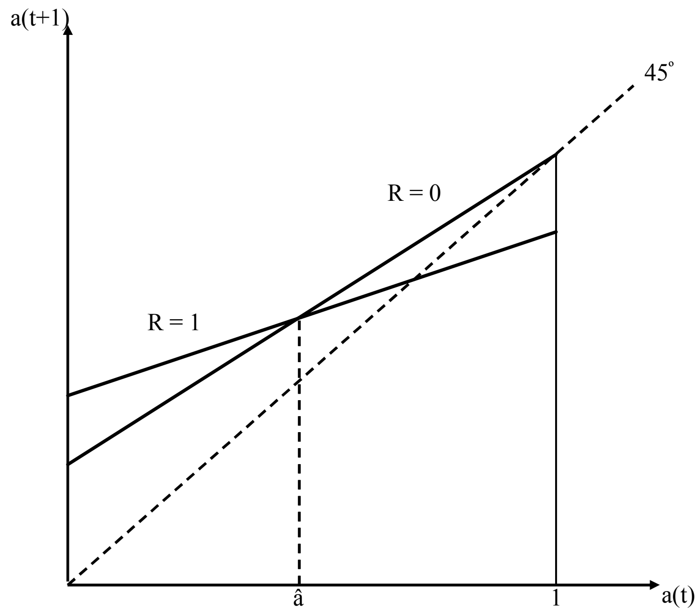

Figure 21.4. The growth-maximizing threshold and the dynamics of the distance to frontier in the growth-maximizing equilibrium.

Figure 21.4 draws equation (21.39), and shows that the economy with long-term contracts ($R = 1$) achieves greater growth (higher level of $a(t)$ for given $a(t-1)$) through imitation channel, but lower growth through the innovation channel. The figure also shows that which regime maximizes the growth rate of the economy depends on the level of $a(t-1)$, that is, on the distance of the economy to the world technology frontier. In particular, inspection of (21.39) is sufficient to establish that there exists a threshold

$$\hat{a} \equiv \frac{\bar{\eta} - \underline{\eta}}{\bar{\gamma} - \underline{\gamma}} \in (0, 1) \tag{21.41}$$

such that when $a(t-1) < \hat{a}$, the imitation-based strategy, $R = 1$ leads to greater growth, and when $a(t-1) > \hat{a}$, the innovation-based strategy, $R = 0$, achieves higher growth. Thus if the economy were to pursue a growth-maximizing sequence of strategies, it would start with $R = 1$ and then switch to an innovation-based strategy, $R = 0$, once it is sufficiently close to the world technology frontier. In the imitation-based regime, incumbent entrepreneurs are sheltered from the competition of younger entrepreneurs and this may enable the economy to make better use of the experience of older entrepreneurs or to finance greater investments out of the retained earnings of incumbent entrepreneurs. In contrast, the innovation-based regime is based on an organizational form relying on greater selection of entrepreneurs and places greater emphasis on maximizing innovation at the expense of experience, imitation and investment.

Figure 21.4 describes the law of motion of technology in an economy as a function of the organization of firms (markets), captured by R. It does not specify what the equilibrium sequence of $\{R(t)\}_{t=0}^{\infty}$ is. To determine this equilibrium sequence, we need to specify the equilibrium behavior, which involves the selection of entrepreneurs as well as the functioning of credit markets. Space restrictions preclude me from providing a full analysis of the equilibrium in such a model. Instead, I will informally discuss some of the main insights of such an analysis.

Conceptually, one might want to distinguish among four configurations, which may arise as equilibria under different institutional settings.

(1) *Growth-maximizing equilibrium*: the first and the most obvious possibility is an equilibrium that is growth maximizing. In particular, if markets and entrepreneurs are able to solve the agency problems, have the right decision-making horizon and are able to internalize the pecuniary and non-pecuniary externalities, we would obtain an efficient equilibrium. This equilibrium will take a simple form:

$$R(t) = \begin{cases} 1 & \text{if} \quad a(t-1) < \hat{a} \\ 0 & \text{if} \quad a(t-1) \geq \hat{a} \end{cases}$$

so that the economy achieves the upper envelope of the two lines in Figure 21.4. In this case, there is no possibility of outside intervention to increase the growth rate of the economy.[1] Moreover, an economy starting with $a(0) < 1$ always achieves a growth rate greater than g, and will ultimately converge to the world technology frontier, i.e., $a(t) \to 1$. In this growth-maximizing equilibrium, the economy first starts with a particular set of organizations/institutions, corresponding to $R = 1$. Then, consistent with Kuznets' vision of a structural transformation emphasized above, the economy undergoes a change in its organizational form and growth strategy, and switches from $R = 1$ to $R = 0$. In our simple economy, this structural transformation takes the form of long-term relationships disappearing and being replaced by shorter-term relationships, by greater competition among entrepreneurs and firms, and by better selection of entrepreneurs.

[1] However, recall that growth-maximization is not necessarily the same as welfare-maximization. Depending on how preferences and investments are specified, the growth-maximizing allocation may not be welfare-maximizing.

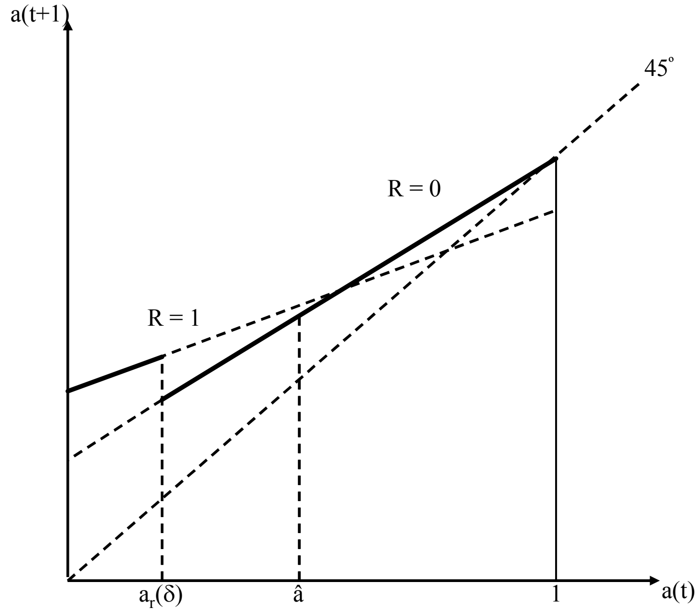

Figure 21.5. Dynamics of the distance to frontier in the underinvestment equilibrium.

(2) *Underinvestment equilibrium*: the second potential equilibrium configuration involves the following equilibrium organizational form:

$$R(t) = \begin{cases} 1 & \text{if} \quad a(t-1) < a_r(\delta) \\ 0 & \text{if} \quad a(t-1) \geq a_r(\delta) \end{cases}$$

where $a_r(\delta) < \hat{a}$. Figure 21.5 depicts this visually, with the thick black lines corresponding to the equilibrium law of motion of the distance to the frontier, a. How is $a_r(\delta)$ determined? Acemoglu, Aghion and Zilibotti (2006) shows that when investments by young and old entrepreneurs are important for innovation and credit markets are imperfect, then the retained earnings of old (experienced) entrepreneurs enable them to undertake greater investments. However, because of monopolistic competition, there is the standard *appropriability effect*, whereby an entrepreneur that undertakes a greater investment does not capture all the surplus generated by this investment because some of it accrues to consumers in the form of greater consumer surplus. The appropriability effect always discourages investments, and in this context since greater investments are associated with more experienced, older entrepreneurs, it discourages the regime in which long-term contracts keep entrepreneurs in place for two periods in a row. This description also explains why this equilibrium is referred to as the "underinvestment equilibrium"; in the range $a \in (a_r(\delta), \hat{a})$, the economy could reach a higher growth rate (as shown in the figure) by choosing

$R(t) = 1$, but because the appropriability effect discourages investments, there is a switch to the innovation-based equilibrium and the associated organizational forms earlier than the growth-maximizing threshold.

A notable feature is that although the equilibrium is different from the previous case, it again follows the sequence of $R = 1$ followed by a structural transformation and a switch to greater competition among and selection of entrepreneurs with the innovation-based regime. Therefore, this equilibrium also exhibits the feature that the process of growth and economic development is associated with structural transformation. Moreover, the economy will still ultimately converge to the world technology frontier, i.e., $a(t) = 1$ is reached as $t \to \infty$. The only difference is that the structural transformation from $R = 1$ to $R = 0$ happens too soon at $a(t-1) = a_r(\delta)$ rather than at $\hat{a}$.

Consequently, in this case, a *temporary* government intervention may increase the growth rate of the economy. The temporary aspect is important here, since the best that the government can do is to increase the growth rate while $a \in (a_r(\delta), \hat{a})$. How can the government achieve this? Subsidies to investment would be one possibility. Acemoglu, Aghion and Zilibotti (2006) show that the degree of competition in the product market also has an indirect effect on the equilibrium, as emphasized by the notation $a_r(\delta)$. In particular, a higher level of δ, which corresponds to lower competition in the product market (i.e., higher χ), will increase $a_r(\delta)$, and thus may close the gap between $a_r(\delta)$ and $\hat{a}$. Nevertheless, it has to be noted that reducing competition will create other, static distortions (because of higher markups). Moreover and more importantly, we will see in the next two configurations that reducing competition can have much more detrimental effects on economic growth.

3. *Sclerotic equilibrium*: the third possibility is a sclerotic in which $a_r(\delta) > \hat{a}$, so that incumbent low-skill, low-productivity firms survive even when they are potentially damaging to economic growth. Acemoglu, Aghion and Zilibotti (2006) show that this configuration can also arise in equilibrium because the retained earnings of incumbent entrepreneurs act as a *shield* protecting them against the creative destruction forces brought about by new entrepreneurs. Consequently, in general, the retained earnings or other advantages of experienced entrepreneurs both have (social) benefits and costs, and which of these will dominate will depend on the details of the model and the parameter values. When the benefits dominate, the equilibrium may feature too rapid a switch to the innovation-based strategy, and when the costs dominate, the economy may experience sclerosis with the imitation-based strategy and excessive protection of incumbents.

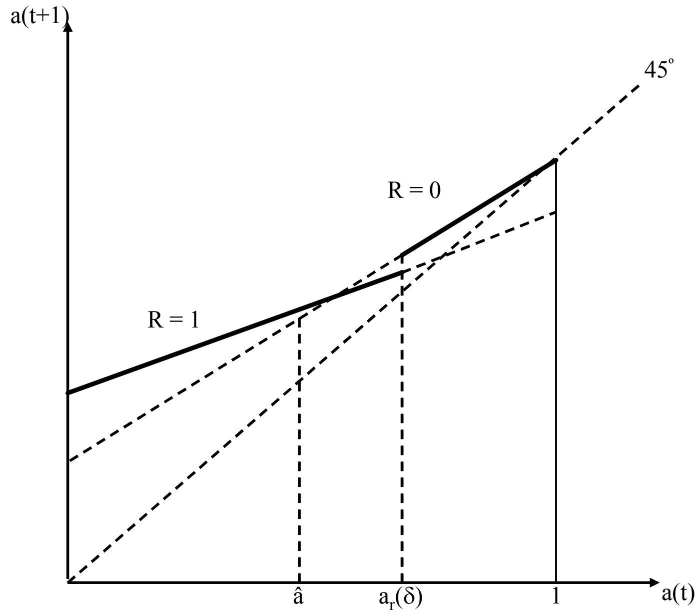

Figure 21.6. Dynamics of the distance to frontier in the sclerotic equilibrium.

The resulting pattern in this case is drawn in Figure 21.6. Now the economy fails to achieve the maximum growth rate for a range of values of a such that $a \in (\hat{a}, a_r(\delta))$. In this range, the innovation-based regime would be growth-maximizing, but the economy is stuck with the imitation-based regime because of the retained earnings and the power of the incumbents prevent the transition to the more efficient organizational forms.

An interesting feature is that, as Figure 21.6 shows, this economy also follows a pattern in line with Kuznets's vision; it starts with a distinct set organizations, represented by $R = 1$, and then switches to a different set of arrangements, $R = 0$. Like the previous two types of equilibria, this case also features convergence to the world technology frontier, i.e., to $a = 1$.

4. *Non-convergence trap equilibrium:* the fourth possibility is related to the third one and also involves $a_r(\delta) > \hat{a}$. However, now the gap between $a_r(\delta)$ and $\hat{a}$ is even larger as depicted in Figure 21.7, and includes the level of a, a_{trap}, such that

$$a_{trap} \equiv \frac{\bar{\eta}}{1 + g - \underline{\gamma}}.$$

Inspection of (21.39) immediately reveals that if $a(t-1) = a_{trap}$ and $R(t) = 1$, the economy will remain at a_{trap}. Therefore, in this case, the retained earnings or the experience of incumbent firms afford them so much protection that the economy

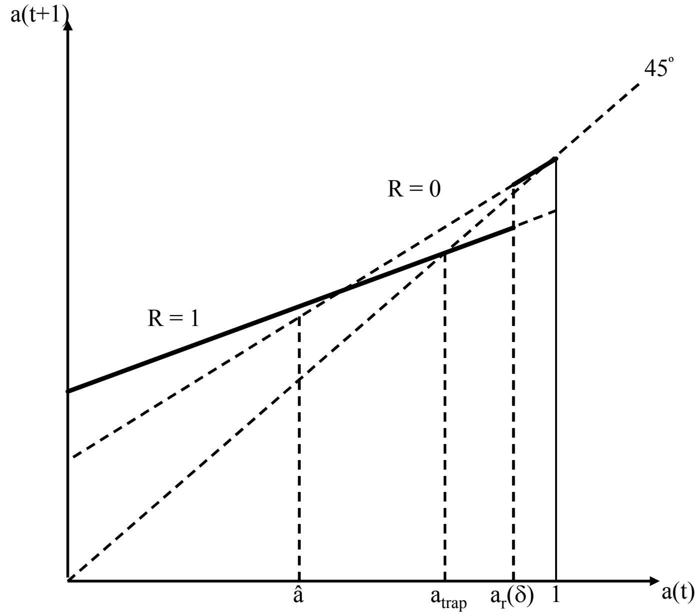

FIGURE 21.7. Dynamics of the distance to frontier in a non-convergence trap. If the economy starts with $a(0) < a_{trap}$, it fails to converge to the world technology frontier and instead converges to a_{trap}.

never transitions to the innovation-based equilibrium. This not only retards growth for a temporary interval but pushes the economy into a non-convergence trap. In particular, this is the only equilibrium pattern in which the economy fails to converge to the frontier; with the imitation-based regime, $R = 1$, the economy does not grow beyond a_{trap}, and at this distance to frontier, the equilibrium keeps choosing $R = 1$.

This equilibrium therefore illustrates the most dangerous scenario—that of non-convergence. Encouraging imitation-based growth—by supporting existing, incumbent firms, may appear as a good policy,[2] but in fact it condemns the economy to non-convergence. This is also the only case in which the Kuznetsian structural transformation does *not* occur because the economy remains trapped. In many ways, this is in line with Kuznets' vision; the resulting economy is an underdeveloped one, unable to realize the structural transformation necessary for the process of economic development.

Taken together the four scenarios suggest that depending on the details of the model, there should be no presumption that the efficient or the growth-maximizing sequence of growth

[2]The reader may notice that this type of policy to encourage growth has many of the features of "industrial policy" pursued by many less-developed economies. The evidence is that experiments with industrial policy also seem to have backfired, and created powerful incumbents and low growth in most instances.

strategies will be pursued. Thus, some degree of government intervention might be useful. However, the third and the fourth cases also emphasize that government intervention can have fairly negative unintended consequences. Such intervention will improve growth performance during a limited period of time (in the second scenario when $a \in (a_r(\delta), \hat{a})$, but it can create much more substantial costs by leading to a non-convergence trap as shown in Figure 21.7. Therefore, unless there is very precise information and some way of reversing policies protecting incumbents that are once implemented (a very difficult practice because of political economy reasons, which will be discussed in greater detail in Chapter 22), government interventions to spearhead economic development might backfire.

Even though the implications of these four scenarios for government intervention are mixed, their implications for changes in the structure of organization over the development process are clearer; irrespective of which scenario applies, the economy starts with a distinct organization of production, where longer-term contracts, the incumbent producers, experience and imitation are more important, and then, except in the non-convergence trap equilibrium, it ultimately switches to an equilibrium with greater creative destruction, shorter-term relationships, younger entrepreneurs and more innovation. This type of transformation is another facet of the structural transformations emphasized by Kuznets as part of the process of economic development. The framework presented here, though reduced-form, can also be used to study other aspects of the transformation of the production of organization. Exercise 21.7 shows how the ideas in this section can be used to study the changes in other aspects of the internal organization of the firm through the course of the process of development.

21.5. Multiple Equilibria From Aggregate Demand Externalities and the Big Push

I now present a simple model of multiple equilibria arising from aggregate demand externalities. The model is a version of Murphy, Shleifer and Vishny's (1989) model of "big push", which formalized ideas proposed by Rosenstein-Rodan (1943), Hirschman and Nurske, hat economic development can be viewed a move from one (Pareto inefficient) equilibrium to another, more efficient equilibrium. Moreover, these early development economists argued that this type of move requires coordination among different individuals and firms in the economy, thus a *big push.* As already discussed in Chapter 4, multiple equilibria, literally interpreted, are unlikely to be the root cause of persistently low levels of development, since if there is indeed a Pareto improvement—a change that will make *all* individuals better off—it is unlikely that the necessary coordination cannot be achieved for decades or even centuries. Nevertheless, the forces leading to multiple equilibria highlight important economic mechanisms that can be associated with market failures slowing down, or even preventing, the process of development. Moreover, dynamic versions of models of multiple equilibria can lead to multiple state states, whereby once an economy ends up in a steady state with low economic activity, it may get stuck there (and there is no possibility of a coordination to jump to the other steady state). Models with multiple steady states will be discussed in the next section, where I will return to a further discussion of the difference between multiple equilibria and multiple steady states.

Murphy, Shleifer and Vishny consider the following two-period economy, $t = 1$ and 2. The economy admits a representative household with preferences given by

$$U = \frac{C(1)^{1-\theta} - 1}{1-\theta} + \beta \frac{C(2)^{1-\theta} - 1}{1-\theta}$$

where $C(1)$ and $C(2)$ denote consumption at the two dates; β is the discount factor of the households; and θ plays a similar to before; $1/\theta$ is the intertemporal elasticity of substitution and determines how willing individuals are to substitute consumption between date 1 and date 2. The representative household supplies labor inelastically and the total labor supply is denoted by L.

The resource constraint for the economy is

$$\begin{aligned} C(1) + I(1) &\leq Y(1) \\ C(2) &\leq Y(2), \end{aligned}$$

where $I(1)$ denotes investment in the first date, $Y(t)$ is total output at date t, and investment is only possible in the first date.

Households can borrow and lend, so their budget constraint can be represented as

$$C(1) + \frac{C(2)}{R} \leq w(1) + \pi(1) + \frac{w(2) + \pi(2)}{R},$$

where $\pi(t)$ denotes the profits accruing to the representative household, and $w(t)$ is the wage rate at time t. R is the gross interest rate between periods 1 and 2. Although individuals can borrow and lend, in the aggregate the resource constraints have to hold, so R will be determined in equilibrium to ensure this.

As in the endogenous technological progress models in Part 4 and in the model of the previous section, the final good is assumed to be a constant elasticity of substitution (Dixit-Stiglitz) aggregate of a continuum 1 of differentiated intermediate goods, and is thus given as

$$Y(t) = \left[\int_0^1 y(\nu, t)^{\frac{\varepsilon-1}{\varepsilon}} d\nu \right]^{\frac{\varepsilon}{\varepsilon-1}},$$

where $y(\nu, t)$ is the output level of intermediate ν at date t. The fact that many features of the model here are similar to the baseline endogenous technological change model highlights that the aggregate demand externalities that may lead to development traps here are already present in our workhorse endogenous growth models. As usual ε is the elasticity of substitution between intermediate goods within a given period and is assumed to be strictly greater than one, i.e., $\varepsilon > 1$.

The production functions of intermediate goods in the two periods are as follows:

$$y(\nu, 1) = l(\nu, 1)$$

and

$$y(\nu, 2) = \begin{cases} l(\nu, 2) & \text{with old technology} \\ \alpha l(\nu, 2) & \text{with new technology} \end{cases} \tag{21.42}$$

where $\alpha > 1$ and $l(\nu, t)$ denotes labor devoted to the production of intermediate good ν at time t. Labor market clearing, naturally, requires

$$\int_0^1 l(\nu, t)\, d\nu \leq L. \tag{21.43}$$

At date 1, there is a designated producer for each intermediate, but a competitive fringe can also enter and produce each good as productively as the designated producer. At date 1, the designated producer can also invest in the new technology, which costs F per firm. If this investment is undertaken, this producer's productivity at date 2 will be higher by a factor α as indicated by equation (21.42). In contrast, the fringe will not benefit from this technological improvement, thus the designated producer will have some degree of monopoly power. The profits from intermediate producers are naturally allocated to the representative household.

Since this is a two-period economy, we will be looking for a subgame perfect equilibrium. Moreover, to simplify the discussion, let us focus on symmetric subgame perfect equilibria, SSPE. An SSPE consists of an allocation of labor across firms, investment decisions for firms, wages for both periods and an interest rate linking consumption between the two periods.

First, since all goods are symmetric, the first period labor market clearing is straightforward and we will have

$$l(\nu, 1) = L \text{ for all } \nu \in [0, 1]$$

(recall that the measure of sectors and firms is normalized to 1). This implies that

$$Y(1) = L.$$

At date 2, the equilibrium will depend on how many firms have adopted the new technology. Since we are looking at the symmetric equilibrium (SSPE), we only consider the two extremes where all firms adopt and no firm adopts. In either case, again the marginal productivity of all sectors are the same, so labor will be allocated equally, i.e.,

$$l(\nu, 2) = L \text{ for all } \nu \in [0, 1].$$

Consequently, when the technology is not adopted, we have

$$Y(2) = L$$

and when the technology is adopted by all the firms, we have

$$Y(2) = \alpha L.$$

We now turn to the pricing decisions. In the first date, the designated producers have no monopoly power because of the competitive fringe, thus they charge price equal to marginal cost, which is $w(1)$, and make zero profits. Since total output is equal to $Y(1) = L$, this also implies that the equilibrium wage rate is equal to

$$w(1) = 1.$$

In the second date, if the technology is not adopted, the same situation repeats, and we have

$$w(2) = 1$$

and thus no profits. In this case there is also no investment, so consumption at both dates is equal to L, thus the interest rate that makes individuals happy to consume this amount in both periods is

$$\hat{R} = \beta^{-1}. \tag{21.44}$$

To see this more formally, recall that the standard Euler equation in this case is

$$C(1)^{-\theta} = R\beta C(2)^{-\theta}, \tag{21.45}$$

which can only be satisfied with $C(1) = C(2)$, if the gross interest rate is $\hat{R}$ as given in (21.44).

Next consider the situation in which the designated producers have invested in the advanced technology. Now they can produce α units of output with one unit of labor, while the fringe of competitive firms still produces one unit of output with one unit of labor. This implies that the designated producers have some monopoly power. The extent of this monopoly power depends on the comparison of ε and α.

Let us first determine the demand facing each producer, which is given as a solution to the following program of profit maximization for the final good sector:

$$\max_{[y(\nu,2)]_{\nu\in[0,1]}} \left[\int_0^1 y(\nu,2)^{\frac{\varepsilon-1}{\varepsilon}} d\nu\right]^{\frac{\varepsilon}{\varepsilon-1}} - \int_0^1 p(\nu,2)\, y(\nu,2)\, d\nu,$$

where $p(\nu,2)$ is the price of intermediate ν at date 2. The first-order condition to this program implies

$$y(\nu,2)^{-1/\varepsilon} Y(2)^{1/\varepsilon} = p(\nu,2),$$

or

$$y(\nu,2) = p(\nu,2)^{-\varepsilon} Y(2). \tag{21.46}$$

This expression is useful in laying the foundations for the aggregate demand externalities, which we will discuss soon; the demand for intermediate ν depends on the total amount of production, $Y(2)$.[3] The familiar feature of the demand curve (21.46) is that it is iso-elastic. To make further progress, first imagine the situation in which there is no fringe of competitive producers. In that case, each designated producer will act as an unconstrained monopolist and maximize its profits given by price minus marginal cost times quantity, i.e.,

$$\pi(\nu,2) = \left(p(\nu,2) - \frac{w(2)}{\alpha}\right) y(\nu,2).$$

substituting from (21.46), the firm maximization problem is

$$\max_{p(\nu,2)} \pi(\nu,2) = \left(p(\nu,2) - \frac{w(2)}{\alpha}\right) p(\nu,2)^{-\varepsilon} Y(2),$$

which has a first-order condition

$$p(\nu,2)^{-\varepsilon} Y(2) - \varepsilon \left(p(\nu,2) - \frac{w(2)}{\alpha}\right) p(\nu,2)^{-\varepsilon-1} Y(2) = 0,$$

which implies

$$p(\nu,2) = \frac{\varepsilon}{\varepsilon - 1}\frac{w(2)}{\alpha}.$$

This is the standard monopoly price formula of a markup related to demand elasticity over the marginal cost, $w(2)/\alpha$. Here the markup is constant because the demand elasticity is constant.

However, the monopolist can only charge this price if the competitive fringe could not enter and make profits stealing the entire market at this price. Since the competitive fringe can produce one unit using one unit of labor, the monopolist can only charge this price if $\varepsilon/((\varepsilon-1)\alpha) \leq 1$. Otherwise, the price would be too high and the competitive fringe would enter. Let us assume that α is not so high as to make the monopolist unconstrained. In other words, we assume that

$$\frac{\varepsilon}{\varepsilon - 1}\frac{1}{\alpha} > 1. \tag{21.47}$$

Under this assumption, the monopolist will be forced to charge a *limit price*. It is straightforward to see that this equilibrium limit price would be

$$p^* = w(2).$$

If it were any higher, the competitive fringe would enter, steal the whole market and make positive profits. If it were any lower, the monopolist could increase its price without losing

[3] The reader may wish to ask why there is an *externality* here. Recall that even with perfectly competitive markets, the demand for goods supplied by a particular producer depends on the supply of other goods in the economy. So why is there an externality here? The answer, which may already be clear to some readers, will be discussed further below.

the market, and thus increase its profits. This implies that under (21.47), each monopolist would make per unit profits equal to

$$w(2) - \frac{w(2)}{\alpha} = \frac{\alpha - 1}{\alpha} w(2).$$

The profits of firms are then obtained from substituting from (21.46) as:

$$\pi(2) = \frac{\alpha - 1}{\alpha} w(2)^{1-\varepsilon} Y(2). \tag{21.48}$$

The wage rate can be determined from income accounting. Total production will be equal to $Y(2) = \alpha L$, and this has to be distributed between profits and wages, thus

$$\frac{\alpha - 1}{\alpha} w(2)^{1-\varepsilon} \alpha L + w(2) L = \alpha L,$$

which has a solution of

$$w(2) = 1,$$

as in the case without the technological investments. Therefore, in this economy the increased marginal product does not translate into higher wages. Instead, it leads to profits for firms. Nevertheless, all of these profits are redistributed to the agents, who are the owners of the firms. Thus $C(2) = \alpha L$. However, because there was investment in the new technology at date 1, $C(1) = L - F$. Again the interest rate has to adjust so that individuals are happy to consume these amounts, i.e., so that they have a steep consumption profile without wanting to borrow. The Euler equation, (21.45), now implies

$$(L - F)^{-\theta} = \tilde{R}\beta(\alpha L)^{-\theta}, \tag{21.49}$$

which solves for

$$\tilde{R} = \beta^{-1}\left(\frac{\alpha L}{L - F}\right)^{\theta} > \hat{R}.$$

Consequently, the interest rate in this case is higher than the one in which there is no investment. This is natural, since investment implies that individuals are being asked to forgo date 1 consumption for date 2 consumption. Note also that the greater is θ, the higher is $\tilde{R}$, since with a greater θ, there is less intertemporal substitution. Also a higher F, meaning a greater consumption sacrifice at date 1 implies a higher interest rate.

The question is whether firms will find it profitable to undertake the investment at date 1. The reason for the possibility of multiplicity is that the answer to this question will depend on whether other firms are undertaking the investment or not. Let us first consider a situation in which no other firm is undertaking the investment, and consider the incentives of a single designated firm to undertake such an investment.

In this case total output at date 2 is equal to L (since the firm considering investment is infinitesimal), and the market interest rate is given by $\hat{R}$. Moreover, from (21.48) and the fact that $w(2) = 1$, profits at date 2 are

$$\pi^N(2) = \frac{\alpha - 1}{\alpha} L.$$

where the superscript N denotes that no other firm is undertaking the investment. Therefore, the net discounted profits at date 1 for the firm in question is

$$\begin{aligned} \Delta\pi^N &= -F + \frac{1}{\hat{R}} \frac{\alpha - 1}{\alpha} L \\ &= -F + \beta \frac{\alpha - 1}{\alpha} L. \end{aligned}$$

Next consider the case in which all other firms are undertaking the investment. In this case, profits at date 2 are

$$\pi^I(2) = (\alpha - 1) L,$$

where the superscript I designates that all other firms are undertaking the investment. Consequently, the profit gain from investing at date 1 is

$$\begin{aligned} \Delta\pi^I &= -F + \frac{1}{\tilde{R}} (\alpha - 1) L \\ &= -F + \beta \left(\frac{\alpha L}{L - F} \right)^{-\theta} (\alpha - 1) L. \end{aligned}$$

As discussed above, the idea of the paper by Murphy, Shleifer and Vishny (1989), similar to the ideas of many economists writing on economic development before them, was to generate multiple equilibria, where one of the equilibria corresponds to backwardness, while the other one corresponds to industrialization. In this context, this means that for the same parameter values both no investment in the new technology and all firms investing in the new technology should be equilibria. This is only possible if we have

$$\Delta\pi^N < 0 \text{ and } \Delta\pi^I > 0, \tag{21.50}$$

that is, when nobody else invests, investment is not profitable, and when all other firms invest, investment is profitable. This is clearly possible as of the *aggregate demand externality* ensures that $\pi^I > \pi^N$; when other firms invest, they produce more, there is more aggregate demand, and therefore profits from having invested in the new technology are higher. Counteracting this effect is the fact that the interest rate is also higher when all firms invest. Therefore, the existence of multiple equilibria requires the interest rate effect not to be too strong. For example, in the extreme case where preferences are linear, i.e., $\theta = 0$, we have that

$$\Delta\pi^I = -F + \beta(\alpha - 1) L > \Delta\pi^N = -F + \beta \frac{\alpha - 1}{\alpha} L,$$

so (21.50) is certainly possible. More generally, the condition for the existence of multiple equilibria is that:

$$\beta \left(\frac{\alpha L}{L - F} \right)^{-\theta} (\alpha - 1) L > F > \beta \frac{\alpha - 1}{\alpha} L. \tag{21.51}$$

It is also straightforward to see that whenever both equilibria exist, the equilibrium with investment Pareto dominates the one without investment, since condition (21.51) implies that all households are better-off with the upward sloping consumption profile giving them higher consumption at date 2 (see Exercise 21.8). Therefore, this analysis establishes that when condition (21.51) is satisfied, there will exist two pure strategy SSPE. In one of these, all firms undertake the investment at date 1 and consumers are better off, while in the other one there are no investments in new technology and greater market failures. Intuitively, multiple equilibria emerge in this model because of *aggregate demand externalities*; investing in the new technology at date 1 is profitable only when there is sufficient demand at date 2 and there will be sufficient aggregate demand at date 2 when all firms invest in the new technology. This is at the root of the aggregate demand externalities, since the investment decision of a particular firm creates a positive (pecuniary) externality on other firms by increasing the level of demand facing their products. The reason why pecuniary externalities, which are present in all models, play a more important role here and lead to Pareto-ranked multiple equilibria is that each firm does not realize the full increase in the social product created by its investment, because the monopoly markup implies that at the margin further increases in output create a first-order gain for consumers. The presence of the markup means that the monopolist does not internalize this first-order gain, thus turning the demand linkages into aggregate demand externalities.

The interpretation for this result suggested by Murphy, Shleifer and Vishny is to consider the equilibrium with no investment in the new technology as representing a "development trap," where the economy remains in "underdevelopment" because no firm undertakes the investment in new technology and this behavior implies that the demand necessary to make such investments profitable is absent. In contrast, the equilibrium with investment in new technology is interpreted as corresponding to "industrialization". According to this interpretation, societies that can somehow *coordinate* on the equilibrium with investment (either because private expectations are aligned or because of some type of government action) will industrialize and realize both economic growth and Pareto improvement. As such, this model is argued to provide a formalization of the "big push" type industrialization described by economists such as Nurske or Rosenstein-Rodan. Although the idea of the big push and the aggregate demand externalities are attractive, the model here suffers from a number of obvious shortcomings. First, even though the process of industrialization is a dynamic one, the model here is static. Therefore, it does not allow a literal interpretation of a society being first in the no investment equilibrium and then changing to the investment equilibrium and industrializing. Second, as already discussed in Chapter 4, models with multiple equilibria do not provide a satisfactory model of development, since it is difficult to imagine a society remaining unable to coordinate on a simple range of actions that would make all households (and firms) better off. Instead, it is much more likely that the ideas related to aggregate demand externalities (or other potential forces leading to multiple equilibria) are more important as sources of persistence or as mechanisms generating multiple steady states (while still maintaining a unique equilibrium path). In the next section, we will discuss how certain factors can lead to multiple steady states in dynamic models instead of the multiple equilibria emphasized by Murphy, Shleifer and Vishny in the context of a static model. I will illustrate these issues focusing on another set of topics that appear to be important in the context of development, the interaction between the distribution of income and human capital investments.

21.6. Inequality, Credit Market Imperfections and Human Capital

The previous section illustrated how aggregate demand externalities can generate development traps. Investment by different firms may require coordination, leading to multiple equilibria. Underdevelopment may be thought to correspond to a situation in which the coordination is on the bad equilibrium, and the development process starts with the "big push," ensuring coordination to the high-investment equilibrium.

Here I will illustrate these issues focusing on how the distribution of income and the organization of financial markets affect human capital investments. The models presented in this section will not only show the possibility of multiple steady states, but also shed light on more substantive questions related to the role of inequality and credit markets in the process of development. Although in this section I focus on human capital investments, the interaction between inequality and credit market problems influences not only human capital investments, but also occupational choices and other aspects of the organization of production. Nevertheless, the models focusing on the link between inequality and human capital are both more tractable and also constitute a natural continuation of the models of human capital investments we studied in Chapter 10.

21.6.1. A Simple Case With No Borrowing. When credit markets are imperfect, a major determinant of human capital investments will be the distribution of income (as well as the degree of imperfection in the credit markets). I start with a discussion of the simplest case in which there is no borrowing or lending, which introduces an extreme form of credit market problems. I will then enrich this model by introducing credit markets that allow borrowing and lending, but introduce credit market imperfections by making the cost of borrowing greater than the interest rate received by households engaged in saving.

The economy consists of continuum 1 of dynasties. Each individual lives for two periods, childhood and adulthood, and gets an offspring in his adulthood. There is consumption only at the end of adulthood. Preferences are given by

$$(1-\delta)\log c_i(t) + \delta \log e_i(t+1)$$

where c is consumption at the end of the individual's life, and e is the educational spending on the offspring of this individual. The budget constraint is

$$c_i(t) + e_i(t+1) \le w_i(t),$$

where w is the wage income of the individual. Notice that preferences here have the "warm glow" type altruism which we encountered in Chapter 9 and in Section 21.2 above. In particular, parents do not care about the utility of their offspring, but simply about what they bequeath to them, here education. As we have already seen, this significantly simplifies the analysis. Moreover, preferences are logarithmic, which as we have already seen, will imply a constant saving rate, here in terms of educational investments.

The labor market is competitive, and wage income of each individual is simply a linear function of his human capital:

$$w_i(t) = Ah_i(t)$$

Human capital of the offspring of individual i of generation t in turn is given by

$$h_i(t+1) = \begin{cases} e_i(t)^\gamma & \text{if } e_i(t) \ge 1 \\ \bar{h} & \text{if } e_i(t) < 1 \end{cases}, \tag{21.52}$$

where $\gamma \in (0,1)$ and $\bar{h} \in (0,1)$ is some minimum level of human capital that the individual will attain even without any educational spending. Once spending exceeds a certain level (here set equal to 1), the individual starts benefiting from the additional spending and accumulates further human capital (though with diminishing returns since $\gamma < 1$).

This equation introduces a crucial feature necessary for models of credit market imperfections to generate multiple equilibria or multiple steady states; a *nonconvexity* in the technology of human capital accumulation. Exercise 21.9 shows that this nonconvexity plays a crucial role in the results of this subsection.

Given this description, the equilibrium is straightforward to characterize. Each individual will choose the spending on education that maximizes its own utility. This immediately implies the following "saving rate":

$$e_i(t) = \delta w_i(t) = \delta A h_i(t). \tag{21.53}$$

This rule has one unappealing feature (not crucial for any of the results), which is that because parents derive utility from educational spending on their children, they will spend on education even when $e_i(t) < 1$, in which case educational spendings are in fact wasted (do not translate into higher human capital of the offspring).

To obtain stark results let us also assume that

$$\delta A > 1 > \delta A \bar{h}. \tag{21.54}$$

Now, let us look at the dynamics of human capital for a particular dynasty i. If at time 0, we have $h_i(0) < (\delta A)^{-1}$, then (21.53) implies that $e_i(t) < 1$, so the offspring will have $h_i(1) = \bar{h}$. Given (21.54), we have $h_i(1) = \bar{h} < (\delta A)^{-1}$, and repeating this argument, we have $h_i(t) = \bar{h} < (\delta A)^{-1}$ for all t. Therefore, a dynasty that starts with $h_i(0) < (\delta A)^{-1}$ will never reach a human capital level greater than $\bar{h}$.

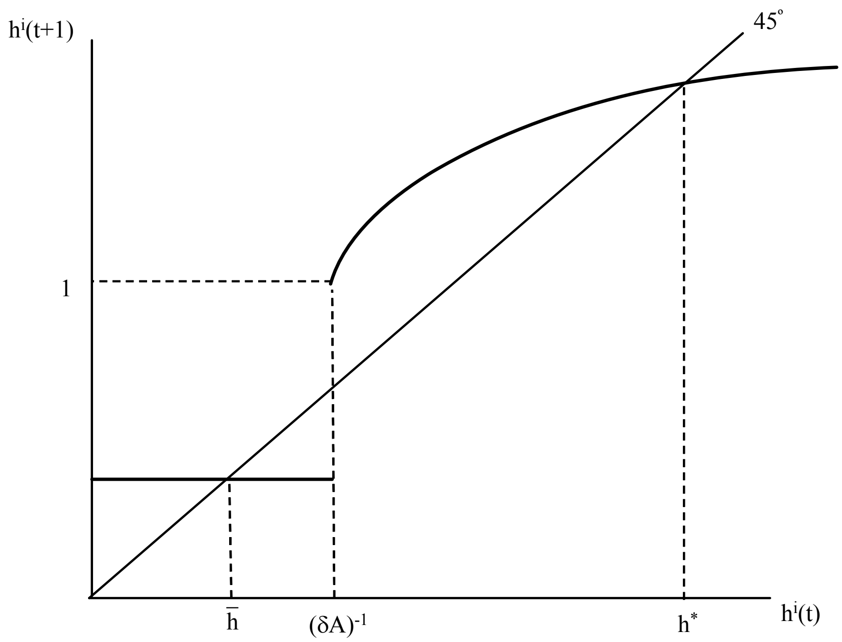

FIGURE 21.8. Dynamics of human capital with nonconvexities and no borrowing.

Next consider a dynasty with $h_i(0) > (\delta A)^{-1}$. Then from (21.54), we have $h_i(1) = (\delta A h_i(0))^{\gamma} > 1$, so this dynasty will gradually accumulate more and more human capital over generations and ultimately reach the "steady state" given by $h^* = (\delta A h^*)^{\gamma}$ or

$$h^* = (\delta A)^{\frac{\gamma}{1-\gamma}} > 1.$$

Naturally, this description applies to a dynasty with $h_i(0) \in ((\delta A)^{-1}, h^*)$. If $h_i(0) > h^*$, then the dynasty would have started with too much human capital and would decumulate human capital. Figure 21.8 illustrates the dynamics of individual human capital decisions. It shows that there are two steady-state levels of human capital for individuals, $\bar{h}$ and $h^* > \bar{h}$. It also plots the basins of attraction of these two steady states, showing that dynasties with $h_i(0) < (\delta A)^{-1}$ will tend to the lower steady state level of human capital, $\bar{h}$, while those with $h_i(0) > (\delta A)^{-1}$ will tend to the higher level, h^*. This figure also reveals why the analysis of the dynamics in this model is so simple; the dynamics of the human capital of a single individual contains all the information relevant for the dynamics of the human capital and income of the entire economy. This is because there are no prices (such as the rate of return to human capital or the interest rate) that are being determined in equilibrium here. For this reason, dynamics in this type of models are sometimes described as "Markovian"—because they are summarized by a Markov process without any general equilibrium interactions. Markovian dynamics are much more tractable than dynamics of inequality depending on equilibrium prices. An example of this richer type of model is given in Exercise 21.13.

The most important implication of this analysis is that this simple model features poverty traps due to the nonconvexities created by the credit market problems. This is most clearly illustrated by contrasting two economies subject to the same technology and the same credit market problems, but starting out with different distributions of income. For example, imagine an economy with two groups starting at income levels h_1 and $h_2 > h_1$ such that $(\delta A)^{-1} < h_2$. Now if inequality (poverty) is high so that $h_1 < (\delta A)^{-1}$, a significant fraction of the population will never accumulate much human capital. In contrast, if inequality is limited so that $h_1 > (\delta A)^{-1}$, all agents will accumulate human capital, eventually reaching h^*. This example also illustrates that there are (many) multiple steady states in this economy. Depending on the fraction of dynasties that start with initial human capital $(\delta A)^{-1}$, any fraction of the population may end up at the low level of human capital, $\bar{h}$. The greater is this fraction, the poorer will the economy be. At some level, there is a parallel between the multiplicity of steady states here and the multiple equilibria highlighted in the model of the previous section. Nevertheless, the differences are also noteworthy. In the model of the previous section, there are multiple equilibria in a static model. Thus nothing determines which equilibrium the economy will be in. At best, we can appeal to "*expectations*," arguing that the better equilibrium will emerge when everybody expects the better equilibrium to emerge. One can informally appealed to the role of "*history,*" for example, suggesting that if an economy has been in the low investment equilibrium for a while, it is likely to stay there, but this argument is somewhat misleading. First of all, the model is a static one, thus a discussion of an economy "that has been in the low equilibrium for a while" is not quite meaningful. Secondly, even if the model were turned into a dynamic one by repeating it over time, the history of being in one equilibrium for a number of periods will have no effect on the existence of multiple equilibria at the next instant. In particular, each static equilibrium would still remain an equilibrium in the "dynamic" environment, and the economy could suddenly jump from one equilibrium to another. This highlights that models with multiple equilibria have the degree of indeterminacy that are both theoretically awkward and empirically difficult to map to reality. Instead, models with multiple steady states avoids these thorny issues. The equilibrium is *unique* (as in the model we have just seen) but the initial conditions determine where the dynamical system will end up eventually. Because the equilibrium is unique, there is no issue of indeterminacy or expectations affecting the path of the economy. But also, because multiple steady states are possible, the model can be useful for thinking about potential development traps.

Aside from providing us with a simple example of multiple steady states, this model shows the importance of the distribution of income in an economy with imperfect credit markets (here with no credit markets). In particular, the distribution of income affects which individuals will be unable to invest in human capital accumulation and thus influences the long-run income level of the economy. For this reason, models of this sort (including the one with imperfect capital markets we will study in the next section) are sometimes interpreted as implying that an unequal distribution of income will lead to lower output (and growth). In fact, the above example with two classes seems to support this conclusion. However, this is not a general result and it is important to emphasize that this class of models does not make specific predictions about relationship between inequality and growth. To illustrate this, consider the same economy with two classes, now starting with $h_1 < h_2 < (\delta A)^{-1}$. In this case, neither group will accumulate human capital, but redistributing resources away from group 1 to group 2 (thus increasing inequality), so that we push group 2 to $h_2 > (\delta A)^{-1}$ would increase human capital accumulation. This is a general feature: in models with nonconvexities, there are *no unambiguous general* results about whether greater inequality is good or bad for accumulation and economic growth; it depends on whether greater inequality pushes more people below or above the critical thresholds. Somewhat sharper results can be obtained about the effect of inequality on human capital accumulation and development under additional assumptions. Exercise 21.10 presents a parameterization of inequality in the model here, which delivers the results that greater inequality leads to lower investments in human capital and lower output per capita in relatively rich economies, but to greater investments in human capital in poorer economies.

21.6.2. Human Capital Investments with Imperfect Credit Markets. We now enrich the model of the previous subsection by introducing credit markets. The model I present here is a simplified version of the first Galor-Zeira model (Galor and Zeira, 1993). Each individual still lives for two periods. In his youth, he can either work or acquire education. The utility function of each individual is

$$(1-\delta)\log c_i(t) + \delta \log b_i(t),$$

where again c denotes consumption at the end of the life of the individual. The budget constraint is

$$c_i(t) + b_i(t) \leq y_i(t),$$

where $y_i(t)$ is individual i's income at time t. Note that preferences still take the "warm glow" form, but the utility of the parent now depends on monetary bequest to the offspring rather than the level of education expenditures. It will now be the individuals themselves who will use the monetary bequests to invest in education. Also, the logarithmic formulation will once again ensure a constant saving rate equal to δ.

Education is a binary outcome, and educated (skilled) workers earn wage w_s while uneducated workers earn w_u. The required education expenditure to become skilled is h, and workers acquiring education do not earn the unskilled wage, w_u, during the first period of their lives. The fact that education is binary introduces the nonconvexity in human capital investment decisions. As demonstrated in Exercise 21.9, such nonconvexities are important for models with imperfect credit markets to generate multiple steady states.[4]

Imperfect capital markets are modeled by assuming that there is some amount of monitoring required for loans to be paid back. The cost of monitoring creates a wedge between the borrowing and the lending rates. In particular, assume that there is a linear savings technology open to all agents, which fixes the lending rate at some constant r. However, the borrowing rate is $i > r$, because of costs of monitoring necessary to induce agents to pay back the loans (see Exercise 21.12 for a more micro-founded version of these borrowing costs).

Also assume that

$$w_s - (1 + r) h > w_u (2 + r) \tag{21.55}$$

which implies that investment in human capital is profitable when financed at the lending rate r.

Let us now consider an individual with wealth x. If $x \geq h$, assumption (21.55) implies that individual will invest in education. If $x < h$, then whether it is profitable to invest in education will depend on the wealth of individual and the borrowing interest rate, i.

Let us now write the utility of this agent (with $x < h$) in the two scenarios, and also the bequest that he will leave to his offspring. These utility levels and bequests are given by

$$\begin{aligned} U_s(x) &= \log(w_s + (1+i)(x-h)) + \log(1-\delta)^{1-\delta}\delta^{\delta} \\ b_s(x) &= \delta(w_s + (1+i)(x-h)), \end{aligned}$$

when he invests in education. And when he chooses not to invest, his utility and bequests are given by

$$\begin{aligned} U_u(x) &= \log((1+r)(w_u + x) + w_u) + \log(1-\delta)^{1-\delta}\delta^{\delta} \\ b_u(x) &= \delta((1+r)(w_u + x) + w_u). \end{aligned}$$

[4] An alternative to nonconvexities in human capital investments is presented in Galor and Moav (2004), who show that multiple steady states are possible when there are no nonconvexities, credit markets are imperfect, and the marginal propensity to save is higher for richer dynasties. This assumption is motivated by Kaldor's (1957) paper and was discussed in Exercise 2.11 in Chapter 2. Galor and Moav (2004) also show that this "Kaldorian" assumption combined with credit market imperfections can help reconcile the emphasis of the recent literature, which is on the adverse effects of inequality, with the earlier literature (e.g., Lewis, 1954, Kaldor, 1957), which stressed the beneficial effects of inequality via greater savings.

Comparing these expressions we obtain that an individual likes to invest in education if and only if

$$x \geq f \equiv \frac{(2+r)\,w_u + (1+i)\,h - w_s}{i-r}$$

The dynamics of the system can then be obtained simply by using the bequests of unconstrained, constrained-investing and constrained-non-investing agents.

More specifically, the equilibrium correspondence describing equilibrium dynamics is

$$x(t+1) = \begin{cases} b_u(x(t)) = \delta\left((1+r)(w_u + x(t)) + w_u\right) & \text{if } x(t) < f \\ b_s(x(t)) = \delta\left(w_s + (1+i)(x(t)-h)\right) & \text{if } h > x(t) \geq f \\ b_n(x(t)) = \delta\left(w_s + (1+r)(x(t)-h)\right) & \text{if } x(t) \geq h \end{cases} \tag{21.56}$$

Equilibrium dynamics can now be analyzed diagrammatically by looking at the graph of (21.56), which is shown in Figure 21.9. As emphasized in the context of the model of the previous subsection, equation (curve) (21.56) describes both the behavior of the wealth of each individual and the behavior of the wealth distribution in the economy. This is again a feature of the "Markovian" nature of the current model.

Now define x^* as the intersection of the equilibrium curve (21.56) with the 45 degree line, when the equilibrium correspondence is steeper than the 45 degree line. Such an intersection will exist when the borrowing interest rate, i, is large enough.

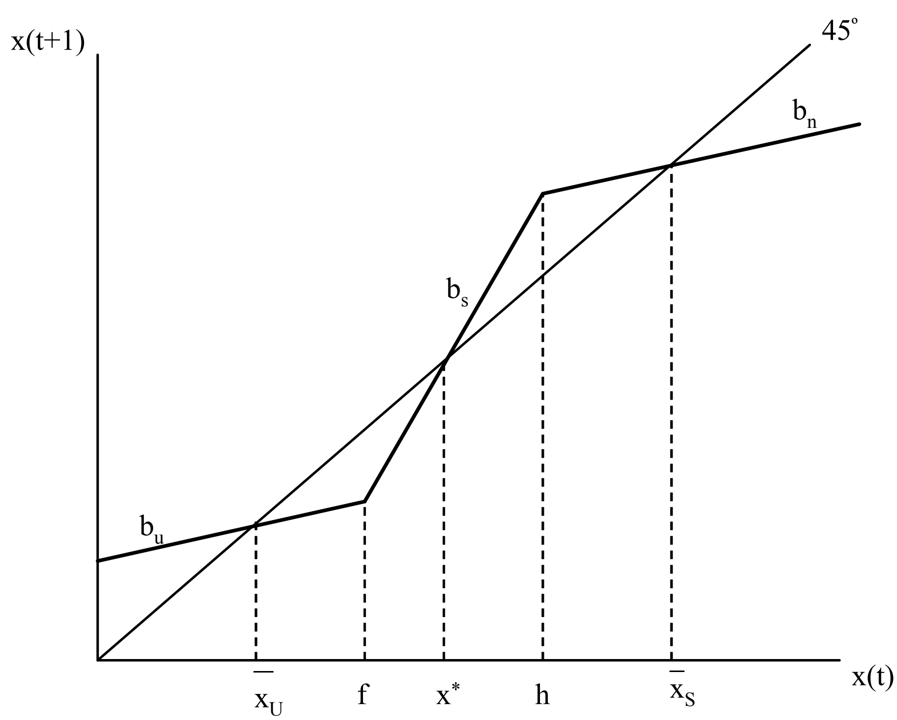

Figure 21.9. Multiple steady-state equilibria in the Galor and Zeira model.

All individuals with $x(t) < x^*$ converge to the wealth level $\bar{x}_U$, while all those with $x(t) > x^*$ converge to the greater wealth level $\bar{x}_S$. As in the example without credit markets, there is a "poverty trap," which attracts agents with low initial wealth. The distribution of income again has a potentially first-order effect on the income level of the economy. If the majority of the individuals start with $x(t) < x^*$, the economy will have low productivity, low human capital and low wealth. Therefore, this model extends all of the insights of the simple model with no borrowing from the previous subsection to a richer environment in which individuals make forward-looking human capital investments. The key is again the interaction between credit market imperfections (which here make the interest rate for borrowing greater than the interest rate for saving) and inequality. As in the model of the previous subsection, it is straightforward to construct examples where an increase inequality can lead to either worse or better outcomes depending whether it pushes more individuals into the basin of attraction of the low steady state.

An important feature of the model of this subsection is that because it allows individuals to borrow and lend in financial markets, it enables us to investigate the implications of financial development for human capital investments. In an economy with better financial institutions, we may expect the wedge between the borrowing rate and the lending rate to be smaller, i.e., i to be smaller given r. With a smaller i, more agents will escape the poverty trap, and in fact the poverty trap may not exist (there may not be an intersection between (21.56) and the 45 degree line where (21.56) is steeper). This shows that financial development not only improves risk sharing has demonstrated in Section 21.1, but in addition, by relaxing credit market constraints, it contributes to human capital accumulation.

Although the model in this section is considerably richer than that in the previous subsection, a number of its shortcomings should also be noted. The most important shortcoming of the model is that, like the one in the previous subsection, it is essentially a partial equilibrium model. Multiple steady states are possible for different individuals as a function of their initial level of human capital (or wealth), but individual dynamics are not affected by general equilibrium prices. Models such as the second model presented in Galor and Zeira (1993), and those in Banerjee and Newman (1994), Aghion and Bolton (1997) and Piketty (1997) consider richer environments in which income dynamics of each dynasty (individual) is affected by general equilibrium prices (such as the interest rate or the wage rate), which are themselves a function of the income inequality at the time. Exercise 21.11 shows that the type of multiple steady states generated by the model presented here may not robust to the addition of noise in income dynamics—instead of multiple steady states, the long-run equilibrium then corresponds to a stationary distribution of human capital levels, though this stationary distribution will exhibit a large amount of persistence.[5] In contrast, models in which prices determined in general equilibrium affect wealth (income) dynamics generate more robust multiplicity of steady states. The second potential shortcoming of the current model is that it focuses on human capital investments. Some development economists, such as Banerjee and Newman (1994), believe that the effect of income inequality on occupational choices is potentially more important than its effect on human capital investments. Exercise 21.13 presents a simplified version of the Banerjee-Newman model that emphasizes the role of occupational choice.

21.6.3. Heterogeneity, Stratification and the Dynamics of Inequality . The models in the previous two sections investigated the implications of credit market imperfections and income distribution on human capital investments. In this subsection, I consider a slightly more general framework due to Benabou (1996a), which enables a study of the dynamics of inequality and its costs for the efficiency of production resulting from its effect on human capital investments as a function of both the technology of production and how much stratification and segregation there is in the society. In particular, let me use a simplified version of Benabou's (1996a) model where aggregate output in the economy at time t is given by

$$Y(t) = H(t),$$

where $H(t)$ is an aggregate of the human capital of all the individuals in the society. In particular, normalizing the total population to 1 and denoting the distribution of human capital at time t by $\mu_t(h)$, we have that

$$H(t) \equiv \left(\int_0^\infty h^{\frac{\sigma-1}{\sigma}} d\mu_t(h) \right)^{\frac{\sigma}{\sigma-1}}, \tag{21.57}$$

where the parameter σ measures the degree of complementarity or substitutability in the human capital of different individuals. When $\sigma \to \infty$, the human capital of different individuals become perfect substitutes and $H(t)$ is simply equal to the mean of the distribution of human capital. For any value of $\sigma \in (0, \infty)$, there is some amount of complementarity between the human capital levels of different individuals. For example, each individual performs a different task and overall output is a combination of these tasks. If some individuals have low human capital and are not very successful in the tasks they are supposed to perform, this reduces the productivity of other individuals in the society. The effect of heterogeneity of human capital on aggregate productivity, for a given mean level of human capital in the society, is most severe when the parameter σ is close to 0. Nevertheless, this formulation is also

[5]The reader will note that this is related to the "Markovian" nature of the model. Markovian models can generate multiple steady states because the Markov chain or the Markov process implied by the model is not ergodic, e.g., poor individuals can never accumulate to become rich. A small amount of noise then ensures that different parts of the distribution "communicate," making the Markov process ergodic and removing the multiplicity of steady states.

general enough to allow for the case in which greater inequality is productivity-enhancing. In particular, even though this aggregator looks like the constant elasticity of substitution production function we have used many times in this book before, in contrast to that production function, it is defined for $\sigma < 0$ as well (whereas recall that the Dixit-Stiglitz aggregator is only defined for $\sigma \geq 0$, see Exercise 21.14). When $\sigma < 0$, greater inequality for a given mean level of human capital increases the level of $H(t)$ and thus productivity. For example, in the extreme case where $\sigma \to -\infty$, we obtain that $H(t) = \max_i \{h_i(t)\}$, that is, it is only the human capital of the highest human capital individual in the society that influences output. Since our interest here is on the potential costs of inequality on human capital investments, we will focus on the case where $\sigma \geq 0$. In this case, a mean preserving spread of the human capital distribution μ will lead to a lower level of $H(t)$ and a lower level of output.

The human capital of an individual from dynasty i at time $t+1$ is given by

$$h_i(t+1) = \xi_i(t) B (h_i(t))^{\alpha} (N_i(t))^{\beta} (H(t))^{\gamma}, \tag{21.58}$$

where B is a positive constant, $h_i(t)$ is the human capital of the individual's parent, $\xi_i(t)$ is a random shock affecting the individual's human capital, and $N_i(t)$ is the "average" human capital in the individual's neighborhood. The human capital of the offspring is affected by his parent's human capital either because of natural spillovers within the family or because the parent devotes some of his time to the rearing of his offspring and his time is more valuable in child-rearing because of his higher human capital. In addition, the neighborhood and aggregate human capital levels, $N_i(t)$ and $H(t)$, affect the human capital of the individual through learning spillovers. For example, when the average human capital in the neighborhood is high, this may make it easier for the individual to acquire human capital. Aggregate human capital also enters this accumulation equation because the total (or average) level of human capital in the society may affect the type of knowledge that is available for the children to learn. The presence of this type of aggregate spillover means that a low level of $H(t)$, for example because of high inequality, will not only reduce income today, but will also slow down further human capital accumulation.

Following Benabou (1996a), let us assume that the neighborhood human capital is also a constant elasticity of substitution aggregator, this time with an elasticity ε, i.e.,

$$N_i(t) \equiv \left(\int_0^{\infty} h^{\frac{\varepsilon-1}{\varepsilon}} d\mu_t^i(h) \right)^{\frac{\varepsilon}{\varepsilon-1}},$$

where now $\mu_t^i(h)$ denotes the distribution of human capital in the neighborhood of individual i at time t. The presence of neighborhood human capital in the accumulation equation (21.58) implies that greater heterogeneity in the composition of a neighborhood might also have a negative effect on human capital accumulation. For example, presuming that $\varepsilon \in (0, \infty)$, a mean preserving spread of neighborhood human capital will reduce the human capital of all the offsprings. This structure of spillovers may be viewed as quite plausible, for example, because the presence of some low human capital children will slow down learning by those with higher potential (or because one "bad apple" will spoil the pack). This type of neighborhood spillovers may then suggest that segregation of high and low human capital parents in different neighborhoods might be beneficial for human capital accumulation. Whether or not this is so is one of the key questions that a model like this can answer.

The multiplicative structure in equation (21.58) gives a tractable evolution of the human capital distribution provided that the initial distribution of human capital is log normal and that the random shocks captured by $\xi(t)$s are log normal. In particular, let us assume the following log normal distributions for initial human capital and shocks

$$\begin{aligned} \ln h_i(0) &\sim \mathcal{N}\left(m_0, \Delta_0^2\right) \\ \ln \xi_i(t) &\sim \mathcal{N}\left(-\frac{\omega^2}{2}, \omega^2\right), \end{aligned} \tag{21.59}$$

where $\mathcal{N}$ denotes the normal distribution, the draws of $\xi_i(t)$ are independent across time and across individuals, and the distribution of $\ln \xi$ is assumed to have mean $-\omega^2/2$ so that ξ has a mean equal to 1 (that is independent of its variance). It can then be established that the distribution of human capital within every generation will remain log normal, that is,

$$\ln h_i(t) \sim \mathcal{N}\left(m_t, \Delta_t^2\right), \tag{21.60}$$

for some endogenous mean m_t and variance Δ_t, which will depend on parameters and the organization of society, for example, on the extent of segregation and integration (see Exercise 21.15). Consequently, the analysis of output and inequality dynamics in this economy boils down to characterizing the law of motion of m_t and Δ_t.

Let us now consider two alternative organizations. The first features full segregation so that each parent is in a neighborhood with identical parents. In that case, (21.58) becomes

$$h_i(t+1) = \xi_i(t) B (h_i(t))^{\alpha+\beta} (H(t))^{\gamma}, \tag{21.61}$$

because the neighborhood human capital is the same as the parent's human capital. The second features full mixing, so that each neighborhood is a mirror image of the entire society and thus for all neighborhoods we have $N^i(t) = N(t) \equiv \left(\int_0^\infty h^{\frac{\varepsilon-1}{\varepsilon}} d\mu_t(h)\right)^{\frac{\varepsilon}{\varepsilon-1}}$, where notice that μ_t refers to the aggregate distribution. In this case, the accumulation equation becomes

$$h_i(t+1) = \xi_i(t) B (h_i(t))^{\alpha} N(t)^{\beta} H(t)^{\gamma}. \tag{21.62}$$

The intuition above suggests that segregation might be preferable because it will prevent the adverse effects of neighborhood inequality on the human capital accumulation process. We will see, however, that this intuition is not entirely accurate, because whether there is segregation also affects the overall level of inequality in this society, and lack of segregation may reduce long-run inequality leading to a better distribution of income and thus to better economic outcomes as a result.

With full segregation, it is straightforward to see that (see Exercise 21.16)

$$
\begin{aligned}
m_{t+1} &= \ln B - \frac{\omega^2}{2} + (\alpha+\beta+\gamma)\, m_t + \gamma \left(\frac{\sigma-1}{\sigma}\right) \frac{\Delta_t^2}{2} \\
\Delta_{t+1}^2 &= (\alpha+\beta)^2\, \Delta_t^2 + \omega^2
\end{aligned} \tag{21.63}
$$

whereas with full integration, we have

$$
\begin{aligned}
\hat{m}_{t+1} &= \ln B - \frac{\omega^2}{2} + (\alpha+\beta+\gamma)\, \hat{m}_t + \left[\gamma \left(\frac{\sigma-1}{\sigma}\right) + \beta \left(\frac{\varepsilon-1}{\varepsilon}\right)\right] \frac{\hat{\Delta}_t^2}{2} \\
\hat{\Delta}_{t+1}^2 &= \alpha^2 \hat{\Delta}_t^2 + \omega^2,
\end{aligned} \tag{21.64}
$$

where $\hat{m}_t$ and $\hat{\Delta}_t^2$ refer to the values of the mean in the variance of the distribution under full integration.

A number of features about both of these equations are noteworthy. First, the expression for the mean of the distribution shows that there will be persistence in the distribution of human capital. This is because the human capital of the offsprings reflects the human capital of the parents (either through the direct affect on their own parent or through neighborhood and aggregate spillovers). This explains the autoregressive nature of the behavior of m_t. In addition, the dispersion of the parents' human capital affects the mean of the distribution. In particular, when $\sigma < 1$ or when $\varepsilon < 1$, so that the degree of complementarity in the aggregate or the neighborhood spillovers is high, greater dispersion reduces the mean of the distribution of human capital. More interesting is the behavior of the variance of the distribution. When there is full segregation, the costs of heterogeneity in human capital accumulation resulting from neighborhood spillovers are avoided. But in return, the variance of log human capital is more persistent under segregation than full integration. In particular, it is straightforward to verify that when $\varepsilon < 1$, starting with the same m_t and Δ_t, we will have

$$
\hat{m}_{t+1} < m_{t+1} \text{ and } \hat{\Delta}_{t+1}^2 < \Delta_{t+1}^2,
$$

so that human capital in the next period is higher under segregation. But counteracting this, inequality is also higher and we know from the functional form in (21.57) that inequality has efficiency costs. So whether in the long run segregation or integration will generate greater output and a higher efficiency of production will depend on the dynamics of inequality and the exact structure of spillovers. To determine this, let us first find the long-run level of inequality under segregation and integration. Equations (21.63) and (21.64) immediately imply that these variances are given by

$$
\Delta_\infty^2 = \frac{\omega^2}{1-(\alpha+\beta)^2} > \hat{\Delta}_\infty^2 = \frac{\omega^2}{1-\alpha^2},
$$

confirming that there will be greater inequality of human capital and income in this society with segregation of neighborhoods. The mean of the two distributions will also be different however. Let us suppose that $\alpha+\beta+\gamma<1$, so that this steady state distribution exists under both full segregation and full integration. Then we have

$$m_\infty=\frac{1}{1-(\alpha+\beta+\gamma)}\left[\ln B-\frac{\omega^2}{2}+\gamma\left(\frac{\sigma-1}{\sigma}\right)\frac{\omega^2}{2\left(1-(\alpha+\beta)^2\right)}\right],$$

and

$$\hat{m}_\infty=\frac{1}{1-(\alpha+\beta+\gamma)}\left[\ln B-\frac{s^2}{2}+\left[\gamma\left(\frac{\sigma-1}{\sigma}\right)+\beta\left(\frac{\varepsilon-1}{\varepsilon}\right)\right]\frac{s^2}{2\left(1-\alpha^2\right)}\right].$$

The comparison of these two expressions shows that the mean level of human capital in the long run may be higher or lower under full integration or full segregation. Using the production function above, taking logs on both sides of (21.57) and using log normality, we obtain that

$$\ln Y(t)=\ln H(t)=m_t+\left(\frac{\sigma-1}{\sigma}\right)\frac{\Delta_t^2}{2},$$

so that long-run income levels under full segregation and full integration are

$$\begin{aligned}\ln Y(\infty) &= m_\infty+\left(\frac{\sigma-1}{\sigma}\right)\frac{\Delta_\infty^2}{2}\\ \ln \hat{Y}(\infty) &= \hat{m}_\infty+\left(\frac{\sigma-1}{\sigma}\right)\frac{\hat{\Delta}_\infty^2}{2}.\end{aligned}$$

Consequently, depending on parameters long-run income levels may be higher or lower under full segregation and full integration (see Exercise 21.17).

This model therefore provides a richer framework for the analysis of the dynamics of income inequality than the models we have seen in the previous two subsections and also highlights various different costs arising from income inequality. Counterbalancing this rich structure is that the costs of inequality in this model are introduced in a reduced-form way. While the aggregator in (21.57) is plausible, one may wonder why there could not be segregation in production, so that high human capital individuals produce with other high human capital individuals, preventing the costs of inequality. One answer to this provided in Acemoglu (1997b), where individuals with different levels of human capital are matched with firms via a imperfect matching technology (see Exercise 21.18). Other, more technology-based justifications for (21.57) can also be provided. Another advantage of this framework is that its relative tractability makes it attractive for the study of political economy decisions, such as a voting over education budgets, and also for the analysis of issues such as education reform. These topics are addressed in Benabou (1996a,b).

21.7. Towards a Unified Theory of Development and Growth?

There has been a unified theme to the models discussed in this chapter (and even between those discussed in this and the previous chapter). They have either emphasized the transformation of the economy and the society over the process of development or potential reasons for why such a transformation might be halted. This transformation takes the form of the structure of production changing, the process of industrialization getting underway, a greater fraction of the population migrating from rural areas to cities, financial markets becoming more developed, mortality and fertility rates changing via health improvements and the demographic transition, and the extent of inefficiencies and market failures becoming less pronounced over time. In many instances this driving force is self-reinforced by the structural transformation that it causes. For example, in Section 21.1 and in the model of Section 17.6 in Chapter 17, economic development leads to financial deepening and this in turn enables a better allocation of resources and contributes to further growth and development. In all of the models, economic development is associated with capital deepening, that is, with greater use of capital instead of human labor (or combined with labor). Thus we can also approximate the growth process with an increase in the capital-labor ratio of the economy, $k(t)$. This does not necessarily mean that capital accumulation is the engine of economic growth. In fact, previous chapters have emphasized how technological change is often at the root of the process of economic growth (and economic development) and thus capital deepening may be the result of technological change. Moreover, Section 21.4 emphasized how the crucial variable capturing the stage of development might be the distance of an economy's technology to the world technology frontier. Since technological progress appears to play a crucial role in economic growth, we may also wish to take at least certain aspects of the technological changes taking place during the process of development as endogenous, especially when the link between development and changes in the extent of market failures is highlighted. Nevertheless, even in these cases, an increase in capital-labor ratio will take place along the equilibrium path and can thus be used as a proxy for the stage of development (though in this case one must be careful not to confuse increasing the capital-labor ratio with ensuring economic development). With this caveat in mind, in this section we take the capital-labor ratio as the proxy for the stage of development and for analytical convenience, we use the Solow model to represent the dynamics of the capital-labor ratio.

With the capital-labor ratio as the proxy for development, can we then construct a unified model where a single force drives the process of development and the structural transformations spurred by this force contribute to the evolution of this driving force? Developing such a unified theory of development is certainly worthwhile. But I will not offer such a unified theory here. This is for two reasons. First, an attempt to pack many different aspects of development into a single model will lead to a framework that is complicated and involved and I believe that relatively abstract representations of reality are more insightful. Second, the economic growth and development literatures have not made great progress towards such unified model. So while I believe there is room for thinking and constructing such unified theories of economic development, I do not think that one (or at least I) can do justice to this challenging task at this point in time and in this limited space.

Instead, I will provide a very reduced-form canonical model of development and structural change. This model is neither meant to be a unified theory of development and growth nor is it meant to be a model that will be informative about the details of the process of development. My purpose is different and more modest. I would like to bring out the common features of the models we have seen in this chapter, albeit in a very stylized and reduced-form manner.

Consider a continuous-time economy. Suppose that output per capita is given by

$$y(t) = f(k(t), x(t)), \tag{21.65}$$

where $k(t)$ is capital-labor ratio and $x(t)$ is some "social variable," such as financial development, urbanization, structure of production, the structure of the family etc. As usual, f is assumed to be twice continuously differentiable and also increasing in concave in k. Moreover, the social variable x potentially affects the efficiency of the production process and thus is part of the per capita production function in (21.65). As a convention we think of an increase in x as corresponding to structural change, such as a move from the countryside to cities, and thus suppose that f is not only increasing in k, but also in x. Naturally, not all structural change is beneficial, and certain aspects of the structural changes, such as pollution, may reduce productivity. But here for simplicity's sake I focus on the case in which f is increasing in x, that is, the partial derivative $f_x \geq 0$.

Let us assume a highly reduced-form model of social change represented by the differential equation

$$\dot{x}(t) = g(k(t), x(t)), \tag{21.66}$$

where g is also assumed to be twice continuously differentiable. Since x corresponds to structural change associated with development, g should be increasing in k, and in particular we assume that its partial derivative with respect to k is strictly positive, that is, $g_k > 0$. Moreover, standard mean reversion type reasoning suggests that g_x should be negative. If x is above its "natural level," it should decline and if it is below its natural level, it should increase. Motivated by this, let us also assume that $g_x < 0$.

Capital accumulates according to the most basic Solow growth model is in Chapter 2, so that

$$\dot{k}(t) = sf(k(t), x(t)) - \delta k(t), \tag{21.67}$$

where I have suppressed population growth and there is no technological change for simplicity. For a fixed x, capital naturally accumulates in an identical fashion to that in the basic Solow model. The structure of this economy is slightly more involved because $x(t)$ also changes. Differential equations (21.66) and (21.67) provide a simple reduced-form representation of structural change driven by economic growth (capital accumulation).

To illustrate the types of dynamics and insights implied by this representation, first consider the case in which $f_x(k,x) \equiv 0$ so that the social variable x has no effect on productivity. Dynamics in this case are shown in Figure 21.10. The thick vertical line corresponds to the locus for $\dot{k}(t)/k(t) = 0$, i.e., it represents the zero of the differential equation (21.67). This locus takes the form of a vertical line, since only a single value of $k(t)$, k^*, is consistent with steady state. The upward sloping line, on the other hand, corresponds to (21.66) and shows the locus of the values of k and x such that $\dot{x}(t)/x(t) = 0$. It is upward sloping, since g is increasing in k and decreasing in x. The laws of motion represented by the arrows follow straightforwardly from (21.66) and (21.67). For example, when $k(t) < k^*$, (21.67) implies that $k(t)$ will increase. Similarly, when $x(t)$ is above the $\dot{x}(t)/x(t) = 0$ locus, (21.66) implies that $x(t)$ will decrease. Given the laws of motion implied by the arrows, it is straightforward to see that the dynamical system representing the equilibrium of this model is globally stable and starting with any $k(0) > 0$ and $x(0) > 0$, the economy will travel towards the unique steady state (k^*, x^*). Now consider the dynamics of a less-developed economy, that is, an economy that starts with a low level of capital-labor ratio, $k(0)$, and a low level of the social variable, $x(0)$. Then development in this economy will take place with gradual capital deepening and a corresponding increase in $x(t)$ towards x^*, which can be viewed as a reduced-form representation of development-induced structural change.

Next, consider the more interesting case in which $f_x(k,x) > 0$. In this case, the locus for $\dot{k}(t)/k(t) = 0$ will also be upward sloping, since $f_x > 0$ and the right-hand side of (21.67) is decreasing in k by the standard arguments (in particular, because of the fact that by the strict concavity of $f(k,x)$ in k, $f(k,x)/k > f_k(k,x)$ for all k and x, see Exercise 21.19) A steady state is again given by the intersection of the loci for $\dot{k}(t)/k(t) = 0$ and $\dot{x}(t)/x(t) = 0$. Since both of these are now upward sloping, multiple steady states are possible as shown in Figure 21.11. These multiple steady states capture, in a very reduced-form way, the potential multiple equilibria arising from aggregate demand externalities or from the interaction between non-convexities and imperfect credit markets in Sections 21.5 and 21.6. The low steady state (k', x') corresponds to a situation in which the social variable x is low and thus productivity is low, and this makes the economy settle into an equilibrium with a low capital-labor ratio. In contrast in the high steady state (k^*, x^*), the high level of x supports greater productivity and thus a greater capital-labor ratio consistent with steady state. Moreover, it can be verified that both the low and the high steady states are typically

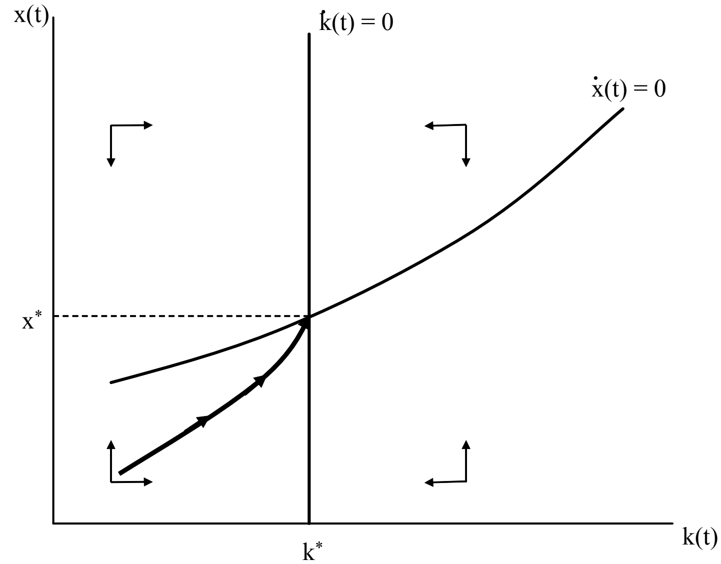

Figure 21.10. Capital accumulation and structural transformation without any effect of the "social variable" x on productivity.

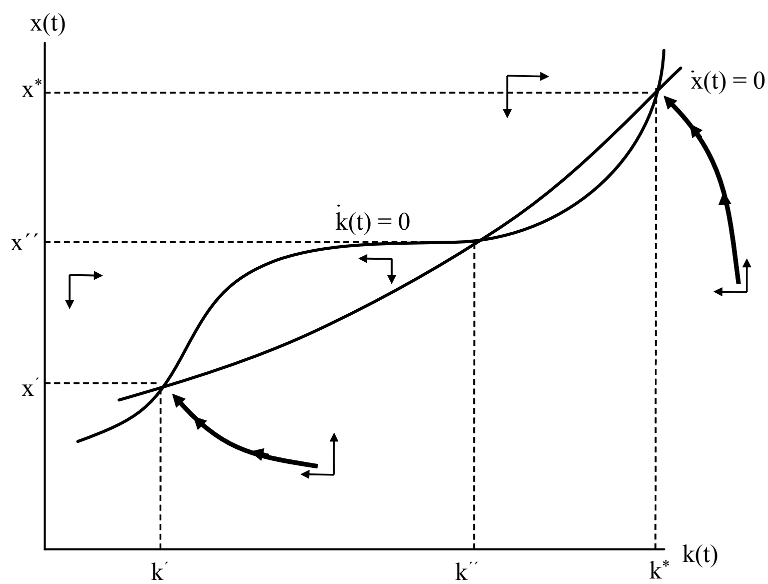

Figure 21.11. Capital accumulation and structural transformation with multiple steady states.

locally stable, so that starting from the neighborhood of one, the economy will converge to the nearest steady state and will tend to stay there. This highlights the importance of *historical factors* in the development process. If historical factors or endowments placed the economy in the neighborhood of the low steady state, the economy will converge to this steady state corresponding to a "development trap". Interestingly, this development trap is, at least in part, caused by lack of structural change (i.e., a low value of the social variable x).

Figure 21.11 makes it clear that such multiplicity requires the locus for $\dot{k}(t)/k(t) = 0$ to be relatively flat, at least over some range. Inspection of equation (21.67) shows that this will be the case when $f_x(k, x)$ is large, at least over some range. Intuitively, multiple steady-state equilibria can only arise when the social variable x has a large effect on productivity, so that the extent of structural change that the economy has already undergone should have a large effect on productivity.

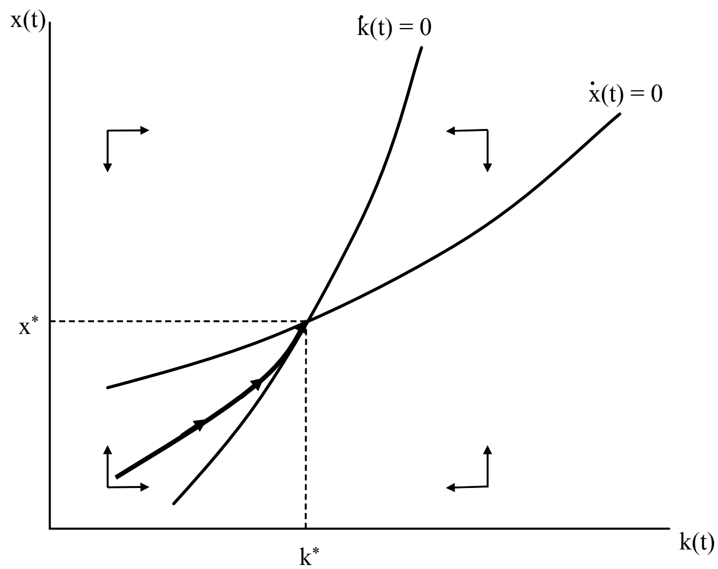

Figure 21.12. Capital accumulation and structural transformation when the "social variable" x affects but there exists a unique steady state.

More interesting than multiple steady states is the situation in which the same forces are present, but a unique steady state exists (like the models discussed in Section 21.6). The same reasoning suggests that this will be the case when $f_x(k, x)$ is relatively small. In this case, the locus for $\dot{k}(t)/k(t) = 0$ will be everywhere steeper than the locus for $\dot{x}(t)/x(t) = 0$. This case is plotted in Figure 21.12 and the unique steady state is given by (k^*, x^*). The laws of motion represented by the arrows again follow from the inspection of the differential equations (21.66) and (21.67). This figure shows that the unique steady state is globally stable (see Exercise 21.19 for a formal proof). Consider, once again, a less-developed economy starting with a low level of capital-labor ratio, $k(0)$, and a low level of the social variable, $x(0)$. The dynamics in this case are qualitatively similar to those in Figure 21.10. However, the economics is slightly different. Capital accumulation (capital deepening) leads to an increase in $x(t)$ as before, but now this structural change also improves productivity as in the models we have analyzed in Section 17.6 in Chapter 17 and in Sections 21.3 and 21.1 of this chapter. This increase in productivity leads to faster capital accumulation and there is a *self-reinforcing* ("cumulative") process of development, with economic growth leading to structural changes facilitating further growth. However, since the effect of x on productivity is limited, this process ultimately takes us towards a unique steady state.

This reduced-form representation of structural change, therefore, captures some of the salient features we have emphasized in this chapter. It is not meant to be a unified model; on the contrary, rather than combining multiple dimensions of structural change, it presents an abstract representation emphasizing how the process of development, corresponding to capital accumulation, can go hand-in-hand with structural change, which may in turn increase productivity and facilitate further capital accumulation. The development of a truly unified model of economic development and the structural change associated with it is an area for future work.

21.8. Taking Stock

This chapter provided a large number of models focusing on various aspects of the structural transformation accompanying economic development. As emphasized in the previous section, there is no single framework unifying all these distinct aspects, even though there are many common themes across these models. The previous section was an attempt to bring out these common themes. Instead of repeating these commonalities, I would like to conclude by pointing out that many of the topics covered in this chapter are at the frontier of current research and much still remains to be done. Economic development is intimately linked to economic growth, but it may require different, even specialized, models that do not just focus on balanced growth and the orderly growth behavior captured by the neoclassical and endogenous technology models. These models may also need to take market failures and how these market failures might change over time more seriously. This view stems from the recognition that the essence of economic development is the process of structural transformation, including financial development, the demographic transition, migration, urbanization, organizational change and other social changes. Another important aspect of economic development, again less prominent in the neoclassical growth models, is the possibility that the inefficiencies in the organization of production, credit markets and product markets may culminate in potential development traps. These inefficiencies may stem from lack of coordination in the presence of aggregate demand externalities or from the interaction between imperfect credit markets and human capital investments. These areas not only highlight some of the questions that need to be addressed for understanding the process of economic development, but they also bring a range of issues that are often secondary in the standard growth literature to the forefront of analysis. These include, among other things, the organization of financial markets, the distribution of income and wealth, and issues of incentives, such as problems of moral hazard, adverse selection and incomplete contracts both in credit markets and in production relationships; unfortunately, space restrictions have precluded me from providing a satisfactory discussion of these issues, and instead, I had to incorporate these in simple growth models in reduced-from ways.

The recognition that the analysis of economic development necessitates a special focus on these topics also opens the way for a more constructive interaction between empirical development studies and the theories of economic development surveyed in this chapter. As already noted above, there is now a large literature on empirical development economics, documenting the extent of credit market imperfections, the impact of inequality on human capital investments and occupation choices, the process of social change and various other market failures in less-developed economies. By and large, this literature is about market failures in less-developed economies and sometimes also focuses on how these market failures can be rectified. The standard models of economic growth do not feature these market failures. A fruitful area for future research is then the combination of theoretical models of economic growth and development (that pay attention to market failures) with the rich empirical evidence on the incidence, characterization and costs of these market failures. This combination will have the advantage of being theoretically rigorous, empirically grounded, and perhaps most importantly focusing on what I believe to be the essence of development economics—the questions of why some countries are less developed, how they can grow more rapidly, and how they can jumpstart the process of structural transformation necessary for economic development.

21.9. References and Literature

By its nature, this chapter has covered a large amount of material. My selection of topics and approaches corresponding to these topics has reflected my own interests and was also motivated by my desire to keep this chapter from becoming even longer than it already is. To obtain an in-depth understanding of the issues in the literature for any one of the topics covered here, the reader would need to study a large literature.

Section 21.1 scratches the surface of a rich literature on financial development and economic growth. On the theoretical side important papers include Townsend (1979), Greenwood and Jovanovic (1990), Bencivenga and Smith (1991), which focus on the interaction between financial development on the one hand and risk sharing and the allocation of funds across different tasks and individuals on the other. Obstfeld (1994), Saint-Paul (1992) and Acemoglu and Zilibotti (1997), which was discussed in Section 17.6, focus on the relationship between financial development and the diversification of risks. There is also a large empirical literature looking at the effect of financial development on economic growth. An excellent survey of this literature is provided in Levine (2005). Some of the most well-known empirical papers include King and Levine (1993), which documents the cross-country correlation between measures of financial development and economic growth, Rajan and Zingales (1998), which shows that lack of financial development has particularly pernicious effects on sectors that have a greater external borrowing needs, and Jayaratne and Strahan (1996), which documents how banking deregulation that lead to greater competition in US financial markets lead to more rapid financial and economic growth within the United States. In discussing financial development, I also mentioned the literature on the Kuznets curve. As mentioned in the text, there is no consensus on whether there is a Kuznets curve. Work that focuses on historical data such as Lindner and Williamson (1976) or Bourguignon and Morrison (2004) report aggregate patterns consistent with a Kuznets curve, while studies using panels of countries in the postwar era, such as Fields (1994) do not find a consistent pattern resembling the Kuznets curve.

The literature on economic growth and fertility and on the demographic transition is even larger than the literature on financial development. The main trends in world population and cross-country differences in population growth are summarized in Livi-Bacci (1997) and Maddison (2003). The idea that parents face a tradeoff between the numbers and the human capital of their children—i.e., the quality and quantity tradeoff—was proposed by Becker (1981). The aggregate patterns in Livi-Bacci (1997) are consistent with this idea, though there is little micro evidence supporting this tradeoff. Recent work on microdata by Black, Devereux and Salvanes (2005), Angrist, Lavy and Schlosser (2006), and Qian (2007) looks at evidence from Norway, Israel and China, but does not find support for the quality-quantity tradeoff. Fertility choices were first introduced into growth models by Becker and Barro (1988) and Barro and Becker (1989). Becker, Murphy and Tamura (1990) provide the first endogenous growth model with fertility choice. More recent work on the demographic transition and the transition from a Malthusian regime to one of sustained growth include Goodfriend and McDermott (1995), Galor and Weil (1996, 1999, 2000), Hansen and Prescott (2002), Tamura (2002), Lagerlof (2003), and Doepke (2004). Kalemli-Ozcan (2002) and Villaverde (2003) focus on the effect of declining mortality on fertility choices in a growth context. A recent series of papers by Galor and Moav (2002, 2004) combine fertility choice, quality-quantity tradeoff and natural selection. Galor (2005) provides an excellent overview of this literature. The first model presented in Section 21.2 is a simplified version of Malthus's classic model in his (1798) book, while the second model is a simplified version of Becker and Barro (1988) and Galor and Weil (1999).

Urbanization is another major aspect of the process of economic development. Bairoch (1988) provides an overview of the history of urbanization and and insightful discussion of some of the economic literature in this area. The first model in Section 21.3 builds on Arthur Lewis's (1954) classic, which argued that early development can be viewed as a situation in which there is surplus labor available to the modern technology, thus growth is constrained by capital and technology but not by labor. A formalization of Lewis's ideas naturally takes us to the realm of the dual economy, since surplus labor for the modern technology can remain only when there is limited interaction between the modern technology (cities) and rural areas. Another well-known model by Harris and Todaro (1970) also emphasize the importance of model of migration, though it features free migration between rural and urban areas and suggests that unemployment in urban areas will be the key equilibriatingvariable.

The second model, presented in subsection 21.3.2, is inspired by Banerjee and Newman (1998) and Acemoglu and Zilibotti (1999). Banerjee and Newman emphasize the advantage of smaller rural communities in reducing moral hazard problems in credit relations and show how this interacts with the process of urbanization, which involves individuals migrating to areas where they are marginal product is higher. Acemoglu and Zilibotti argue that development—capital accumulation—leads to "information accumulation," in particular, as more individuals perform similar tasks, more socially useful information is revealed and relative performance of valuations can be used more effectively to filter out common shocks. They show that greater information enables individuals to write more sophisticated contracts and draw the implications of these more complex contractual relations on technology choice, financial development and social transformations associated with economic development. Though inspired by these two papers, the model presented in subsection 21.3.2 was designed to be more reduced-form and simpler so as to communicate the basic ideas in the most economical way. In particular, it did not incorporate credit market relations in villages and urban areas as in Banerjee and Newman or risk-sharing contracts as in Acemoglu and Zilibotti. Instead, it emphasized another important aspect of social and economic relations in less-developed economies, the importance of community enforcement. The sociologist Clifford Geertz (1963), for example, emphasizes the importance of community enforcement mechanisms and how they may sometimes conflict with markets.

Section 21.4 builds on Acemoglu, Aghion and Zilibotti (2004, 2006). Evidence consistent with organizational changes related to the distance to the frontier are provided in Acemoglu, Aghion, Lelarge , Van Reenen and Zilibotti (2007).

Section 21.5 is based on Murphy, Shleifer and Vishny's famous (1989) paper, which formalizes ideas first proposed by Rosenstein-Rodan (1943). Other models that demonstrate the possibility of multiple equilibria in monopolistic competition models featuring nonconvexities include Kiyotaki (1988), who derives a similar result in a model with endogenous labor supply choices as well as investment decisions. Matsuyama (1996) provides an excellent overview of these models, as well as other approaches that can lead to multiple equilibria in multiples the states. Matsuyama (1996) also provides a very clear discussion of why pecuniary externalities can lead to multiple equilibria in the presence of monopolistic competition.

The distinction between multiple equilibria and multiple steady states is discussed in Krugman (1994) and Matsuyama (1994). Both of these papers highlight that in models with multiple equilibria, expectations determine which equilibrium will be played, while with multiple steady states, there can be (or there often is) a unique equilibrium and initial conditions (history) determines where the economy will end up.

Section 21.6 covers the enormous literature on the role of inequality in human capital investments and occupational choices. The model in subsection 21.6.2 is based on the first model in Galor and Zeira's well-known (1993) paper. Similar ideas are investigated in Banerjee and Newman (1994) in the context of the effect of inequality on occupational choice, and Aghion and Bolton (1998) and Piketty (1998) in the context of the interaction between inequality and entrepreneurial investments. The model in subsection 21.6.3 is based on Benabou (1996a,b), which investigates the dynamics of inequality, how inequality affects productive efficiency and the implications of different forms of community structures. Other papers that investigate similar questions include Loury (1981), Tamura (1991, 2001), Durlauf (1996), Fernandez and Rogerson (1996, 1998), Glomm and Ravikumar (1992), and Acemoglu (1997). An excellent survey of this set of papers, together with extensions to analyze the interaction between political economy and inequality and between endogenous technology choices and inequality is contained in Benabou (2005). There is also a large literature on inequality, human capital and taxation that incorporates political economy features. This literature will be discussed in the next chapter.

21.10. Exercises

Exercise 21.1. Analyze the equilibrium of the economy in Section 21.1 relaxing the assumption that each individual has to invest either all or none of their wealth in the risky saving technology. Does this generalization affect the qualitative results derived in the text?

Exercise 21.2. Consider the economy in Section 21.1.

(1) Show that in equation (21.5), $K(t+1)$ is everywhere increasing in $K(t)$ and that there exists some $\bar{K}$ such that the capital stock will grow over time when $K(t) > \bar{K}$.
(2) Can there be more than one steady state level of capital stock in this economy? If so provide an intuition for this type of multiplicity.
(3) Provide sufficient conditions for the steady state level of capital stock, K^*, to be unique. Show that in this case $K(t+1) > K(t)$ whenever $K(t) > K^*$.

Exercise 21.3. In the model of subsection 21.2.1, suppose that the population growth equation takes the form

$$L(t+1) = \varepsilon(t)(n(t+1) - 1) L(t)$$

instead of (21.8), where $\varepsilon(t)$ is a random variable that takes one of two values, $1-\overline{\varepsilon}$ or $1+\overline{\varepsilon}$, reflecting random factors affecting population growth. Characterize the stochastic equilibrium. In particular, plot the stochastic correspondence representing the dynamic equilibrium behavior and analyze how shocks affect population growth and income dynamics.

Exercise 21.4. Characterize the full dynamics of migration, urban capital-labor ratio and wages in the the model of subsection 21.3.1 (that is, consider the cases in which conditions 1, 2 and 3 in that subsection do not all hold together).

Exercise 21.5. Consider the model of subsection 21.3.2 and suppose that all individuals have utility given by the standard CRRA preferences, i.e.,

$$U(0) \equiv \int_0^\infty \exp(-\rho t) \frac{c(t)^{1-\theta}-1}{1-\theta} dt.$$

Taking the equilibrium path in that subsection is given, find a level of community enforcement advantage ξ that would maximize $U(0)$. What happens if the actual comparative advantage of community enforcement of villages is greater than this level?

Exercise 21.6. Consider the maximization problem given in (21.30).

(1) Explain why this maximization problem characterizes the equilibrium allocation of workers to tasks. What kind of price system will support this allocation?
(2) Derive the first-order conditions given in (21.31).
(3) Provide sufficient conditions such that the solution to this problem involves all skilled workers employed at technology $\bar{h}$.
(4) Provide an example in which no worker will be employed at technology $\bar{h}$ even though $A_{\bar{h}} > A_h$ for all $h \in [0, \bar{h}]$.
(5) Can there be a solution where more than two technologies are being used in equilibrium? If so, explain the conditions for such an equilibrium to arise.

Exercise 21.7. Consider a variant of the model in Section 21.4, where firms have an organizational form decision, in particular, they decide whether or not to vertically integrate. For this purpose, consider a slight modification of equation (21.37) where

$$A(\nu,t) = \eta\bar{A}(t-1) + \gamma(\nu,t) A(t-1),$$

with

$$\gamma(\nu,t) = \underline{\gamma} + \theta(\nu,t).$$

Suppose that entrepreneurial effort increases $\theta(\nu,t)$, and the internal organization of the firm affects how much effort the entrepreneur devotes to innovation activities. In particular, suppose that $\theta(\nu,t)=0$ if there is vertical integration, because the entrepreneur is overloaded and has limited time for innovation activities. In contrast, with outsourcing $\theta(\nu,t)=\theta>0$. However, when there is outsourcing, the entrepreneur has to share a fraction $\beta>0$ of the profits with the manager (owner) of the firm to which certain tasks have been outsourced (whereas in a vertically integrated structure, he can keep the entire revenue).

(1) Determine the profit-maximizing outsourcing decision for an entrepreneur as a function of the (inverse) distance to frontier $a(t)$. In particular, show that there exists a threshold $\bar{a}$ such that there will be vertical integration for all $a(t) \leq \bar{a}$ and outsourcing for all $a(t) > \bar{a}$.
(2) Contrast this equilibrium behavior with the growth-maximizing internal organization of the firm.

Exercise 21.8. Show that when multiple equilibria exist in the model of Section 21.5, the equilibrium with investment Pareto dominates the one without.

Exercise 21.9. Consider the model of subsection 21.6.1 and remove the nonconvexity in the accumulation equation, (21.52), so that the human capital of the offspring of individual i is given by

$$h_i(t+1) = e_i(t)^{\gamma},$$

for any level of $e_i(t)$ and $\gamma \in (0,1)$. Show that there exists a unique level of human capital to which each dynasty will converge to. Based on this result, explain the role of nonconvexities in generating multiple steady states.

Exercise 21.10. Consider the model of subsection 21.6.1 and suppose that initial inequality is given by a uniform distribution with mean human capital of $h(0)$ and support over $[h(0) - \psi, h(0) + \psi]$. Clearly an increase in ψ corresponds to greater inequality.

(1) Show that when $h(0)$ is sufficiently small, an increase in ψ will increase long-run average human capital and income, whereas when $h(0)$ is sufficiently large, an increase in ψ will reduce long run human capital and income. [Hint: use Figure 21.8 or Figure 21.9].
(2) What other types of distributions (instead of uniform) would lead to the same result?
(3) Show that the same result generalizes to the model of 21.6.2.
(4) On the basis of this result, discuss whether we should expect greater inequality to lead to higher income in poor societies and lower income in rich societies. (If your answer is no, then sketch an environment in which this will not be the case).

Exercise 21.11. Consider the model presented in subsection 21.6.2. Make the following two modifications. First, the utility function is now

$$(1-\delta)^{-(1-\delta)}\,\delta^{-\delta} c^{1-\delta} b^{\delta} \tag{21.68}$$

and second, unskilled agents receive a wage of $w_u + \varepsilon$ where ε is a mean-zero random shock.

(1) Suppose that ε is distributed with support $[-\psi, \psi]$, and show that if ψ is sufficiently close to 0, then the multiple steady states characterized in 21.6.2 "survive" in the

sense that depending on their initial conditions some dynasties become high skilled and others become low skilled.

(2) Now suppose that ε is distributed with support $[-\psi, \infty)$, where $\psi \leq w_u$. Show that in this case there is a unique ergodic distribution of wealth and no poverty trap (in the sense that every dynasty will become skilled at some point with probabability 1). Explain why the results here are different from those presented in subsection 21.6.2?

(3) Why was it convenient to change the utility function from the log form used in the text to (21.68)?

Exercise 21.12. We now discuss potential microfoundations for the borrowing constraints in the model of subsection 21.6.2.

(1) Suppose that each individual can run away without paying his debts, and if he does so, he will never be caught. However, a bank that lends to the individual can make sure that the individual is unable to run away by paying a monitoring cost per unit of borrowing equal to m. Suppose that there are many banks competing for lending opportunities, so that Bertrand competition among them will drive them to zero profits. Under these assumptions, show that all bank lending will be accompanied with monitoring, and the lending rate will satisfy $i = r + m$. Show that in this case all of the results in the text apply.

(2) Next suppose that the bank can prevent individual from running away by paying a fixed monitoring cost of M. Under the same assumptions as in part 1 above show that in this case the interest rate charged to an individual that borrows an amount $x - h$ will be $i\,(x-h) = r + M/\,(x-h)$. Given this assumption, characterize the equilibrium of the model in subsection 21.6.2. How do the conclusions change in this case?

(3) Next suppose that there is no way of preventing running away by individuals, but if an individual runs away, he will be caught with probability p, and in this case, a fraction $\lambda \in (0,1)$ of his income will be confiscated. Given this assumption, characterize the equilibrium of the model in subsection 21.6.2. How do the conclusions change in this case?

(4) Now consider an increase in w_s (for a given level of w_u) so that the skill premium in the economy increases. In which on the three scenarios outlined above will this have the largest effect on human capital investments?

Exercise 21.13. In this exercise, we study Banerjee and Newman's (1994) model of occupational choice, which leads to similar results to the Galor-Zeira model, though with richer dynamics. The utility of each individual again depends on consumption and bequest, with

$$(1-\delta)^{-(1-\delta)}\,\delta^{-\delta}c^{1-\delta}b^{\delta}-z$$

where z denotes whether the individual is exerting effort, with cost of effort normalized to 1. Each agent chooses one of four possible occupations. These are (1) subsistence and no work, which leads to no labor income and has a rate of return on assets equal to $\hat{r}<1/\delta$; (2) work for a wage v; (3) self-employment, which requires investment I plus the labor of the individual; and (4) entrepreneurship, which requires investment μI plus the employment of μ workers, and individual himself becomes the boss, monitoring the workers (and does not take part in production). All occupations other than subsistence involve effort. Let us assume that both entrepreneurship and self-employment generate a rate of return greater than subsistence, i.e., the mean return for both activities is $\bar{r}>\hat{r}$.

(1) Derive the indirect utility function associated with the preferences above. Show that no individual will work as a worker for a wage less than 1.

(2) Assume that $\mu\left[I\left(\bar{r}-\hat{r}\right)-1\right]-1>I\left(\bar{r}-\hat{r}\right)-1>0$. Interpret this assumption. [Hint: it relates the private profitability of entrepreneurship and self-employment at the minimum possible wage of 1].

(3) Suppose that only agents that have wealth $w\geq w^{*}$ can borrow enough to become self-employed and only agents that have wealth $w\geq w^{**}>w^{*}$ can borrow μI to become an entrepreneur. Explain why this type of borrowing constraints may be present.

(4) Now compute the expected indirect utility from the for occupations. Show that if

$$v>\bar{v}\equiv\frac{\mu-1}{\mu}\left(\bar{r}-\hat{r}\right)I,$$

than self-employment is preferred to entrepreneurship.

(5) Suppose wealth distribution of time t is given by $G_t\left(w\right)$. On the basis of the results in part 4, showed that the demand for labor in this economy is given by

$$\begin{array}{ll} x=0 & \text{if } v>\bar{v} \\ x\in\left[0,\mu\left(1-G_t\left(w^{**}\right)\right)\right] & \text{if } v=\bar{v} \\ x=\mu\left(1-G_t\left(w^{**}\right)\right) & \text{if } v<\bar{v} \end{array}.$$

(6) Let $\tilde{v}\equiv\left(\bar{r}-\hat{r}\right)I>\bar{v}$. Then show that the supply of labor is given by

$$\begin{array}{ll} s=0 & \text{if } v<1 \\ s\in\left[0,G_t\left(w^{*}\right)\right] & \text{if } v=1 \\ s=G_t\left(w^{*}\right) & \text{if } 1<v<\tilde{v} \\ s\in\left[G_t\left(w^{*}\right),1\right] & \text{if } v=\tilde{v}, \\ s=1 & \text{if } v>\tilde{v}. \end{array}$$

(7) Show that if $G_t\left(w^{*}\right)>\mu\left[1-G_t\left(w^{**}\right)\right]$, there will be an excess supply of labor and the equilibrium wage rate will be $v=1$.

(8) Show that if $G_t(w^*) < \mu[1 - G_t(w^{**})]$, there will be an excess demand for labor and the equilibrium wage rate will be $v = \bar{v}$.

(9) Now derive the individual wealth (bequest) dynamics (for a worker with wealth w) as follows: (1) subsistence and no work $b(t) = \delta \hat{r} w$; (2) worker: $b(t) = \delta(\hat{r}w + v)$; (3) self-employment: $b(t) = \delta(\bar{r}I + \hat{r}(w - I))$; (4) entrepreneurship: $b(t) = \delta(\bar{r}\mu I + \hat{r}(w - \mu I) - \mu v)$. Explain the intuition for each of these expressions.

(10) Now using these wealth dynamics show that multiple steady states with different wealth distributions and occupational choices are possible. In particular, show that the steady-state wealth level of a worker when the wage rate is v will be $w_w(v) = \delta v/(1 - \delta \hat{r})$, while the steady-state wealth level of a self-employed individual will be $w_{se} = \delta(\bar{r} - \hat{r}) I/(1 - \delta \hat{r})$, and the wealth level of an entrepreneur will be $w_e(v) = \delta(\bar{r}\mu I - \hat{r}\mu I - \mu v)/(1 - \delta \hat{r})$. Now show that when $w_w(v = 1) < w^*$ and $w_e(v = \bar{v}) > w^{**}$, a steady state in which the equilibrium wage rate is equal to $v = 1$ would involve workers not accumulate sufficient wealth to become self-employed, while entrepreneurs accumulate enough wealth to remain entrepreneurs. Explain why this is the case. [Hint: it depends on the equilibrium wage rate].

(11) Given the result in part 10, show that if we start with a wealth distribution such that $\mu(1 - G(w^{**})) < G(w^*)$, the steady state will involve an equilibrium wage $v = 1$ and no self-employment, on whereas if we start with $\mu(1 - G(w^{**})) > G(w^*)$, the equilibrium wage would be $v = \bar{v}$ and there will be self-employment. Contrast the level of output in these two steady states.

(12) Is the comparison of the steady states in terms of output in this model plausible? Is it consistent with historical evidence? What are the pros and cons of this model relative to the Galor-Zeira model we studied in subsection 21.6.2?

Exercise 21.14. Explain why the aggregator in (21.57) could not be the production function of a final good producer, with each $h(t)$ corresponding to intermediates, but it can be used as an aggregator of the human capital levels of different individuals in the society.

Exercise 21.15. Given (21.59), derive (21.60). [Hint: take logs in (21.61) and (21.62)].

Exercise 21.16. Derive equations (21.63) and (21.64).

Exercise 21.17. In the model of subsection 21.6.3, determine conditions under which the long run income level is higher under full integration than full segregation.

Exercise 21.18. Consider the following non-overlapping generations model, with population normalized 1, where each individual i lives for one period and then begets an offspring. Each individual has has preferences given by

$$c_i(t)^{1-\delta} e_i(t+1)^{\delta},$$

where $c_i(t)$ is the consumption of the individual and $e_i(t+1)$ is educational investment in the human capital of the offspring. Each individual has some earned income $w_i(t)$ and is subject to the budget constraint $c_i(t)+e_i(t+1) \leq w_i(t)$. The human capital of the offspring, $h_i(t+1)$, is given by equation (21.52) in subsection 21.6.1 with $\bar{h}=1$ and $\gamma=1$. There is also to continue 1 of firms, each with the production function

$$y_{i,j} = A(k_j)^{\alpha}(h_i)^{1-\alpha},$$

where worker i has been matched with firm j. Firms choose the level of physical capital investment at some cost R before matching with workers. Let us also assume that matching is random, so that any worker has the same likelihood of matching any firm, and in particular, there is no selective process of high human capital workers being allocated to high physical capital firms. If the firm is not happy with the worker that it matches with, it can fire the worker and resample another worker from the remaining (potentially unmatched) distribution of workers. If it does so, it will have lost a fraction $1-\eta$ of the time devoted for production, so its output will be only a fraction η of the expression given above. Conditional on matching, workers and firms bargain on the wage. Let us suppose that when the distribution of human capital is given by $\mu_t(h)$ and the distribution of physical capital is given by $\nu_t(k)$, the wage of a worker of human capital h_i matched with a firm of physical capital k_j is given by

$$\begin{aligned} w(h_i, k_j, \mu_t, \nu_t) &= \beta A(k_j)^{\alpha}(h_i)^{1-\alpha} - \beta(1-\beta)A(k_j)^{\alpha}\int h^{1-\alpha}d\mu_t(h) \\ &\quad +\beta(1-\beta)A(h_i)^{1-\alpha}\int k^{\alpha}d\nu_t(k). \end{aligned}$$

(1) Interpret this wage equation. Could you derive it from Nash bargaining? If so, be specific about what assumptions are necessary, particularly concerning what type of worker the worker will match with if it separates from its first partner and vice versa.
(2) In view of this wage equation, show that there exists some η^* so that for all $\eta < \eta^*$, all firms will choose the same level of physical capital investment at time t given by $k^*[\mu_t, \eta, \beta, R]$. Show that in this case, a mean-preserving spread of the human capital distribution will reduce aggregate output. Provide an intuition for this result.
(3) Suppose that $\eta < \eta^*$ as determined in part 2 and that the economy starts at time $t=0$ with two groups of workers, a fraction λ with human capital $h_1(0)$ and a fraction $1-\lambda$ with human capital $h_2(0) > h_1(0) > \bar{h} = 1$, where $\bar{h}=1$ is the minimum human capital level defined in (21.52). Let $\phi(t) \equiv h_1(t)/h_2(t)$. Show that the economy will always have two groups of workers and thus its low of motion can be summarized by $\phi(t)$.

(4) Derive a difference equation for $\phi(t)$ using the optimal capital investment level of firms $k^*[\mu_t, \eta, \beta, R]$ derived in part 2 and the preferences of individuals regarding investments in their offspring's human capital.

(5) Prove that there exists some $\bar{\phi} \in (0,1)$, such that if $\phi(0) > \bar{\phi}$, then the dynamic equilibrium involves $\phi(t) \to 1$ and the economy achieves a constant growth rate. In contrast, if $\phi(0) < \bar{\phi}$, then $h_1(t) \to \bar{h}$ and $h_2(t) \to \tilde{h}$ for some $\tilde{h} > 1$, and the economy converges to no growth. Explain the intuition for this result.

(6) Compare the model here with the model in subsection 21.6.3. What are its pros and cons? How would you generalize or make this model more realistic?

EXERCISE 21.19. This exercise asks you to analyze the dynamics of the reduced-form model in Section 21.7 more formally.

(1) Show that when $f_x > 0$, the locus for $\dot{k}/k = 0$ implied by (21.66) is an upward sloping curve.

(2) Consider the differential equations (21.66) and (21.67), and a steady state (k^*, x^*). By linearizing the two differential equations around (k^*, x^*), show that if $f_x(k^*, x^*)$ is sufficiently small, the steady state is locally stable.

(3) Provide a bound on $f_x(k, x)$ over the entire domain so that there exists a unique steady state. Show that when this bound applies, the unique steady state is globally stable.

(4) Construct a parameterized example where there are multiple steady states. Interpret the conditions necessary for this example. Do you find them economically likely?

Part 8

Political Economy of Growth

In this part of the book, I turn from the *mechanics* of economic growth to an investigation of potential *causes* of economic growth. Almost all of the models we have studied so far take economic institutions (such as whether property rights are enforced and what types of contracts can be written), policies (such as tax rates, distortions, subsidies) and often the market structure as given. They then derive implications for economic growth and cross-country income differences. While these models constitute the core of growth theory, they leave some of the central questions raised in Chapters 1 and 4 unanswered: why do some societies choose institutions and policies that discourage growth, while others choose growth-enhancing social arrangements? In this part of the book, I will make a first attempt to provide some answers to these questions based on political economy—that is, on differences in institutions and policies arising from different ways of aggregating individual preferences across societies and on differences in the type and nature of social conflict. In particular, I will emphasize a number of key themes and attempt to provide a tractable and informative formalization of these issues. The main themes are:

(1) Different institutions and policies almost always generate winners and losers. In other words, there are a few economic or political reforms that would benefit all members of the society. Consequently, there will be *social conflict* concerning the types of policies and institutions that a society should adopt.

(2) Aggregating the preferences of different individuals to arrive to collective choices is nontrivial in the presence of social conflict. Two interrelated factors will be central for the aggregation of these preferences: the form of *political institutions* and the *political power* of different groups. Individuals and groups with significant political power are more likely to be influential and sway policies towards their preferences. Exactly how political power is distributed within the society and how individuals can exercise their political power (resulting from their votes, connections or brute force) will depend on political institutions. For example, a dictatorship that concentrates political power in the hands of a small group will imply a different distribution of political power in the society than a democracy, which corresponds to a society with a greater degree of political equality. We expect that these different political regimes will induce different sets of economic institutions and policies, and thus lead to different economic outcomes. The purpose of the next two chapters is to investigate this process of collective decision-making and the implications of different choices of institutions and policies on economic growth.

(3) The technology, the nature of the endowments and the distribution of income and endowments in the economy will influence both the preferences of different individuals and groups towards policies (or specific institutions) and also their political

powers. For example, the nature of political conflict and the resulting political economy equilibrium is likely to be different in a society where much of the land and the capital stock is concentrated in the hands of a few individuals and families than one in which there is a more equitable distribution of resources. We would also expect politics to function differently in a society where the major assets are in the form of human capital vested in individuals than in a society where natural resources, such as diamonds or oil, are the major assets.

The issues raised and addressed in this part of the book are central to the field of political economy. Since this is a book on economic growth not on political economy, I will not try to do justice to the large and growing literature in this area. Instead, I will focus on topics and models that I deem to be most important for the questions posed above. I will also save space and time by focusing, whenever possible, on the neoclassical growth model rather than some of the richer models we have seen so far. This might at first appear as an odd choice. Why should we focus on the neoclassical growth model, which does not generate growth (other than via exogenous technological change), to study the political economy of growth? My answer is that the neoclassical growth model offers two significant advantages: first, it provides the most tractable framework to analyze the main political economy conflicts. Second, because the competitive equilibrium in the neoclassical growth model is Pareto optimal, it will make the role of political economy distortions more transparent. Naturally, once the basic forces are understood, it is relatively straightforward to incorporate them into endogenous growth models or other richer structures. Some of the exercises will consider these extensions.

I have organized the material on political economy of growth into two chapters. The first chapter takes political institutions as given and focuses on the implications of distributional conflict under different scenarios. In this chapter, I try to highlight why and when distributional conflict can lead to distortionary policies retarding growth. I will also offer various complementary frameworks for the analysis of these questions. Chapter 23 then turns to the implications of different political regimes for economic growth and also includes a brief discussion of how political institutions themselves are determined endogenously.

Before presenting this material, it is useful to start with an abstract discussion of the relationship between economic institutions, political institutions, and economic outcomes, and individual preferences over economic and political institutions. The distinction between economic and political institutions has already been highlighted in Chapter 4 and will be discussed again below. For now I take this distinction as given. The political science literature often posits that individuals have (direct) preferences over political institutions (and perhaps also over economic institutions). For example, individuals might derive utility from living under a democratic system. While this is plausible, the approach we have developed so far in this book emphasizes another, potentially equally important, reason for individuals to have preferences over political institutions. However, economic institutions and policies also have a direct effect on economic outcomes (for example, as illustrated by the effects of tax policies, regulation and contracting institutions in previous chapters). Thus we may plausibly expect that the major determinant of individual preferences on economic institutions (and policies) will be the allocations that result from these arrangements. Based on this viewpoint, throughout I will focus on these *induced* preferences on economic institutions.

The same applies to political institutions. These determine the *political rules* under which individuals interact. In direct democracy, for example, key decisions are made by majoritarian voting. In representative democracy, majorities choose representatives who then make the policy choices and face the risk of being removed from office if they pursue policies that are not in line with the preferences of the electorate. In contrast, in nondemocratic regimes, such as dictatorships or autocracies, a small clique, an oligarchy of rich individuals, or a junta of generals make the key decisions. These differences in political rules imply that different political institutions lead to the different distributions of *de jure political power*, meaning that the institutionally-sanctioned distribution of political power, and thus the decision-making capacity, is distributed differently within the society. As a result, we would expect different policies and economic institutions to emerge in different political systems. For example, democracies are more likely to choose redistributive policies as we have already seen in the previous chapter, whereas a society that is dominated by the elite or by a small group of individuals is likely to choose policies, that will further the economic interests of this narrow group. This reasoning suggests that since different political institutions will lead to different economic institutions and policies and via this channel to different economic allocations, individuals will similarly have induced preferences over political institutions.

To emphasize this point, let us represent the chain of causation described above by a set of mappings. Let $\mathcal{P}$ denote the set of political regimes or institutions, $\mathcal{R}$ be the set of feasible economic institutions (or policies), and $\mathcal{X}$ denote the set of feasible allocations (which include different levels of consumption of all goods and services by all individuals in the society). Ignoring any stochasticity in outcomes for simplicity, we can think of each political institution in the set $\mathcal{P}$ leading to some specific set of economic institutions in the set $\mathcal{E}$. Let this be represented by the mapping $\pi(\cdot)$. Similarly, each different set of economic institutions will lead to a different allocation (ignoring again stochastic elements and multiple equilibria), and let this be represented by the mapping $\rho(\cdot)$. Schematically, we can write

$$\mathcal{P} \xrightarrow{\pi(\cdot)} \mathcal{R} \xrightarrow{\rho(\cdot)} \mathcal{X}.$$

Now suppose that each individual i has a utility function $u_i : \mathcal{X} \to \mathbb{R}$, representing his preferences over possible allocations in $\mathcal{X}$. Suppose also that individuals do not care about economic or political institutions beyond these institutions' influences on allocations. In other words, we presume that individuals are purely *consequentialist* (and thus ignore any direct benefits they may obtain from institutions). Then their preferences over some economic institution $R \in \mathcal{R}$ is simply given by $u_i(\rho(R)) \equiv u_i \circ \rho : \mathcal{R} \to \mathbb{R}$. This mapping therefore captures their *induced* preferences over economic institutions (as a function of the economic allocations that these institutions will lead to). Preferences on political institutions are also induced in the same manner. The utility that individual i will derive from some political institution $P \in \mathcal{P}$ is simply given by $u_i(\rho(\pi(P)))$, where clearly $u_i \circ \rho \circ \pi : \mathcal{P} \to \mathbb{R}$. These preferences on institutions are important, since an equilibrium framework must, at least to some degree, explain the emergence and change of political institutions as a function of the preferences of the members of the society over these objects.

Throughout, the next two chapters, I will adopt this consequentialist view and define individual political preferences, over economic institutions or political institutions, purely in terms of these *induced preferences*—that is, preferences according to economic allocations that will ultimately result from these institutions. The interesting part of the analysis is to understand how economic institutions affect economic outcomes, how this shapes individual attitudes towards different economic institutions and policies, and which political institutions will lead to what types of economic institutions. We have already seen in Chapter 4 some empirical approaches to howthe cluster of economic and political institutions affect economic outcomes (including growth and distribution of resources). Much on my focus in the next two chapters will be to investigate the same linkages theoretically.

This brief section has therefore laid two types of foundations for the rest of this part of the book. (1) Our first task will be to understand how different types of economic institutions (and policies) affect economic outcomes, including economic performance and distribution of resources, which, here, is summarized by the mapping $\rho(\cdot)$. Then based on this, we should analyze the preferences of different groups over these economic institutions (policies) and determine the conditions under which different groups will have a preference for distortionary, non-growth-enhancing economic arrangements. This will be the topic of the next chapter. (2) In order to understand political change and how it interacts with economic decisions and economic growth, we next need to understand induced preferences over political institutions, that is, understand the mapping $\pi(\cdot)$ and combine it with our insights about $\rho(\cdot)$. This will inform us on how political institutions affect economic arrangements, how economic arrangements influence economic allocations, and on the basis of this, how different groups value different sets of political institutions.

CHAPTER 22

Institutions, Political Economy and Growth

This chapter will make a first attempt towards answering the following fundamental question that has been in the background of much of what we have done so far: *why do similar societies choose different institutions and policies, leading to very different economic growth outcomes?*

Our analysis so far has highlighted the role of capital accumulation, human capital and technology in economic growth. We have investigated the incentives to accumulate physical capital and human capital, the process via which technology progresses and how different societies transfer and adopt technologies from others (or from the world technology frontier). Throughout we have emphasized that the level of physical capital, the extent of human capital and even the technology of societies should be thought of as endogenous and respond to incentives. This brings us to the fundamental question: why do different societies end up with different levels of physical capital, human capital and technology (or organize their production differently)? While we cannot provide a full and comprehensive answer to this question, the recent literature has made considerable progress in the study of why societies differ in their choices. Chapter 4 has argued against the importance of geographic and cultural factors and has instead suggested that differences in institutions are likely to be the most important fundamental cause of differences in economic performance. The purpose of this and the next chapter is to investigate this claim in greater detail and provide models that can help us understand why institutions might have such an effect and why institutions themselves differ across societies.

This chapter starts with a brief informal discussion of *political economic* analysis of institutions and policies. I will emphasize that different constellations of institutions and policies will typically create different winners and losers, thus social conflict over collective choices (institutions and policies) will be ubiquitous. Political economy concerns the analysis of how societies resolve—or fail to resolve—these conflicts.

The rest of the chapter will present a number of models to shed light on the impact of distributional conflicts. I will start with the simplest environments, highlighting why societies choose distortionary policies. I will then enrich these environments both to investigate the robustness of the channels they highlight and also how these mechanisms interact with each other. Special emphasis will be placed on distortionary policies arising because of two complementary reasons: (1) the desire of individuals or social groups to transfer resources to themselves using limited fiscal instruments, and (2) the potential conflict between different social groups in the marketplace or in the political arena. I will try to highlight why these two sources of distortionary policies are distinct and also argue that the second source of distortions is typically more costly for growth.

I will conclude the chapter by pointing out that in addition to economic policies (such as taxes) and economic institutions such as the security of property rights and the regulation of entry, the provision of public goods by the government is an important topic in the political economy of growth.

22.1. The Impact of Institutions on Long-Run Development

22.1.1. Institutions and Growth. As already emphasized in Chapter 4 "institutions" matter—at least when we look at clusters ofeconomic and political institutions over long horizons. Moreover, most of the models we have seen so far in the book highlight how economic institutions and policies should matterincorporate this feature. For example, we have already seen how tax and subsidy policies and market structure may affect physical capital accumulation, human capital investments and technological progress, and how contracting institutions and the structure of the credit markets will influence technology choices and the efficiency of production.Perhaps even more important, all of the models studied so far assume a relatively orderly working of the market economy. Add to these models some degree of insecurity of property rights or entry barriers preventing activities by the more productive firms, they will imply major inefficiencies. Both theory and casual empiricism suggest that these factors must be important. We are unlikely to explain the differences in economic growth or income per capita between the United States and much of sub-Saharan African by small differences in taxes on capital or in subsidies to R&D. Even differences in discount factors or exogenous technology are unlikely to lead to the huge differences in income per capita and growth we have documented in Chapter 1. Instead, we have to recognize and understand that doing business is very different in the United States than in sub-Saharan Africa. Entrepreneurs and businessmen in the United States (or pretty much everywhere in the OECD) face relatively secure property rights, and individuals or corporations that wish to create new businesses face relatively few barriers. The situation is very different in much of the rest of the world, for example, in sub-Saharan Africa, in the Caribbean, and in large parts of Central America and Asia. Similarly, the lives of the majority of the population are radically different across these societies. In much of OECD, most citizens have access to a wide variety of public goods and the ability to invest in their human capital, while the situation is different in many less-developed economies.

We summarize these variations across societies as "*institutional differences*" (or differences in institutions and policies). The term is slightly unfortunate, but is one that is widely used and accepted in the literature. Institutions mean many different things in different contexts, and none of these exactly correspond to the meaning intended here. As already emphasized in Chapter 4, by institutional differences, we are referring to differences in a broad cluster of social arrangements including the security of property rights for businesses as well as for regular citizens and the ability of firms and individuals to write contracts to facilitate their transactions (contracting institutions), the entry barriers faced by new firms, the socially-imposed costs and barriers facing individual decisions in human capital accumulation, and incentives of politicians and individuals in providing or contributing to the provision of public goods. This definition of institutions is quite encompassing. To make theoretical and empirical progress, one typically needs a narrower definition of institutions. Towards this goal, I have already distinguished between *economic institutions*, which correspond to the security of property rights, contracting institutions, entry barriers and other economic arrangements, and *political institutions*, which correspond to the rules and regulations affecting political decision-making, including checks and balances against presidents, prime ministers or dictators as well as methods of aggregating the different opinions of individuals in the society (for example, electoral laws). In terms of the notation introduced in the introduction on to this part, the effect of economic institutions is summarized by the mapping $\rho(\cdot)$, while the implications of political institutions for the types of economic institutions and policies that will arise is captured by the mapping $\pi(\cdot)$.

It is also useful to note at this point that the difference between economic institutions and policies is not always clear, so it is often the combination of economic institutions and policies that matter not simply the "institutions". For example, we can refer to security of property rights or to the quality of contracting institutions as economic institutions but we would not typically refer to tax rates as institutions. Yet insecure property rights and 100% taxation of all income have much in common. One difference might be that institutions are more durable than policies. Motivated by this, in Section 22.4, I will make a distinction between economic institutions and policies whereby economic institutions provide a framework in which policies are set. The role played by the durability of political institutions will be further studied in the next chapter. Finally, in Section 22.8, I will discuss another potential reason why taxation and security of property rights might be different. Nevertheless, the contrast of insecure property rights and 100% taxation illustrates the large overlap between economic institutions and policies, and when the distinction between the two is unclear or unimportant, I will typically refer to "institutions and policies".

The evidence presented in Chapter 4 suggests that institutional differences do matter for economic growth. The focus of this section is not to review this evidence but build on it and ask the next question: if economic institutions matter so much for economic growth, why do some societies chooses institutions that do not encourage growth? In fact, based on available historical evidence we can go further: why do some societies choose institutions and policies that specifically block technological and economic progress? The rest of this chapter and much of the next chapter will try to provide a framework for answering these questions. I start with an informal discussion of the main building blocks towards an answer.

The *first important element* of the political economy approach is *social conflict*. There are few (if any) economic changes that would benefit all agents in the society. Thus every change in institutions and policies will create winners and losers relative to the status quo. Take the simplest example: removing entry barriers so that a previously monopolized market becomes competitive. Economic theory tells us that this is desirable in the sense that it removes distortions and creates a “potential Pareto improvement”. In the context of growth, we often focus on the implications of changes in institutions and policies on the level of income or the rate of economic growth. In this respect as well, removing entry barriers is likely to be a beneficial reform, since the removal of monopoly power will increase the quantity transacted in the market and raise real incomes. Nevertheless, not all parties in the economy will be winners from the removal of entry barriers. While consumers will benefit because of lower prices, the monopolist, who was previously enjoying a privileged position and high profits, will be a “loser”. The effect on workers depends on the exact market structure. If the labor market is competitive, workers will also benefit, since the demand for labor will increase with the entry of new firms. But if there are labor market imperfections, so that the employees of the monopolist were previously sharing some of the rents accruing to this firm, they will also be potential losers from the reform. Thus if we start with the status quo of a monopoly and consider the reform of liberalizing markets (removing entry barriers), there will not be unanimous support for this proposal. Put differently, there will be social conflict over the policy of “market liberalization”.

The presence of social conflict over institutions and policies is not specific to reform. If instead of starting with the status quo we were deciding how markets should be organized without reference to any past arrangements, the same conflicts would be present. Many firms would prefer arrangements in which they are the monopolist protected by entry barriers, while consumers and potential entrants would prefer a more competitive arrangement.

Therefore, because of the different allocations that they will induce, individuals will have different, conflicting preferences over economic institutions. So if there are conflicting preferences over collective choices in general (and over institutions and policies in particular), how do societies make decisions? Political economy is the formal analysis of this process of collective decision-making. If there is social conflict between a monopolist that wishes to retain entry barriers and consumers that wish to dismantle them, it will be the equilibrium of a political process that decides the outcome. This process may be "orderly" as in democracies, or disorderly or even chaotic as in other political regimes as illustrated by the all too frequent civil wars throughout human history. Whether it is a democratic or a non-democratic process that will lead to the equilibrium policy, the *political power* of the parties with conflicting interests will play a central role. Put simply, if two individuals disagree over a particular choice (for example, about how to divide a dollar), how powerful each is will play an important role on the ultimate choice. In the political arena, this corresponds to the political power of different individuals and groups. For example, in the monopoly example, we may expect the monopolist to have political power because it has already amassed income and wealth and may be able to lobby politicians. In a non-democratic society where the rule of law is tenuous, we can even imagine the monopolist utilizing thugs and paramilitaries to quash the opposition. On the other hand, in a democracy, consumers may have sufficient political power to overcome the interests and wishes of the monopolist through the ballot box or by forming their own lobbying groups. Whatever the outcome, political power will play an important role.

The *second key element* of the political economy approach is *commitment problems*, which will act both as a source of inefficiency and also augment the distortions created by social conflict. Political decisions at each date are made by the political process at that date (for example, by those holding political power at that point); commitment to future sequences of political and economic decisions are not possible unless they happen to be "equilibrium commitments" arising as part of the equilibrium (here, we will see that whether we use the concept of subgame perfect equilibrium or Markov perfect equilibrium will play a role in shaping the extent of available commitments).

At this point, it is important to distinguish between non-growth-enhancing policies (or distortionary policies) and Pareto inefficiency. Many political economy models will not lead to Pareto inefficiency (though some will). This is because in some reduced-form way their equilibrium can be represented as a solution to a weighted social welfare function (see Section 22.6). The resulting allocation, by virtue of maximizing this weighted social welfare function given the set of available instruments, will be a point along the constrained Pareto frontier of the economy. Nevertheless, many such allocations will involve distortionary and non-growth-enhancing policies (think, for example, of an allocation in which a dictator such as Mobutu in Zaire expropriates all the investors in the country; it is possible to change policies to increase investment and growth, but this will typically imply taking resources and power away from Mobutu). Interestingly, when commitment problems are present, and especially when we focus on Markovian equilibria, the political equilibrium will typically lead to a constrained

Pareto inefficient allocation, because there will often exist *future* policies that can make all parties better-off, but those policies will not arise as part of the equilibrium.

Consider a situation in which political power is in the hands of a specific group or an individual—the *political elite*. To simplify the thought experiment, let us ignore for now any constraints on the exercise of this political power (this is essentially where we will begin in the next section). Then the elite can set policies in order to induce allocations that are most beneficial for themselves, and thus the political equilibrium can be thought of as the solution to the maximization of a social welfare function giving all the weight to the elite. Even though the resulting equilibrium will not necessarily be Pareto inefficient, it will typically involve non-growth-enhancing policies. Why and when will the exercise of political power by the elite lead to such distortionary policies?

I will argue that there are two broad reasons for why those with political power will choose distortionary policies. The first is *revenue extraction*, that is, the attempt by the elite to extract resources from other members of the society using a limited menu of fiscal instruments. Central to this source of distortionary policies is two aspects of the society: (1) a decoupling between political power (which is here in the hands of the the elite) and economic power (which lies with the entrepreneurs and the workers); (2) a limited set of fiscal instruments. These two aspects combined imply that the elite will use the available fiscal instruments to transfer resources from the rest of the society to themselves, which is the first potential reason for distortionary policies. We will also see that the same type of distortionary policies emerge even when there is no political elite, but decisions are made democratically (Section 22.7). The restriction to a limited set of fiscal instruments, such as linear taxes that discourage investment or work effort is important here. Had there been non-distortionary taxes, such as lump-sum taxes, the elite could extract resources from the rest of the society without discouraging economic growth. But lump-sum taxes are often not possible, and more generally, most forms of redistribution do create distortions by reducing incentives for work effort or by discouraging investment.

I will argue, however, that the second reason for the use of distortionary policies by the political elite is potentially more damaging to economic growth. The elite will also choose distortionary policies because it will often be *in competition* with other social groups in society. This competition may be economic. For example, the elite may also engage in production and understand that by taxing and creating distortions on other entrepreneurs, they will be able to reduce their demand for factors (for example labor) and thus increase their profits. I will refer to this as the *factor price manipulation* motive for distortionary policies. The competition between the elite and other social groups may also be political. The elite might foresee that enrichment by other groups will pose a threat to their political power and to their ability to use and benefit from their power in the future. When this is the case, they will use distortionary policies to impoverish their political competitors. I will refer to this as the *political replacement* motive for distortionary policies. The rest of the chapter will illustrate how distortionary policies can be adopted for extracting resources from different social groups and for factor price manipulation and political replacement motives. An interesting implication of the models I will present will be that factor price manipulation and political replacement motives will often lead to greater distortions and will be more damaging to the growth potential of a society than the revenue extraction motive.

Next, this basic framework also enables us to illustrate the additional inefficiencies created by commitment problems. In particular because the elite cannot commit to future policies, there will be a *holdup problem*, whereby investments, once undertaken, may be taxed at prohibitively high rates or expropriate. Holdup problems are likely to be important in a wide range of circumstances, for example, when the relevant investments are in long-term projects and assets, so that a range of policies will be decided after these investments are undertaken.

Much of the current chapter is devoted to understanding how the potential conflict over different economic allocations leads to different preferences over economic institutions and policies. The next two sections focus on distributional conflict in a simple society, consisting of different social groups. Throughout this chapter, I will take the distribution of political power as given and in the next few sections, political power—and thus the authority to decide policy than economic institutions—will be in the hands of that group of individuals to whom I will refer to as "the elite". I will investigate how the desire of this group of individuals to influence the allocation of resources in their favor may lead to distortionary policies that reduce investment and output. I will also highlight how these problems can become more severe in the presence of commitment/holdup problems.

Section 22.4 starts the investigation of how inefficiencies in policies might translate into inefficient (economic) institutions. In particular, in this section I will show how the same forces leading to distortionary policies will affect two aspects of economic institutions, whether effective constitutional limits on taxation and expropriation arise endogenously, and whether there will be regulation (blocking) of new technologies. Economic institutions preventing future high taxes may emerge if holdup problems are important and the main source of distortionary policies is revenue extraction. In contrast, when factor price manipulation or political replacement motives are important, economic institutions limiting distortionary policies are unlikely to emerge. On the contrary, in this case, economic institutions that explicitly block the adoption of more efficient technologies may emerge. These results underlie the claim above that factor price manipulation and political replacement motives are typically more damaging to economic growth than the revenue extraction motive.

Throughout this chapter, I will focus on *comparative statics* that illustrate which types of societies are more likely to adopt growth-enhancing policies, and which others are likely to try to block economic growth. The major comparative static exercises will look at the effects of the nature of production technology, the distribution of resources within the society, whether the politically powerful compete with economically productive agents in factor markets, the extent of holdup problems, the importance of natural resources and whether or not political power is contested. While the field of the political economy of growth is still in its infancy, it is only by developing such comparative statics that it can contribute to a systematic analysis of why some societies grow and become rich, and others stagnate.

22.2. Distributional Conflict and Economic Growth in a Simple Society

The discussion in the previous section illustrated the complex set of forces that might affect collective choices concerning economic institutions and policies. The rest of this chapter and the next will focus on various dimensions of social conflict that will make societies adopt different economic institutions and policies, leading to different growth trajectories.

While my ultimate purpose is to present a relatively comprehensive framework, it is useful to start with a minimalist setup. For this reason, in this and the next two sections, I will discuss the implications of distributional conflict for economic growth in *a simple society*, in which individuals are permanently allocated to certain groups (such as producers, landowners, workers) and the main distributional conflict is among groups. The latter feature is ensured by assuming that individuals within each group are ex ante identical and by restricting the set of fiscal instruments such that it is not possible to redistribute resources from one member of the group to another (at least not along the equilibrium path). The former feature, on the other hand, rules out issues of occupational choice and social mobility, which will be discussed in the next chapter. The main advantage of a simple society for our purposes here is that, thanks for the combination of a limited set of fiscal instruments and the symmetry of individuals within social groups, it enables a tractable aggregation of political preferences of individuals. We will analyze models in which there is a non-degenerate distribution of endowments (e.g., wealth) across individuals, such as the canonical political economy model, in Section 22.7 below. While these models would be significantly richer than the simple society studied here, the economic forces that shape the political economy equilibrium will be similar, which motivates my choice of presenting a detailed analysis of political economy equilibrium in a simple society in the next few sections.

Consider a model in which there are three groups of individuals. The first is workers who supply their labor inelstically. The second group consists of entrepreneurs who have access to a production technology and make the investment decisions. The third group is the elite who make the political decisions. In particular, below I will assume that the political system is an *oligarchy*, dominated by the elite. The assumption that the elite make the political decisions, while the most important economic decisions, the level of investment, are made by entrepreneurs will highlight the impact of distributional conflict (given the set of fiscal instruments) on equilibrium policies and production in the sharpest possible way. The presence of three groups is important for the modeling of the effect of competition between the elite and other producers in the labor market in Sections 22.3 and 22.4. The model will then be enriched in various different ways in this and the next chapter by introducing additional heterogeneity, incorporating occupational choice, and also endogenizing the distribution of political power among the various members of the society.

The baseline model is designed to be as similar to the standard neoclassical model in discrete time studied in Chapters 6 and 8 as possible. The focus on the neoclassical growth model was justified above (so as to abstract from imperfections other than those due to the political economy interactions). The focus on discrete time facilitates both the exposition and the analysis of game-theoretic interactions that are inherent in political-economy situations.

22.2.1. The Basic Environment. The economy is populated by a continuum $1+\theta^e+\theta^m$ of risk neutral agents, each with a discount factor equal to $\beta<1$. There is a unique non-storable final good denoted by Y. The expected utility of agent i at time 0 is given by:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t C_i(t), \tag{22.1}$$

where $C_i(t) \in \mathbb{R}$ denotes the consumption of agent i at time t and $\mathbb{E}_t$ is the expectations operator conditional on information available at time t. The most important feature about these preferences is their linearity (risk-neutrality). The gain in simplicity from the linear preferences more than makes up for the loss of generality—linear preferences remove some interesting transitional dynamics, but in return, enable a complete characterization of the political economy equilibrium. The next section will show that concave preferences complicate the analysis even in the most basic environment and often make it impossible to obtain a theoretical characterization of equilibria.

There is a continuum of workers, with measure normalized to 1, who supply their labor inelastically. The elite, denoted by e, initially hold political power in this society. There is a total of θ^e elites. As a starting assumption, we suppose that the elite do not take part in productive activities. Political economy interactions become considerably richer and more interesting (but also somewhat more involved) when the elite are also competing with other groups in product or factor markets. This issue will be discussed in Section 22.3. Finally, there are θ^m "middle class" agents, denoted by m, who are the entrepreneurs in the economy with access to the production technology. The label of "middle class" for the entrepreneurs is motivated by some historical examples that will be discussed in the next chapter and plays no role in the formal analysis. The sets of elite and middle class producers are denoted by S^e and S^m respectively. With a slight abuse of notation, I will use i to denote either individual or group (though when referring to groups, I will use i as superscript, and when referring to individuals, as subscript). The identity of the agents (their social group membership) does not change over time.

Each entrepreneur (middle-class agent) $i \in S^m$ has access to the following production technology for producing the final good:

$$Y_i(t) = F(K_i(t), L_i(t)), \tag{22.2}$$

where $Y_i(t)$ is final output produced by entrepreneur i, $K_i(t)$ and $L_i(t)$ are the total amount of capital and labor he uses in production. We assume that F satisfies the standard neoclassical assumptions from Chapter 2, Assumptions 1 and 2, which in particular means that F exhibits constant returns to scale. Without further restrictions, a single entrepreneur can employ the entire labor force and the capital stock of the economy. In this section, whether this is the case or not has no bearing on the results. But in some of the models below, it will be important to have a dispersed distribution of entrepreneurial activity. To ensure this, we also assume that there is a maximum scale for each entrepreneur (for example, because each entrepreneur has a limited span of control when it comes to managing his employees). In particular, we impose $L_i(t) \in [0, \bar{L}]$ for some $\bar{L} > 0$. This implies that at least after a certain level of employment, there will be diminishing returns to additional capital investments by each entrepreneur. Since the total workforce in the economy is equal to 1, labor market clearing at time t in this economy requires

$$\int_{S^m} L_i(t)\, di \leq 1 \tag{22.3}$$

with $L_i(t) \leq \bar{L}$. As in the standard neoclassical model, we assume that a fraction δ of capital depreciates.

The equilibrium of this economy without "political economy" is straightforward. Imagine that there are no taxes and labor markets are competitive. Let $k \equiv K/L$ denote the capital-labor ratio as usual and $f(k)$ be the per capita production function again defined as usual (i.e., $f(k) \equiv F(K/L, 1)$). This immediately implies that each entrepreneur will choose the capital-labor ratio given by

$$k_i(t) = k^* \equiv (f')^{-1}(\beta^{-1} + \delta - 1) \tag{22.4}$$

for each t, where $(f')^{-1}(\cdot)$ is the inverse of the marginal product of capital (the derivative of the f function). Equation (22.4) is identical to the standard steady-state equilibrium condition from Chapters 6 and 8, which equates the gross marginal product of capital in steady state, $f'(k^*) + 1 - \delta$, with the inverse of the discount factor, β^{-1} (for example, recall equation (6.38) in Chapter 6). The difference here is that this equation applies at all points in time, not only in steady state. This is a consequence of linear preferences, and implies that there are no transitional dynamics; irrespective of initial conditions, each entrepreneur will immediately choose the capital-labor ratio k^* as in (22.4).

Another special feature of this economy is that it may fail to achieve full employment. Recall that the total labor force is equal to 1. However, equation (22.5) shows that the level of employment of each employer may be strictly less than $1/\theta^m$ because of the maximum size constraint on firms. When this is the case, $1 - \theta^m \bar{L}$ workers will be unemployed and wages will be equal to 0. When there is excess supply of labor, each entrepreneur $i \in S^m$ will employ $\bar{L}$ workers and total employment will fall short of the total supply. When there is no excess supply, the entire labor force will be employed and the allocation of these workers across the entrepreneurs is arbitrary (since all entrepreneurs would be making zero profits). To simplify the exposition, I assume, without loss of any generality, that even in this case, all entrepreneurs will employ the same number workers, so that the equilibrium labor allocation satisfies

$$L_i(t) = L^* \equiv \min\left\{\bar{L}, \frac{1}{\theta^m}\right\} \tag{22.5}$$

for each $i \in S^m$ at each t.

In addition, in this section I assume that

$$\theta^m \bar{L} > 1, \tag{22.6}$$

which insures that there will be full employment and thus $L^* = 1/\theta^m$. Under this assumption, the equilibrium wage rate at every date will also be given by the usual expression (which follows from Theorem 2.1 in Chapter 2)

$$w(t) = w^* \equiv f(k^*) - k^* f'(k^*), \tag{22.7}$$

where k^* is given in (22.4) above. We will refer to the equilibrium without political economy (with capital-labor ratio k^* and wage rate given by w^*) as the *first-best equilibrium* and contrast it to political economy equilibria.

22.2.2. Policies and Economic Equilibrium. Before we can characterize political equilibria, we need to specify the set of available fiscal instruments (policies), and then define an economic equilibrium for given sequences of policies. Different economic equilibria will involve different levels of welfare for different agents, thus implicitly defining induced preferences over the policies and economic institutions leading to different economic equilibria (this point is further discussed in the next chapter). The political equilibrium then aggregates these preferences over different sequences of policies, taking into account the economic equilibrium that they will induce, to arrive to collective choices. In the current model, this last step is simplified given our focus on sequences of policies that maximize the utility of the elite. The characterization of the economic equilibrium is, in turn, much simplified thanks to linear preferences. Nevertheless, it is useful to go through each of these steps in order.

As for policies, we assume that the society has access to four different policy instruments at each date t:

- a linear tax rate on output $\tau(t) \in [0, \bar{\tau}]$, where $\bar{\tau} \in (0, 1]$ is a maximum tax rate that may be imposed constitutionally or technologically (for example, when the tax rate is above this level, all activity flees into the informal sector). In this and the next two sections, we take $\bar{\tau} = 1$, so that any tax rate is allowed. We will later analyze whether "constitutional" limit on taxes may be desirable and may emerge as part of the equilibrium.
- lump-sum transfers to each of the three groups (workers, middle-class entrepreneurs and the elite), $T^w(t) \geq 0$, $T^m(t) \geq 0$, and $T^e(t) \geq 0$.

Notice that we have assumed the lump-sum transfers are nonnegative. This rules out lump-sum taxes that could raise revenues without creating distortions. Instead revenues can only be raised using the linear tax on output, which, as we will see, will be distortionary. While lump-sum taxes might sometimes be possible, the ability of individuals to move into the informal sector or stop working puts limits on the use of lump-sum taxes. Nevertheless, the restriction to a simple linear tax rate is quite restrictive and there might often exist more efficient ways of raising revenues. In political economy models such restrictions are sometimes made so as to be able to characterize the equilibrium (for example, when using the Median Voter Theorem, see Section 22.6). Here we are imposing these restrictions to emphasize how the interaction between the decoupling of political and economic power and a limited menu of fiscal instruments can lead to distortionary policies. We will return to the question of why, even with efficient means of raising revenues, political economy motives can lead to non-growth-enhancing policies (see Section 22.4).

We also need to specify the timing of events within each date, and especially we have to be specific about when taxes are set (and this is the main reason why discrete time models are slightly more convenient in this context). We assume that there is one period commitment to taxes. In other words, the timing of events is such that at each t, we start with a pre-determined tax rate on output $\tau(t)$, as well as the capital stocks $[K_i(t)]_{i \in S^m}$ of the entrepreneurs. Then entrepreneurs decide how much labor to hire $[L_i(t)]_{i \in S^m}$ (and in the process the labor market clears). Output is produced and a fraction $\tau(t)$ of the output is collected as tax revenue. After the tax revenue is observed, the political process (for example the politically powerful elite) decides the transfers, $T^w(t) \geq 0$, $T^m(t) \geq 0$, and $T^e(t) \geq 0$ subject to the government budget constraint

$$T^w(t) + \theta^m T^m(t) + \theta^e T^e(t) \leq \tau(t) \int_{S^m} F(K_i(t), L_i(t))\, di, \tag{22.8}$$

where the left-hand side denotes total government expenditure in transfers and the right-hand side is the pre-determined tax rate times the output of all entrepreneurs. Next, the political process announces the tax rate $\tau(t+1)$ that will apply at the next date and entrepreneurs, after observing this tax rate, choose their capital stocks for the next date, $[K_i(t+1)]_{i\in S^m}$. The important feature in this timing of events is that entrepreneurs know exactly what tax rate they will face when choosing their capital stock. The alternative, where the capital stock is chosen before the tax rate, will be discussed in Section 22.3. For now it suffices to say that this alternative will lead to greater distortions because of *holdup problems*. It is therefore more natural to start with the timing of events specified here.

Let us also denote the policy or tax sequence starting at time t by $p^t = \{\tau(s), T^w(s), T^m(s), T^e(s)\}_{s=t}^{\infty}$, which specifies a *feasible* infinite sequence of policies starting at time t. One has to be a little careful about feasibility here, because whether a policy sequence is feasible or not cannot be determined without reference to the actions of the entrepreneurs (for example, any policy sequence with positive transfers cannot be feasible if all entrepreneurs choose zero capital stock). For our purposes, this is not important, since with linear preferences, only the tax rate sequence matters for capital and production decisions, and the transfers can be determined as residuals to satisfy the government budget constraint (22.8). It should nonetheless be noted that each individual, in particular, each entrepreneur, is infinitesimal, thus ignores his impact on total tax revenues and on the government budget constraint.

We can now define *an economic equilibrium* from time t onwards given a pre-determined distribution of capital stock among the entrepreneurs, $[K_i(t)]_{i\in S^m}$ and a feasible policy sequence p^t. This economic equilibrium corresponds to a sequence of capital stock and labor decisions for each entrepreneur, $\left\{[K_i(s+1), L_i(s)]_{i\in S^m}\right\}_{s=t}^{\infty}$ and wage rates, $\{w(s)\}_{s=t}^{\infty}$, such that given $[K_i(t)]_{i\in S^m}$, p^t and $w^t \equiv \{w(s)\}_{s=t}^{\infty}$, $\{K_i(s+1), L_i(s)\}_{s=t}^{\infty}$ maximizes the utility of entrepreneur i for each $i \in S^m$, and such that given $\left\{[L_i(s)]_{i\in S^m}\right\}_{s=t}^{\infty}$, the labor market clears. While an economic equilibrium appears as a complicated object, the linearity of the preferences leads to a major simplification, enabling us to focus on the main political economy interactions.

Since workers supply labor inelastically, the only nontrivial decisions are by the entrepreneurs. As a first step towards characterizing the equilibrium, note that given *any* feasible policy sequence p^t and equilibrium wages w^t, the utility of an entrepreneur starting with capital stock $K_i(t)$ at time t as a function of these policies can be written as

(22.9)

$$\begin{aligned}\mathbf{U}_i\left(\{K_i(s), L_i(s)\}_{s=t}^{\infty} \mid p^t, w^t\right) &= \sum_{s=t}^{\infty} \beta^{s-t}[(1-\tau(s))F(K_i(s), L_i(s)) \\ &\quad -(K_i(s+1)-(1-\delta)K_i(s)) - w(s)L_i(s) + T^m(s)].\end{aligned}$$

This expression makes use of the fact that preferences are linear, thus the value of the entrepreneur can be written simply in terms of the discounted sum of his consumption. His consumption, on the other hand, is simply given by the term in square brackets, since output is taxed at the rate $\tau(t)$ at time t and moreover, a fraction $(1-\delta)$ of last period's capital stock is left, so an additional investment of $(K_i(t+1)-(1-\delta)K_i(t))$ is made for next period. Finally, the labor costs at the current wage are subtracted and the lump-sum transfer to middle-class entrepreneurs is added. A special feature concerning (22.9) should be noted. It is formulated for a given sequence of policies p^t. Loosely speaking, this could be thought of as the case if the sequence of policies were specified and committed to at some date. Although we are interested in political economy equilibria, where there is no commitment to future policies, we can think of the sequence of policies p^t as given from the viewpoint of an individual entrepreneur. Nevertheless, this way of writing the maximization problem of the entrepreneur does not give information about how he would react if the political process (here the elite) deviated from p^t, since this might also be associated with a change in the remainder of the policy sequence. Nevertheless, linear preferences again ensure that we do not need to worry about this issue, since, as we will see momentarily, entrepreneurial decisions will only depend on current taxes. This issue of off-the-equilibrium path behavior becomes important when preferences are not linear and will be discussed in the next section.

Maximizing (22.9) with respect to the sequences of capital stock and labor choices, we obtain the following simple first-order condition:

$$\beta\left[(1-\tau(t+1))f'(k_i(t+1))+(1-\delta)\right]=1, \tag{22.10}$$

where $k_i(t+1)$ denotes the capital-labor ratio chosen by entrepreneur i for time $t+1$ given the tax rate $\tau(t+1)$, which has already been announced (and committed to) at the time of the investment decision. Thanks to the Inada conditions in Assumption 2, this first-order condition holds as equality for any $\tau(t+1)\in[0,1)$ and Exercise 22.1 shows that there will never be 100% taxation. Thus we do not need to spell out complementary slackness conditions.

Equation (22.10) determines the equilibrium capital-labor ratio. Given (22.6), i.e., $\theta^m\bar{L}>1$, there will be full employment of the total mass 1 of workers, and thus the total capital stock is also given by (22.10).

It can be verified easily that if all taxes were equal to zero, i.e., $\tau(t)=0$ for all t, the unique solution to (22.10) would be identical to the steady-state capital-labor ratio k^* in (22.4) given in the previous subsection. Naturally, when there are positive taxes, the level of capital-labor ratio will be less than k^* (this follows immediately since $f(\cdot)$ is strictly concave; see (22.12) below). It is also worth noting that while in the equilibrium "without political economy" (i.e., without taxes and transfers) the capital stock exhibited no transitional dynamics and immediately jumped to its state-state value, this may no longer be the case in an economic equilibrium given the policy sequence p^t, since the policy sequence may involve time-varying taxes.

The most noteworthy feature of the equilibrium capital-labor ratio given in (22.10) is that, thanks to linear preferences, the choice of the capital-labor ratio by each entrepreneur at time $t+1$ only depends on the tax rate $\tau(t+1)$, and not on future taxes. We can therefore write the equilibrium capital-labor ratio at time t for all entrepreneurs as $\hat{k}(\tau(t))$ or as $\hat{k}(\tau)$:

$$\hat{k}(\tau) \equiv (f')^{-1}\left(\frac{\beta^{-1}+\delta-1}{1-\tau}\right). \tag{22.11}$$

The fact that the equilibrium capital-labor ratio depends only on one tax rate and is the same for all entrepreneurs will simplify the analysis of political economy considerably. For future reference, note also that since $F(\cdot,\cdot)$, and thus $f(\cdot)$, is twice continuously differentiable, $\hat{k}(\tau)$ is also differentiable, with derivative

$$\hat{k}'(\tau) = \frac{f'\left(\hat{k}(\tau)\right)}{(1-\tau) f''\left(\hat{k}(\tau)\right)} < 0, \tag{22.12}$$

which follows by directly differentiating (22.11) and is negative in view of the fact that $f'(k) > 0$ and $f''(k) < 0$ for all k (from Assumption 1).

Given the expression for the equilibrium capital-labor ratio in (22.11) and full employment as implied by (22.6), the equilibrium wage at time t is given by the usual expression:

$$\hat{w}(\tau(t)) = (1-\tau(t))\left[f\left(\hat{k}(\tau(t))\right) - \hat{k}(\tau(t)) f'\left(\hat{k}(\tau(t))\right)\right], \tag{22.13}$$

which is similar to (22.7) except for the presence of the tax rate in the front.

While (22.11) gives a very simple expression for the capital stock as a function of the tax rate, without knowing more about the sequence of policies we cannot ascertain whether the sequence of equilibrium capital stocks will converge to some steady-state value (for example, this will not be the case if taxes periodically fluctuate between different levels). Nevertheless, our analysis so far has established the following proposition:

Proposition 22.1. *Suppose that (22.6) holds. Then for any initial distribution of capital stocks among entrepreneurs, $[K_i(0)]_{i\in S^m}$ and for any feasible sequence of policies, $p^t = \{\tau(s), T^w(s), T^m(s), T^e(s)\}_{s=0}^{\infty}$, there exists a unique equilibrium in which the sequence of capital-labor ratios for each entrepreneur is $\left\{\hat{k}(\tau(s))\right\}_{s=0}^{\infty}$ and the equilibrium wage sequence is $\{\hat{w}(\tau(s))\}_{s=0}^{\infty}$ where $\hat{k}(\tau(t))$ is given by (22.11) and $\hat{w}(\tau(t))$ is given by (22.13).*

This proposition is convenient not only because the form of the equilibrium is particularly simple, but also because for any given sequence of policies, the aggregate equilibrium allocation is unique. If some policy sequences led to multiple equilibrium allocations, then expectations concerning which of these equilibrium would be played conditional on these policy choices would complicate the analysis.[1]

22.2.3. Political Economy under Elite Control. As noted at the beginning of this section, our task of characterizing the political economy equilibrium here is considerably simplified by the assumption that political power is entirely in the hands of the elite. There is no issue of political power changing hands or the elites choosing policies in order to appease voters or other social groups. Moreover, there are no fiscal instruments that would redistribute income among the elite. Thus the political economy choices here just involve the choice of fiscal policies at each date that would maximize the net present discounted utility of a representative elite agent.

Throughout this section, we will look at the political equilibria using the notion of *Markov Perfect Equilibria* (MPE). This notion, which was first used in Chapter 14, requires the policy sequence p^t be such that policies dated t only depend on the dated t *payoff-relevant* variables (see the Appendix Chapter C for a formal definition of MPE). Here the only payoff relevant variables are the capital stocks of the entrepreneurs. Thus most generally, current policies can depend on the current distribution of capital stocks. Linear preferences again simplify the analysis, and imply that we do not need to keep track of a complicated distribution of capital stocks as the relevant state variable. Since the MPE are a subset of the Subgame Perfect Equilibria (SPE) that do not condition on past history except through payoff-relevant variables, they rule out repeated game punishments (such as those relying on trigger strategies). We return to a discussion of situations in which the SPE are different from the MPE below. We will also refer to the MPE in this chapter as Markov Perfect Political Economy Equilibrium.

For the characterization of the political economy choices of the elite, recall that the elite also care about their level of consumption (discounted with the discount factor β). From this observation, it is straightforward to see that they would never choose to redistribute to workers or to the middle class, thus in what follows we can restrict attention to sequences of policies that involve $T^w(t) = T^m(t) = 0$ for all t. Next, let us combine this fact with the government budget constraint, (22.8), which must hold as equality (since otherwise the elite

[1]Notice the slight abuse of notation in this proposition, which I will make throughout this and the next chapter: the equilibrium would not be "unique" in general, since the allocation of capital and labor across middle-class entrepreneurs is not pinned down. What is typically unique is the aggregate allocation and also the fact that any entrepreneur who is active must have a capital-labor ratio of $\hat{k}(\tau(t))$ at time t. In the present context, "uniqueness" is achieved by the assumption above that, when indifferent, all firms employ the same amount of labor. Throughout this chapter when the issue arises again, rather than explicitly state that the aggregate allocation implied by the equilibrium is unique, I will refer to the equilibrium as "unique".

could increase their consumption and utility by increasing transfers to themselves), to obtain

$$
\begin{aligned}
T^{e}(t) &= \frac{1}{\theta^{e}}\tau(t)\int_{S^{m}} F\left(K_{i}(t), L_{i}(t)\right) di \\
&= \frac{1}{\theta^{e}}\tau(t) f\left(\hat{k}(\tau(t))\right),
\end{aligned}
\tag{22.14}
$$

where the first line simply uses the government budget constraint (22.8), while the second line uses the equilibrium characterization in Proposition 22.1 together with the fact that with full employment, the total number of workers is equal to 1.

The problem of maximizing the utility of the elite agents can then be written as

$$
\max_{\{\tau(t), T^{e}(t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^{t} T^{e}(t)
\tag{22.15}
$$

subject to (22.14) at each t. Notice again that although it appears from (22.15) as if the elite were choosing the tax sequences at date $t = 0$, since there is no commitment to future policies, they are in fact only setting taxes for time $t+1$ at time t. But this way of writing the elite's program will characterize the MPE since middle-class entrepreneurs capital-labor ratio decisions at time $t+1$ only depend on the tax rate announced for time $t+1$, and not on future or past taxes.

To characterize the equilibrium tax sequence, note that $T^{e}(t)$ only depends on the tax rate at time t and involves the choice of the tax rate that would maximize tax revenue (i.e., that would put the elite at the peak of the "Laffer curve") at each date. Then the utility-maximizing tax rate for the elite, $\hat{\tau}$, can be obtained as a solution to the following first-order condition:

$$
f\left(\hat{k}(\hat{\tau})\right) + \hat{\tau} f'\left(\hat{k}(\hat{\tau})\right) \hat{k}'(\tau) = 0.
$$

Substituting for $\hat{k}'(\tau)$ from (22.12), we obtain the following expression for $\hat{\tau}$:

$$
f\left(\hat{k}(\hat{\tau})\right) + \frac{\hat{\tau}}{1-\hat{\tau}} \frac{\left(f'\left(\hat{k}(\hat{\tau})\right)\right)^{2}}{f''\left(\hat{k}(\hat{\tau})\right)} = 0.
\tag{22.16}
$$

Intuitively, the utility-maximizing tax rate for the elite trades off the increase in revenues resulting from a small increase in the tax rate, $f\left(\hat{k}(\hat{\tau})\right)$, against the loss in revenues that will result because the increase in the tax rate will reduce the equilibrium capital-labor ratio, $\hat{\tau} f'\left(\hat{k}(\hat{\tau})\right) \hat{k}'(\tau)$. It can be verified that this tax rate $\hat{\tau}$ is always between 0 and 1 (see Exercise 22.1), though the maximization problem of the elite is not necessarily concave and (22.16) may have more than one solution. If this is the case, we always refer to the solution $\hat{\tau}$ that corresponds to a global maximum for the elite.

Notice that (22.16) implies a constant tax rate across different dates, and moreover, if we were to consider the maximization problem of the elite (22.15) after some arbitrary date t', exactly the same tax sequence would result. This is the reason why we could, without loss of any generality, focus on the maximization problem in (22.15). We will see in the next section that this is not always the case and we need to take into account the sequential nature of the decision-making by the elite and the entrepreneurs.

This analysis so far has thus established the following characterization of the political economy equilibrium:

Proposition 22.2. *Suppose that (22.6) holds. Then for any initial distribution of capital stocks among entrepreneurs, $[K_i(0)]_{i \in S^m}$, there exists a unique Markov Perfect Political Economy Equilibrium, where at each $t = 0, 1, ...,$ the elite sets the tax $\hat{\tau} \in (0, 1)$ as given in (22.16), and all entrepreneurs choose the capital-labor ratio $\hat{k}(\hat{\tau})$ as given by (22.11) and the equilibrium wage rate is $\hat{w}(\hat{\tau})$ as given by (22.13). We have that $\hat{k}(\hat{\tau}) < k^*$, where k^* is given by (22.4) and $\hat{w}(\hat{\tau}) < w^*$, where w^* is given by (22.7).*

This proposition shows that a unique well-defined political equilibrium exists and involves positive taxation of entrepreneurs by the elite. Consequently, the capital-labor ratio and the wage rate are strictly lower than they would be in an economy without taxation. Strictly speaking, the equilibrium distortionary policies do not change "growth," since we are focusing on a neoclassical economy without technological progress. As explained above, this is the only for convenience and it is straightforward to extend the framework here to incorporate endogenous growth, so that the distortionary policies do affect the equilibrium growth rate of the economy (see Exercise 22.2).

We can now return to the fundamental question raised at the beginning of this chapter: why would a society impose distortionary taxes on businesses/entrepreneurs? The model in this section gives a simple answer. Political power is in the hands of the elite, who would like to extract revenues from the entrepreneurs. Given the available tax instruments, here linear taxes on output, the only way they can achieve this is by imposing distortionary taxes. Thus the source of "inefficiencies" in this economy is the combination of *revenue extraction* motive by the politically powerful combined with *a limited menu of fiscal instruments.*

While the analysis so far shows how distortionary policies can emerge and reduce the level of investment and output below the "first-best" level, it is important to emphasize that the equilibrium here is *not* Pareto inefficient. In fact, given the set of fiscal instruments,the equilibrium allocation is the solution to maximizing a social welfare function that puts all the weight on the elite. Pareto inefficiency requires that, *given the set of instruments and informational constraints*, there should exist an alternative feasible allocation that would make each agent either better-off or at least as well of as they were in the initial allocation. Such an allocation can be found if we allowed lump-sum taxes. But given the restriction to linear taxes, which in the current economy are "technological," there is no way of improving the utility of the middle-class entrepreneurs and the workers without making the elite worse-off.[2] This is an important observation, since it implies that when we explicitly incorporate political economy aspects into the analysis, there are typically no "free lunches"—that is, no way of making all agents better-off. This is the reason why political economy considerations typically involve tradeoffs between losers and winners in the process of various different changes in institutions and policies. Since the allocation in Proposition 22.2 involves distortionary policies and reduces output relative to the first-best allocation, we might want to refer to this outcome as "inefficient" (despite the fact that it is not "Pareto inefficient"). In fact, this label is often used for such allocations in the literature and I will follow this practice. When using these terms, it is important to bear in mind that such "inefficiencies" do not mean Pareto inefficiencies.

As a preliminary answer, Proposition 22.2 is a useful starting point. However, it leaves a number of important questions unanswered. First, it does not provide useful comparative statics regarding when we should expect higher rates of distortion of the taxes. Second, it takes the distribution of political power as given, and it appears important for the results that political power rests in the hands of the non-productive elite, who are using the fiscal instruments to extract resources from middle-class producers. If political power were in the hands of the middle-class entrepreneurs rather than the non-productive elite, the choice of fiscal instruments would be very different. A first intuition might be that the entrepreneurs would never tax themselves. However, Exercise 22.3 shows that this is not necessarily the case and the middle-class entrepreneurs may prefer to tax themselves as a way of indirectly changing equilibrium wages. This is important because the role of fiscal policies in changing factor prices is often underappreciated and we will argue in Section 22.3 that it is one of the more important sources of political distortions in the process of growth. Third, this analysis takes the menu of available fiscal instruments as given. If the elite had access to lump-sum taxes, it could extract revenue from the entrepreneurs without creating distortions. We will extend the current framework to provide answers to all of these questions in this and the next chapter. Before doing this, we will first consider a more specific version of the economy analyzed so far where the production function is Cobb-Douglas. This Cobb-Douglas economy, by virtue of its tractability, will be a workhorse model for our analysis in Sections 22.3 and 22.4. Finally, in Section 22.5, we will consider a generalized version of the environment here where individuals have concave preferences.

[2]In a slightly modified environment there exist "mechanisms" that would lead to Pareto improvements, but these mechanisms could not be supported as MPE (but could be supported as SPE). For example, suppose that there is a finite number of entrepreneurs, who can make voluntary donations to a pot of money that is then redistributed to the elite. It can be proved that as the discount rate approaches 1, there exists an SPE in which each entrepreneur makes sufficient donations and chooses the first-best capital-labor ratio and the elite refrain from distortionary taxation. This example suggests that the MPE could easily lead to Pareto inefficient equilibria, even though this is not the case in our baseline economy.

22.2.4. The Canonical Cobb-Douglas Model of Distributional Conflict. Consider a slightly specialized version of the economy analyzed so far, with two differences. First, the production function of each entrepreneur takes the following Cobb-Douglas form:

$$Y_i(t) = \frac{1}{\alpha}(K_i(t))^{\alpha}(A_i(t)\,L_i(t))^{1-\alpha}, \tag{22.17}$$

where $A_i(t)$ is a labor-augmenting group-specific or individual-specific productivity term, which will be used later in this chapter. For now, we can set $A_i(t) = A^m$ for all $i \in S^m$. The term $1/\alpha$ in the front is included as a convenient normalization. We will see that the Cobb-Douglas form will enable an explicit-form characterization of the political equilibrium and will also link the elasticity of output with respect to capital to the equilibrium taxes. This is the reason why I refer to this model is the "canonical model" of distributional conflict. Second, the analysis so far has shown that with linear preferences, incomplete depreciation of capital plays no qualitative role, so I will also simplify the notation by assuming full depreciation of capital, i.e., $\delta = 1$. This assumption is without any substantive implications.

The Cobb-Douglas production function in (22.17) implies that the per capita production function is given by

$$f(k_i) = \frac{1}{\alpha}(A^m)^{1-\alpha}\,k_i^{\alpha}.$$

Combining this production function with the assumption that $\delta = 1$, equation (22.10) above implies shows that at date $t+1$ each entrepreneur will choose a capital-labor ratio $k(t+1)$ such that

$$k(t+1) = [\beta(1-\tau(t+1))]^{1/(1-\alpha)}\,A^m. \tag{22.18}$$

The utility-maximizing tax policy of the elite is still given by equation (22.16), which combined with the Cobb-Douglas form here implies that the utility-maximizing tax for the elite at each date is given by

$$\hat{\tau} = 1-\alpha.$$

This formula is both simple and economically intuitive. When α is high, the production function is nearly linear in capital. This implies that the demand for capital as a function of its effective price is highly elastic. With such an elastic demand for capital, high taxes would lead to a large decline in the capital stock. Thus by charging high taxes, the elite would be reducing their own revenues. Put differently, with an elastic demand for capital (which in turn follows from production function that is not very concave in capital), the peak of the Laffer curve for the elite is at a low tax rate. On the other hand, if α is low, the production function is highly concave in capital, thus even a significant tax rate will not lead to a large decline in the equilibrium capital-labor ratio choice of the entrepreneurs. In this case, the elite will find it profitable to charge very high taxes.

Both the tractability afforded by the Cobb-Douglas production function and the link between the concavity of the production function and equilibrium taxes that it highlights make this a very useful framework, which we will next use in a number of applications below.

22.3. Distributional Conflict and Competition

In this and the next section, I will use the canonical framework with Cobb-Douglas production functions and full depreciation of capital ($\delta = 1$) to illustrate two important issues. I will first investigate how competition (in the marketplace or in the political arena) between those with political power and the rest can lead to significantly more distortionary policies than the revenue extraction motive discussed so far in this chapter. In the next section I will use the same framework to derive some preliminary insights on how distributional conflict can provide perspectives on equilibrium economic institutions regulating the formation of policies.

The model and the setup are essentially identical to the canonical Cobb-Douglas model in Section 22.2. In particular, the elite, of size θ^e, are in political power and decide all the policies. The timing of events is identical to that in Section 22.2. There are three differences. First, the elite as well as the middle class can become entrepreneurs. The productivity of each middle-class agent in terms of this production function is A^m (i.e., $A_i = A^m$ for all $i \in S^m$) and the productivity of each elite agent is A^e (i.e., $A_i = A^e$ for all $i \in S^e$). Productivity of the two groups may differ, for example, because they are engaged in different economic activities (e.g., agriculture versus manufacturing, old versus new industries), or because they have different human capital or talent. Workers do not have access to these production functions and supply their labor inelastically. As in Section 22.2, each entrepreneur can hire at most $\bar{L}$ workers, and we no longer impose (22.6). Second, I reintroduce the constitutional maximum on the tax rate, $\bar{\tau}$, so that $\tau(t) \in [0, \bar{\tau}]$ for all t. Finally, I now allow group-specific taxes so that the elite will choose two tax rates, $\tau^e(t)$, applying to the output of elite entrepreneurs, and $\tau^m(t)$, applying to middle-class entrepreneurs. The government budget constraint then takes the form

$$T^w(t) + \theta^m T^m(t) + \theta^e T^e(t) \leq \phi \int_{S^m \cup S^e} \tau^i(t) F(K_i(t), L_i(t))\, di + R^N \tag{22.19}$$

where $\phi \in [0, 1]$ is a parameter that captures how much of tax revenue can be redistributed (with the remaining $1 - \phi$ being wasted). This parameter can be thought of as a measure of "state capacity"—with high ϕ, the state has the capacity to raise and redistribute significant revenues. R^N denotes rents from natural resources or from other sources unrelated to the production activities of the elite and the middle class. In Section 22.2, the government budget constraint, (22.8) involved $\phi = 1$ and $R^N = 0$. These parameters will be useful for comparative static exercises below.

Since there are entrepreneurs both from the elite and the middle class, the condition for full employment is different from (22.6). In particular, we assume throughout that $\theta^e \bar{L} < 1$ and $\theta^m \bar{L} < 1$, so that neither of the two groups generates enough labor demand by itself to employ the entire labor force. The following condition then determines whether the elite and the middle class together will generate enough labor demand for the entire labor force:

Condition 22.1. $(\theta^e + \theta^m)\, \bar{L} > 1$.

When this condition holds, there will be full employment. When it does not (by which I mean $(\theta^e + \theta^m)\, \bar{L} < 1$, excluding the knife-edge case $(\theta^e + \theta^m)\, \bar{L} = 1$, where there could be multiple equilibrium wage levels), there is a shortage of labor demand and equilibrium wages will be equal to 0. Whether this condition holds or not will affect the nature of the political equilibrium.

The analysis in Section 22.2, in particular, equation (22.18), implies that the capital-labor ratio choice of each entrepreneur $i \in S^m \cup S^e$ will be given by

$$k_i\,(t+1) = \hat{k}_i\,(\tau\,(t+1)) \equiv (\beta\,(1-\tau\,(t+1)))^{1/(1-\alpha)}\, A_i, \tag{22.20}$$

where the expression $\hat{k}_i\,(\tau)$ is implicitly defined by the second equality. This expression is very similar to equation (22.11), but is adapted to the Cobb-Douglas production function, with labor-augmenting productivity of entrepreneur i equal to A_i. Substituting $\hat{k}_i\,(\tau)$ into the production function for each entrepreneur and subtracting the cost of investment, we obtain that the net revenue per worker is $(1-\alpha)\,(\beta\,(1-\tau\,(t+1)))^{1/(1-\alpha)}\, A_i/\alpha$. This implies that the labor demand for each entrepreneur at time t as a function of the wage rate $w\,(t)$ will take the form

$$L_i\,(t) \begin{cases} = 0 & \text{if } w\,(t) > (1-\alpha)\,(\beta\,(1-\tau\,(t+1)))^{1/(1-\alpha)}\, A_i/\alpha \\ \in \left[0, \bar{L}\right] & \text{if } w\,(t) = (1-\alpha)\,(\beta\,(1-\tau\,(t+1)))^{1/(1-\alpha)}\, A_i/\alpha \\ = \bar{L} & \text{if } w\,(t) < (1-\alpha)\,(\beta\,(1-\tau\,(t+1)))^{1/(1-\alpha)}\, A_i/\alpha \end{cases}. \tag{22.21}$$

This expression states that if the wage exceeds the net profitability for the entrepreneur, the entrepreneur would hire zero labor and shut down the firm. If the wage is strictly less than this net profitability, then the entrepreneur would like to hire up to the maximum possible amount of labor, $\bar{L}$. The following proposition is immediate:

Proposition 22.3. *Consider the canonical elite-dominated politics model with Cobb-Douglas technology. Let the taxes on output of the elite and middle-class entrepreneurs at time t be $\tau^e\,(t)$ and $\tau^m\,(t)$, then the equilibrium capital-labor ratio of each entrepreneur is uniquely given by (22.20). In addition, if Condition 22.1 holds, then the equilibrium wage at time t is*

$$w\,(t) = \min\left\langle \frac{1-\alpha}{\alpha}\,(\beta(1-\tau^e\,(t)))^{1/(1-\alpha)}\, A^e, \frac{1-\alpha}{\alpha}\,(\beta(1-\tau^m\,(t)))^{1/(1-\alpha)}\, A^m \right\rangle. \tag{22.22}$$

If Condition 22.1 does not hold, then $w\,(t) = 0$.

22.3.1. Competition in the Marketplace: The Factor Price Manipulation Effect. The next proposition is the equivalent of Proposition 22.2, except that it now applies when Condition 22.1 fails to hold. The reason for this is that, when this condition holds, there will also be the competition effect, changing the policy preferences of the elite. Propositions 22.4 and 22.5 below will focus on the implications of competition in the factor market.

Proposition 22.4. *Consider the canonical elite-dominated politics model with Cobb-Douglas technology. Suppose that Condition 22.1 does not hold and $\phi > 0$, then the unique Markov Perfect Political Economy Equilibrium features*

$$\tau^m(t) = \tau^{RE} \equiv \min\{1-\alpha, \bar{\tau}\} \text{ and } \tau^e(t) = T^m(t) = T^w(t) = 0 \tag{22.23}$$

for all t. $T^e(t)$ is then determined from (22.19) holding as equality.

Proof. See Exercise 22.4. □

This proposition thus shows that as in the version of the economy with Cobb-Douglas technology discussed in Section 22.2, the elite would like to set a tax rate of $1-\alpha$ on middle-class entrepreneurs. If this tax is less than the constitutionally allowed maximum $\bar{\tau}$, the political equilibrium will involve $\tau^m = 1-\alpha$. If on the other hand, $\bar{\tau} < 1-\alpha$, the utility maximizing tax rate for the elite is $\tau^m = \bar{\tau}$ (this follows because the maximization program of the elite is strictly concave, see Exercise 22.4). Notice, however, that this proposition is stated under the assumption that Condition 22.1 fails to hold—so that the equilibrium wage rate is $w(t) = 0$ for all t. If this were not the case, the elite would also recognize the effect of their taxation policy on equilibrium wages. This would introduce the competition motive in the choice of policies, which is our next focus. An extreme form of this competition effect is shown in the next proposition. The state this proposition, I introduce one more condition:

Condition 22.2. *The maximum tax rate $\bar{\tau}$ is such that $A^e > (1-\bar{\tau}))^{1/(1-\alpha)} A^m$.*

The role of this condition will be discussed below.

Proposition 22.5. *Consider the canonical elite-dominated politics model with Cobb-Douglas technology. Suppose that Condition 22.1 and 22.2 hold and $\phi = 0$, then the unique Markov Perfect Political Economy Equilibrium features $\tau^m(t) = \tau^{FPM} \equiv \bar{\tau}$ and $\tau^e(t) = T^m(t) = T^w(t) = 0$ for all t.*

Proof. See Exercise 22.5. □

In this case, ϕ is set equal to 0, so that there is no revenue extraction motive in taxation. Instead, the only reason why the elite might want to use taxes is in order to affect the equilibrium wage rate as given in (22.22). Clearly, for this we need Condition 22.1 to hold; otherwise, the wage rate would be equal to zero and there would be known motive to manipulate factor prices. Condition 22.2 is necessary, since otherwise even at the maximal tax rate $\bar{\tau}$, the middle class entrepreneurs are more productive than the elite and the elite make zero profits. The noteworthy conclusion of Proposition 22.5 is that the equilibrium tax rate in this case, τ^{FPM}, is greater than the tax rate when the only motive for taxation was revenue extraction (τ^{RE}). This might at first appear paradoxical, but is quite intuitive. With the factor price manipulation mechanism, the objective of the elite is to reduce the profitability of the middle class as much as possible, whereas for revenue extraction, the elite would like the middle class to invest and generate revenues. Consequently, τ^{RE} puts the elite at the top of the Laffer curve, while τ^{FPM} tries to harm middle-class entrepreneurs as much as possible so as to reduce their labor demand (and thus equilibrium wages). It is also worth noting that, differently from the pure revenue extraction case, the tax policy of the elite is indirectly extracting resources from the workers, whose wages are being reduced because of the tax policy.

The role of the assumption that $\phi = 0$ in this context also needs to be emphasized. Taxing the middle class at the highest rate is clearly inefficient. Why is there not a more efficient way of transferring resources to the elite? The answer again relates to the limited fiscal instruments available to the elite. In particular, $\phi = 0$ implies that they cannot use taxes to extract revenues from the middle class, so they are forced to use inefficient means of increasing their consumption, by directly impoverishing the middle class. The absence of any means of transferring resources from the middle class to the elite is not essential for the factor price manipulation mechanism, however. This will be illustrated next by combining the factor price manipulation motive with revenue extraction (though the absence of non-distortionary lump-sum taxes is naturally important).

Naturally, the assumption that $\phi = 0$ is extreme. The next proposition derives the equilibrium when Condition 22.1 holds and $\phi > 0$, so that both the factor price manipulation and the revenue extraction motives are present. Proposition 22.5 showed that by itself the factor price manipulation motive leads to the extreme result that the tax on the middle class should be as high as possible. Revenue extraction, though typically another motive for imposing taxes on the middle class, will serve to reduce the power of the factor price manipulation effect. The reason is that high taxes also reduce the revenues extracted by the elite (moving the economy *beyond the peak* of the Laffer curve), and are costly to the elite. To derive the political equilibrium in this case, first note that the elite will again not tax themselves, i.e., $\tau^e(t) = 0$ for all t. Next the maximization problem of the elite at time $t-1$ for setting the tax rate $\tau^m(t)$ can be written as:

(22.24)

$$\max_{\tau^m(t)} \left[\frac{1-\alpha}{\alpha}\beta^{1/(1-\alpha)}A^e - w(t)\right] L^e(t) + \frac{1}{\theta^e}\left[\frac{\phi}{\alpha}\tau^m(t)\left(\beta(1-\tau^m(t))\right)^{\alpha/(1-\alpha)} A^m\theta^m L^m(t) + R^N\right],$$

subject to (22.22) and

$$\theta^e L^e(t) + \theta^m L^m(t) = 1, \text{ and} \tag{22.25}$$

$$L^m(t) = \bar{L} \qquad \text{if} \quad (1-\tau^m(t))^{1/(1-\alpha)} A^m \geq A^e, \tag{22.26}$$

where $L^m(t)$ denotes equilibrium employment by a middle-class entrepreneur and $L^e(t)$ is equilibrium employment by an elite entrepreneur. The first term in (22.24) is the elite's net revenues and the second term is the transfer they receive. Equation (22.25) is the labor market clearing constraint, while (22.26) ensures that middle class producers employ as much labor as they wish provided that their net productivity is greater than those of elite producers.

The solution to this problem can take two different forms depending on whether (22.26) holds at the optimal solution. If it does, then $w = (1-\alpha)\beta^{1/(1-\alpha)} A^e/\alpha$, and elite producers make zero profits and their only income is derived from transfers. Intuitively, this corresponds to the case where the elite prefer to let the middle class producers undertake all of the profitable activities and maximize tax revenues. In this case, the equilibrium will be clearly identical to that in Proposition 22.4. If, on the other hand, (22.26) does not hold, then the elite generate revenues both from their own production and from taxing the middle class producers. In this case, the equilibrium wage will be $w(t) = (1-\alpha)(\beta(1-\tau^m(t)))^{1/(1-\alpha)} A^m/\alpha$. The next proposition focuses on this case:

Proposition 22.6. *Consider the canonical elite-dominated politics model with Cobb-Douglas technology. Suppose that Condition 22.1 holds, $\phi > 0$, and*

$$A^e \geq \phi \alpha^{\alpha/(1-\alpha)} A^m \frac{\theta^m}{\theta^e}. \tag{22.27}$$

Then the unique Markov Perfect Political Economy Equilibrium features

$$\tau^m(t) = \tau^{COM} \equiv \min\left\{ \frac{\kappa(\bar{L}, \theta^e, \alpha, \phi)}{1 + \kappa(\bar{L}, \theta^e, \alpha, \phi)}, \bar{\tau} \right\}, \tag{22.28}$$

for all t, where

$$\kappa(\bar{L}, \theta^e, \alpha, \phi) \equiv \frac{1-\alpha}{\alpha}\left(1 + \frac{\theta^e \bar{L}}{(1-\theta^e \bar{L})\phi}\right). \tag{22.29}$$

Proof. See Exercise 22.6. □

A number of features about this proposition are worth noting. First, $\kappa(\bar{L}, \theta^e, \alpha, \phi)$ is always less than ∞, so that the most preferred tax rate by the elite is always less than 1. Recall that with the pure factor price manipulation motive, the elite preferred a tax rate of 100% (though their actual tax policy may have been constrained by $\bar{\tau}$). Proposition 22.6 therefore shows that the prospect of raising revenues from the middle class reduces the desired tax rate by the elite. On the other hand, $\kappa(\bar{L}, \theta^e, \alpha, \phi)$ is always strictly greater than $(1-\alpha)/\alpha$, so that τ^{COM} is always greater than $1-\alpha$, the desired tax rate with pure resource extraction. Therefore, the factor price manipulation motive always increases taxes above the pure revenue maximizing level, and thus beyond the peak of the Laffer curve (though never to as high as 100%). Naturally, if this level of tax is greater than $\bar{\tau}$, the equilibrium tax will be $\bar{\tau}$.

Second, since Proposition 22.6 incorporates both the revenue extraction and the factor price manipulation motives, it contains the main comparative static results of interest for us. First, the equilibrium tax rate is *decreasing* in ϕ, because as ϕ increases, revenue extraction becomes more efficient and this has a moderating effect on the tax preferences of the elites. Loosely speaking, this shows the positive side of "state capacity"; with greater state capacity, the politically powerful can raise revenues through taxation, thus their motives to impoverish competing groups become weaker (we will see a potentially negative or "dark" side of state capacity below). Second, the equilibrium tax rate is increasing in θ^e. The reason for this is again the interplay between the revenue extraction and factor price manipulation mechanisms. When there are more elite producers, reducing factor prices becomes more important relative to raising tax revenues. This comparative static thus reiterates that when the factor price manipulation effect is more important, there will typically be greater distortions. Third, a decline in α raises equilibrium taxes for exactly the same reason as in the pure revenue extraction case; taxes create fewer distortions and this increases the revenue-maximizing tax rate. Finally, for future reference, note that rents from natural resources, R^N, have no effect on equilibrium policies.

22.3.2. Political Competition: The Political Replacement Effect. The previous subsection illustrated how competition in the factor market between the elite and the middle class induces the elite to choose distortionary policies to reduce the labor demand from the middle class. In this section, I will briefly outline the implications of competition in the political arena for equilibrium taxes. The main difference from the models studied so far is that we will allow for endogenous switches of political power. Institutional change and the implications of different political regimes for economic growth will be discussed in greater detail in the next chapter. For now, let us denote the probability that in period t political power permanently shifts from the elite to the middle class by $\eta(t)$. Once they come to power, the middle class will pursue the policies that maximize their own utility. We can easily derive what these policies would be using the same analysis as in the previous subsection. Since the analysis is identical to that above, this is left to Exercise 22.7. Denote the utility of the elite when they are in control of politics and when the middle class are in control of politics by $V^e(E)$ and $V^e(M)$ respectively.

When the probability of the elite losing power to the middle class, η, is exogenous, the analysis in the previous subsection applies without any significant change. New political economy effects arise, however, when the probability that the elite will lose power is endogenous. To save space while communicating the main ideas, I use a very reduced-form model and assume that the probability that the elite will lose power to the middle class is a function of the net income level of the middle class, in particular,

$$\eta(t) = \eta(\theta^m C^m(t)) \in [0,1], \tag{22.30}$$

where $C^m(t)$ is the net income of a representative middle-class entrepreneur, which is also equal to his consumption. I assume that η is continuously differentiable and strictly increasing, with derivative $\eta'(\cdot) > 0$. This assumption implies that when the middle class producers are richer, they are more likely to gain power, which may be because with greater resources, they may be more successful in solving their collective action problems or they may increase their military power. The assumption that when the middle class are better-off they are more likely to replace the elite is not just reduced-form but also only approximates certain situations. One can also imagine environments in which groups that are better-off are less likely to take action against the existing regime. This issue will be discussed in the next chapter (using a more microfounded model) in the context of endogenous changes in political institutions.

To simplify the discussion, let us focus on the case in which Condition 22.1 fails to hold, so that equilibrium wage is equal to 0 and there is no factor price manipulation motive. Thus in the absence of the political replacement motive, the only reason for taxation will be revenue extraction (resulting in an equilibrium tax rate of τ^{RE}). Given these assumptions and the definitions of $V^e(E)$ and $V^e(M)$, we can write the maximization problem of the elite when choosing the tax rate $\tau^m(t)$ at $t-1$ as

$$\begin{aligned} V^e(E) &= \max_{\tau^m}\{\beta^{\alpha/(1-\alpha)}A^e\bar{L}/\alpha + \left[\phi\beta^{\alpha/(1-\alpha)}\tau^m(1-\tau^m)^{\alpha/(1-\alpha)}A^m\theta^m\bar{L}/\alpha + R^N\right]/\theta^e \\ &\quad +\beta\left[(1-\eta[\tau^m])V^e(E) + \eta[\tau^m]V^e(M)\right]\}, \end{aligned}$$

where I wrote $\eta[\tau^m]$ to emphasize the dependence of the replacement probability on the tax rate on the middle class (while economizing on notation by not explicitly spelling out the argument of the $\eta(\cdot)$ function).

The first-order condition for an interior solution for the tax rate τ^m is:

$$\begin{aligned} &\frac{\phi\beta^{\alpha/(1-\alpha)}(1-\tau^m(t))^{\alpha/(1-\alpha)}A^m\theta^m\bar{L}}{\alpha\theta^e}\left(1-\frac{\alpha}{1-\alpha}\frac{\tau^m(t)}{1-\tau^m(t)}\right) - \\ &\beta\frac{d\eta\left((\beta(1-\tau^m))^{1/(1-\alpha)}A^m\theta^m\bar{L}/\alpha\right)}{d\tau^m}\left(V^e(E)-V^e(M)\right) = 0. \end{aligned}$$

The first-term in this first-order condition corresponds to the revenue extraction motive, while the second term relates to the political replacement effect. Inspection of this condition shows that when $\eta'(\cdot) = 0$, we obtain $\tau^m = \tau^{RE} \equiv \min\{1-\alpha, \bar{\tau}\}$ as above. However, when

$\eta'(\cdot) > 0$ and $V^e(E) - V^e(M) > 0$, we have $\tau^m(t) = \tau^{PC} > \tau^{RE} \equiv \min\{1-\alpha, \bar{\tau}\}$. The result that $V^e(E) - V^e(M) > 0$ follows from Exercise 22.7.

The important point here is that, as with the factor price manipulation mechanism, the elite tax *beyond the peak* of the Laffer curve. Their objective is not to increase their current revenues, but to consolidate their political power (in fact, taxes beyond the peak of the Laffer curve *reduce* the current income of the elite). However, higher (more distortionary) taxes are useful for the elite because they reduce the income of the middle class and their political power. Consequently, there is a higher probability that the elite remain in power in the future, enjoying the benefits of controlling the fiscal policy.

A number of new comparative static results follow from the possibility that the elite might lose political power. First, as R^N increases, it is straightforward to verify that the gap between $V^e(E)$ and $V^e(M)$ increases (see Exercise 22.7). This immediately translates into a higher equilibrium tax rate on the middle class. Intuitively, the party in power receives the revenues from natural resources, R^N and when these revenues are higher, *political stakes*—defined as the value of controlling political power—are greater. Consequently, the elite are more willing to sacrifice tax revenue (by overtaxing the middle class) in order to increase the probability that they remain in power (because remaining in power has now become more valuable). This contrasts with the results so far where R^N had no effect on taxes. Moreover, in this case a higher state capacity, ϕ, also increases the gap between $V^e(E)$ and $V^e(M)$ (because this enables the group in power to raise more tax revenues, see Exercise 22.7) and thus creates a force towards higher equilibrium taxes (though this effect might be dominated by the tax-reducing effect of ϕ emphasized in the previous subsection). This comparative static result therefore shows the potential dark side of greater state capacity; when there is no political competition, greater state capacity, by allowing more efficient forms of transfers, improves the allocation of resources. In contrast, in the presence of political competition, a greater state capacity, increases the political stakes and may induce more distortionary policies.

Finally, when the replacement of the elite by the middle class is very likely, i.e., $\eta \approx 1$ or when such political replacement is very unlikely, i.e., $\eta(\cdot) \approx 0$, we will have that $\eta'(\cdot)$ will be uniformly low. In these cases, there is only limited increase in the tax rate above the revenue maximizing level. It is only when η takes intermediate values that depend on the wealth level of the middle class that $\eta'(\cdot)$ is high and the political replacement effect leads to substantially more distortionary taxes. Therefore, we expect the elite to choose more distortionary policies when they have an intermediate level of security (rather than when they are entirely secure in their political power, i.e., $\eta(\cdot) \approx 0$, or when they definitely expect to replaced, i.e., $\eta(\cdot) \approx 1$). This is the sense in which the political replacement effect here is very similar to the replacement effect pointed out by Arrow in the context of innovation (recall Chapter 12).

22.3.3. Subgame Perfect Versus Markov Perfect Equilibria. I have so far focused on Markov Perfect Equilibria (MPE). In general, such a focus can be restrictive. A natural question is whether the set of Subgame Perfect Equilibria (SPE) is larger than the set of the MPE and whether some of the SPE can lead to more efficient allocation of resources (see Appendix Chapter C for formal definitions of MPE and SPE and differences between the two concepts). We will first see that in the setup analyzed so far the set of SPE and MPE coincide (this is of course not always the case, for example, as suggested by the discussion in footnote 2). We will then turn to potential holdup problems, exacerbating the commitment problems involved in the economy, and see that the SPE can lead to a more efficient allocation of resources than the MPE because it allows for greater "equilibrium commitment" on the part of the elite.

Essentially, the MPE are generally a subset of the SPE, because the latter include equilibria supported by some type of "history-dependent punishment strategies". If there is no room for such history dependence, SPEs will coincide with the MPEs. In the models analyzed so far, such punishment strategies are not possible even in the SPE. Intuitively, each individual is infinitesimal and makes its economic decisions to maximize profits. Therefore, (22.20) and (22.21) determine the factor demands uniquely in any equilibrium. Given the factor demands, the payoffs from various policy sequences are also uniquely pinned down. This means that the returns to various strategies for the elite are *independent of history.* Consequently, there cannot be any SPEs other than the MPE characterized above. Therefore, we have:

Proposition 22.7. *The MPEs characterized in Propositions 22.4-22.6 are the unique SPEs.*

Proof. See Exercise 22.9. □

In addition, Exercise 22.10 shows that the MPE in the model of subsection 22.3.2 is also the unique SPE. This last result, however, depends on the assumption that there is only one possible power switch (from the elite to the middle class). If, instead, there were continuous power switches, potential punishment strategies could be constructed and the set of SPEs could include non-Markovian equilibria.

22.3.4. Lack of Commitment—Holdup. The models discussed so far featured full commitment to one-period ahead taxes by the elites. In particular, at the end of period t, the elite (and in the model of Section 22.7, the median voter) could commit to the tax rate on output that would apply at time $t+1$. Using a term from organizational economics, this corresponds to the situation without any "holdup". *Holdup*, on the other hand, corresponds to a situation without commitment to taxes or policies, so that after entrepreneurs have undertaken their investments they can be "held up" by higher rates of taxation or by expropriation. These types of holdup problems are endemic in political economy situations, since commitments to future policies is difficult or impossible. Those who have political power at a certain point in time are likely to make the relevant decisions at that point. Moreover, when the key investments are long-term (so that once an investment is made, it is irreversible), there will be a holdup problem even if there is a one period commitment (since there will be taxes that will affect this investment after the investment decisions are sunk).

The problem with holdup is that the elite will be unable to commit to a particular tax rate before middle class producers undertake their investments (taxes will be set after investments). This lack of commitment will generally increase the amount of taxation and distortions. Moreover, in contrast to the allocations that we have seen so far, which featured distortions but were Pareto optimal, the presence of commitment problems will lead to Pareto inefficiency. To illustrate the main issues that arise in the presence of commitment problems in the simplest possible way, I consider the same model as above, but change the timing of events such that taxes on output at time t are decided in period t, that is, after the capital investments for this period have already been made (instead of at $t-1$, before these capital investments, as we have assumed so far). The economic equilibrium is essentially unchanged, and in particular, (22.20) and (22.21) still determine factor demands, with the only difference that τ^m and τ^e now refer to "expected" taxes. Naturally, in equilibrium expected and actual taxes coincide.

What is different is the calculus of the elite in setting taxes. Previously, they took into account that higher taxes on output at date t would discourage investment for production at date t. Since, now, taxes are set after investment decisions are sunk, this effect is absent. As a result, in the MPE, the elite will always want to tax at the maximum rate, so in all cases, there is a unique MPE where $\tau^m(t) = \tau^{HP} \equiv \bar{\tau}$ for all t. This establishes:

Proposition 22.8. *With holdup, there is a unique Markov Perfect Political Economy Equilibrium with $\tau^m(t) = \tau^{HP} \equiv \bar{\tau}$ for all t.*

It is clear that this holdup equilibrium is more inefficient than the equilibria characterized above. For example, imagine a situation in which Condition 22.1 fails to hold, so that with the original timing of events (without holdup), the equilibrium tax rate is $\tau^m(t) = 1-\alpha$. Consider the extreme case where $\bar{\tau} = 1$. Now without holdup, $\tau^m(t) = 1-\alpha$ and there is positive economic activity by the middle class producers. In contrast, with holdup, the equilibrium tax is $\tau^m(t) = 1$ and the middle class stop producing. This is not only costly for the middle-class entrepreneurs, but also for the elite since they lose all their tax revenues.

In this model, it is no longer true that the unique MPE is the only SPE, since there is room for an implicit agreement between different groups whereby the elite (credibly) promise a different tax rate than $\bar{\tau}$. Relatedly, the MPE in this model, provided in Proposition 22.8 is Pareto inefficient, and a social planner with access to exactly the same fiscal instruments can improve the utility of all agents in the economy.

To illustrate the difference between the MPE and the SPE (and the associated Pareto inefficiency of the MPE), consider the example where Condition 22.1 fails to hold and $\bar{\tau} = 1$. Recall that the history of the game is the complete set of actions taken up to that point. In the MPE, the elite raise no tax revenue from the middle class producers. Instead, consider the following trigger-strategy profile: the elite set $\tau^m(t) = 1 - \alpha$ for all t and the middle class producers invest according to (22.20) with $\tau^m(t) = 1 - \alpha$ as long as the history consists of $\tau^m = 1 - \alpha$ and investments have been consistent with (22.20). If there is any other action in the history, the elite set $\tau_m = 1$ and the middle class producers invest zero. Does this strategy constitute a SPE? First, it is clear that the middle class have no profitable deviation, since at each t, they are choosing their *best response* to taxes along the equilibrium path as implied by (22.20). To check whether the elite have a profitable deviation, note that with this strategy profile, they are raising a tax revenue of $\phi(1-\alpha)\alpha^{\alpha/(1-\alpha)}\beta^{\alpha/(1-\alpha)}A^m\theta^m\bar{L}/\alpha$ in every period, thus receiving transfers worth

$$\frac{\phi}{(1-\beta)}(1-\alpha)\alpha^{-(1-2\alpha)/(1-\alpha)}\beta^{\alpha/(1-\alpha)}A^m\theta^m\bar{L}. \tag{22.31}$$

If, in contrast, they deviate at any point, the most profitable deviation for them is to set $\tau^m = 1$, and they will raise a tax revenue of

$$\phi\alpha^{-(1-2\alpha)/(1-\alpha)}\beta^{\alpha/(1-\alpha)}A^m\theta^m\bar{L} \tag{22.32}$$

at that period. Following such a deviation, the continuation equilibrium involves switching to the unique MPE (which is here the worst possible continuation SPE). We have seen above that, with $\bar{\tau} = 1$ the continuation value of the elite in this case is equal to 0. Therefore, the trigger-strategy profile will be an equilibrium as long as (22.31) is greater than or equal to (22.32), which requires $\beta \geq \alpha$. Therefore we have:

Proposition 22.9. *Consider the holdup game, and suppose that Condition 22.1 holds and that $\bar{\tau} = 1$. Then for $\beta \geq \alpha$, there exists a SPE where $\tau^m(t) = 1 - \alpha$ for all t.*

Proof. See Exercise 22.11. □

An important implication of this result is that in societies where there are greater holdup problems, for example, because typical investments involve longer horizons, the MPE leads to a Pareto inefficient equilibrium allocation and there is room for coordinating on a SPE supported by an implicit agreement (trigger strategy profile) between the elite and the rest of the society that can make all the agents in the society better-off. This analysis also shows that whether we use the MPE or the SPE equilibrium concept has important implications for the the structure of the equilibrium and its efficiency properties. While the use of the equilibrium concept is a choice for the modeler, different equilibrium concepts approximate different real-world situations. For example, MPE may be much more appropriate when the institutional structure, the frequency of interactions or the past history make coordination and mutual trust unlikely, while SPE may be be useful in modeling equilibria in societies where some degree of mutual trust can be developed among the different parties with conflicting interests.

22.3.5. Technology Adoption. Another source of holdup comes from the technology adoption decisions of entrepreneurs, which may, in practice, be more important than the timing of taxes. Many important technology adoption decisions are made with the long horizon in mind, thus future tax rates matter for these decisions. The analysis earlier in the book highlighted the importance of technology adoption decisions for economic growth, thus the new types of political economy interactions that arise in the presence of such decisions are of practical as well as theoretical interest.

To illustrate the main issues raised by the presence of technology adoption decisions, let us go back to the original timing where taxes for time $t+1$ are set and committed to at time t (so that the source of holdup in the previous subsection is now removed). Instead, suppose that at time $t=0$ before any economic decisions or policy choices are made, middle class agents can invest to increase their productivity. In particular, suppose that there is a cost $\Gamma(A^m)$ of investing in productivity A^m. The function Γ is non-negative, continuously differentiable and convex. This investment is made once and the resulting productivity A^m applies forever after.

Once investments in technology are made, the game proceeds as before. Since investments in technology are sunk after date $t=0$, the equilibrium allocations are the same as those presented above. The interesting question is whether the presence of the technology adoption decisions creates additional inefficiencies (including Pareto inefficiencies). One way of answering this question is to ask whether, if they could, the elite would prefer to commit to a tax rate sequence at time $t=0$ different from the MPE or the SPE tax sequence characterized above. The following proposition answers this question in the case of pure factor price manipulation affect:

Proposition 22.10. *Consider the game with technology adoption and suppose that Condition 22.1 and 22.2 hold and $\phi=0$. Then the unique MPE and the unique SPE involve $\tau^m(t)=\tau^{FPM}\equiv\bar{\tau}$ for all t. Moreover, if the elite could commit to a tax sequence at time $t=0$, then they would still choose $\tau^m(t)=\tau^{FPM}\equiv\bar{\tau}$.*

The result that the allocation described in the proposition is the unique MPE follows immediately from the analysis so far. The fact that it is also the unique SPE follows from Proposition 22.7 and implies that the elite would choose exactly this tax rate even if they could commit to a tax rate sequence at time $t = 0$. The reason is intuitive: in the case of pure factor price manipulation, the only objective of the elite is to reduce the middle class' labor demand, so they have no interest in increasing the productivity of middle class producers.

The situation is quite different, however, when the elite would also like to extract revenues from the middle class. To illustrate this in the starkest possible way, let us next consider the pure revenue extraction case, where Condition 22.1 fails to hold (so that the equilibrium wage is equal to 0 and there is no factor price manipulation). Once again, the MPE is identical to before and involves a tax of τ^{RE} as in (22.23) at each date. As a result, the first-order condition for an interior solution to the middle class producers' technology choice is:

$$\Gamma'(A^m) = \frac{1-\alpha\beta}{(1-\beta)\alpha\beta}\left(\beta(1-\tau^{RE})\right)^{1/(1-\alpha)}\bar{L}. \tag{22.33}$$

Once again, Proposition 22.7 implies that a tax rate of $\tau^m = \tau^{RE} \equiv \min\{1-\alpha, \bar{\tau}\}$ and technology choice given by (22.33) is also the unique SPE. Intuitively, once again after the middle class producers have made their technology decisions, there is no history-dependent action left, and it is impossible to create history-dependent punishment strategies to support a tax rate different than the static optimum for the elite. However, in this case this equilibrium allocation is Pareto inefficient and in fact, if the elite could commit to a tax rate sequence at time $t = 0$, they would choose lower taxes. To illustrate this, suppose that the elite can indeed commit to a constant tax rate at $t = 0$ (it is straightforward to show that they will in fact choose a constant tax rate even without this restriction, but this restriction saves on notation). Therefore, the optimization problem of the elite is to maximize tax revenues taking the relationship between taxes and technology as in (22.33) as given. In other words, they will maximize $\phi\tau^m\left(\beta(1-\tau^m)\right)^{\alpha/(1-\alpha)}A^m\theta^m\bar{L}/\alpha$ subject to (22.33). The constraint (22.33) incorporates the fact that (expected) taxes affect technology choice.

The first-order condition for an interior solution can be expressed as

$$A^m - \frac{\alpha}{1-\alpha}\frac{\tau^m}{1-\tau^m}A^m + \tau^m\frac{dA^m}{d\tau^m} = 0$$

where $dA^m/d\tau^m$ takes into account the effect of future taxes on technology choice at time $t = 0$. This expression can be obtained by differentiating (22.33) written with τ^m instead of τ^{RE} as:

$$\frac{dA^m}{d\tau^m} = -\frac{\beta^{1/(1-\alpha)}}{1-\beta}\frac{1}{\alpha}\frac{(1-\tau^m)^{\alpha/(1-\alpha)}}{\Gamma''(A^m)} < 0.$$

This immediately implies that the solution to the maximization problem of the elite when they can commit to a tax rate sequence at $t = 0$ has a solution $\tau^m = \tau^{TA} < \tau^{RE} \equiv \min\{1-\alpha, \bar{\tau}\}$ (provided that $\tau^{TA} < \bar{\tau}$, for example, because $\bar{\tau}$ is sufficiently close to 1). Hence, if they could, the elite would commit to a lower tax rate in the future in order to encourage the middle class producers to undertake technological improvements. Their inability to commit to such a tax policy leads to more distortionary policies (and in fact in this case to Pareto inefficiency). The next proposition states this result and to simplify the statement, I assume $\bar{\tau} = 1$.

Proposition 22.11. *Consider the game with technology adoption, and suppose that Condition 22.1 fails to hold, that Condition 22.2 holds, that $\phi > 0$ and that 1. Then the unique MPE and the unique SPE involve $\tau^m(t) = \tau^{RE} \equiv 1 - \alpha$ for all t. If the elite could commit to a tax policy at time $t = 0$, they would prefer to commit to a tax level $\tau^{TA} < \tau^{RE}$ at $t = 0$.*

An important feature is that in contrast to the pure holdup problem where SPE could prevent the additional inefficiency (when $\beta \geq \alpha$, recall Proposition 22.9), with the technology adoption game, the inefficiency survives the SPE. The reason is that, since middle class producers invest only once at the beginning, there is no possibility of using history-dependent punishment strategies. This illustrates the limits of implicit agreements to keep tax rates low. Such agreements not only require a high discount factor ($\beta \geq \alpha$), but also frequent investments by the middle class, so that there is a credible threat against the elite if they deviate from the promised policies. When such implicit agreements fail to prevent the most inefficient policies, there is greater need for economic institutions to play the role of placing limits on future policies.

22.4. Inefficient Economic Institutions: A First Pass

I will now use the framework from the previous section to make a first attempt to understand (i) the conditions under which equilibrium economic institutions might put limits on distortionary policies and (ii) the conditions under which economic institutions might go on to the other extreme, involving the elite using inefficient instruments to reduce output and *block* economic development. To communicate the ideas in the simplest possible way, I will consider two prototypical economic institutions that affect the policy choices by the elite: (1) *Security of property rights;* there may be constitutional or other limits on the extent of redistributive taxation and/or other policies that reduce profitability of producers' investments. In terms of the model above, we can think of this as determining the level of $\bar{\tau}$. (2) *Regulation of technology*, which concerns direct or indirect factors affecting the productivity of producers, in particular middle class producers.

The analysis of factor price manipulation in the previous subsection already provides a partial answer to one of the questions raised above: why would the political system use *inefficient instruments*? A full analysis to this question requires a setup with a richer menu of fiscal instruments, such as lump-sum taxes. A glimpse of how such an analysis might go is provided in Exercise 22.15 below. For now, note that the analysis in Propositions 22.5 and 22.6 already provide the beginning of an answer, since they show that the equilibrium tax rate would be strictly above the revenue-maximizing level. Our first task is to derive some implications from these observations about constitutional limits on taxation by the elite.

22.4.1. Emergence of Secure Property Rights. The environment is the same as in the previous section, with the only difference that at time $t = 0$, before any decisions are taken, the elite can change the constitution so as to reduce $\bar{\tau}$, say from $\bar{\tau}^H$ to some level in the interval $[0, \bar{\tau}^H]$, thus creating an upper bound on taxes and providing greater security of property rights to the middle class. Here, we are thinking of $\bar{\tau}^H$ as technologically imposed, for example, because if the tax rate were above $\bar{\tau}^H$, middle-class entrepreneurs would flee to the informal sector. Naturally, a key question is how such a constitution would be made credible. For now, we do not address this question and take it as given that such a constitutional limits on future taxes can be imposed (though this, to some degree, goes against the presumption we have made so far that commitment to future policies is not possible; in some sense, we are relaxing this somewhat by assuming that commitment to an upper bound on policies is possible). The key question is whether the elite would like to do so, i.e., whether they prefer $\bar{\tau} = \bar{\tau}^H$ or $\bar{\tau} < \bar{\tau}^H$. Also, to start with, we take the natural benchmark in which economic institutions (here constitutional limits on taxation) are decided by the elite, who hold political power at $t = 0$ when these restrictions are introduced.

The next three propositions answer this question in various different versions of the environment studied so far in this section:

Proposition 22.12. *Without holdup and technology adoption, the elite prefer $\bar{\tau} = \bar{\tau}^H$.*

The proof of this result is immediate, since without holdup or technology adoption, putting further restrictions on the taxes can only reduce the elite's utility. This proposition implies that when economic institutions are decided by the elite, who will hold political power in the future as well, they will have no interest in introducing constitutional limits on their future taxes and will not introduce security of property rights to other producers.

The results are different when there are holdup problems. To illustrate this, let us go back to the situation with holdup (where taxes for time t are decided after the capital stock for time t is determined). Let us focus on the general case where both the revenue extraction and factor price manipulation motives are present. Moreover, let us for now focus on the MPE.

Proposition 22.13. *Consider the game with holdup and suppose that Conditions 22.1 and 22.2 hold and $\phi > 0$. Then the unique MPE involves $\tau^m(t) = \bar{\tau}^H$ for all t. If τ^{COM} given by (22.28) is strictly less than $\bar{\tau}^H$, the elite prefer to set $\bar{\tau} = \tau^{COM}$ at $t = 0$.*

Proof. See Exercise 22.12. □

The intuition for this proposition is simple: in the presence of holdup problems, Proposition 22.8 shows that the unique MPE involves $\tau = \bar{\tau}^H$. However, this is (Pareto) inefficient and in fact, if the elite could commit to a tax rate of $\bar{\tau} = \tau^{COM}$, they would increase their consumption (and also the middle class and the workers would achieve greater consumption at each date). If the elite could use economic institutions, for example by setting constitutional limits on taxes, then they would like to use these to manipulate equilibrium taxes. By manipulating economic institutions, the elite may approach their desired policy (in fact, in this simple economy, they can exactly commit to the tax rate that maximizes their utility).

This result shows that the elite may wish to change economic institutions to provide additional property rights protection to producers in the presence of holdup problems. Note however that the restriction to MPE is important in this proposition. If we allow history-dependent punishment strategies and look at the SPE, the elite would be able to improve over the MPE allocation in Proposition 22.9, and depending on parameters, they may even be able to implicitly (and credibly) commit to an equilibrium in which the tax rate at each date is equal to τ^{COM}. If this were the case, there would be less need for changing economic institutions in order to place limits on future taxes. Whether the MPE or the SPE is more relevant in such a situation depends on what the expectations of the different parties are and what degree of coordination can be achieved among the players. It is generally difficult to ascertain whether one or the other equilibrium concept would be more appropriate without specifying other (institutional or historical) details of the situation.

However, interestingly, when the source of additional inefficiency is technology adoption rather than the holdup problem (resulting from the timing of taxes), there will be a need for a change in economic institutions even if we focus on the SPE. This is shown in the next proposition:

Proposition 22.14. *Consider the game with technology adoption, and suppose that Condition 22.1 holds and $\phi > 0$. Then the unique MPE and the unique SPE involve $\tau^m(t) = \tau^{COM}$ given by (22.28) for all t. If τ^{COM} is strictly greater than τ^{TA} defined in Proposition 22.11, then the elite would prefer to set $\bar{\tau} = \tau^{TA}$ at $t = 0$.*

This proposition therefore highlights that when we focus on long-term investments or technology adoption decisions, implicit promises as in Proposition 22.9 will be of little use and explicit guarantees through economic institutions would be the only way of providing incentives to middle-class entrepreneurs to undertake the appropriate technology investments.

Thus, while implicit promises and other informal arrangements might play the role of economic institutions under some circumstances, there will be limits to how well they can perform this role and in many environments, constitutional limits on distortionary policies and expropriation (if feasible) would endogenously emerge in the political equilibrium.

22.4.2. Blocking Economic Development. The focus in the previous subsection was on choosing economic institutions at $t = 0$ to provide more secure property rights and better investment incentives to middle-class entrepreneurs. These types of economic institutions play an important role in practice and variation in the security of property rights for businesses across societies likely explains part of the variation in economic performance. Nevertheless, security of property rights and limits on taxes are only one aspect of the potential effect of institutions on economic activity and economic development. As briefly discussed in Chapter 4, in many societies, rather than encouraging economic activity, the elite actively tries to block economic development. Why would the elite in some societies choose specifically inefficient policies in order to reduce the productivity of entrepreneurs and block economic development?

We now discuss this question and try to shed light on the aspects of equilibrium economic institutions related to the regulation of technology. Once again, to provide the basic ideas in the simplest possible way, we will extend the basic framework in this section in one direction. We will assume that at time $t = 0$, the government (thus the elite controlling political power) chooses a policy affecting the technology choices of producers, denoted by $g \in \{0, 1\}$. This choice can be thought of investment in infrastructure, protection of intellectual property rights, or the provision of law and order (with $g = 0$ corresponding to not making these investments and $g = 1$ corresponding to creating a better business environment). Alternatively, $g = 0$ may directly correspond to actions taken by the elite to block the technology adoption decisions of the entrepreneurs. We assume that $g \in \{0, 1\}$ affects the productivity of middle-class producers in all future periods, and in particular $A^m = A^m(g)$, with $A^m(1) > A^m(0)$. To simplify the discussion, we assume that g has no effect on the productivity of the elite and also $g = 1$ has no direct cost relative to $g = 0$. The key question is this: Will the elite always choose $g = 1$, increasing the middle class producers' productivity, or will they try to block technology adoption by the middle class?

When the only mechanism at work is revenue extraction, the answer is that the elite would like the middle class to have the best technology:

Proposition 22.15. *Suppose that Condition 22.1 fails to hold and $\phi > 0$. Then the economic equilibrium always involves $w(t) = 0$ and the elite always choose $g = 1$.*

Therefore, this proposition shows a range of situations in which the elite would not block the technology adoption decisions of middle-class entrepreneurs. This result follows immediately since $g = 1$ increases the tax revenues and has no other effect on the elite's consumption. Consequently, in this case, the elite benefit from the increase in the productivity of the middle-class entrepreneurs and thus would like them to be as productive as possible. Intuitively, there is no competition between the elite and the middle class (either in factor markets or in the political arena), and when the middle class entrepreneurs are more productive, they generate greater tax revenues for the elite.

However, the situation is different when the elite wish to manipulate factor prices:

Proposition 22.16. *Suppose Condition 22.1 holds and Condition 22.2 holds (with* $A^m(g=0)$ *replacing* A^m*),* $\phi = 0$*, and* $\bar{\tau} < 1$*. Then in any MPE or SPE, the elite choose* $g = 0$*.*

Proof. See Exercise 22.14. □

Intuitively, with $\bar{\tau} < 1$, labor demand from the middle class is high enough to generate positive equilibrium wages. Since $\phi = 0$, taxes raise no revenues for the elite, and their only objective is to reduce the labor demand from the middle class and wages as much as possible. This makes $g = 0$ the preferred policy for the elite. Consequently, the factor price manipulation mechanism suggests that, when it is within their power, the elite will choose economic institutions so as to reduce the productivity of competing (middle class) producers. Proposition 22.16 therefore shows how the elite may take actions to directly reduce the productivity of the (other) entrepreneurs in the economy, thus retarding or blocking economic development.

The next proposition shows that a similar effect applies when the political power of the elite is contested.

Proposition 22.17. *Consider the economy with political replacement. Suppose Condition 22.1 fails to hold and* $\phi = 0$*. Then in any MPE or SPE, the elite prefer* $g = 0$*.*

Proof. See Exercise 22.15. □

In this case, the elite cannot raise any taxes from the middle class since $\phi = 0$. But differently from the previous proposition, there are no labor market interactions, since there is excess labor supply and wages are equal to zero. Nevertheless, the elite would like the profits from middle class producers to be as low as possible so as to consolidate their political power. They achieve this by creating an environment that reduces the productivity of middle class producers.

Overall, this section has demonstrated how the elite's preferences over policies, and in particular their desire to set inefficient policies, translate into preferences over non-growth enhancing (or "inefficient") economic institutions. When there are no holdup problems, introducing economic institutions that limit taxation or put other constraints on policies provides no benefits to the elite. This is intimately related to the fact that in the absence of holdup problems and given the menu of fiscal instruments, the equilibria characterized above corresponded to allocations maximizing a weighted social welfare function (and were thus constrained Pareto efficient). However, when the elite are unable to commit to future taxes (because of holdup problems), the equilibrium is no longer Pareto efficient and equilibrium taxes may be too high even from the viewpoint of the elite. In this case, using economic institutions to manipulate future taxes may be beneficial for the elite who control the political power of the state. Similarly, the analysis reveals that the elite may want to use economic institutions to discourage productivity improvements by the middle class. Interestingly, this never happens when the main mechanism leading to inefficient policies is revenue extraction. Instead, when factor price manipulation and political consolidation effects are present, the elite may want to discourage or block technological improvements by the middle class.

The analysis so far has focused on the basic forces leading to non-growth enhancing policies and economic institutions in the context of a simple society with linear preferences. The rest of this chapter investigates how relaxing these assumptions changes the insights. The next section is concerned with the case in which preferences are concave, while the following two sections introduce a richer structure of heterogeneity among the agents.

22.5. Distributional Conflict and Economic Growth: Concave Preferences*

In this section, I provide a preliminary analysis of an environment similar to the baseline model studied so far, but with concave preferences. My main purpose is to illustrate how to approach the analysis of such an economy and highlight some of the additional conceptual and technical issues that arise in this case. As a byproduct, this analysis will show how much the analysis of political economy was simplified by the assumption of linear preferences.

Relative to the the framework in Section 22.2 I make two different assumptions. First, preferences are now assumed to take the form

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U\left(C_i\left(t\right)\right), \tag{22.34}$$

where $U(\cdot)$ is a strictly increasing, strictly concave and continuously differentiable utility function. This specification assumes that all three groups have the same utility function, though this is not important for the analysis and is simply adopted to reduce notation. The second important assumption is that we close all financial markets. Consequently, entrepreneurs cannot borrow in order to invest in capital, and have to save by reducing their current consumption. Note that in the absence of political economy (and taxes), this would have no effect on the qualitative features of the dynamics of the model, which still closely resemble those of the baseline neoclassical growth model. In particular, without any taxes, there exists a unique globally (saddle-path) stable steady-state equilibrium, which satisfies equation

(6.38) in Chapter 6. All the other assumptions from Section 22.2, especially those regarding the production functions and the timing of policies, continue to apply.

Using exactly the same notation as in Section 22.2, the dynamic optimization of middle-class entrepreneurs for a given sequence of policies and wages, p^t and w^t, can be written as

$$\mathbf{U}_i\left(\{K_i(s), L_i(s)\}_{s=t}^{\infty} \mid p^t, w^t\right) =$$
$$\sum_{s=t}^{\infty} \beta^{s-t} U[(1-\tau(s)) F(K_i(s), L_i(s)) - K_i(s+1) - w(s) L_i(s) + T^m(s)],$$

where I have set the depreciation rate of capital δ equal to 1 to simplify the notation. This expression is similar to (22.9), except that the utility of consumption replaces the level of consumption as the instantaneous return at each date. Note that $\mathbf{U}_i$ is strictly concave in the sequence $\{K_i(s), L_i(s)\}_{s=t}^{\infty}$ for any tax sequence with $\tau(t) < 1$ for all t. We know from the analysis so far that there will never be 100% taxation, thus we can restrict attention to such tax sequences and the maximization problem of each entrepreneur is indeed a strictly concave problem.

The relevant necessary and sufficient first-order conditions for entrepreneur i can be written as

$$U'[C_i(t)] = \beta(1-\tau(t+1)) f'(k_i(t+1)) U'[C_i(t+1)], \tag{22.35}$$

for each t, which looks identical to the Euler equation for the representative consumer given in (6.37) in Chapter 6. This is not surprising, since each entrepreneur solves a similar program to that facing the representative consumer or the social planner in the basic neoclassical growth model. The only difference is the presence of the taxes, which implies that one unit of consumption foregone today does not earn the full marginal product of capital, but only that left over from taxes. This equation implies that the capital-labor ratio are chosen by entrepreneur i will now depend on the entire sequence of taxes, since these taxes will influence the current and future consumption levels. Consequently, we no longer have a simple equation such as (22.11) in Section 22.2. Nevertheless, (22.35) does determine a unique sequence of capital-labor ratio choices for each entrepreneur given the sequence of taxes and their initial capital stock, $K_i(0)$.

To simplify the analysis, let us suppose that all entrepreneurs start with the same initial capital stock, i.e., $K_i(0) = K(0)$. Given the symmetric initial conditions and the strict concavity of the problem, the equilibrium will be symmetric as well and each entrepreneur will choose exactly the same capital-labor ratio sequence. Now if the sequence of policies p^0 were indeed given, then we could define a single-valued mapping Φ: $\mathcal{P} \to \mathcal{K}$, where $\mathcal{P}$ is the set of all feasible policy sequences with less than 100% taxation at each point and $\mathcal{K}$ is the set of equilibrium capital-labor ratios. The political economy problem would then be for the elite to choose some $p^0 \in \mathcal{P}$ to maximize their discounted utility. This would indeed be the solution to the political economy problem if the elite could commit to a sequence of policies at date $t = 0$. But the assumption we have made so far, which is a natural approximation to reality, is that political decisions are made sequentially, and commitment to a future sequence of policies is *not* possible. This is exactly where my treatment in the Section 22.2 cut some corners. With linear utility, it did not matter whether the elite chose the sequence of policies at date $t = 0$ or sequentially as specified in the timing of events. To simplify the discussion there, I did not dwell on this distinction. This distinction now becomes crucial.

The right way to approach this problem is to specify the payoff-relevant state variables and then at each date have the elite make their utility-maximizing policy choices (as a function of the payoff-relevant state variables). The major difference from the analysis in this chapter so far is that once the elite undertake a deviation the future sequence of policies should not remain fixed but also change, because the deviation will have affected the evolution of the state variables and the evolution of the state variables will induce a different set of preferred policies for the elite. With linear preferences the deviation had no effect on future equilibrium policies, thus the analysis in the previous sections did not explicitly specify the effect of a deviation on the future sequence of policies. To show how this can be done in general and what its implications will be are the main focus of this section.

Let us now start developing the notation and the language for such an analysis. Most generally, the relevant state variable at time t would be a distribution of capital stocks or capital-labor ratios across all entrepreneurs denoted by $[K_i(t)]_{i \in S^m}$. This would significantly complicate the analysis, since working with entire distributions as the state variable is difficult. Fortunately, we can circumvent this problem. The same type of argument used above for a specific sequence of policies implies that, even taking the potential changes in future policies into account, the maximization problem of each entrepreneur is strictly concave. In addition, each entrepreneur recognizes that he has no effect on aggregates and also all entrepreneurs start with the same initial condition. Thus we can restrict attention to a situation in which at all dates all entrepreneurs will choose the same capital-labor ratio, and the state variable at time t can be represented by the capital-labor ratio of the "representative" entrepreneur, $k(t) \in \mathbb{R}_+$. Moreover, as in Chapter 6, the Inada conditions in Assumption 2 imply that we can restrict attention to state variables in a compact set $k(t) \in [0, \bar{k}]$.[3]

[3] It may appear that we are cutting some corners here as well. All entrepreneurs choose the same capital-labor ratio *along-the-equilibrium path*. What happens if an entrepreneur takes a deviation? It would appear that at that point, the state variable can no longer be represented by a one-dimensional object, and to take care of behavior off-the-equilibrium path properly, we would need to consider state variables of much higher dimension. Fortunately, this is not an issue, thanks to the fact that there is a continuum of entrepreneurs. If a single entrepreneur takes a deviation, this will have no effect on aggregates. Thus both along-the-equilibrium path and for one-step-ahead deviations from the equilibrium (which are the only ones that matter, see the Mathematical Appendix), focusing on the one-dimensional state variable is sufficient.

Given this state variable, the policy choice of the elite can be represented by a *policy function* denoted by

$$P : \left[0, \bar{k}\right] \to [0,1],$$

which determines the utility-maximizing tax rate for the elite at the next date, $\tau(t+1)$, as a function of the current capital-labor ratio $k(t)$. We could extend this function so that it also determines the amount of transfers. But this is not necessary, since, as in the previous subsection, the elite will always choose $T^w(t) = T^m(t) = 0$ for all t and $T^e(t)$ will be given by the government budget constraint, (22.8).

Let us next write the payoff function of a representative entrepreneur recursively. In his optimization problem, each entrepreneur takes five objects as given: first its own capital-labor ratio, k_i; second, the capital-labor ratio of all other entrepreneurs, k (this will naturally be equal to its own capital-labor ratio in equilibrium, but the entrepreneur does not control this variable himself); third, tax rate for today, τ; fourth, the tax rate announced for the next date, τ'; and finally, the policy function P of the elite. The fact that the entrepreneur is taking the policy function P as given implies that he is presuming that even if the elite take a deviation today, from tomorrow onwards, they will follow the policy function P that maximizes their discounted utility. This is simply an application of the *one step ahead deviation principle* (see Theorem C.1 in Appendix Chapter C). Let us also introduce the best-response function of the entrepreneurs at this point. From the viewpoint of entrepreneur i, whose behavior we are looking at now, this is the best-response function of all other entrepreneurs, which he takes as given. In equilibrium, his best response function must coincide with this, thus we will be looking for a fixed point. In particular, let

$$\kappa : \left[0, \bar{k}\right] \times [0,1]^2 \to \left[0, \bar{k}\right]$$

be this best response function, where the the first argument is today's capital stock and the next two arguments are today's and tomorrow's tax rates, so that the function takes the form $\kappa(k, \tau, \tau')$, with τ denoting the current tax rate and τ' denoting the tax rate announced for next period.

With this preparation, we can write the maximization problem of a representative entrepreneur i as follows:

$$V_i\left(k_i, k, \tau, \tau' \mid P, \kappa\right) = \max_{k' \in \left[0, \bar{k}\right]} U(C) + \beta V_i\left(k', \kappa\left(k, \tau, \tau'\right), \tau', P\left(\kappa\left(k, \tau, \tau'\right)\right) \mid P, \kappa\right) \tag{22.36}$$

subject to

$$C = \left[(1-\tau) f(k) - k' - w\right] / \theta^m, \tag{22.37}$$

where I have suppressed expectations, since there will be no uncertainty in this environment (because there are no exogenous shocks and we are focusing on pure strategies). Note that $V_i(k_i, k, \tau, \tau' \mid P, \kappa)$ denotes the value function of entrepreneur i, when his capital stock (capital-labor ratio) is given by k_i, those of other entrepreneurs is k, today's tax rate is τ, and tomorrow's tax rate has been announced as τ'. In all of this he takes the policy functions of the elite, P, and the best-response function of other entrepreneurs, κ, as given. The continuation value is therefore $\beta V_i\left(k', \kappa\left(k, \tau, \tau'\right), \tau', P\left(\kappa\left(k, \tau, \tau'\right)\right) \mid P, \kappa\right)$. Here, k' is his choice of next period's capital stock, so it will be the first element of the state variable entering his value function. The capital stock of other entrepreneurs will be given as a function of announced tax rate τ' and according to their best-response function, this is $\kappa\left(k, \tau, \tau'\right)$. Then the tax rate announced yesterday becomes the current tax rate, so the third element is τ', and finally, the policy function of the elite implies that they will choose a tax rate for the day after tomorrow as a function of the capital-labor ratio of entrepreneurs then, i.e., $P\left(\kappa\left(k, \tau, \tau'\right)\right)$. The entrepreneur's current level of consumption is then given by (22.37) by standard arguments, with w denoting the equilibrium wage rate. I did not condition on this equilibrium wage rate to reduce the notation which is already quite plentiful.

The maximization by entrepreneur i at each stage simply involves the choice of next date's capital stock, k'. Let us denote the best response function corresponding to this choice by $k'\left(k, \tau, \tau'\right)$. The following proposition can be established using the tools from Chapter 6:

Proposition 22.18. *Consider the maximization problem in (22.36). For any $\kappa(\cdot)$ and $P(\cdot)$ functions, the value function V is uniquely defined, continuous in all its arguments, and differentiable in the interior of its domain. The optimal policy $k'\left(k, \tau, \tau'\right)$ is defined uniquely and is continuous in all of its arguments.*

Proof. See Exercise 22.16. □

While the analysis in this section is considerably more complicated than in the models presented so far, Proposition 22.18 is a significant step towards characterizing the equilibrium of this more general economy. In particular, once we have the optimal policy of individual entrepreneur i, $k'\left(k, \tau, \tau'\right)$, it becomes apparent that this optimal policy of individual entrepreneur is the same as the best response function of all entrepreneurs, i.e.,

$$\kappa\left(k, \tau, \tau'\right) \equiv k'\left(k, \tau, \tau'\right).$$

Therefore we have managed to characterize the behavior of the entrepreneurs in a Markov Perfect Political Economy Equilibrium. Our next step is to take this best response function as given and solve the problem of the elite in setting taxes. For this purpose, let us now write the payoff to the elite recursively. Let $W\left(k, \tau \mid P, \kappa\right)$ be the value function of the elite when the current capital-labor ratio chosen by the entrepreneurs is k and the current tax rate is τ. Those are the only two states variables relevant for the elite. In addition, we condition this value function on κ, since this determines how entrepreneurs will react to different tax rates. To simplify the analysis, let us assume that the elite do not have access to any saving technology, thus they have to consume their current tax revenue and also normalize $\theta^e = 1$ without loss of generality. Then their value function can be written as

$$W(k, \tau \mid \kappa) = \max_{\tau'} U(\tau f(k)) + \beta W(\kappa(k, \tau, \tau'), \tau' \mid \kappa). \quad (22.38)$$

Intuitively, in the current period, the elite receive a pre-determined amount of tax revenue given by the tax rate, τ, announced in the previous period times output produced by the the capital stock on the economy (which is also predetermined). The capital stock of the economy is equal to the capital-labor ratio of the representative entrepreneur, since the total labor force is equal to 1 (and we have assumed (22.6) so that there is full employment). Next period's value is then given by the tax rate announced now, τ', and the capital-labor ratio choice of the entrepreneurs given by $\kappa(k, \tau, \tau')$.

The next proposition is again established using the tools from Chapter 6.

Proposition 22.19. *The value function given in (22.38) is uniquely defined and is continuous in k and τ.*

Proof. See Exercise 22.17. □

Unfortunately, this proposition does not establish that the value function W is concave or differentiable. This is because it does depend on the function $\kappa(k, \tau, \tau')$, which itself may be non-convex. Nevertheless, to make progress with the analysis in the simplest possible way, let us suppose that W is indeed differentiable in both k and τ, and also that $\kappa(k, \tau, \tau')$ is differentiable in all three of its arguments. To write the first-order condition for the choice of the tax rate by the elite, let us denote the partial derivatives of the W and κ functions with respect to their jth argument by W_j and κ_j, so that, for example, the derivative of W with respect to τ is denoted by W_2, etc. Then the first-order condition takes the form

$$W_2(\kappa(k, \tau, \tau'), \tau' \mid \kappa) + W_1(\kappa(k, \tau, \tau'), \tau' \mid \kappa)\, \kappa_3(k, \tau, \tau') = 0.$$

As in Chapter 6, to make further progress, we need to evaluate the two derivatives of the value function in the first-order condition, and we do this by differentiating the value function with respect to k and τ, which gives

$$W_1(k, \tau \mid \kappa) = \tau f'(k)\, U'(\tau f(k)) + \beta W_2(\kappa(k, \tau, \tau'), \tau' \mid \kappa)\, \kappa_1(k, \tau, \tau'),$$

and

$$W_2(k, \tau \mid \kappa) = f(k)\, U'(\tau f(k)) + \beta W_1(\kappa(k, \tau, \tau'), \tau' \mid \kappa)\, \kappa_2(k, \tau, \tau').$$

Now taking both expressions to the next period (from t to $t+1$), and substituting for these, we obtain the following condition for the utility-maximizing tax rate choice of the elite:

$$\left[f(\kappa(k, \tau, \tau') + \tau' f'(\kappa(k, \tau, \tau')))\, \kappa_3(k, \tau, \tau')\right] U'(\tau f(k)) +$$
$$\beta W_3(\kappa(\kappa(k, \tau, \tau'), \tau', \tau'') \mid \kappa)\left[\kappa_2(\kappa(k, \tau, \tau'), \tau', \tau'') + \kappa_1(\kappa(k, \tau, \tau'), \tau', \tau'')\, \kappa_3(k, \tau, \tau')\right] = 0,$$

with $\tau'' \equiv P(\kappa(k, \tau, \tau'))$. This first-order condition has some similarity to (22.16) from Section 22.2, but is clearly much more complicated. As with that expression, it trades off the gain from additional taxation, $f(\kappa(k, \tau, \tau'))$, against the loss that additional taxation will induce by reducing the equilibrium capital-labor ratio (the second term in the first bracket in first line). The second line represents the discounted future change in value arising from the fact that a different tax rate changes the capital-labor ratio tomorrow . Notice that these terms depend both on the best response function of the entrepreneurs, κ, and also on the policy function that the elite will use in the continuation game, P. In general, it is not possible to obtain closed-form solutions for the equilibrium tax rate. The presence of the current capital-labor ratio, k, indicates that the utility-maximizing tax rate will not be a constant. Instead, the equilibrium taxes will evolve over time together with the equilibrium capital-labor ratio.

Unfortunately, a further characterization of equilibrium is not possible without imposing further structure. Typically these types of models are solved under a variety of simplifying assumptions (such as quadratic utility) or the equilibrium is characterized numerically. Even though this more general model does not yield an explicit characterization of the Markov Perfect Political Economy Equilibrium, it highlights the new forces that arise once we incorporate the transitional dynamics in individual entrepreneurs' investment decisions, which, in turn, make it optimal for the elite to choose a non-constant path of taxes. However, since there is full depreciation of capital here, some simple cases still enable explicit solutions and Exercise 22.18 discusses a special case with logarithmic preferences and a Cobb-Douglas production function.

22.6. Heterogeneous Preferences, Social Choice and the Median Voter*

My next objective is to relax the focus on simple societies, which ensured that the social conflict was between the elite and the entrepreneurs. Instead, I wish to illustrate how a richer and more realistic form of heterogeneity among the members of the society will influence policy choices. I will do this in two steps. In this section, I will provide a brief overview of how to deal with aggregation of preferences in a society with heterogeneous agents. The celebrated Arrow's Impossibility Theorem, which we will see shortly, states that this is not possible in general. Nevertheless, under some further assumptions on the structure of preferences (and limits on the menu of available policy options) such aggregation is possible. The main tool in this context, which has wide-ranging applications in political economy models, is the *Median Voter Theorem*, and its cousin, the *Downsian Policy Convergence Theorem*. I will show that these two theorems together provide a useful characterization of democratic politics under (limited) heterogeneity among agents. Then, in the next section, using these results I will show that the qualitative results derived in Section 22.2 generalize to a model with heterogeneity among entrepreneurs. The bottom line of the analysis in the next section will be that the source of distortionary ("inefficient") policies that arise from the desire of the political system to extract revenues from a subset of the population is quite a bit more general than in the simple society investigated in Section 22.2. But before doing this, we will get a modicum of basic social choice theory. Strictly speaking, only a simple form of the Median Voter Theorem is necessary for next section, and some of the results here are abstract, hence this section has a "*".

The Median Voter Theorem (MVT) has a long pedigree in economics and has been applied in many different contexts. Given its wide use in political economy models, I will start with a section stating and outlining this theorem. I will also take this opportunity to provide a brief statement and proof of Arrow's Theorem, because this theorem makes the value of the MVT more transparent. I will then emphasize that the MVT, despite its simplicity and elegance, is of limited use, because it only applies to situations in which the menu of policies can be reduced such that the disagreement among all the individuals and society is over a one-dimensional (or essentially one-dimensional) policy choice. In situations where the society has to make multiple-dimensional decisions, such as tax on capital and labor or nonlinear income taxation, we cannot use the MVT. I will end this section by outlining some alternative ways of aggregating heterogeneous preferences in such cases, which will also illustrate why in many circumstances the determination of political equilibria can be represented as the maximization of a weighted social welfare function.

22.6.1. Basics. This subsection gives an introductory treatment of the large area of social choice theory. Social choice theory is concerned with the fundamental question of political economy already discussed at the beginning of this chapter: how to aggregate the preferences of heterogeneous agents over policies (collective choices). Differently from the most common political economy approaches, however, social choice theory takes an axiomatic approach to this problem. Nevertheless, a quick detour into social chose theory as an introduction to the Median Voter Theorem is useful.

Let us consider an abstract economy consisting of a finite set of individuals denoted by $\mathcal{H}$. We denote the number of individuals by H. Individual $i \in \mathcal{H}$ has a utility function

$$u\left(x_i, Y\left(x, p\right), p \mid \alpha_i\right).$$

Here x_i is his action, with a set of feasible actions denoted by X_i; p denotes the vector of political choices (institutions, policies, other collective choices etc.), with the menu of policies denoted by $\mathcal{P}$; and $Y\left(x, p\right)$ is a vector of general equilibrium variables, such as prices or externalities that result from all agents' actions as well as policy, and x is the vector of the x_i's. Instead of writing a different utility function u_i for each agent, I have parameterized the differences in preferences by the variable α_i. This is without loss of any generality (simply define $u_i(\cdot) \equiv u_i(\cdot \mid \alpha_i))$ and is convenient for some of the analysis that will follow. Clearly, the general equilibrium variables, such as prices, represented by $Y(x,p)$ here, need not be uniquely defined for a given set of policies p and vector of individual choices x. Since multiple equilibria are not our focus here, we ignore this complication and assume that $Y(x,p)$ is uniquely defined.

We also assume that, given aggregates and policies, individual objective functions are strictly quasi-concave so that each agent has a unique optimal action $x_i(p, Y(x,p), \alpha_i) = \arg\max_{x \in X_i} u(x_i, Y(x,p), p \mid \alpha_i)$. Substituting this maximizing choice of individual i into his utility function, we obtain his *indirect utility function* defined over policy as $U(p;\alpha_i)$. Next define the *preferred policy*, or the (political) bliss point, of voter i as

$$p(\alpha_i) = \arg\max_{p \in \mathcal{P}} U(p;\alpha_i)$$

In addition, we can think of a more primitive concept of individual preference orderings, which captures the same information as the utility function $U(p;\alpha_i)$. In particular, if individual individual i weakly prefers p to p', we write $p \succeq_i p'$ and if he has a strict preference, we write $p \succ_i p'$. Under the usual assumptions on individual preferences (completeness, which allows any two choices to be compared, reflexivity, so that $z \succeq_i z$, and transitivity, so that $z \succeq_i z'$ and $z' \succeq_i z''$ implies $z \succeq_i z''$), we can equivalently represent individual preferences by the ordering $\succeq_i$ or by the utility function $U(p;\alpha_i)$ (see Exercise 22.19). Throughout, we assume that individual preferences are transitive.

In this context, we can also think of a "political system" as a way of aggregating the set of utility functions, $U(p;\alpha_i)$'s, to a social welfare function $U^S(p)$ that ranks policies for the society. Put differently, a political system is a *mapping* from individual preference orderings to a social preference ordering. Arrow's Theorem shows that if this mapping satisfies some relatively weak conditions, then social preferences have to be "dictatorial" in the sense that they will exactly reflect the preferences of one of the agents. We first present this theorem.

22.6.2. Arrow's (Im) Possibility Theorem. Let us simplify the discussion by assuming that the set of feasible policies, $\mathcal{P}$, is finite and moreover it is a subset of the Euclidean space, i.e., $\mathcal{P} \subset \mathbb{R}^K$ where K is an integer. Let $\Re$ be the set of all weak orders on $\mathcal{P}$, that is, $\Re$ contains information of the form $p_1 \succeq_i p_2 \succeq_i p_3$ etc, and imposes the requirement of transitivity. An individual ordering R_i is an element of $\Re$, , i.e., $R_i \in \Re$. This statement reiterates that we are only considering individuals with well-defined transitive preferences.

Since our society consists of H individuals, we define $\rho = (R_1, ..., R_H) \in \Re^H$ as the society's *preference profile.* That is, ρ gives the preference ordering of each individual $i \in \mathcal{H}$. Also $\rho_{|\mathcal{P}'} = \left(R_{1|\mathcal{P}'}, ..., R_{H|\mathcal{P}'}\right)$ is the society's preference profile when alternatives are restricted to some subset $\mathcal{P}'$ of $\mathcal{P}$.

Let $\Im$ be the set of all reflexive and complete binary relations on $\mathcal{P}$ (but notice *not necessarily* transitive). A social ordering is $R^S \in \Im$, i.e., it is a reflexive complete binary relation over all the policy choices in $\mathcal{P}$. Thus, a social ordering can be represented as

$$\phi : \Re^H \to \Im.$$

This mathematical formalism implies that $\phi(\rho)$ gives the social ordering for the preference profiles in ρ. We can alternatively think of ϕ as a political system mapping individual preferences into a social choice. A trivial example of ϕ is the dictatorial ordering making agent 1 the dictator, so that for any preference profile $\rho \in \Re^H$, ϕ induces a social order that entirely coincides with R_1.

Note that our formulation already imposes the condition of "*unrestricted domain*," which says that in constructing a social ordering we should consider all possible (transitive) individual orderings. Therefore, we are not limiting ourselves to a special class of individual orderings, such as those with "single-peaked" preferences as we will do later in this section.

We say that a social ordering is *weakly Paretian* if $[p \succ_i p'$ for all $i \in \mathcal{H}] \Longrightarrow p \succ^S p'$, that is, if all individuals in the society prefer p to p', than the social ordering must also rank p ahead of p'. This is weakly Paretian (rather than strongly), since we require all agents to strictly prefer p to p'.

Next we say that, for a preference profile $\rho \in \Re^H$, a subset $\mathcal{D}$ of $\mathcal{H}$ is *decisive between* $p, p' \in \mathcal{P}$, $[p \succeq_i p'$ for all $i \in \mathcal{D}$ and $p \succ_{i'} p'$ for some $i' \in \mathcal{D}] \Longrightarrow p \succ^S p'$ (given ρ). If $\mathcal{D}' \subset \mathcal{H}$ is decisive between for $p, p' \in \mathcal{P}$ for *all* preference profiles $\rho \in \Re^H$, then it is *dictatorial between* for $p, p' \in \mathcal{P}$. We say that $\mathcal{D} \subset \mathcal{H}$ is *decisive* if it is decisive between any $p, p' \in \mathcal{P}$, and $\mathcal{D}' \subset \mathcal{H}$ is *dictatorial* if it is dictatorial between any $p, p' \in \mathcal{P}$, If $\mathcal{D}' \subset \mathcal{H}$ is dictatorial and a singleton, then its unique element is a *dictator*, meaning that social choices will exactly reflect his preferences regardless of the preferences of the other members of the society. In this case, we say that a social ordering ϕ is *dictatorial.*

Next a social ordering satisfies *independence from irrelevant alternatives* if for any ρ and $\rho' \in \Re^H$ and any p, $p' \in \mathcal{P}$, we have that

$$\rho_{|\{p,p'\}} = \rho'_{|\{p,p'\}} \Longrightarrow \phi(\rho)_{|\{p,p'\}} = \phi(\rho')_{|\{p,p'\}}.$$

The axiom of independence from irrelevant alternatives is essential for Arrow's Theorem. It states that if two preference profiles have the same choice over two policy alternatives, the social orderings that derive from these two preference profiles must also have identical choices over these two policy alternatives, irrespective of how these two preference profiles differ for "irrelevant" alternatives. While this condition (axiom) at first appears plausible, it is in fact a reasonably strong one. In particular, it rules out any kind of interpersonal "cardinal" comparisons—i.e., it excludes information on how strongly an individual prefers one outcome versus another.

The main theorem of the field of social choice theory is the following:

Theorem 22.1. ***(Arrow's (Im)Possibility Theorem)*** *If a social ordering, ϕ, is transitive, weakly Paretian and satisfies independence from irrelevant alternatives, then it is dictatorial.*

Proof. Suppose to obtain a contradiction that there exists a non-dictatorial and weakly Paretian social ordering, ϕ, satisfying independence from irrelevant alternatives. We will derive a contradiction in two steps.

Step 1: Let a set $\mathcal{J} \subset \mathcal{H}$ be *strongly decisive* between $p_1, p_2 \in \mathcal{P}$ if for any preference profile $\rho \in \Re^H$ with $p_1 \succ_i p_2$ for all $i \in \mathcal{J}$ *and* $p_2 \succ_j p_1$ for all $j \in \mathcal{H} \backslash \mathcal{J}$, we have that $p_1 \succ^S p_2$ ($\mathcal{H}$ itself is strongly decisive since ϕ is weakly Paretian). We first prove that if $\mathcal{J}$ is strongly decisive between $p_1, p_2 \in \mathcal{P}$, then $\mathcal{J}$ is dictatorial (and hence decisive for all $p, p' \in \mathcal{P}$ and for all preference profiles $\rho \in \Re^H$). To prove this, consider the restriction of an arbitrary preference profile $\rho \in \Re^H$ to $\rho_{|\{p_1,p_2,p_3\}}$ and suppose that we also have $p_1 \succ_i p_3$ for all $i \in \mathcal{J}$. Next consider an alternative profile $\rho'_{|\{p_1,p_2,p_3\}}$, such that $p_1 \succ'_i p_2 \succ'_i p_3$ for all $i \in \mathcal{J}$ and $p_2 \succ'_i p_1$ and $p_2 \succ'_i p_3$ for all $i \in \mathcal{H} \backslash \mathcal{J}$. Since$\mathcal{J}$ is strongly decisive between p_1 and p_2, $p_1 \succ'^S p_2$. Moreover, since ϕ is weakly Paretian, we also have $p_2 \succ'^S p_3$, and thus $p_1 \succ'^S p_2 \succ'^S p_3$. Notice that $\rho'_{|\{p_1,p_2,p_3\}}$ did not specify the preferences of individuals $i \in \mathcal{H} \backslash \mathcal{J}$ between p_1 and p_3, but we have established $p_1 \succ'^S p_3$ for $\rho'_{|\{p_1,p_2,p_3\}}$. We can then invoke independence from irrelevant alternatives and conclude that the same holds for $\rho_{|\{p_1,p_2,p_3\}}$, i.e., $p_1 \succ^S p_3$. But then since the preference profiles and p_3 are arbitrary, it must be the case that $\mathcal{J}$ is dictatorial between p_1 and p_3. Next repeat the same argument for $\rho_{|\{p_1,p_2,p_4\}}$ and $\rho'_{|\{p_1,p_2,p_4\}}$, except that now $p_4 \succ_i p_2$ and $p_4 \succ'_i p_1 \succ'_i p_2$ for $i \in \mathcal{J}$, while $p_2 \succ'_j p_1$ and $p_4 \succ'_j p_1$ for all $j \in \mathcal{H} \backslash \mathcal{J}$. Then the same chain of reasoning, using the facts that $\mathcal{J}$ is strongly decisive, $p_1 \succ'^S p_2$, ϕ is weakly Paretian and satisfies independence from irrelevant alternatives, implies that $\mathcal{J}$ is dictatorial between p_4 and p_2 (that is, $p_4 \succ^S p_2$ for any preference profile $\rho \in \Re^H$). Now once again using independence from irrelevant alternatives and also transitivity, we have that for any preference profile $\rho \in \Re^H$, $p_4 \succ_i p_3$ for all $i \in \mathcal{J}$. Since $p_3, p_4 \in \mathcal{P}$ were arbitrary, this completes the proof that $\mathcal{J}$ is dictatorial (i.e., dictatorial for all $p, p' \in \mathcal{P}$).

Step 2: Given the result in Step 1, if we prove that some individual $h \in \mathcal{H}$ is strongly decisive for some $p_1, p_2 \in \mathcal{P}$, we will have established that it is a dictator and thus ϕ is dictatorial. Let $\mathcal{D}_{ab}$ be the strongly decisive set between p_a and p_b. Such a set always exists for any p_a, $p_b \in \mathcal{P}$, since $\mathcal{H}$ itself is a strongly decisive set. Let $\mathcal{D}$ be the minimal strongly decisive set (meaning the strongly decisive set with the fewest members). This is also well-defined, since there is only a finite number of individuals in $\mathcal{H}$. Moreover, without loss of generality, suppose that $\mathcal{D} = \mathcal{D}_{12}$ (i.e., let the strongly decisive set between p_1 and p_2 be the minimal strongly decisive set). If $\mathcal{D}$ a singleton, then Step 1 applies and implies that ϕ is dictatorial, completing the proof. Thus suppose that $\mathcal{D} \neq \{i\}$. Then by unrestricted domain, the following preference profile (restricted to $\{p_1, p_2, p_3\}$) is feasible

$$\begin{array}{cc} \text{for } i \in \mathcal{D} & p_1 \succ_i p_2 \succ_i p_3 \\ \text{for } j \in \mathcal{D} \backslash \{i\} & p_3 \succeq_j p_1 \succ_j p_2 \\ \text{for } k \notin \mathcal{D} & p_2 \succ_k p_3 \succ_k p_1. \end{array}$$

By hypothesis, $\mathcal{D}$ is strongly decisive between p_1 and p_2 and therefore $p_1 \succ^S p_2$. Next if $p_3 \succ^S p_2$, then given the preference profile here, $\mathcal{D} \backslash \{i\}$ would be strongly decisive between p_2 and p_3, and this would contradict that $\mathcal{D}$ is the minimal strongly decisive set. Thus, we must have $p_2 \succsim^S p_3$. Combined with $p_1 \succ^S p_2$, this implies $p_1 \succ^S p_3$. However, given the preference profile here, this implies that $\{i\}$ is strongly decisive, yielding another contradiction. Therefore, the minimal strongly decisive set must be a singleton $\{h\}$ for some $h \in \mathcal{H}$. Then from Step 1 $\{h\}$ is a dictator and ϕ is dictatorial, completing the proof. □

An immediate implication of this theorem is that any set of minimal decisive individuals $\mathcal{D}$ within the society $\mathcal{H}$ must either be a singleton, that is, $\mathcal{D} = \{i\}$, so that we have a dictatorial social ordering, or we have to live with intransitivities.

While this theorem is often referred to as Arrow's Possibility Theorem, it is really an "Impossibility Theorem". An alternative way of stating the theorem is that there exists no social ordering that is transitive, weakly Paretian, consistent with independence from irrelevant alternatives and non-dictatorial. Viewed in this light, an important implication of this theorem is that there is no way of avoiding the issue of conflict in preferences of individuals by positing a social welfare function. A social welfare function, respecting transitivity, can only replace the actual political economic process of decision making when it is dictatorial. Naturally, who will become the dictator in the society fundamentally brings back the issue of *political power*, which is also essential for any positive political economy analysis of collective decision-making. In addition, from a modeling point of view, Arrow's theorem means that, if we are interested in non-dictatorial (and transitive) outcomes, we have to look at political systems that either restrict choices or focus on more concrete situations, where we have to be more specific about the distribution of political power and the vertical institutions regulating the decision-making process. This will be the basis of our analysis for the rest of this chapter and for the next chapter.

Often, economic models restrict the policy space and/or preferences of citizens in order to ensure that this impossibility theorem does not apply. Unfortunately, such restrictions on the policy space have more than technical implications. For example, they often force the modeler to restrict agents to use inefficient methods of redistribution. As a result, some of the inefficiencies that are found in political economy models are not a consequence of the logic of these models, but a consequence of the technical assumptions that the modelers make in restricting the policy space to a single policy. In some circumstances, limits on fiscal instruments might be justified on economic grounds. For example, the assumption that there was only a linear tax on output in Section 22.2 was justified with the argument that lump-sum taxes were not possible. Whether or not this is the case, it is important to recognize that the limits on the set of fiscal instruments is often responsible for the potential distortions resulting from political economy (as was the case in Section 22.2).

One reaction to Arrow's Theorem might be that the problem of creating individual preferences in this theorem arises because we are not looking at more relevant mechanisms such as voting. The next subsection shows that the same problems arise when collective choices are made by voting. In fact Arrow's Theorem applies to any possible way of aggregating individual preferences, and if voting were able to solve the problems raised by the theorem, it would be a contradiction to the theorem! Nevertheless, voting can be useful in situations whether we put more structure on preferences and on how individuals vote, which will essentially amount to either giving up the "unrestricted domain" assumption on choices or relaxing the independence from irrelevant alternatives.

22.6.3. Voting and the Condorcet Paradox. Let us illustrate how voting also runs into exactly same problems as those highlighted by Arrow's Theorem by using a well-known example, *the Condorcet paradox.* The underlying reason for this paradox is related to Arrow's Theorem and will also illustrate why, to obtain the Median Voter Theorem below, we will have to introduce reasonably strong restrictions.

Example 22.1. Imagine a society consisting of three individuals, 1, 2, and 3 and three choices. The individuals' preferences are as follows:

$$\begin{array}{ll} 1 & a \succ c \succ b \\ 2 & b \succ a \succ c \\ 3 & c \succ b \succ a \end{array}$$

Moreover, let us make the political mechanism somewhat more specific, and assume that it satisfies the following three requirements, which together make up the "open agenda direct democracy" system.

A1. *Direct democracy.* The citizens themselves make the policy choices via majoritarian voting.

A2. *Sincere voting.* In every vote, each citizen votes for the alternative that gives him the highest utility according to his policy preferences (indirect function) $U(p;\alpha_i)$. This requirement is now adopted for simplicity. In many situations, individuals may vote for the outcome that they do not prefer, anticipating the later repercussions of this choice (we refer to this type of behavior as "*strategic voting*"). Whether they do so or not is important in certain situations, but not for the discussion at the moment.

A3. *Open agenda.* Citizens vote over pairs of policy alternatives, such that the winning policy in one round is posed against a new alternative in the next round and the set of alternatives includes all feasible policies. Later, we will replace the open agenda assumption with parties offering policy alternatives, thus moving away from direct democracy some way towards indirect/representative democracy. For now it is a good starting point.

Now, using the three assumptions, consider a contest between policies a and b. In this contests, agents 2 and 3 will vote for b over a, so b is the majority winner. Next, by the open agenda assumption, the other policy alternative c will run against b. Now agents 1 and 3 prefer c to b, which is the new majority winner. Next, c will run against a, but now agents 1 and 2 prefer a, so a is the majority winner. Therefore, in this case we have "cycling" over the various alternatives, or put differently there is no "equilibrium" of the voting process that selects a unique policy outcome.

For future reference, let us now define a Condorcet winner as a policy choice that does not lead to such cycling. In particular,

DEFINITION 22.1. *A Condorcet winner is a policy p^* that beats any other feasible policy in a pairwise vote.*

In light of this definition, there is no Condorcet winner in the example of the Condorcet paradox.

22.6.4. Single-Peaked Preferences. Suppose now that the policy space is unidimensional, so that p is a scalar, i.e., $\mathcal{P} \subset \mathbb{R}$. In this case, a simple way to rule out the Condorcet paradox is to assume that preferences are *single peaked* for all voters. We will see below that the restriction that $\mathcal{P}$ is unidimensional is very important and single-peaked preferences are not well defined when there are multiple policy dimensions.

We say that voter i has single-peaked preferences if his preference ordering for alternative policies is dictated by their relative distance from his bliss point, $p(\alpha_i)$: a policy closer to $p(\alpha_i)$ is preferred over more distant alternatives. Specifically:

DEFINITION 22.2. *Consider a finite set of $\mathcal{P} \subset \mathbb{R}$ and let $p(\alpha_i) \in \mathcal{P}$ be individual i's unique bliss point over $\mathcal{P}$. Then, the policy preferences of citizen i are single peaked iff:*

$$\begin{aligned} \text{For all } p'', p' &\in \mathcal{P}, \text{ such that } p'' < p' \leq p(\alpha_i) \text{ or } p'' > p' \geq p(\alpha_i), \\ \text{we have } U(p''; \alpha_i) &< U(p'; \alpha_i). \end{aligned}$$

Note that strict concavity of $U(p'; \alpha_i)$ is sufficient for it to be single peaked, but is not necessary. In fact, single-peakedness is equivalent to strict quasi-concavity. This definition could be weakened so that the bliss point of the individual is not unique (i.e., from strict quasi-concavity to quasi-concavity). But this added generality is not important for our purposes.

We can easily verify that in the Condorcet paradox, not all agents possessed single-peaked preferences. For example, taking the ordering to be a, b, c, agent 1 who has preferences $a \succ c \succ b$ does not have single-peaked preferences (if we took a different ordering of the alternatives, then the preferences of one of the other two agents would violate the single-peakedness assumption, see Exercise 22.21).

The next theorem shows that with single-peaked preferences there always exists a Condorcet winner. Before stating this theorem, let us define the *median voter* of the society. Given the assumption that each individual has a unique bliss point over $\mathcal{P}$, we can rank all individuals according to their bliss points, the $p(\alpha_i)$'s. Also, to remove uninteresting ambiguities, let us imagine that H is an odd number (i.e., $\mathcal{H}$ consists of an odd number of individuals). Then the median voter is the individual who has exactly $(H-1)/2$ bliss points to his left and $(H-1)/2$ bliss points to his right. Put differently, his bliss point is exactly in the middle of the distribution of bliss points. We denote this individual by α_m , and his bliss point (ideal policy) is denoted by p_m.

Theorem 22.2. ***(The Median Voter Theorem)*** *Suppose that H is an odd number, that A1 and A2 hold, and that all voters have single-peaked policy preferences over a given ordering of policy alternatives, $\mathcal{P}$. Then, a Condorcet winner always exists and coincides with the median-ranked bliss point, p_m. Moreover, p_m is the unique equilibrium policy (stable point) under the open agenda majoritarian rule, that is, under A1-A3.*

Proof. The proof is by a "separation argument". Order the individuals according to their bliss points $p(\alpha_i)$, and label the median-ranked bliss point by p_m. By the assumption that H is an odd number, p_m is uniquely defined (though α_m may not be uniquely defined). Suppose that there is a vote between p_m and some other policy $p'' < p_m$. By definition of single-peaked preferences, for every individual with $p_m < p(\alpha_i)$, we have $U(p_m; \alpha_i) > U(p''; \alpha_i)$. By A2, these individuals will vote sincerely and thus also vote for p_m. The coalition voting for supporting p_m thus constitutes a majority. The argument for the case where $p'' > p_m$ is identical. □

The assumption that the society consists of an odd number of individuals was made only to shorten the statement of the theorem and the proof. Exercise 22.23 asks you to generalize the theorem and its proof to the case in which H is an even number.

More important than whether there are an odd or even number of individuals in the society is the assumption of sincere voting. Clearly, rational agents could deviate from truthful reporting of their preferences (and thus from truthful voting) when this is beneficial for them. So an obvious question is whether the MVT generalizes to the case in which individuals do not vote sincerely? The answer is yes. To see this, let us modify the sincere voting assumption to strategic voting:

A2$'$. *Strategic voting.* Define a *vote function* of individual i in a pairwise contest between p' and p'' by $v_i(p', p'') \in \{p', p''\}$. Let a voting (counting) rule in a society with H citizens be $V{:}\{p', p''\}^H \to \{p', p''\}$ for any $p', p'' \in \mathcal{P}$. (For example, the majoritarian voting rule V^M picks p' over p'' when this policy receives more votes than p''). Let $V(v_i(p', p''), v_{-i}(p', p''))$ be the policy outcome from voting rule V applied to the pairwise contest $\{p', p''\}$, when the remaining individuals cast their votes according to the vector $v_{-i}(p', p'')$, and when individual i votes $v_i(p', p'')$. Strategic voting means that

$$v_i\left(p', p''\right) \in \arg\max_{\tilde{v}_i(p', p'')} U\left(V\left(\tilde{v}_i\left(p', p''\right), v_{-i}\left(p', p''\right)\right); \alpha_i\right).$$

In other words, strategic voting implies that each individual chooses the voting strategy that maximizes utility given the voting strategies of other agents.

Finally, recall that a *weakly-dominant* strategy for individual i is a strategy that gives weakly higher payoff to individual i than any of his other strategies irrespective of the strategy profile played by all other players

Theorem 22.3. ***(The Median Voter Theorem With Strategic Voting)*** *Suppose that H is an odd number, that A1 and A2$'$ hold, and that all voters have single-peaked policy preferences over a given ordering of policy alternatives, $\mathcal{P}$. Then, there exists a weakly-dominant strategy for each player to vote sincerely and in this equilibrium, the median-ranked bliss point, p_m is the Condorcet winner.*

Proof. The vote counting rule (the political system) in this case is majoritarian, denoted by V^M. Consider two policies $p', p'' \in \mathcal{P}$ and fix an individual $i \in \mathcal{H}$. Assume without loss of any generality that $U(p'; \alpha_i) \geq U(p''; \alpha_i)$. Suppose first that for any $v_i \in \{p', p''\}$ we have that $V^M(v_i, v_{-i}(p', p'')) = p'$ or $V^M(v_i, v_{-i}(p', p'')) = p''$, that is, individual i is *not* pivotal. This implies that $v_i(p', p'') = p'$ is a best response for individual i. Suppose next that individual i is pivotal, that is, $V^M(v_i(p', p''), v_{-i}(p', p'')) = p'$ if $v_i(p', p'') = p'$ and $V^M(v_i(p', p''), v_{-i}(p', p'')) = p''$ otherwise. In this case, the action $v_i(p', p'') = p'$ is clearly a best response for i. Since this argument applies for each $i \in \mathcal{H}$, it establishes that voting sincerely is a weakly-dominant strategy and the conclusion of the theorem follows from Theorem 22.2. □

Notice that the second part of the Theorem 22.2, which applied to open agenda elections, is absent in Theorem 22.3. This is because the open agenda assumption does not lead to a well defined game, so a game theoretic analysis and thus an analysis of strategic voting is no longer possible.

In fact, there is no guarantee that sincere voting is optimal in dynamic situations even with single-peaked preferences. The following example illustrates this:

Example 22.2. Consider three individuals with the following preference orderings.

$$\begin{array}{ll} 1 & a \succ b \succ c \\ 2 & b \succ c \succ a \\ 3 & c \succ b \succ a \end{array}$$

These preferences are clearly single peaked (order them alphabetically to see this). In a one round vote, b will beat any other policy. But now consider the following dynamic voting set up: first, there is a vote between a and b. Then, the winner goes against c, and the winner of this contest is the social choice. Sincere voting will imply that in the first round players 2 and 3 will vote for b, and in the second round, players 1 and 2 will vote for b, which will become the social choice.

Is such sincere voting "equilibrium behavior"? Exactly the same argument as above shows that in the second round, sincere voting is a weakly dominant strategy. But not necessarily in round one. Suppose players 1 and 2 are playing sincerely. Now if player 3 deviates and votes for a (even though she prefers b), then a will advance to the second round and would lose to c. Consequently, the social choice will coincide with the bliss point of player 3. Exercise 22.24 asks you to characterize the subgame perfect equilibrium of this game under strategic voting by all players.

Dynamic voting issues become more interesting, and open the way for agenda setting, when there are no Condorcet winners. The following example illustrates this.

Example 22.3. Consider the preference profile in Example 22.1 and the following political mechanism. First, all individuals vote between a and b, and then they vote over the winner of this contest and c. With sincere voting, b will win the first round, and then c wins the second round against b. Now consider agent 2. If he changes his vote in the first round to a (thus does not vote sincerely), the first-round winner will be a, which will also win against c, and player 2 prefers this outcome to the outcome of sincere voting, which was c.

This example can also be used to illustrate the role of "agenda setting". Suppose that in the above game, agent 1 decides the exact orderings of voting. In particular, he has to choose between three options (a vs. b first, a vs. c first, and b vs. c, first). Anticipating strategic voting by player 2, he will choose the first option and will ensure that his most preferred alternative becomes the political choice of the society. In contrast, if agent 3 chose the ordering, he would go for a vs. c first, which would induce agent 1 to vote strategically for c, and lead to c as the ultimate outcome.

22.6.5. Party Competition and the Downsian Policy Convergence Theorem. The focus so far has been on voting between two alternative policies or on open agenda voting, which can be viewed as an extreme form of "direct democracy". The MVT becomes potentially more relevant and more powerful when applied in the context of indirect democracy, that is, when combined with a simple model of party competition. We now give a brief overview of this situation and derive the Downsian Policy Convergence Theorem, which is the basis of much applied work in political economy.

Suppose that we have a situation in which there is a Condorcet winner, and there are two parties, A and B, competing for political office. Assume that the parties do not have an ideological bias, and would like to come to power (for example, they receive some utility from being in power). In particular, they both maximize the probability of coming to power, for example, because they receive a rent or utility of $Q > 0$ when they are in power.

Assume also that parties simultaneously announce their policy, and are committed to this policy. This implies that the behavior of the two parties can be represented by the following pair of maximization problems:

$$\begin{aligned} \text{Party } A &: \quad \max_{p_A} \; P(p_A, p_B)Q \\ \text{Party } B &: \quad \max_{p_B} \; (1 - P(p_A, p_B))\, Q \end{aligned} \tag{22.39}$$

where Q denotes the rents of being in power and $P(p_A, p_B)$ is the probability that party A comes to power when the two parties' platforms are p_A and p_B respectively. When the median voter theorem applies, and denoting the bliss point of the median voter by p_m, we have

$$\begin{aligned} P(p_A, p_B &= p_m) = 0, \; P(p_A = p_m, p_B) = 1, \text{ and} \\ P(p_A &= p_m, p_B = p_m) \in [0, 1]\,. \end{aligned} \tag{22.40}$$

This last statement follows since when both parties offer exactly the same policy, it is the best response for all citizens to vote for either party. However, the literature typically makes the following assumption:

A4. *Randomization:*

$$P(p_A = p_m, p_B = p_m) = 1/2.$$

This assumption can be rationalized by arguing that when indifferent individuals, randomize between the two parties, and since there are many many individuals, by the law of large numbers, each party obtains exactly half of the vote.

We then have the following result:

Theorem 22.4. ***(Downsian Policy Convergence Theorem)*** *Suppose that there are two parties that first announce a policy platform and commit to it and a set of voters $\mathcal{H}$ that vote for one of the two parties. Assume that A4 holds and that all voters have single-peaked policy preferences over a given ordering of policy alternatives, and denote the median-ranked bliss point by p_m. Then both parties will choose p_m as their policy platform.*

Proof. The proof is by contradiction. Suppose not, then there is a profitable deviation for one of the parties. For example, if $p^A > p^B > p_m$, one of the parties can announce p_m

and win the election for sure. When $p^A \neq p_m$ and $p^B = p_m$, party A can also announce p_m and increase its chance of winning to 1/2. □

Exercise 22.25 asks you to provide a generalization of this theorem without Assumption A4.

This theorem is important because it demonstrates that there will be policy convergence between the two parties and that party competition will implement the Condorcet winner among the voters. Therefore, in situations in which the MVT applies, the democratic process of decision making with competition between two parties will lead to a situation in which both parties will choose their policy platform to coincide with the bliss point of the median voter. Thus the MVT and the Downsian Policy Convergence Theorem together enable us to simplify the process of aggregating the heterogeneous preferences of individuals over policies and assert that, under the appropriate assumptions, democratic decision-making will lead to the most preferred policy of the median voter. The Downsian Policy Convergence Theorem is useful in this context, since it gives a better approximation to "democratic policymaking" in practice than open agenda elections.

There is a sense in which Theorem 22.4 is slightly misleading, however. While the theorem is correct for a society with two parties, it gives the impression of a general tendency towards policy convergence in all democratic societies. Many democratic societies have more than two parties. A natural generalization of this theorem would be to consider three or more parties. Unfortunately, as Exercise 22.26 shows the results of this theorem do not generalize to three parties. Thus some care is necessary in applying the Downsian Policy Convergence Theorem without regards to existing political institutions of the society.

Another obvious question is what would happen in the party competition game when there is no Condorcet winner. Theorem 22.4 does not generalize to this case either. In particular, if we take a situation in which there is "cycling," like the above Condorcet paradox example, it is straightforward to verify that there is no pure strategy equilibrium in the political competition game. This is further discussed in Exercise 22.27.

22.6.6. Beyond Single-Peaked Preferences. Single-peaked preferences played a very important role in the results of Theorem 22.2 by ensuring the existence of a Condorcet winner. However, single peakedness is a very strong assumption and does not have a natural analog in situations in which voting is over more than one policy choice. When there are multiple policy choices (or even voting over "functions" such as nonlinear taxation), much more structure needs to be imposed over voting procedures and agenda setting to determine equilibrium policies. Those issues are beyond the scope of our treatment here. Nevertheless, it is possible to relax the assumption of single-peaked preferences, and also introduce a set of preferences that are "close" to single-peaked in multidimensional spaces. The latter task would take us too far afield from our focus, so will be left to Exercise 22.28. Here we introduce the useful concept of *single-crossing property*, which will enable us to prove a version of Theorem 22.2 under somewhat weaker assumptions.

Definition 22.3. *Consider an ordered policy space $\mathcal{P}$ and also order voters according to their α_i's. Then the preferences of voters satisfy the single-crossing property over the policy space $\mathcal{P}$ when the following statement is true:*

$$\begin{aligned} \text{if } p &> p' \text{ and } \alpha_{i'} > \alpha_i, \text{ or if } p < p' \text{ and } \alpha_{i'} < \alpha_i, \text{ then} \\ U(p;\alpha_i) &> U(p';\alpha_i) \text{ implies that } U(p;\alpha_{i'}) > U(p';\alpha_{i'}). \end{aligned}$$

Example 22.4. To see why single-crossing property is weaker than single-peaked preferences, consider the following example:

1	$a \succ b \succ c$
2	$a \succ c \succ b$
3	$c \succ b \succ a$

It can be verified easily that these preferences are not single peaked. The natural ordering is $a > b > c$, but in this case the preferences of player 2 have two peaks, at a and c. To see why these preferences satisfy single crossing, take the same ordering, and also order players as 1, 2, 3. Now we have

$$\begin{aligned} \alpha &= 2\text{: } c \succ b \Longrightarrow \alpha = 3\text{: } c \succ b \\ \alpha &= 2\text{: } \begin{array}{c} a \succ c \\ a \succ b \end{array} \Longrightarrow \alpha = 1\text{: } \begin{array}{c} a \succ c \\ a \succ b \end{array}. \end{aligned}$$

Notice that while single peakedness is a property of preferences only, the single-crossing property refers to a set of preferences over a given policy space $\mathcal{P}$. It is therefore a joint property of preferences and choices. The following theorem generalizes Theorem 22.2 to a situation with single crossing.

Theorem 22.5. ***(Extended Median Voter Theorem)*** *Suppose that A1 and A2 hold and that the preferences of voters satisfy the single-crossing property. Then a Condorcet winner always exists and coincides with the bliss point of the voter with the median value α_m.*

Proof. The proof works with exactly the same separation argument as in the proof of Theorem 22.2. Consider the median voter with α_m, and bliss policy p_m. Consider an alternative policy $p' > p_m$. Naturally, $U(p_m;\alpha_m) > U(p';\alpha_m)$. Then, by the single crossing property, we have that for all $\alpha_i > \alpha_m$, $U(p_m;\alpha_i) > U(p';\alpha_i)$. Since α_m is the median, this implies that there is a majority in favor of p_m. The same argument for $p' < p_m$ completes the proof □

Given this theorem, the following result is immediate:

Theorem 22.6. ***(Extended Downsian Policy Convergence)*** *Suppose that there are two parties that first announce a policy platform and commit to it and a set of voters that vote for one of the two parties. Assume that A4 holds and that all voters have preferences that satisfy the single-crossing property, and denote the median-ranked bliss point by* p_m*. Then both parties will choose* p_m *as their policy.*

Proof. See Exercise 22.27. □

Despite this generalization, which is quite useful in many applications, and the extension of the MVT presented in Exercise 22.28, the MVT-type results do not apply in many situations with multi-dimensional policies. Exercise 22.29 gives a simple example, which illustrates how widespread the failure of the MVT will be in practice.

22.6.7. Equilibrium Social Welfare Functions. The MVT and the Downsian Policy Convergence Theorems are powerful for the analysis of many models of political economy. However, as Exercise 22.29 illustrates, the assumptions necessary for these theorems do not apply in many interesting (even simple) models. The political economy literature has thus considered a variety of other plausible ways of aggregating heterogeneous preferences within democratic contexts. Three particularly popular approaches are (1) the "probabilistic voting" models, which essentially add some noise in the voting behavior of individuals (for example, because individuals care about some other non-policy characteristic of the parties that are competing for office); (2) models without policy commitment, such as the citizen-candidate models, in which voters elect a politician, who then decides the policies after election; (3) lobbying models, in which some of the individuals or groups in the society can spend money in order to influence the outcome of democratic politics. A full analysis of these models is beyond the scope of the current book. Nevertheless, one feature of many of these formulations is worth noting, especially in light of the discussion of the issue of Pareto efficiency above. Many simple versions of these models lead to equilibria that are equivalent to maximizing a "reduced-form weighted social welfare function". The form of this social welfare function is derived from the political economy equilibrium and depends on the specific assumptions made in these models. For our purposes, the noteworthy point is that in some situations, the political economy equilibrium involves maximizing a weighted social welfare function (given the set of policy instruments), as we have done in Sections 22.2-22.4. We now discuss two standard models that leads to this type of equilibrium (weighted) social welfare functions.

22.6.7.1. *Probabilistic Voting and Swing Voters.* Let the society consist of G distinct groups of voters, with all voters within a group having the same economic characteristics and preferences. As in the Downsian model, there is electoral competition between two parties, A and B, and let π_P^g be the fraction of voters in group g voting for party P where $P = A, B$, and let λ^g be the share of voters in group g and naturally $\sum_{g=1}^{G} \lambda^g = 1$. Then the expected vote share of party P is

$$\pi_P = \sum_{g=1}^{G} \lambda^g \pi_P^g.$$

In our analysis so far, all voters in group g would have cast their votes identically (unless they were indifferent between the two parties). The idea of probabilistic voting is to smooth out this behavior by introducing other considerations in the voting behavior of individuals. Put differently, probabilistic voting models will add "noise" to equilibrium votes, smoothing the behavior relative to models we analyzed so far. In particular, suppose that individual i in group g has the following preferences:

$$\tilde{U}_i^g(p, P) = U^g(p) + \tilde{\sigma}_i^g(P) \tag{22.41}$$

when party P comes to power, where p is the vector of economic policies chosen by the party in power. We assume that $p \in \mathcal{P} \subset \mathbb{R}^K$, where K is an integer, possibly greater than 1. Thus $p \equiv \left(p^1, ..., p^K\right)$ is a potentially multi-dimensional vector of policies. In addition, $U^g(p)$ is the indirect utility of agents in group g as before (previously denoted by $U(p; \alpha_i)$ for individual i) and captures their economic interests. In addition, the term $\tilde{\sigma}_i^g(P)$ captures the non-policy related benefits that the individual will receive if party P comes to power. The most obvious source of these preferences would be ideological. So this model allows individuals within the same economic group to have different ideological preferences.

Let us normalize $\tilde{\sigma}_i^g(A) = 0$, so that

$$\tilde{U}_i^g(p, A) = U^g(p), \text{ and } \tilde{U}_i^g(p, B) = U^g(p) + \tilde{\sigma}_i^g \tag{22.42}$$

In that case, the voting behavior of individual i can be represented as

$$v_i^g(p_A, p_B) = \begin{cases} 1 \text{ if } U^g(p_A) - U^g(p_B) > \tilde{\sigma}_i^g \\ \frac{1}{2} \text{ if } U^g(p_A) - U^g(p_B) = \tilde{\sigma}_i^g \\ 0 \text{ if } U^g(p_A) - U^g(p_B) < \tilde{\sigma}_i^g \end{cases}, \tag{22.43}$$

where $v_i^g(p_A, p_B)$ denotes the probability that the individual will vote for party A, p_A is the platform of party A and p_B is the platform of party B, and as above, we have assumed that if an individual is indifferent between the two parties (inclusive of the ideological benefits), he randomizes his vote.

Let us now assume that the distribution of non-policy related benefits $\tilde{\sigma}_i^g$ for individual i in group g is given by a smooth cumulative distribution function H^g defined over $(-\infty, +\infty)$, with the associated probability density function h^g. The draws of $\tilde{\sigma}_i^g$ across individuals are independent. Consequently, the vote share of party A among members of group g is

$$\pi_A^g = H^g(U^g(p_A) - U^g(p_B)).$$

Furthermore, to simplify the exposition here, suppose that parties maximize their expected vote share. In this case, party A sets this policy platform p_A to maximize:

$$\pi_A = \sum_{g=1}^{G} \lambda^g H^g(U^g(p_A) - U^g(p_B)). \tag{22.44}$$

Party B faces a symmetric problem and maximizes π_B, which is defined similarly. In particular, since $\pi_B = 1 - \pi_A$, party B's problem is exactly the same as minimizing π_A. Equilibrium policies will then be determined as the Nash equilibrium of a (zero-sum) game where both parties make simultaneous policy announcements to maximize their vote share. Let us first look at the first-order condition of party A with respect to its own policy choice, p_A, taking the policy choices of the other party, p_B, as given. This is:

$$\sum_{g=1}^{G} \lambda^g h^g(U^g(p_A) - U^g(p_B)) DU^g\left(p_A\right) = 0,$$

where $DU^g\left(p_A\right)$ is the gradient of $U^g\left(\cdot\right)$ given by

$$DU^g\left(p_A\right) = \left(\frac{\partial U^g\left(p_A\right)}{\partial p_A^1}, ..., \frac{\partial U^g\left(p_A\right)}{\partial p_A^K}\right)^T,$$

with p_A^k corresponding to the kth component of the policy vector p_A. Since the problem of party B is symmetric, it is natural to focus on pure strategy symmetric equilibria. In fact, if the maximization problems of both parties are strictly concave, such a symmetric equilibrium will exist (see Exercise 22.30). Clearly in this case, we will have policy convergence with $p_A = p_B = p^*$, and thus $U^g(p_A) = U^g(p_B)$. Consequently, symmetric equilibrium policies, announced by both parties, must be given by

$$\sum_{g=1}^{G} \lambda^g h^g(0) DU^g\left(p^*\right) = 0. \tag{22.45}$$

It is now straightforward to see that equation (22.45) also corresponds to the solution to the maximization of the following weighted utilitarian social welfare function:

$$\sum_{g=1}^{G} \chi^g \lambda^g U^g\left(p\right), \tag{22.46}$$

where $\chi^g \equiv h^g(0)$ are the weights that different groups receive in the social welfare function. This analysis therefore establishes:

Theorem 22.7. ***(Probabilistic Voting Theorem)*** *Consider a set of policy choices $\mathcal{P}$, let $p \in \mathcal{P} \subset \mathbb{R}^K$ be a policy vector and let preferences be given by (22.42), with the distribution function of $\tilde{\sigma}_i^g$ as H^g. Then if a pure strategy symmetric equilibrium exists, equilibrium policy is given by p^* that maximizes (22.46).*

The important point to note about this result is its seeming generality: as long as a pure strategy symmetric equilibrium in the party competition game exists, it will correspond to a maximum of some weighted social welfare function. This generality is somewhat exaggerated, however, since such a symmetric equilibrium does not always exists. In fact, conditions to guarantee existence of pure strategy symmetric equilibria are rather restrictive and are discussed in Exercise 22.30.

22.6.7.2. *Lobbying.* Consider next a very different model of policy determination, a lobbying model. In a lobbying model, different groups make campaign contributions or pay money to politicians in order to induce them to adopt a policy that they prefer. With lobbying, political power comes not only from voting, but also from a variety of other sources, including whether various groups are organized, how much resources they have available, and their marginal willingness to pay for changes in different policies. Nevertheless, the most important result for us will be that even with lobbying, equilibrium policies will look like the solution to a weighted utilitarian social welfare maximization problem.

To see this, we will quickly review the lobbying model due to Grossman and Helpman (1996). Imagine again that there are G groups of agents, with the same economic preferences. The utility of an agent in group g, when the policy that is implemented is given by the vector $p \in \mathcal{P} \subset \mathbb{R}^K$, is equal to

$$U^g(p) - \gamma^g(p)$$

where $U^g(p)$ is the usual indirect utility function, and $\gamma^g(p)$ is the per-person lobbying contribution from group g. We will allow these contributions to be a function of the policy implemented by the politician, and to emphasize this, it is written with p as an explicit argument.

Following Grossman and Helpman, let us assume that there is a politician in power, and he has a utility function of the form

$$V(p) \equiv \sum_{g=1}^{G} \lambda^g \gamma^g(p) + a \sum_{g=1}^{G} \lambda^g U^g(p), \tag{22.47}$$

where as before λ^g is the share of group g in the population. The first term in (22.47) is the monetary receipts of the politician, and the second term is utilitarian aggregate welfare. Therefore, the parameter a determines how much the politician cares about aggregate welfare. When $a = 0$, he only cares about money, and when $a \to \infty$, he acts as a utilitarian social planner. One reason why politicians might care about aggregate welfare is because of electoral politics (for example, they may receive rents or utility from being in power as in the last subsection and their vote share might depend on the welfare of each group).

Now consider the problem of an individual j in group g. By contributing some money, he might be able to sway the politician to adopt a policy more favorable to his group. But he is one of many members in his group, and there is a natural free-rider problem. He might let others make the contribution, and simply enjoy the benefits. This will typically be an outcome if groups are unorganized (for example, there is no effective organization coordinating their lobbying activity and excluding non-contributing members from some of the benefits etc.). On the other hand, organized groups might be able to collect contributions from their members in order to maximize group welfare.

We will think that out of the G groups of agents, $G' < G$ of those are organized as lobbies, and can collect money among their members in order to further the interests of the group. The remaining $G - G'$ are unorganized, and will make no contributions. Without loss of any generality, let us rank the groups such that groups $g = 1, ..., G'$ to be the organized ones.

The lobbying game takes the following form: every organized lobby g simultaneously offers a schedule $\gamma^g(p) \geq 0$ which denotes the payments they would make to the politician when policy $p \in \mathcal{P}$ is adopted. After observing the schedules, the politician chooses p. Notice the important assumption here that contributions to politicians (campaign contributions or bribes) can be conditioned on the actual policy that's implemented by the politicians. This assumption may be a good approximation to reality in some situations, but in others, lobbies might simply have to make up-front contributions and hope that these help the parties that are expected to implement policies favorable to them get elected.

This is a potentially very complex game, since various different agents (here lobbies) are choosing functions (rather than scalars or factors). Nevertheless, the equilibrium of this lobbying game takes a relatively simple form.

Proposition 22.20. ***(Lobbying Equilibrium)*** *In the lobbying game described above, contribution functions for groups $g = 1, 2...J$, $\{\hat{\gamma}^g(\cdot)\}_{g=1,2..J}$ and policy p^* constitute a subgame perfect Nash equilibrium if:*

(1) *$\hat{\gamma}^g(\cdot)$ is feasible in the sense that $0 \leq \hat{\gamma}^g(p) \leq U^g(p)$.*

(2) *The politician chooses the policy that maximizes its welfare, i.e.,*

$$p^* \in \arg\max_p \left(\sum_{g=1}^{G'} \lambda^g \hat{\gamma}^g(p) + a \sum_{g=1}^{G} \lambda^g U^g(p) \right).$$

(3) *There are no profitable deviations for any lobby, $g = 1, 2, .., G'$, i.e.,*

(22.48)
$$p^* \in \arg\max_p \left(\lambda^g \left(U^g(p) - \hat{\gamma}^g(p) \right) + \sum_{g'=1}^{G'} \lambda^{g'} \hat{\gamma}^{g'}(p) + a \sum_{g'=1}^{G} \lambda^{g'} U^{g'}(p) \right) \text{ for all } g = 1, 2, .., G'.$$

(4) *There exists a policy p^g for every lobby $g = 1, 2, .., G'$ such that*

$$p^g \in \arg\max_p \left(\sum_{g'=1}^{G'} \lambda^{g'} \hat{\gamma}^{g'}(p) + a \sum_{g'=1}^{G} \lambda^{g'} U^{g'}(p) \right)$$

and satisfies $\hat{\gamma}^g(p^g) = 0$. *That is, the contribution function of each lobby is such that there exists a policy that makes no contributions to the politician, and gives her the same utility.*

Proof. (Sketch) Conditions 1, 2 and 3 are easy to understand. No group would ever offer a contribution schedule that does not satisfy Condition 1. Condition 2 has to hold, since the politician chooses the policy. If Condition 3 did not hold, then the lobby could change its contribution schedule slightly and improve its welfare. In particular suppose that this condition does not hold for lobby $g = 1$, and instead of p^*, some $\hat{p}$ maximizes (22.48). Denote the difference in the values of (22.48) evaluated at these two vectors by $\Delta > 0$. Consider the following contribution schedule for lobby $g = 1$:

$$\tilde{\gamma}^1(p) = \lambda_1^{-1}\left[\sum_{g=1}^{G'}\lambda^g\hat{\gamma}^g(p^*) + a\sum_{g=1}^{G}\lambda^g U^g(p^*) - \sum_{g=2}^{G'}\lambda^g\hat{\gamma}^g(p) - a\sum_{g=1}^{G}\lambda^g U^g(p) + \varepsilon c^1(p)\right]$$

where $c^1(p)$ is an arbitrary function that reaches its maximum at $p = \hat{p}$. Following this contribution offer by lobby 1, the politician would choose $p = \hat{p}$ for any $\varepsilon > 0$. To see this note that by part (1), the politician would choose policy $\tilde{p}$ that maximizes

$$\lambda^1\tilde{\gamma}^1(p) + \sum_{g=2}^{G'}\lambda^g\hat{\gamma}^g(p) + a\sum_{g=1}^{G}\lambda^g U^g(p) = \sum_{g=1}^{G'}\lambda^g\hat{\gamma}^g(p^*) + a\sum_{g=1}^{G}\lambda^g U^g(p^*) + \varepsilon c^1(p).$$

Since for any $\varepsilon > 0$ this expression is maximized by $\hat{p}$, the politician would choose $\hat{p}$. The change in the welfare of lobby 1 as a result of changing its strategy is $\Delta - \varepsilon c^1(\hat{p})$. Since $\Delta > 0$, for small enough ε, the lobby gains from this change, showing that the original allocation could not have been an equilibrium.

Finally, condition 4 ensures that the lobby is not making a payment to the politician above the minimum that is required. If this condition were not true, the lobby could reduce its contribution function by a constant, still induce the same behavior, and obtain a higher payoff. □

Next suppose that these contribution functions are differentiable.[4] Then it has to be the case that for every policy choice, p^k, within the vector p^*, we must have from the first-order condition of the politician that

$$\sum_{g=1}^{G'}\lambda^g\frac{\partial\hat{\gamma}^g(p^*)}{\partial p^k} + a\sum_{g=1}^{G}\lambda^g\frac{\partial U^g(p^*)}{\partial p^k} = 0 \text{ for all } k = 1, 2, .., K$$

[4]But there is nothing in the analysis that implies that these functions have to be differentiable; in fact, equilibria with non-differentiable functions are easy to construct. Nevertheless, it is generally thought that equilibria with non-differentiable functions are more "fragile" and thus less relevant. See the discussion in Grossman and Helpman (1994) and Bernheim and Whinston (1986).

and from the first-order condition of each lobby that

$$\lambda^g \left(\frac{\partial \hat{\gamma}^g (p^*)}{\partial p^k} - \frac{\partial U^g (p^*)}{\partial p^k} \right) + \sum_{g'=1}^{G} \lambda^g \frac{\partial \hat{\gamma}^{g'} (p^*)}{\partial p^k} + a \sum_{g'=1}^{G} \lambda^g \frac{\partial U^{g'} (p^*)}{\partial p^k} = 0 \text{ for all } k = 1, 2, .., K \text{ and } g = 1, 2, .., G'.$$

Combining these two first-order conditions, we obtain

$$\frac{\partial \hat{\gamma}^g (p^*)}{\partial p^k} = \frac{\partial U^g (p^*)}{\partial p^k} \tag{22.49}$$

for all $k = 1, 2, .., K$ and $g = 1, 2, .., G'$. Intuitively, at the margin each lobby is willing to pay for a change in policy exactly as much as this policy will bring them in terms of marginal return.

But then this implies that the equilibrium can be characterized as a solution to maximizing the following function

$$p^* = \arg\max_p \left(\sum_{j=1}^{G'} \lambda^g U^g (p) + a \sum_{j=1}^{G} \lambda^g U^g (p) \right).$$

Consequently, the lobbying equilibrium can also be represented as a solution to the maximization of a weighted social welfare function, with individuals in unorganized groups getting a weight of a and those in organized group receiving a weight of $1+a$. Intuitively, $1/a$ measures how much money matters in politics, and the more money matters, the more weight groups that can lobby receive. As $a \to \infty$, we converge to the utilitarian social welfare function.

22.7. Distributional Conflict and Economic Growth: Heterogeneity and the Median Voter

Let us now return to the model of Section 22.2 with linear preferences, but relax the assumption that political power is in the hands of an elite. Instead, we will now introduce heterogeneity among the agents and then apply the tools from the previous section, in particular, the Median Voter Theorem, Theorems 22.2 and 22.5, to analyze the political economy of this model. Recall that these theorems show that if there is a one-dimensional policy choice and individuals have single-peaked preferences (or preferences over the menu of policies that satisfy the single-crossing property), then the political equilibrium will coincide with the most preferred policy of the median voter.

To focus on the main issues in the simplest possible way, I will modify the environment from Section 22.2 slightly. First, there are no longer any elites. Instead, economic decisions will be made by democratic voting among all the agents. Second, to abstract from political conflict between entrepreneurs and workers, I will also assume that there are no workers (recall Exercise 22.3 for why having only entrepreneurs simplifies the analysis; see Exercise 22.31 for an economy where individuals differ both in terms of their productivity and occupation).

Instead, the economy consists of a continuum 1 of *yeoman-entrepreneurs*, each denoted by $i \in [0,1]$ and with access to a neoclassical production function

$$Y_i(t) = F(K_i(t), A_i L_i(t)),$$

where A_i is a time-invariant labor-augmenting productivity measure and will be the only source of heterogeneity among the entrepreneurs. In particular, F satisfies Assumptions 1 and 2. We assume that A_i has a distribution given by $\mu(A)$ among the entrepreneurs. The yeoman-entrepreneur assumption means that each entrepreneur can only employ himself as the worker, so $L_i(t) = 1$ for all $i \in [0,1]$ and for all t. This assumption is important, since otherwise the most productive entrepreneur would hire the entire labor force. Since heterogeneity is the main focus, some way of introducing diminishing returns for each entrepreneur is important, and the yeoman-entrepreneur assumption achieves this in a simple way. I also set the depreciation rate of capital δ equal to 1 to simplify notation.

As noted previously, all agents have linear preferences given by (22.1). Linear preferences again simplify the analysis, by separating the political decisions at different periods. As in Section 22.2, the investment decisions at time $t+1$ will depend only on the tax rate announced for time $t+1$. This latter feature is particularly important here, since we know from the previous section that the Median Voter Theorem does not generally apply with multi-dimensional policy choices. The fact that at each point in time there is only one relevant tax policy will enable us to use the Median Voter Theorem.

More specifically, the timing of events is very similar to that in Section 22.2. At each date t:

(1) there is voting over a linear tax rate on output $\tau(t+1) \in [0,1]$ that will apply to all entrepreneurs in the next period (at $t+1$). We assume that the voting is between two parties with policy commitment, so that Theorems 22.2 and 22.5 (and Theorems 22.4 and 22.6) apply.

(2) the proceeds of the taxation from time $t+1$ are redistributed as a lump-sum transfer to all agents, denoted by $T(t+1) \geq 0$.

We will focus on the Markov Perfect Political Economy Equilibrium of this game.

The important assumption here is that at each stage voting is over the tax rate that will apply in the next period only (with the lump-sum transfer determined from the budget constraint). Moreover, given the linear preferences, each individual takes future taxes as given (independent of current tax decisions and the current capital stock) and only cares about the current tax rate when making its current decisions. Thus, despite the fact that the economy involves an infinite sequence of taxes, the MVT can be applied to the tax decision at each date, provided the other conditions of the theorem are satisfied. We next show that this is the case.

Let us define $k_i(t) \equiv K_i(t)/A_i$ as the effective capital-labor ratio (i.e., the ratio of capital to "effective labor") of entrepreneur i (bearing in mind that its employment level is equal to 1) and recall that p^t determines the sequence of taxes starting from time t. With this definition, we can write the value of each entrepreneur recursively as

(22.50)
$$V_i\left(k_i(t) \mid p^t\right) = \max_{k_i(t+1)\geq 0}\{(1-\tau(t))A_i f(k_i(t)) - A_i k_i(t+1) + T(t) + \beta V_i\left(k_i(t+1) \mid p^{t+1}\right)\},$$

where the fact that total output is equal to $A_i f(k_i(t))$ at time t follows from the constant returns to scale property of F (Assumption 1) and the total amount of capital invested is, by definition, $K_i(t+1) = A_i k_i(t+1)$.

Applying the same type of reasoning as in Section 22.2 to the maximization problem in (22.50), we obtain that the capital-labor ratio by entrepreneur i at time t will satisfy the following first-order condition (see Exercise 22.32):

$$\beta(1-\tau(t+1)) f'(k_i(t+1)) = 1. \tag{22.51}$$

The noteworthy feature is that the choice of the effective capital-labor ratio $k_i(t+1)$ is independent of A_i. This intuitive result implies that all entrepreneurs will choose the same effective capital-labor ratio irrespective of their exact productivity. This is stated in the next proposition:

Proposition 22.21. *Let the tax rate announced for date $t+1$ be τ. Then in any MPE, each entrepreneur $i \in [0,1]$ chooses the effective capital labor ratio $\hat{k}(\tau)$ for date $t+1$ given by*

$$\hat{k}(\tau) = \left(f'\right)^{-1}\left((\beta(1-\tau))^{-1}\right), \tag{22.52}$$

where $(f')^{-1}(\cdot)$ denotes the inverse of the marginal product of capital.

Now given the result in Proposition 22.21, we can calculate total tax revenues, and thus the lump-sum transfer from the government budget constraint, at time $t+1$ as

$$\begin{aligned} T(t+1) &= \int_0^1 \tau(t+1) A_i f\left(\hat{k}(\tau(t+1))\right) di \\ &= \tau(t+1) \bar{A} f\left(\hat{k}(\tau(t+1))\right), \end{aligned} \tag{22.53}$$

where $\bar{A} \equiv \int_0^1 A_i di$ is the mean productivity among the entrepreneurs and $\hat{k}(\cdot)$ is given by (22.52). The first line simply uses the definition of total tax revenue (and per capita lump-sum transfer) as the sum (integral) of output over all entrepreneurs and uses the fact that all entrepreneurs will choose the effective capital-labor ratio $\hat{k}(\tau(t+1))$. The second line takes the terms that do not depend on the identity of the entrepreneur out of the integral and uses the definition of mean productivity $\bar{A}$.

Let us next determine the political bliss point of each entrepreneur, i.e., their most preferred tax rate. To do this, let us write their continuation utility from the end of period t. Ignoring all the terms that are bygone by this point and substituting for best responses (i.e., for the effective capital-labor ratio from (22.52)), the expected discounted utility of entrepreneur i from (22.50) can be written as

$$\tilde{V}_i\left(\tau' \mid p^{t+1}\right) = -A_i\hat{k}\left(\tau'\right) + \beta\left[\left(1-\tau'\right)A_i f\left(\hat{k}\left(\tau'\right)\right) + \tau'\bar{A}f\left(\hat{k}\left(\tau'\right)\right) + \tilde{V}_i\left(p^{t+2}\right)\right], \tag{22.54}$$

where τ' denotes the tax rate announced for date $t+1$ and I have used the notation $\tilde{V}_i$ to distinguish this value function defined over the current tax rate from the value function V_i defined in (22.50). In addition, $\tilde{V}_i\left(p^{t+1}\right)$ is defined as the continuation value from the end of date $t+1$ onwords and I have substituted for the transfer $T(t+1)$ from (22.53).

We can obtain the most preferred tax rate for entrepreneur i, from the expression for $\tilde{V}_i\left(\tau' \mid p^{t+1}\right)$. However, it can be verified easily that $\tilde{V}_i\left(\tau' \mid p^{t+1}\right)$ is not necessarily quasi-concave in τ', thus preferences are not single peaked (see Exercise 22.33). However, we have:

Proposition 22.22. *Preferences given by $\tilde{V}_i\left(\tau' \mid p^{t+1}\right)$ in (22.54) over the policy menu $\tau' \in [0,1]$ satisfy the single crossing property in Definition 22.3.*

Proof. See Exercise 22.33. □

In view of Proposition 22.22, we can apply Theorems 22.5 and 22.6, and conclude that at each date, the tax rate most preferred by the entrepreneur with the median productivity will be implemented. Let this median productivity be denoted by A^m. From (22.54), this most preferred tax rate satisfies the following first-order condition:

$$\left(\bar{A} - A^m\right) f\left(\hat{k}\left(\tau'\right)\right) + \tau'\bar{A}\frac{\left(f'\left(\hat{k}\left(\tau'\right)\right)\right)^2}{\left(1-\tau'\right)f''\left(\hat{k}\left(\tau'\right)\right)} \leq 0 \text{ and } \tau' \geq 0, \tag{22.55}$$

with complementary slackness. In writing this expression, we have made use of condition (22.52) to simplify the expression and also to express the derivative of $\hat{k}'\left(\tau'\right)$, as

$$\hat{k}'\left(\tau'\right) = \frac{f'\left(\hat{k}\left(\tau'\right)\right)}{\left(1-\tau'\right)f''\left(\hat{k}\left(\tau'\right)\right)}.$$

This derivative is strictly negative since $f'' < 0$. Therefore, as in Section 22.2, higher taxes lead to lower capital-labor ratios and lower output (higher distortions). The emphasis on complementary slackness in (22.55) is important here, since the most preferred tax rate of the median voter (entrepreneur) may not satisfy the first-order condition as equality, instead corresponding to a corner solution of $\tau' = 0$. The next proposition shows that this is in fact relevant for a range of distributions of productivity among the entrepreneurs.

Proposition 22.23. *Consider the above-described model. Then there exists $\tau^m \in [0,1)$ such that the unique Markov Perfect Political Economy Equilibrium involves $\tau(t) = \tau^m$ for all t. Moreover,:*

- *if the distribution of productivity among the entrepreneurs, $\mu(A)$, is such that $A^m \geq \bar{A}$, then $\tau^m = 0$;*
- *if $A^m < \bar{A}$, then $\tau^m > 0$;*
- *suppose that $A^m < \bar{A}$. Then, for given $\bar{A}$, τ^m is strictly decreasing in A^m.*

Proof. The argument preceding the proposition combined with Theorems 22.5 and 22.6 shows that the tax rate most preferred by the entrepreneur with the median productivity will be chosen at every period. Moreover, clearly $\tau' = 1$ cannot be preferred by any entrepreneur, since it would lead to zero output by each entrepreneur and thus to zero tax revenues (Exercise 22.1), thus the result that there exists $\tau^m \in [0,1)$ such that $\tau(t) = \tau^m$ for all t follows, with τ^m a solution to (22.55). Note that this equation might have more than one solution, and if so, τ^m corresponds to the global maximizer of (22.54) with productivity evaluated at A^m.

Next suppose that $A^m = \bar{A}$, then the first expression in (22.55) is equal to zero, and the left-hand side of the equation is unambiguously negative for any $\tau' > 0$ and exactly equal to zero for $\tau' = 0$. This establishes that in this case $\tau^m = 0$. If, on the other hand, $A^m > \bar{A}$, then the first expression is strictly negative and the left-hand side of (22.55) is unambiguously negative and the conclusion that $\tau^m = 0$ follows from the complementary slackness conditions.

Finally, suppose that $A^m < \bar{A}$. In this case, the first expression strictly positive. Suppose, to obtain a contradiction, that $\tau^m = 0$. Then the second term is exactly equal to 0 (since τ' is in the numerator). Consequently, the left-hand side of (22.55) is strictly positive and $\tau^m = 0$ cannot be a solution. Hence the unique equilibrium tax rate must be $\tau^m > 0$. To obtain the comparative static result, simply apply the Implicit Function Theorem to (22.55) and use the fact that since τ^m is a global maximum, the derivative of (22.55) with respect to τ^m is negative. □

There are a number of important results in this proposition. First, it shows that linear preferences guarantee the existence of a well-defined Markov Perfect Political Economy Equilibrium even when there is heterogeneity among the individuals in terms of their productivity. The important role played by linear preferences in this result cannot be overstated (recall Section 22.5). As discussed at the end of this chapter, there are a number of results in the literature similar to this proposition, but they do not correspond to well-defined Markov Perfect Equilibria because they do not feature linear preferences (instead, they can be thought of as the equilibria in which individuals vote once at the beginning of time and have no option to change taxes thereafter).

Second, this proposition shows that if the productivity of the median voter is above the average, there will be no redistributive taxation. This is intuitive. As the first term in (22.55) makes it clear, the benefits of taxation are proportional to the average productivity in the economy, while the cost (to the median voter) is related to his productivity. If the median entrepreneur is more productive than the average, there are two forces making him oppose redistributive taxation; he is effectively redistributing away from himself, and there is also the distortionary effect of taxation captured by the second term in (22.55).

Third and more important, in the case in which the productivity of the median voter is below the average, the political equilibrium will involve positive (distortionary) taxation on all entrepreneurs. To obtain the intuition for this result, recall that taxes revenues are equal to zero at $\tau = 0$. A small increase in taxes starting at $\tau = 0$ has a second-order loss for each entrepreneur and when $A^m < \bar{A}$, a first-order redistributive gain for the median voter. This result is important in part because most real-world wealth and income distributions appear to be skewed to the left (with the median lower than the mean), thus this configuration is more likely in practice. Furthermore, this result is most interesting in comparison with those in Section 22.2, which also involved positive distortionary taxation, but in an environment in which a non-productive elite was in power. Proposition 22.23 shows that the same *qualitative* result generalizes to the case in which there is democratic politics and the decisive (median) voter is a productive entrepreneur himself, but is less productive than average. This implies that the essence of the results obtained in our analysis of elite-dominated politics applies much more generally.

Finally, Proposition 22.23 gives a new comparative static result. It shows that, holding average productivity constant, a decline in the productivity of the median entrepreneur (voter) leads to an increase in the amount of distortionary taxes. Since as in Section 22.2, higher taxes correspond to lower output and the larger gap between the mean and the median of the productivity distribution can be viewed as a measure of inequality, this result suggests a political mechanism via which greater inequality may translate into higher distortions and lower output.

Nevertheless, some care is necessary in interpreting this last result, since the gap between the mean and the median is *not* a (formally appropriate) measure of inequality. Exercise 22.34 gives an example in which a mean preserving spread of the distribution lead to a smaller gap between the mean and the median. This caveat notwithstanding, the literature interprets this last result as suggesting that greater inequality should lead to lower output and lower growth. Exercise 22.35 presents a version of the model here where taxes affect the equilibrium growth rate.

22.8. The Provision of Public Goods: Weak Versus Strong States

The analysis so far has emphasized the distortionary effects of taxation and expropriation. This paints a picture whereby the major determinants of poor economic performance are high taxes or some type of expropriation, and political economy does (or should) focus on the determination of the incentives for redistributive taxation. While the disincentive effects of taxation cannot be denied, whether or not taxes are high is only one of the dimensions of policy that might affect economic growth. For example, in many of the endogenous growth models appropriate R&D or industrial policy might encourage faster growth (even if it involves some taxation of capital and labor). More generally, public good provision, investment in infrastructure and provision of law and order are important roles of a government, and the failure of the government to perform these roles may also have negative consequences for economic performance. In fact, existing evidence does not support the view that growth (or high levels of output) are strongly associated with taxation. On the contrary, poor economies typically have lower levels of tax revenues and government spending. This is most stark if we compare the OECD to sub-Saharan Africa. Consequently, the political economy of growth must also pay attention to whether governments perform the roles that they are supposed to. The standard non-political-economy (for example the traditional public finance) approach to this question starts by positing the existence of a benevolent government and looks for policy combinations that would maximize social welfare. Once we incorporate political economy considerations, however, we recognize that the political elite that control the government may not have an interest in investing in public goods. If public good investments (or investments in infrastructure or law and order) are an important determinant of the growth performance of an economy, it then becomes essential to investigate under what circumstances governments will undertake such investments.

In this section, I present the simplest model that can shed light on this topic, which is based on Acemoglu (2005). The economy consists of a political elite controlling the government and a set of citizens with access to production opportunities. Productivity depends on public good investments by the government. The government, on the other hand, will only undertake these investments, if these are beneficial to the political elite. In this section, I therefore investigate the conditions under which a greater amount of investment in public good will be undertaken in a well-defined political equilibrium. Not surprisingly, the extent of public good provision will depend on the future returns that the political elite can secure by undertaking such investments, which is related to the issue of weak versus strong states. If the state is very weak, the elite will be unable to raise taxes in the future and reap the benefits of their investments. Anticipating this, they will be unwilling to invest in public goods. On the other hand, if there are no checks on the ability of the elite to impose taxes on the population, then the state is too "strong," and private investment will be stifled. Thus states that have intermediate levels of strength are most conducive to economic growth.

This model will also enable us to discuss the potential distinction between expropriation and taxes, an issue raised at the beginning of this chapter.

22.8.1. The Model. All agents have again linear preferences given in (22.1). The population consists of a set of yeoman-entrepreneurs (citizens), with mass normalized to 1, and a political elite. The elite do not engage in production but control the government and thus will decide the levels of taxation and public good provision. Without loss of any generality, we also normalize the size of the political elite to 1.

Each citizen i has access to the following Cobb-Douglas production technology to produce the unique final good in this economy:

$$Y_i(t) = \frac{1}{\alpha} K_i(t)^{\alpha} \left(A(t) L_i(t)\right)^{1-\alpha}, \tag{22.56}$$

which only differs from (22.17) because here $A(t)$ is time-varying. We will assume that $A(t)$ will be determined by the public good investments of the government. Given the assumption that citizens correspond to yeoman-entrepreneurs, $L_i(t) = 1$ for all $i \in [0, 1]$ and for all t.

The timing of events is similar to the baseline model with holdup, in that taxes on output are set at time t, whereas capital investments for time t are decided at $t-1$. There is again a maximum tax rate $\bar{\tau}$. However, instead of the constitutional limits on taxation, here we suppose that this maximum tax rate arises from the possibility that producers might hide their output (or move to the informal sector) if they face very high taxes. For example, if they do so, they lose a fraction $\bar{\tau}$ of their output, so that with a tax rate above $\bar{\tau}$, all producers would move to the informal sector and tax revenues would be equal to zero. Consequently, the tax rate at any point in time has to be $\tau(t) \in [0, \bar{\tau}]$. With this interpretation, $\bar{\tau}$ corresponds to the (economic) *strength of the state*. When $\bar{\tau}$ is high, we have a strong state, which can raise high taxes. When it is low, the state is unable to raise high taxes.

As usual, given a tax rate $\tau(t) \in [0, \bar{\tau}]$, tax revenues are

$$\text{Tax}(t) = \tau(t) \int_0^1 Y_i(t)\, di = \tau(t) Y(t), \tag{22.57}$$

where $Y(t)$ is total output. Naturally, if the tax rate is above $\bar{\tau}$, tax revenues are equal to zero, because all production shifts to the informal economy.

The government (the political elite) at time t decides how much to spend on public goods for the next date, thus on $A(t+1)$. I assume that

$$A(t) = \left[\frac{\alpha\phi}{1-\alpha} G(t)\right]^{1/\phi} \tag{22.58}$$

where $G(t)$ denotes government spending on public goods, and $\phi > 1$, so that there are decreasing returns in the investment technology of the ruler (a greater ϕ corresponds to greater decreasing returns). The term $[\alpha\phi/(1-\alpha)]^{1/\phi}$ is included as a convenient normalization. In addition, (22.58) implies full depreciation of $A(t)$, which simplifies the analysis below. The consumption of the elite is given by whatever is left over from tax revenues after expenditure and transfers, thus is equal to $C^E(t) = \text{Tax}(t) - G(t)$.

Let us first characterize the first-best level of public good provision, where there are no distortionary taxes on entrepreneurs and the level of public good provision is chosen to maximize the net present discounted value of a representative entrepreneur. Given the production function and the timing of events here, which corresponds to that of the canonical elite-dominated model with Cobb-Douglas technology, the equilibrium capital-labor ratio of an entrepreneur is the same as in (22.18) above. Consequently, the first-best level of public good investment can be computed as

$$A(t) = A^{fb} \equiv \beta^{1/(\phi-1)(1-\alpha)} \tag{22.59}$$

and the first-best levels of the capital-labor ratio and output are

$$k^{fb} \equiv \beta^{\phi/(\phi-1)(1-\alpha)} \text{ and } Y^{fb} \equiv \frac{1}{\alpha}\beta^{(\phi\alpha+1-\alpha)/(\phi-1)(1-\alpha)}.$$

Let us next focus on the Markov Perfect Equilibrium (MPE) of this game. As usual, a MPE is defined as a set of strategies at each date t, such that these strategies only depend on the current (payoff-relevant) state of the economy, $A(t)$, and on prior actions within the same date according to the timing of events above. Thus, a MPE can be represented by $\left(\tau(A(t)), [k_i(A(t))]_{i\in[0,1]}, G(A(t))\right)$, where, by definition of a MPE, the key actions, which consist of the tax rate on output, τ, the capital-labor ratio decision of each entrepreneur $[k_i]_{i\in[0,1]}$, and the government expenditure on public good, G, are conditioned on the current payoff-relevant state variable, $A(t)$. Clearly, since each yeoman-entrepreneur employs only himself, the capital-labor ratio, k_i, and the total capital stock, K_i, of each entrepreneur are identical.

It is clear that in any MPE, the unique equilibrium tax rate for the political elite will be

$$\tau(t) = \bar{\tau} \text{ for all } t, \tag{22.60}$$

since investment decisions are already sunk at the time the elite set the taxes.

Next, the capital-labor ratio of entrepreneurs is again given by (22.18), and thus can be written as

$$k_i(t) = (\beta(1-\bar{\tau}))^{1/(1-\alpha)} A(t) \text{ for all } i \in [0,1] \text{ and for all } t. \tag{22.61}$$

Combining this expression with (22.56) and (22.57), we obtain equilibrium tax revenue as a function of the level of public goods as:

$$T(A(t)) = \frac{(\beta(1-\bar{\tau}))^{\alpha/(1-\alpha)}\bar{\tau}A(t)}{\alpha}. \tag{22.62}$$

Finally, the elite will choose public investment, $G(t)$ to maximize his consumption. To characterize this, let us write the discounted net present value of the elite as

$$V^e(A(t)) = \max_{A(t+1)}\left\{T(A(t)) - \frac{1-\alpha}{\alpha\phi}A(t+1)^{\phi} + \beta V^e(A(t+1))\right\}, \tag{22.63}$$

which simply follows from writing the discounted payoff of the elite recursively, after substituting for their consumption, $C^E(t)$, as equal to taxes given by (22.62) minus their spending on public goods from equation (22.58).

Since, for $\phi > 1$, the instantaneous payoff of the elite is bounded, continuously differentiable and concave in A, so Theorems 6.3, 6.4 and 6.6 in Chapter 6 imply that the value function $V^e(\cdot)$ is concave and continuously differentiable. Hence, the first-order condition of the ruler in choosing $A(t+1)$ can be written as:

$$\frac{1-\alpha}{\alpha}A(t+1)^{\phi-1} = \beta(V^e)'(A(t+1)), \tag{22.64}$$

where $(V^e)'$ denotes the derivative of the value function of the elite. This equation links the marginal cost of greater investment in public goods to the greater value that will follow from this. To make further progress, I use the standard envelope condition, which is obtained by differentiating (22.63) with respect to $A(t)$:

$$(V^e)'(A(t+1)) = T'(A(t)) = \frac{(\beta(1-\bar{\tau}))^{\alpha/(1-\alpha)}\bar{\tau}}{\alpha}. \tag{22.65}$$

The value of greater public goods for the elite is the additional tax revenue that this will generate, which is given by the expression in (22.65).

Combining these conditions, we obtain the unique Markov Perfect Equilibrium choice of the elite as:

$$A(t+1) = A[\bar{\tau}] \equiv \left(\beta^{1/(1-\alpha)}(1-\alpha)^{-1}(1-\bar{\tau})^{\alpha/(1-\alpha)}\bar{\tau}\right)^{\frac{1}{\phi-1}}, \tag{22.66}$$

which also defines $A[\bar{\tau}]$ as an expression that will be useful below. Substituting (22.66) into (22.63) yields a simple form of the elite's value function:

$$V^e(A(t)) = \frac{(\beta(1-\bar{\tau}))^{\alpha/(1-\alpha)}\bar{\tau}A(t)}{\alpha} + \frac{\beta^{1/(1-\alpha)}(\phi-1)(1-\bar{\tau})^{\alpha/(1-\alpha)}\bar{\tau}}{(1-\beta)\phi\alpha}A[\bar{\tau}]. \tag{22.67}$$

The second term in (22.67) follows since the level of public with spending implied by (22.66) is equal to a fraction $1/\phi$ of tax revenue. The value of the elite naturally depends on the current state of public goods, $A(t)$, inherited from the previous period, and from this point on, the equilibrium involves investment levels given by (22.61) and (22.66).

Proposition 22.24. *In the above-described economy, there exists a unique MPE where $\tau(A) = \bar{\tau}$ for all A, $A(t)$ is given by $A[\bar{\tau}]$ as in (22.66) for all $t > 0$, and, the capital-labor ratio of each entrepreneur $i \in [0,1]$ and for all t is given by (22.61). For all $t > 0$, the equilibrium level of aggregate output is:*

$$Y(t) = Y[\bar{\tau}] \equiv \frac{1}{\alpha}(\beta(1-\bar{\tau}))^{\alpha/(1-\alpha)} A[\bar{\tau}]. \tag{22.68}$$

Proof. See Exercise 22.36. □

Note that because of linear preferences and full depreciation of public goods, the economy reaches the steady-state level of output in one period (and $Y(0)$ is not given by (22.68), but instead depends on the initial value of the public goods, $G(0)$).

22.8.2. Weak Versus Strong States. The first result implied by Proposition 22.24 is the importance of the strength of the state as parameterized by $\bar{\tau}$, its ability to raise revenues. When $\bar{\tau}$ is high, the state is "economically powerful"—citizens have little recourse against high rates of taxes. In contrast, when $\bar{\tau}$ is low, the state is "economically weak" (and there is "limited government"), since it is unable to raise taxes. With this interpretation, we can now ask whether greater economic strength of the state leads to worse economic outcomes. The answer is ambiguous, as it can be seen from the fact that when $\bar{\tau} = 0$, i.e., when the state is extremely weak, the elite will choose $G(t) = 0$, while with $\bar{\tau} = 1$, the citizens will choose zero investments. In both cases, output will be equal to zero.

It is straightforward to determine the level of $\bar{\tau}$ that maximizes output in the society at all dates after the initial one, i.e., $Y(t)$ for $t > 0$. It is given by $\max_{\bar{\tau}} Y[\bar{\tau}]$, where $Y[\bar{\tau}]$ is given by (22.68). Exercise 22.36 shows that the solution to this program, denoted $\bar{\tau}^*$, is

$$\bar{\tau}^* \equiv \frac{1-\alpha}{1-\alpha+\alpha\phi}. \tag{22.69}$$

If the economic power of the state is greater than $\bar{\tau}^*$, then the state is too powerful, and taxes are too high relative to the output-maximizing benchmark. This corresponds to the standard case on which the political economy models we have studied so far have focused. In contrast, if the economic power of the state is less than $\bar{\tau}^*$, then the state is not powerful enough for there to be sufficient rents in the future to entice the elite to invest in public goods. This corresponds to the case, where contrary to what we have emphasized so far, the state is not too powerful, but too weak. Consequently, the elite do not have the correct incentives to invest in productivity-enhancing public goods. Therefore, we have another example of non-growth-enhancing institutions/policies, but this time resulting from the weakness of the state.

There is an interesting parallel to the theory of the firm here. In the theory of the firm, the optimal structure of ownership and control gives ex post bargaining power to the parties that have more important investments. The same principle applies to the allocation of economic strength as captured by the parameter $\bar{\tau}$; greater power for citizens is beneficial when their investments matter more. When it is the state's investment that is more important for economic development, a higher $\bar{\tau}$ is required (justified).

The above discussion focused on the output-maximizing value of the parameter $\bar{\tau}$. Equally relevant is the level of $\bar{\tau}$, say $\bar{\tau}^e$, which maximizes the beginning-of-period payoff to the elite and $\bar{\tau}^c$, which maximizes the beginning-of-period payoff to the citizens. One might also define a tax rate $\bar{\tau}^{wm}$ that maximizes the beginning-of-period payoff to a fictitious social planner weighing the elite and the citizens equally. The next proposition shows how these tax rates compare to the output-maximizing tax rate $\bar{\tau}^*$:

Proposition 22.25. *Let $\bar{\tau}^*$, $\bar{\tau}^{wm}$, $\bar{\tau}^e$ and $\bar{\tau}^c$ be the values of $\bar{\tau}$ that respectively maximize output, social welfare, the elite's utility and citizens' utility for all $t > 0$. Then*

$$0 < \bar{\tau}^c < \bar{\tau}^* < \bar{\tau}^e < 1 \text{ and } 0 < \bar{\tau}^c < \bar{\tau}^{wm} < \bar{\tau}^e < 1.$$

Proof. See Exercise 22.36. □

The main conclusion from this analysis is that when both the state and the citizens make productive investments, it is no longer necessarily true that limiting the rents that accrue to the state is always good for economic performance. Instead, there needs to be a certain degree of *balance of powers* between the state and the citizens. When the political elite controlling the power of the state expect too few rents in the future, they have no incentive to invest in public goods. Consequently, excessively weak states are likely to be as disastrous for economic development as the unchecked power and expropriation by excessively strong states.

A number of shortcomings of the analysis in this section should be noted at this point. The first is that it relied on economic exit options of the citizens in the informal sector as the source of their control over the state, whereas, in practice, political controls may be more important. The second is that it focused on the MPE, without any possibility of an implicit agreement between the state and the citizens, whereby the state raises sufficient tax revenue to both finance public good investments and also distribute some of it to the elite, and the citizens are willing to put up with this high level of taxation because of the benefits that they receive from public good provision, and the elite prefer not to deviate to higher levels of taxes. Acemoglu (2005) generalizes the results presented here in these directions. First, similar results can be obtained when the constraints on the power of the state are not economic, but political. In particular, we can envisage a situation in which citizens can (stochastically) replace the government if taxes are too high. In this case, when citizens are politically powerful, this will limit the extent of taxation, but also the amount of public good provision. Second, using a model with variable political checks on the state, one can analyze the SPE, where there might be an implicit agreement between the state and the citizens to allow for some amount of taxation and also correspondingly high levels of public good provision. In Acemoglu (2005), I referred to this equilibrium configuration as a "consensually-strong state," since the citizens allow the economic power of the state to be high (partly because they believe they can politically control the state). The configuration with the consensually-strong state might provide an insight into why many OECD countries have higher tax rates and higher levels of public good provision than many less-developed economies.

This perspective also suggests a useful distinction between taxation and expropriation. As we have seen so far, high taxes have similar effects on investment and economic performance as expropriation. One difference between expropriation and taxes might be uncertainty. It can be argued that producers know exactly at what rate they will be taxed, while expropriation is riskier. In the presence of risk aversion, expropriation could be considerably more costly than taxation. However, the analysis here suggests another useful distinction, which comes not from the revenue side but from the expenditure side; expropriation might correspond to the government taking a share of the output the producers for its own consumption, while taxation, along-the-equilibrium path, involves some of the revenues being spent for public goods, which are useful for the producers. If this distinction is important, one of the reasons why taxation is viewed as fundamentally different from expropriation may be because taxation is often associated with some of the proceeds being given back to the citizens in the form of public goods.

Perhaps the most important implication of the analysis in this section is to emphasize different aspects of growth-enhancing institutions. Economic growth not only requires secure property rights and low taxes, but also complementary investments, often most efficiently undertaken by the government. Provision of law and order, investment in infrastructure and public goods are obvious examples. Thus growth-promoting institutions need to provide some degree of security of property rights to individuals, but also incentivize the government to undertake the appropriate public good investments. In this light, excessively weak governments might be as costly as the unchecked power of the government.

22.9. Taking Stock

This chapter made a first attempt at investigating why institutions and policies differ across societies. Even though this question may be viewed as part of the study of political economy rather than economic growth per se, in the absence of satisfactory answers to this question, our understanding of the process of economic growth will be limited. The evidence provided in Chapter 4 suggested that societies often choose different institutions and policies, with very different implications for economic growth. Thus to understand why some countries are poor and some are rich, we need to understand why some countries choose growth-enhancing policies while others choose policies that block economic development.

This chapter emphasized a number of key themes in developing answers to these questions. First, the sources of different institutions (and non-growth-enhancing institutions) must be sought in *social conflict* among different individuals and groups in the society. Social conflict implies that there is no guarantee for the society to adopt economic institutions and policies that will encourage economic growth. Such social arrangements will benefit many individuals in the society, but they will also create losers, groups whose rents will be destroyed or eroded by the introduction of new technologies or by the process of economic growth. When individuals in the society have conflicting preferences over institutions and policies, the distribution of *political power* in the society plays an important role in determining which institutions and policies will be chosen (and whether non-growth-enhancing institutions will be reformed).

I emphasized that non-growth-enhancing policies can emerge even without any significant Pareto inefficiencies. In particular, a range of political economy models lead to equilibrium allocations that would also result from the maximization of a weighted social welfare function. The resulting equilibrium allocation will naturally be constrained Pareto efficient. However, Pareto efficiency does not guarantee high output or growth. I illustrated this first by focusing on a "simple society," where individuals belong to a social group, the conflict of interest is among social groups, and all political power rests in the hands of a political elite. I showed that this environment, combined with linear preferences, implies that even the more restrictive Markov Perfect Equilibrium (MPE) concept leads to constrained Pareto efficient allocations. Despite the Pareto efficiency of the equilibrium allocations, we also saw that there are two distinct sets of reasons for distortionary policies, which will discourage investment and economic growth (suggesting as a byproduct that Pareto efficiency may not be the right concept to focus in the analysis of the political economy of growth). The first is *revenue extraction*, while the second results from competition between the elite and other social groups in the marketplace or in the political arena, and can take the form of *factor price manipulation effects* or *political replacement effects*. Revenue extraction involves the use of distortionary taxes by the elite (because those are the only fiscal instruments available to them) in order to extract revenues from entrepreneurs and workers in the society. Factor price manipulation results when the elite use policies in order to reduce the labor demand (or more generally the factor demand) coming from other social groups, so that they face more favorable factor prices. Finally, the political replacement effect emerges when the elite tries to impoverish social groups that might politically compete with themselves. The analysis demonstrated that revenue extraction, though distortionary, is typically much less harmful to economic growth than factor price manipulation and political replacement effects, because ultimately the elite can only raise revenues if the groups that are being taxed have the incentives (and the ability) to undertake investments and produce. In contrast, the factor price manipulation and the political replacement effects encourage the elite to pursue policies that harm groups that they perceive as their competitors. This typically leads to higher taxes and also to explicit actions to block technology adoption or other productivity-enhancing investments by competing entrepreneurs. The consequences of these types of policies for economic growth can be disastrous. Remarkably, however, all of this can happen while the equilibrium is still constrained Pareto efficient—it is so, because all weight is given to the elite without any regard to the welfare of the other individuals and groups in the society.

In addition to providing a simple and useful framework for the analysis of policy, the framework with political power vested in the elite also leads to a range of comparative static results that shed light on what types of societies will adopt policies that encourage growth and which societies are likely to pursue non-growth-enhancing policies or even try to block economic development. The following are some of the main comparative static results: (1) taxes are likely to be higher when the demand for capital by entrepreneurs is inelastic, because in this case the revenue-maximizing tax rate for the elite is higher; (2) taxes are likely to be higher when the factor price manipulation effect is more important relative to the revenue extraction effect; (3) taxes are higher when the political power of the elite is contested and reducing the income level of the competing groups will lead to political consolidation for the elite; (4) in the absence of the political replacement effect, greater state capacity leads to lower taxes; (5) when the political replacement effect is important, both greater state capacity and greater rents from natural resources lead to more distortionary policies because they increase the political stakes, i.e., the value of holding on to political power.

Pareto efficiency is not a general property of political economy models, however. In particular, once the timing of policies is changed so that taxes by the politically powerful are set after entrepreneurial investments or when long-term investments are important, serious holdup problems emerge. At the simple level, the holdup problem corresponds to a situation in which the politically powerful can not commit to not taxing investments once they have been undertaken. Anticipating these high taxes, entrepreneurs refrain from investment. In such an environment, MPE leads to constrained Pareto inefficient equilibria. Subgame Perfect Equilibria (SPE) then sometimes significantly improves over the MPE. Whether the MPE or the SPE is the appropriate equilibrium concept depends on institutional and historical details, which affect whether individuals and groups can coordinate their actions in equilibrium. In any case, we also saw that even with SPE the equilibrium might involve a lack of commitment to a desirable tax sequence by the elite. In this case, if feasible, the elite may introduce economic institutions providing greater security of property rights to entrepreneurs so as to encourage investments (and thus increase tax revenues). Thus the possibility of commitment problems provides us with one perspective for thinking about the emergence of secure property rights.

To move beyond models in which political power rests with a pre-specified group, here the political elite, we need systematic ways of aggregating heterogeneous political preferences. After reviewing some basic political economy theory, I used the well-known Median Voter Theorem (MVT), which applies in certain economic environments, to investigate the determination of equilibrium policies in an economy with heterogeneous entrepreneurs. One of the most interesting results of this analysis is that the revenue extraction mechanism emphasized in the context of elite-dominated politics is also present in more complex societies with heterogeneity among entrepreneurs. In particular, if the median voter is poorer than the average individual (entrepreneur) in the society, he may want to use distortionary policies to transfer resources to himself. This type of distortionary revenue extraction by the median voter is qualitatively similar to the use of distortionary policies by the elite to extract revenues from middle-class entrepreneurs. Nevertheless, this effect exhibits itself in a more general environment with heterogeneity among the entrepreneurs and also leads to a new comparative static result; when the gap between the mean and the median of the productivity distribution is greater, the incentives to extract revenues are stronger and policies are more likely to be distortionary.

Finally, I emphasized that taxation is not the only relevant policy affecting economic growth. The provision of public goods, in the form of securing law and order, investments in infrastructure or even appropriate subsidies, might also be important for inducing a high rate of economic growth. Will the state provide the appropriate amount and type of public goods? In the context of a political economy model, the answer to this question depends on whether the politically powerful groups controlling the state have the incentives to do so. As already discussed above, the elite may want to block economic development in order to affect the factor prices that it faces or to secure their political position. Beyond this, the elite would only invest in public goods if they expect to reap the benefits of these investments in the future. This raises the issue of weak versus strong states. While an emphasis on taxes suggests that checks on the economic or political power of the state should be conducive to more growth-enhancing policies, weak states will be unwilling to invest in public goods because those controlling the state realize that they will not be able to tax future revenues created by these public good investments. Consequently, an intermediate strength of the state might be most conducive to growth-enhancing policies. The more important point here is that an analysis of the effect of economic institutions and policies on growth should take into account both the incentives to private agents and also the incentives to the government for providing the appropriate amount and type of public goods.

The material in this chapter is no more than an introduction to the exciting and important field of political economy of growth. Many issues have not been addressed. Among those the following appear most important: first, in addition to taxes, expropriation and public goods, whether the society provides a level playing field to a broad cross-section of society is important. For example, broad-based human capital investments, which may be quite important for modern economic growth, require the provision of appropriate incentives not only to a few businesses, but to the entire population. Similarly, security of property rights for existing businesses have to be balanced against the ease of entry for new firms. Second, the entire analysis here took the distribution of political power in the society, and the political institutions that generated this distribution of political power, as given. It is clear, however, that different distributions of power in the society will lead to different policies and thus to different growth trajectories. Consequently, it seems important to understand how the distribution of political power and equilibrium political institutions might evolve endogenously and whether it might interact with the economic equilibrium. Some of these issues will be discussed in the next chapter.

22.10. References and Literature

The material in this chapter draws on the large political economy literature and also on some of the recent work on the political economy of growth. My purpose has not been to provide a balanced survey of these literatures, but to emphasize the most important features pertaining to the sources of differences in economic institutions and policies across societies with the hope of shedding some light on differential cross-country growth performances. As noted above, I focused throughout on the neoclassical growth model and its variants in order to isolate the contribution of political economy mechanisms and also to keep the exposition manageable.

Persson and Tabellini (2000) provides an excellent survey of much of the work done in political economy in the 1980s and 1990s, though does not focus on the political economy of growth. Drazen (2001) also provides an excellent introduction to this work, with slightly more emphasis on growth issues. Eggertsson (2005) provides a non-formal discussion of the same issues as well as a wider set of political economy questions.

The material in Sections 22.2, 22.3 and 22.4 and the discussion of revenue extraction and factor price manipulation effects draw upon Acemoglu (2007a,b), but the setup has been modified to be more consistent with the neoclassical growth model. Versions of the factor price manipulation effect feature in Acemoglu (2008), which will be discussed in the next chapter, and also in Galor, Moav and Vollrath (2005), who emphasize how the land-owning elite may discourage investment in human capital. The political replacement effect is also discussed in Acemoglu (2007a,b), though it originates in Acemoglu and Robinson (2000b).

A detailed analysis of why the political elite may block technological innovations in order to increase the likelihood of their survival is presented in Acemoglu and Robinson (2007a). That paper also shows how both relatively secure elites and elites that are in competitive political environments will not have incentives to block technological change, but those with intermediate levels of security that might be challenged by new technologies are likely to adopt policies that will block economic development. It also provides historical examples of this type of behavior. Models with competitive economic behavior by price-taking agents, but strategic political decisions were first developed by Chari and Kehoe (1990, 1993), though the focus in these papers is on the "time-consistency" of the behavior of a benevolent government.

The material in Section 22.5 builds upon the analysis of MPE in competitive economies with capital accumulation in Krusell and Rios-Rull (1997), Klein, Krusell and Rios-Rull (2004) and Hassler, Krusell, Storesletten and Zilibotti (2004). The first two papers compute the MPE numerically in a related environment, while the last one contains characterization results for an economy with quadratic preferences and linear technology. Acemoglu and Robinson (2000, 2001, 2007a) and Hassler, Rodriguez-Mora, Storesletten and Zilibotti (2003) provide explicit characterizations of MPE in simpler political environments.

The material in Section 22.6 is largely standard. An excellent introduction to social choice theory, with a thorough discussion of Arrow's Theorem, is provided in Austen-Smith and Banks (2000). My proof of the theorem here builds on the somewhat longer proof in Austen-Smith and Banks (2000). Arrow (1951) is still the classic for the basis of social choice theory and for Arrow's Theorem, though similar ideas were also developed in earlier work by Black (1948). The single crossing property is introduced in Roberts (1977) and further developed by Gans and Smart (1996). The notion of intermediate preferences introduced in Exercise 22.28 is due to Grandmont (1978). The Downsian model of political competition is introduced in Downs (1957), and builds heavily on Hotelling's seminal (1929) paper. Austen-Smith and Banks (2000) and Persson and Tabellini (2000) discuss the Downsian party competition model in detail. The probablilistic voting model is due to Lindbeck and Weibull (1987) and Coughlin (1992). Persson and Tabellini (2000) provided detailed treatment of this model. My exposition here was simplified by the assumption that parties care about their vote share, not the probability of coming to power. There are many different lobbying models in the literature. The first one was formulated by Becker (1983). The one presented here builds on Grossman and Helpman (1994), which in turn builds on the menu auctions approach of Bernheim and Winston (1986). Grossman and Helpman (2001) provides a more detailed exposition of various different lobbying models.

Section 22.7 presents one of the most standard models of distributional conflict, which uses the celebrated Median Voter Theorem (which was presented in Section 22.6). The Median Voter Theorem was first applied to an economy with linear redistributive taxes by

Roberts (1977) and Romer (1975). Meltzer and Richard (1981) used the Roberts-Romer model to relate inequality to taxes and more importantly, to draw implications about the extent of the voting franchise on the size of the government. Meltzer and Richard's work is a classic as it can be viewed as the beginning of positive political economy, i.e., the use of political economy models in order to explain cross-country and over-time differences in public policies. A number of authors have since applied the Roberts-Romer model in growth settings. The most notable examples are Alesina and Rodrik (1994), Persson and Tabellini (1994), Saint-Paul and Verdier (1996) and Benabou (2000). The models in Alesina and Rodrik (1994) and Persson and Tabellini (1994) are very similar to the one I developed in Section 22.7, except for some technical details and except that they focus on an economy with endogenous growth, so that differences in taxes lead to differences in equilibrium growth rates (see Exercise 22.35). Both Alesina and Rodrik and Persson and Tabellini emphasize the negative effects of inequality on economic growth, interpreting the gap between the mean and the median as a measure of inequality. They also present cross-country evidence suggesting that inequality is negatively correlated with economic growth. This cross-country growth evidence is difficult to interpret, however, since there are many omitted variables in such growth regressions, and other researchers have found no relationship, and some have even found a positive relationship, between inequality and growth (see, for example, Barro, 2000, Banerjee and Duflo, 2003, and Forbes, 1996). Saint-Paul and Verdier (1996), on the other hand, showed that higher inequality can lead to greater growth, because tax revenues may be invested in human capital accumulation. Benabou's important (2000) paper pushes this idea further, and shows how a negative relationship between inequality and growth is consistent with higher inequality leading to less redistribution in a world in which greater redistribution may be growth-enhancing, again because taxes are partly invested in education. None of these papers characterize the MPE of a dynamic economy, instead assuming that voting is either myopic or is done once at the beginning of time. Thus the model in Section 22.7 is a small contribution to this literature as it derives these results as a well-defined MPE of a simple neoclassical growth model with linear preferences.

Finally, Section 22.8 builds on Acemoglu (2005). The idea that weak states might be an important impediment to economic growth is popular among political scientists and political sociologists, and is most famously articulated in Migdal (1988), Tilly (1990), Wade (1990), Herbst (2000) and Evans (2000). These approaches typically do not incorporate the incentives of the politicians or the government in providing public goods or adopt growth-enhancing strategies. Acemoglu (2005) provides the first formal framework to analyze these issues, and the material in this section embeds the baseline model in that paper into a neoclassical growth model.

22.11. Exercises

Exercise 22.1. Prove that $\hat{\tau}$ given by (22.16) satisfies $\hat{\tau} \in (0,1)$.

Exercise 22.2. Consider the model in Section 22.2, with the only difference that the production technology is as in the Romer (1986) model studied in Chapter 11. In particular, recall that each entrepreneur now has access to the production function $Y_i(t) = F(K_i(t), A(t) L_i(t))$ and $A(t) = B \int_0^1 K_i(t)\, di = BK(t)$. Characterize the Markov Perfect Political Economy Equilibrium in this case and show that distortionary taxes by the elite reduce the equilibrium growth rate of the economy.

Exercise 22.3. Consider the model in Section 22.2 and assume that policies are decided by the middle class. Show that the middle class might prefer positive taxation on themselves (with the proceeds redistributed to themselves as lump-sum transfers). Provide a precise intuition for why such taxation may make political-economic sense for middle-class entrepreneurs. Would the same result apply if the proceeds of taxation were redistributed as a lump-sum transfer to every individual in the society (including workers).

Exercise 22.4. Prove Proposition 22.4. Check in particular that the maximization program of the elite is concave, so that when $\bar{\tau} < 1 - \alpha$, the utility-maximizing tax rate for the elite is $\bar{\tau}$.

Exercise 22.5.
(1) Prove Proposition 22.5.
(2) Explain why the Condition 22.2 is necessary in this proposition.
(3) What happens if Condition 22.2 is not satisfied?

Exercise 22.6. Prove Proposition 22.6.

Exercise 22.7. Consider a model in Section 22.3 and suppose that the middle class are in political power. Characterize the Markov Perfect Political Economy Equilibrium in this case. Derive the discounted utility of the elite when the middle class are in control of politics, denoted by $V_e(M)$, and compare this to their utility when they are in control, $V_e(E)$. [Hint: see Section 23.2 in the next chapter and Exercise 23.2].

Exercise 22.8. In the model with political replacement in subsection 22.3.2, suppose that $\eta'(\cdot) < 0$. Show that in this case the tax rate preferred by the elite is less than $1 - \alpha$ and that when the elite can block technology adoption, they will not choose to do so. Explain the intuition for this result. What types of institutional structures might lead to $\eta'(\cdot) < 0$ as opposed to $\eta'(\cdot) > 0$.

Exercise 22.9. Prove Proposition 22.7.

Exercise 22.10. In the model with political replacement in subsection 22.3.2, show that the unique MPE is also the unique SPE.

Exercise 22.11.
(1) Prove Proposition 22.9.

(2) Explain how Proposition 22.9 needs to be modified if $\bar{\tau} < 1$ and provide an analysis of the best stationary SPE in this case.

EXERCISE 22.12. Prove Proposition 22.13.

EXERCISE 22.13. Prove Proposition 22.14.

EXERCISE 22.14. Prove Proposition 22.16.

EXERCISE 22.15. (1) Prove Proposition 22.17.

(2) Now suppose that in this proposition ϕ is not equal to 0. Provide an example in which in the MPE, the elite would still prefer $g = 0$.

(3) Now suppose that the elite can charge lump-sum taxes to middle-class entrepreneurs. Provide an example in which in the MPE, the elite would still prefer $g = 0$.

(4) In light of your answers to 2 and 3 above, explain why the political equilibrium might involve the use of inefficient fiscal instruments, even when more efficient alternatives exist.

EXERCISE 22.16. * Prove Proposition 22.18.

EXERCISE 22.17. * Prove Proposition 22.19.

EXERCISE 22.18. Consider an environment with concave preferences as in Section 22.5. Assume that there is full depreciation (i.e., $\delta = 1$), citizens are yeoman-producers only using their own labor and have access to a production technology for producing the unique final good given by $Y_i(t) = AK_i(t)^{\alpha}$, where $K_i(t)$ is the capital holding of producer i. Both citizens and elites have logarithmic preferences. Characterize the MPE in this environment. [Hint: conjecture a policy rule that depends only on the current (average) net output, so that the tax rate for next period is $\tau(t+1) = \tau\left(Y^N(t)\right)$, where $Y^N(t) = (1-\tau(t))AK(t)^{\alpha}$, where $K(t)$ is the common capital stock of all producers].

EXERCISE 22.19. * Prove that if individual preferences are reflexive, complete and transitive, then they can be represented by a real-valued utility function.

EXERCISE 22.20. *

(1) Consider a society with two individuals 1 and 2 and three choices, a, b and c. For the purposes of this exercise, only consider strict individual and social orderings (i.e., no indifference allowed). Suppose that the preferences of the first agent are given by abc (short for $a \succ b \succ c$, i.e., a strictly preferred to b, strictly preferred to c). Consider the six possible preference orderings of the second individual, i.e., $s_2 \in \{abc$, acb, $bac, ...\}$, etc.. Define a social ordering as a mapping from the preferences of the second agent (given the preferences of the first) into a social ranking of the three outcomes, i.e., some function ϕ such that the social ranking is $s = \phi(s_2)$. Illustrate the Arrow impossibility theorem using this example [Hint: start as follows: $abc = \phi(abc)$, i.e., when the second agent's ordering is abc, the social ranking must be abc; next, $\phi(acb) = abc$ or acb (why?); then if $\phi(acb) = abc$, we must also have

$\phi(cab) = abc$ (why?); and proceeding this way to show that the social ordering is either dictatorial or it violates one of the axioms].

(2) Now suppose we have the following aggregation rule: individual 1 will (sincerely) rank the three outcomes, his first choice will get 6 votes, the second 3 votes, the third 1 vote. Individual 2 will do the same, his first choice will get 8 votes, the second 4 votes, and the third 0 vote. The three choices are ranked according to the total number of votes. Which of the axioms of the Arrow's Theorem does this aggregation rule violate?

(3) With the above voting rule, show that for a certain configuration of preferences, either agent has an incentive to distort his true ranking (i.e., not vote sincerely).

(4) Now consider a society consisting of three individuals, with preferences given by:

$$\begin{array}{ll} 1 & a \succ b \succ c \\ 2 & c \succ a \succ b \\ 3 & b \succ c \succ a \end{array}$$

Consider a series of pairwise votes between the alternatives. Show that when agents vote sincerely, the resulting social ordering will be "intransitive". Relate this to the Theorem 22.1.

(5) Show that if the preferences of the second agent are changed to $b \succ a \succ c$, the social ordering is no longer intransitive. Relate this to "single-peaked preferences".

(6) Explain intuitively why single-peaked preferences are sufficient to ensure that there will not be intransitive social orderings. How does this relate to Theorem 22.1?

Exercise 22.21. * In the Condorcet paradox example provided in Section 22.6, show that other orderings of the choices a, b and c will also imply that the preferences of at least one of the three individuals is not single peaked.

Exercise 22.22. *

(1) Consider the example of a three-person three-policy society with preferences

$$\begin{array}{ll} 1 & a \succ b \succ c \\ 2 & b \succ c \succ a \\ 3 & c \succ b \succ a \end{array}$$

Voting is dynamic: first, there is a vote between a and b. Then, the winner goes against c, and the winner of this contest is the social choice. Find the subgame perfect Nash equilibrium voting strategy profiles in this two-stage game (recall that each player's strategy has to specify how they will vote in the first round, and how they will vote in the second round as a function of the outcome of the first round).

(2) Suppose a generalization whereby the society $\mathcal{H}$ consists of H individuals and there are finite number of policies, $\mathcal{P} = \{p_1, p_2, ..., p_M\}$. For simplicity, suppose that H is an odd number. Voting takes $M - 1$ stages. In the first stage, there is a vote

between p_1 and p_2. In the second stage, there is a vote between the winner of the first stage and p_3, until we have a final vote against p_M. The winner of the final vote is the policy choice of the society. Prove that if preferences of all agents are single peaked (with a unique bliss point for each), then the unique subgame perfect Nash equilibrium implements the bliss point of the median voter.

Exercise 22.23. * Prove Theorem 22.2 when H is even.

Exercise 22.24. * Characterize the subgame perfect equilibrium of the game in Example 22.2 under strategic voting by all players.

Exercise 22.25. * Modify and prove Theorem 22.4 without Assumption A4.

Exercise 22.26. * This exercise reviews Downsian party competition and then show that Theorem 22.4 does not apply if there are three parties competing. In particular, consider Downsian party competition in a society consisting of a continuum 1 of individuals with single-peaked preferences. The policy space $\mathcal{P}$ is the $[0,1]$ interval and assume that the bliss points of the individuals are uniformly distributed over this space.

(1) To start with, suppose that there are two parties, A and B. They both would like to maximize the probability of coming to power. The game involves both parties simultaneously announcing $p^A \in [0,1]$ and $p^B \in [0,1]$, and then voters voting for one of the two parties. The platform of the party with most votes gets implemented. Determine the equilibrium of this game. How would the result be different if the parties maximized their vote share rather than the probability of coming to power?
(2) Now assume that there are three parties, simultaneously announcing their policies $p^A \in [0,1]$, $p^B \in [0,1]$, and $p^C \in [0,1]$, and the platform of the party with most votes is implemented. Assume that parties maximize the probability of coming to power. Characterize all pure strategy equilibria.
(3) Now assume that the three parties maximize their vote shares. Prove that there exists no pure strategy equilibrium.
(4) In 3, characterize the mixed strategy equilibrium. [Hint: assume the same symmetric probability distribution for two parties, and make sure that given these distributions, the third party is indifferent over all policies in the support of the distribution].

Exercise 22.27. * Prove Theorem 22.6.

Exercise 22.28. * This exercise involves generalizing the idea of single-crossing property used in Theorem 22.5 to multi-dimensional policy spaces. The appropriate notion of preferences of individuals turns out to be "intermediate preferences". Let $\mathcal{P} \subset \mathbb{R}^I$, where I is an integer, and policies p belong to $\mathcal{P}$. We say that voters have *intermediate preferences*, if their indirect utility function $U(p;\alpha_i)$ can be written as

$$U(p;\alpha_i) = J_1(p) + K(\alpha_i)J_2\left(p\right),$$

where $K(\alpha_i)$ is monotonic (monotonically increasing or monotonically decreasing) in α_i, and the functions $J_1(p)$ and $J_2(p)$ are common to all voters. Suppose that A2 holds and voters have intermediate preferences. We define the bliss point (vector) of individual i as in the text, as $p(\alpha_i) \in \mathcal{P}$ that maximizes individual i's utility. Prove that when preferences are intermediate a Condorcet winner always exists and coincides with bliss point of the voter with the median value of α_i, that is, $p_m = p(\alpha_m)$.

EXERCISE 22.29. * Consider a society consisting of three individuals, 1, 2 and 3 and a resource of size 1. The three individuals vote over how to distribute the resource among themselves and each individual prefers more of the resource for himself and does not care about consumption by the other two. Since all of the resource will be distributed among the three individuals, we can represent the menu of policies as $\{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0 \text{ and } x_1 + x_2 \leq 1\}$, where x_i denotes the share of the resource consumed by individual i. A policy vector (x_1, x_2) is accepted if it receives two votes.

(1) Show that individual preferences over policy vectors do not satisfy single crossing or the conditions in Exercise 22.28.
(2) Show that there does not exist a policy vector that it is a Condorcet winner.

EXERCISE 22.30. *

(1) Show that in Theorem 22.7, a necessary condition for a pure strategy symmetric equilibrium, with $p_A = p_B = p^*$, to exist is that the matrix

$$\sum_{g=1}^{G} \lambda^g h^g(0) D^2 U^g(p^*)$$
$$+\sum_{g=1}^{G} \lambda^g \frac{\partial h^g(0)}{\partial \sigma} (DU^g(p^*)) \cdot (DU^g(p^*))^T$$

is negative semidefinite, where $D^2 U^g$ denotes the Jacobian matrix of U^g.
(2) Derive a sufficient condition for such a symmetric equilibrium to exist. [Hint: distinguish between local and global maxima].
(3) Show that without any assumptions on $U^g(\cdot)$'s (beyond concavity), the sufficient condition for a symmetric equilibrium can only be satisfied if all H^g's are uniform.

EXERCISE 22.31. Consider the following one-period economy populated by a mass 1 of agents. A fraction λ of these agents are capitalists, each owning capital k. The remainder have only human capital, with human capital distribution $\mu(h)$. Output is produced in competitive markets, with aggregate production function

$$Y = K^{1-\alpha} H^{\alpha},$$

where uppercase letters denote total supplies. Assume that factor markets are competitive and denote the market clearing rental price of capital by r and that of human capital by w.

(1) Suppose that agents vote over a linear income tax, τ. Because of tax distortions, total tax revenue is

$$Tax = (\tau - v(\tau)) \left(\lambda r k + (1-\lambda) w \int h d\mu(h) \right)$$

where $v(\tau)$ is strictly increasing and convex, with $v(0) = v'(0) = 0$ and $v'(1) = \infty$ (why are these conditions useful?). Tax revenues are redistributed lump sum. Find the ideal tax rate for each agent. Find conditions under which preferences are single peaked, and determine the equilibrium tax rate. How does the equilibrium tax rate change when k increases? How does it change when λ increases? Explain the intuition for these results.

(2) Suppose now that agents vote over capital and labor income taxes, τ_k and τ_h, with corresponding costs $v(\tau_k)$ and $v(\tau_h)$, so that tax revenues are

$$Tax = (\tau_k - v(\tau_k)) \lambda r k + (\tau_h - v(\tau_h)) (1-\lambda) w \int h d\mu(h)$$

Determine the most preferred tax rates for each agent. Suppose that $\lambda < 1/2$. Does a voting equilibrium exist? Explain. How does it change when λ increases? Explain why this would be different from the case with only one tax instrument?

(3) In this model with two taxes, now suppose that agents first vote over the capital income tax, and then taking the capital income tax as given, they vote on the labor income tax. Does a voting equilibrium exist? Explain. If an equilibrium exists, how does the equilibrium tax rate change when k increases? How does it change when λ increases?

EXERCISE 22.32. Derive expression (22.51).

EXERCISE 22.33. (1) Show that $\tilde{V}_i(\tau' \mid p^{t+1})$ defined in (22.54) is not necessarily quasi-concave.

(2) Show that $\tilde{V}_i(\tau' \mid p^{t+1})$ satisfies the single-crossing property in Definition 22.3.

EXERCISE 22.34. Consider an economy consisting of three groups, a fraction θ_p poor agents each with income y_p, a fraction θ_m middle-class agents with income $y_r > y_m$, and the remaining fraction $\theta_r = 1 - \theta_p - \theta_m$ rich agents with income $y_r > y_p$. Suppose that both θ_p and θ_r are less than 1/2, so that the individual with the median income (the "median voter") is a middle-class individual.

(1) Construct a change in incomes that leaves mean income in the society unchanged and increases the gap between the mean and the median but does not constitute a mean preserving spread of the distribution.

(2) Construct a mean preserving spread of the distribution such that the gap between the mean and the median narrows. [Hint: increase y_m and reduce y_p, holding y_r constant].

EXERCISE 22.35. We now consider the model by Alesina and Rodrik, which is similar to the model studied in Section 22.7. There is a continuum 1 of individuals. All individuals have logarithmic instantaneous utility, so that $U_i = \sum_{t=0}^{\infty} \beta^t \ln C_i(t)$, where i denotes the individual and $C_i(t)$ refers to his consumption at time t. Each individual has one unit of labor, which he supplies inelastically. Final output is produced as

$$Y_i(t) = AK_i(t)^{1-\alpha} G(t)^{\alpha} L_i(t)^{\alpha}$$

where K_i and L_i denote capital and labor employed by individual i and G is government investment in infrastructure. The only tax instrument is a linear tax on the capital holdings of all individuals at the rate $\tau(t)$ at time t. All the proceeds of this taxation are spent on government investment in infrastructure, so that

$$G(t) = \tau(t) \bar{K}(t) \quad (22.70)$$

where $\bar{K}(t)$ is the average (total) capital stock in the economy. This specification implies that government's provision of infrastructure creates a Romer-type externality. Denote the initial capital stock of the economy by $\bar{K}(0)$.

(1) Characterize the equilibrium with a constant tax rate $\tau > 0$ at each date and show that with A sufficiently large, the economy will achieve a constant and positive growth rate. Show that the growth rate of the economy is independent of the distribution of the initial capital stock among the individuals. [Hint: note that the net interest rate faced by consumers is equal to the marginal product of capital minus the tax rate, τ].
(2) Let the initial capital stock $\bar{K}(0)$ be distributed among the agents with shares θ_i, i.e., individual i's initial capital holding is $K_i(0) = \theta_i \bar{K}(0)$. Show that in equilibrium we have $K_i(t) = \theta_i \bar{K}(t)$ for any $t = 1, 2, ...$
(3) Suppose that the economy will legislate a constant tax rate τ forever. Determine the most preferred tax rate of individual i as a function of his share of initial capital θ_i at time $t = 0$.
(4) Show that individuals have single-peaked preferences. On the basis of this, appeal to Theorems 22.2 and 22.4 to argue that the tax rate most preferred by the individual with the median capital holdings, θ_m, will be implemented. Show that as this median capital holdings falls, the rate of capital taxation increases. What is the effect of this on economic growth?
(5) Show that the equilibrium characterized in 4 above is *not* a Markov Perfect Political Economy Equilibrium. Explain why not. How would you set up the problem to characterize such an equilibrium? [Hint: just describe how you would set up the problem; no need to solve for the equilibrium].

Exercise 22.36. (1) Prove Proposition 22.24.
(2) Derive the output-maximizing tax rate as in (22.69).
(3) Characterize the tax rates maximizing the utility of the elite and the citizens and establish the results in Proposition 22.25.

CHAPTER 23

Political Institutions and Economic Growth

The previous chapter investigated why societies often choose inefficient economic institutions and policies and consequently fail to take advantage of growth opportunities. It emphasized the importance of social conflict between different groups and lack of commitment to future policies as major sources of non-growth-enhancing policies. Much of the discussion was in the context of a given set of *political institutions*, which shaped both the extent and kind of social conflict between different individuals and groups, and what types of policies were possible or could be committed to. A natural conjecture is that political institutions will influence a society's choices of economic institutions and policies and thus its growth trajectory. This conjecture leads to the following two questions: (1) do certain political institutions mediate social conflict more successfully, thus potentially avoiding non-growth-enhancing policies? (2) why do different societies choose or end up with different political institutions.

This chapter provides some preliminary answers to these two questions. I start with a brief summary of the empirical evidence on the effect of different political regimes and other political factors (such as political instability and civil wars) on economic growth. Section 23.2 then uses the baseline model in Section 22.2 from the previous chapter to illustrate that, once we take the existence of conflicting preferences into account, no political regime is perfect and each will create different types of costs and benefits associated with different losers and winners in the society. Whether a particular set of political institutions leads to growth-enhancing policies then depends on the details of how it will function, on the technology and the factor endowments of the society, and on which groups will benefit from these institutions. Section 23.3 then turns to the dynamic tradeoffs between different regimes, emphasizing how democratic regimes might compensate for the short-run distortions that they create by long-run benefits, both in terms of avoiding sclerotic outcomes and by creating greater flexibility. This section will also emphasize how different political regimes deal with the process of *creative destruction*, which, as we saw in Chapter 14, is one of the engines of modern economic growth. It will suggest that democracies may be better at taking advantage of the forces of creative destruction.

Although Section 23.3 introduces the dynamics of economic allocations under different political regimes, it only gives us a few clues about how political institutions themselves emerge and change. This is a major area of current research in political economy and largely falls beyond the scope of the current book. Sections 23.4 and 23.5 will therefore give a bird's eye view of some of the main issues involved in the analysis of equilibrium political institutions and their implications for economic growth. Section 23.4 starts with a general discussion of what types of models we might want to consider for understanding the dynamics of political institutions and endogenous political change. Section 23.5 illustrates some of these general ideas with a specific example, which shows how we can construct dynamic models featuring changes in political and economic institutions, and how such models can shed light on the empirical patterns discussed in Section 23.1.

23.1. Political Regimes and Economic Growth

The centerpiece of our approach to the political economy of growth are the two mappings introduced at the beginning of Part 8 above, $\rho(\cdot)$ and $\pi(\cdot)$. Recall that these determine how political institutions shape economic institutions and how economic institutions map into allocations. The latter has already been discussed to some degree in the previous chapter, though it would not be unfair to say that we do not yet have a full understanding of the consequences of different economic institutions (consider, for example, the debate on the implications of privatization, or what the implications of different contract enforcement practices will be on the exact investment behavior and welfare levels of different individuals in a society). Our understanding of the implications of different political institutions on economic outcomes is even more imperfect. Thus the mapping $\rho \circ \pi : \mathcal{P} \rightarrow \mathcal{X}$, which determines how various combinations of political and economic institutions lead to different economic allocations is something we would like to (and should) learn more about.

In this section, I will briefly discuss the little that we know on this topic. Many different types of distinctions can be made among political institutions. Most scholars would probably start by thinking of the contrast between democratic and nondemocratic regimes. But there are many different types and shades of democracies. Democracy is typically defined by a set of procedural rules, for instance, by whether there are free and fair elections in which most adults can participate and there is free entry of parties. But this definition of democracy is quite encompassing. First, it leaves many distinctive institutional features of democracies unspecified. Democracies can be parliamentary or presidential. They can use different electoral rules, giving different degrees of voice to minorities. Perhaps more importantly, there are different degrees of "free and fair" and "most adults". In particular, most elections, even those in Europe or the United States, involve some degree of fraud and some restrictions on the entry of parties or candidates. Moreover, many individuals are effectively or sometimes explicitly disenfranchised. Political scientists consider Britain and the United States in the late 19th century to have been democratic, though only males had the right to vote. Few people would consider the United States in the 1960s a nondemocracy, though many blacks were disenfranchised. This creates various different shades of democracy that one might wish to take into account. For example, Colombia has been a democratic country for over half a century according to most political scientists, though in many parts of the country elections are very far from "free and fair" and take place under the threat of explicit violence from paramilitaries. Over the same time period, the entry of a Socialist party into Colombian politics has been blocked by various legal and non-legal means, violating the "free entry" requirement. This discussion alerts us to the fact that we may want to draw distinctions between the degree of democracy and the types of democracies With these caveats in mind, one might be tempted to conclude that the label of "democracy" is so encompassing and nondescript as to become meaningless. This is not my view, however. I maintain that the distinction between democracy and nondemocracy is not meaningless, and in fact, it is a particularly useful starting point for our analysis of the effect of political regimes on economic outcomes and the dynamics of political regimes. Nevertheless, when we wish to understand why different democracies behave differently and also the contrast between democratic and nondemocratic regimes, it will be necessary to delve deeper and make systematic distinctions about the nature and functioning of different types of democracies.

The differences between nondemocratic societies are probably even more pronounced. China under the rule of the Communist Party since 1948 is an undisputed case of a nondemocratic regime, but it is very different in nature from the oligarchic regime in place in Britain before the process of democratization started with the First Reform Act of 1832. In Britain before 1832, there were prime ministers and parliaments, though they were elected by a small minority of the population, those with wealth, education and privilege, who made up less than 10% of the adult population. Furthermore, the powers of the state never rivaled those of the Communist Party in China. The Chinese example is also different from military dictatorships such as that of Chile under General Pinochet or South Korea under General Park. Once we consider regimes based on personal rule, such as that of Mobutu in Zaire, and monarchies, such as the rule of the Saud family in Saudi Arabia, the contrast is even more marked.

Nevertheless, there is an important commonality among these nondemocracies and an important contrast between nondemocratic and democratic regimes, making these categories still useful for conceptual and empirical analysis. Despite all of their imperfections and different shades, democratic regimes, at least when they have a certain minimal degree of functionality, provide greater *political equality* than nondemocratic regimes. Free entry of parties and one-person one-vote in democracy are the foundations of this and ensure some amount of voice for each individual. When democracies are particularly well functioning, majorities will have some (often a significant) influence on policies—though they themselves may be constrained by certain constitutional restrictions. In contrast, nondemocracies, rather than representing the wishes of the population at large, represent the preferences of a subgroup of the population. In the previous chapter, I referred to this subgroup as the "elite," and I will continue to do so here. The identity of the elite differs across nondemocratic societies. In China, it is mainly the wishes of the Communist Party that matters. In Chile under Pinochet, most decisions were taken by a military junta, and it was their preferences, and perhaps the preferences of certain affluent segments of the society supporting the dictatorship, that counted. In Britain before the First Reform Act of 1832, it was the small wealthy minority that was politically influential.

With this cautionary introduction on the distinctions between democracies and nondemocracies, what are the major differences between these political regimes? First, one might imagine that democracies and nondemocracies will have different growth performances. The first place to look for such differences is the postwar era for which we have better data on economic growth. Unfortunately, the picture here is not very clear. Przeworski and Limongi (1993) and Barro (1999) document, using cross-country regression evidence, that democracies do not appear to perform better than nondemocracies. Nevertheless, there is no universal consensus on this matter. For example, Minier (2004) reports results showing some positive effects of democratizations on growth and more robust negative effects of transitions to nondemocracy on growth. Persson and Tabellini (2007) argue that once one distinguishes between actual democracy and the probability of democracy surviving, there is a significant effect of democratic survival probabilities on economic growth. All in all, however, the bulk of the available evidence supports the conclusions of Przeworski and Limongi (1993) and Barro (1999) and suggests that, on average, democracies do not grow much faster than nondemocracies (at least, once one controls for other potential determinants of economic growth). This is, at first, a surprising and even perhaps a disturbing finding. One might have expected significantly worse growth performances among nondemocracies, since this group includes highly unsuccessful countries such as Iraq under Saddam Hussein, Zaire under Mobutu, or Haiti under the Duvaliers. Despite these nondemocratic "basket cases" of growth, there are plenty of unsuccessful democracies, including India until the 1990s and many newly independent former colonies that started their independence period as electoral democracies (though, often quickly falling prey to coups or personal rule of some strongman). There are also many successful nondemocracies, including Singapore under Lee Kwan Yew or South Korea under General Park, or more recently China. Thus to understand how different political institutions affect economic decisions and economic growth we will need to go beyond the distinction between democracy and nondemocracy. One idea, which I will argue is useful in thinking about these distinctions, is that of *dysfunctional democracies*, that is, the possibility that some democracies are functioning in a very different, and inefficient, manner than we typically envisage. I will further argue that a particularly important reason why democracies might be dysfunctional is because they are captured by elites despite the fact that on paper they are supposed to provide majoritarian decision-making and political equality. According to this definition, a democracy will be "captured" when its modus operandi—its purpose of creating greater political equality than a typical nondemocratic regime—fails. *Captured democracies* are one example of the more general typology of dysfunctional democracies. Another one might be highly *populist democracies*, where a strongman, such as Juan Peron in Argentina or Hugo Chávez in Venezuela, receives majority support but pursues policies that are detrimental to economic growth. This discussion therefore suggests that a satisfactory understanding of the relationship between democracy and growth, and more generally, that between political regimes and growth, necessitates an analysis of how different political regimes function and why some democracies might become captured or take the populist route. I will discuss some possible answers to these questions in Section 23.5.

If there are no marked growth differences between democracies and nondemocracies, are there other significant policy or allocational differences? On this question, there is even more controversy than on the effects of democracy on growth. Rodrik (1999) documents that democracies have higher labor shares and interprets this as the outcome of greater redistribution in democracies. He shows that the same result holds both in cross-sectional and in panel data regressions. Acemoglu and Robinson (2006a) summarize a range of case studies showing how democracies do indeed pursue more redistributive policies and how this has an effect on the distribution of income, for example, on the share of capital in national income or on the overall extent of income inequality. Acemoglu (2007b) also shows that nondemocratic regimes often adopt different regulations, for example, erecting greater barriers against entry of new businesses. Finally, Persson and Tabellini (2004) document major differences in fiscal policy between different types of democracies. In contrast, Gil, Mulligan and Sala-i-Martin (2004) use cross-sectional regressions to show that a range of policies, in particular overall government spending and spending on Social Security, do not differ between democracies and dictatorships. Based on this, they argue that the distinctions between democracy and nondemocracy or between different shades of democracy are not useful in understanding policymaking and the extent of redistributive policies that societies adopt. Overall, therefore, there is no consensus in the literature on whether democracies pursue significantly different fiscal policies and whether this has a significant impact on the distribution of resources in the society. Nevertheless, the evidence in Rodrik (1999) and some of the evidence summarized in Acemoglu and Robinson (2006a) do indicate that, at least in some cases, democracies pursue significantly more redistributive policies than nondemocracies, and we can take these differences as our starting point, at least as a working hypothesis. Nevertheless, it is useful to bear in mind that the differences in policy between democracies and nondemocracies, even if present, appear to be much less pronounced than one might have expected on the basis of theory alone. I will argue in Section 23.5 that the same factors that are important in thinking about why democracies do not grow much faster than nondemocracies are likely to be important in understanding why policies in democracies and nondemocracies do not differ by much (for example, captured democracies are unlikely to pursue highly redistributive policies). However, before turning to these issues, we need a more systematic analysis of how political institutions influence the economic organization and the economic outcomes of a society. This will be the topic of the next two sections.

It should also be noted at this point that the comparison of democracies to nondemocracies over the postwar era might be overly restrictive. When we look at a longer time horizon, it seems to be the case that democracies have significantly better economic growth performance. Most of the countries that industrialized rapidly during the 19th century were more democratic than those that failed to do so. The comparison of the United States to the Caribbean or to Peru, or of Britain and France to Russia and Austria-Hungary are particularly informative in this context. For example, the United States, which was one of the most democratic societies at the time, was not any richer than the highly nondemocratic and repressive Caribbean colonies at the end of the 18th century, and a range of evidence in fact indicates that the Caribbean colonies may have been significantly richer than Northeastern United States throughout the 17th and 18th centuries. Even when we compare the United States to Peru, which is another repressive, nondemocratic colony, there is no strong evidence that throughout the 17th and 18th centuries, the United States was much richer, or in fact any richer, than Peru. However, the 19th and 20th centuries witnessed rapid growth in industrialization in the United States and stagnation in the entire Caribbean area and in Peru, as well as in much of the rest of South America. This historical episode therefore suggests that the more democratic societies may have been better at taking advantage of the new investment and growth opportunities that came with the age of industrialization at the beginning of the 19th century. The contrast of Britain and France to Russia and Austria-Hungary is similar. Even though the former two countries were already richer at the beginning of the 19th century than their Russian and Austria-Hungarian counterparts, the income differences were small. Differences in political institutions were much more marked, however. Britain was already on its way to becoming a parliamentary democracy and France had already undergone the Revolution of 1789 and was becoming a much more representative society. Britain and France adopted pro-growth policies throughout much of the 19th century, even when this was costly to existing landowning elites, whereas Russia and Austria-Hungary explicitly blocked industrialization in order to protect the economic and political interests of their landowning aristocracies.

Long-run regressions, such as those discussed in Chapter 4, are also consistent with this pattern and show a significant effect of a broad cluster of institutions on economic growth. While we cannot confidently say that this represents the effect of political institution on growth, this cluster of institutions comprises both political and economic elements and it is likely that the growth-enhancing cluster of institutions could not exist without the political institutions supporting economic institutions encouraging investment and free entry.[1]

Finally, even though the effects of democracy and nondemocracy on growth might be less clear-cut than we would have liked, there are certain other regularities that are worth noting. The evidence seems to indicate quite strongly that political order is much more conducive to economic growth than political instability. A range of papers, for example, Alesina and Perotti (1993), Alesina et al. (1996), and Svensson (1998), find a negative and significant effect of political instability, as measured by assassinations or civil unrest, on economic growth. Even more clear are the negative effects of civil wars on economic growth. Many of the big growth disasters in Africa over the past half century have been associated with civil wars and infighting among different warlords, such as in Angola, Mozambique, Rwanda, Ethiopia, Sudan, Sierra Leone and Liberia. Therefore, even if the effect of the exact shade of democracy on growth is still unknown, there is strong evidence that political factors, at least in their extreme form, have an effect on the economic opportunities available to individuals and thus on economic growth. Exercise 23.1 gives an example of conflict between two groups within a society that creates instability and leads to economic crises and bad economic performance.

I next turn to a theoretical investigation of how we might expect different political institutions to affect economic policies and economic outcomes. I will then enrich this framework to shed light on why the relationship between political regimes and economic growth may be more complex than one might have originally expected.

23.2. Political Institutions and Growth-Enhancing Policies

In this section, I consider the canonical Cobb-Douglas model analyzed in subsection 22.2.4 (and then used again in Sections 22.3 and 22.4 of the previous chapter). In that chapter, this model was analyzed under the assumption that the group of producers which I referred to as the "elite" were in power. I showed how the political equilibrium in this case can lead to various different forms of non-growth-enhancing policies. I will now briefly discuss the equilibrium in the same environment when the middle class or the workers are in power and then contrast the resulting allocations.

[1]Persson (2006), for example, uses some of the approaches in Chapter 4 to instrument for political institutions using historical variables and concludes that democracy has a positive effect on long-run growth.

23.2.1. The Dictatorship of the Middle Class Versus the Dictatorship of the Elite. First, let us suppose that the middle class hold political power, so that we have the *dictatorship of the middle class* instead of the dictatorship of the elites in the previous chapter. The situation is entirely symmetric to that in the previous chapter with the middle class and the elite having exchanged places. In particular, the analysis leading to Proposition 22.6 immediately yields the following result.

Proposition 23.1. *Consider the environment of subsection 22.2.4 with Cobb-Douglas technology, but the middle class instead of the elite holding political power. Suppose that Condition 22.1 holds,* $\phi > 0$*, and*

$$A^m \geq \phi \alpha^{\alpha/(1-\alpha)} A^e \frac{\theta^e}{\theta^m}. \tag{23.1}$$

Then the unique Markov Perfect Political Economy Equilibrium features $\tau^m(t) = 0$ *and*

$$\tau^e(t) = \bar{\tau}^{COM} \equiv \min\left\{ \frac{\kappa\left(\bar{L}, \theta^m, \alpha, \phi\right)}{1 + \kappa\left(\bar{L}, \theta^m, \alpha, \phi\right)}, \bar{\tau} \right\}, \tag{23.2}$$

for all t*, where* $\kappa\left(\bar{L}, \theta^e, \alpha, \phi\right)$ *is defined in (22.29).*

Proof. See Exercise 23.2. □

The notable feature about this equilibrium is the strong parallel to Proposition 22.6. The equilibria under elite control and middle class control are identical, except that the two groups have switched places. We therefore have an example of political institutions having a real effect on both the types of economic policies and economic institutions in place, and on the allocation of resources; in the elite-controlled society, the middle class are taxed both to create revenues for the elite and to reduce their labor demand. In the middle class dominated society, the competing group of producers that are out of political power are the "elite" (even though the name "elite" has the connotation of political power). So now the elite are taxed to generate tax revenue and to create more favorable labor market conditions for the middle class. The contrast between the elite dominated and the middle class dominated politics approximates certain well-known historical episodes. For example, in the context of the historical development of European societies, political power was first in the hands of landowners, who exercised it to keep labor tied to land and to reduce the power and the profitability of merchants and early industrialists (capitalists). In many cases, these policies favoring landowners were detrimental to economic growth. Nevertheless, with economic changes and the constitutional revolutions taking place in the late medieval period, power shifted away from landowning aristocracies towards the merchants and Lists, and it was their turn to adopt policies favorable to their own economic interests and costly for landowners. The repeal of the Corn Law in 1846 illustrates this point, even though the conflict between landowners and capitalists was probably weakest in England because many members of the gentry and the previous landowning class had already transitioned into commercial agriculture and other industrial activities. Nevertheless, there were intense political debates surrounding the Corn Law, with landowners supporting the tariffs imposed by the law, which kept the price of their produce high, and industrialists opposing it so that the implicit tax on their inputs, especially labor, would be removed with the import of cheaper corn from abroad.

So which one of these two sets of political institutions—the dictatorship of the middle class or the dictatorship of the elite—is better? The answer is that they cannot be compared easily. First, as already emphasized in the previous chapter, the equilibrium considered in Section 22.3 was already Pareto optimal; starting from the allocation there, it is not possible to make any member of the society better off without making the elite worse off. In the same way, the current allocation of resources is Pareto optimal, but it has picked a different point along the Pareto frontier—a point that favors middle-class agents instead of the elite. However, in the previous chapter we also emphasized that Pareto optimality may be too weak a concept for a useful analysis of the effect of institutions on economic growth, since two allocations that are Pareto optimal may involve significantly different growth rates. And yet, when we compare the growth rates under these two different political regimes, we also find that there is no straightforward ranking. Either of these two societies may achieve a higher level of income per capita. Which one does so depends on which group has more productive investment opportunities. When the middle class has the more productive investment opportunities, a society in which the elite are in power will create significant distortions. In contrast, if the elite have more profitable and socially beneficial production opportunities, then having political power vested with the elite is more beneficial for economic performance than the dictatorship of the middle class.

The following proposition illustrates a particularly simple case of this result.

Proposition 23.2. *Consider the environment of subsection 22.2.4 with Cobb-Douglas technology. Suppose that Conditions 22.1, (22.27) and (23.1) hold, $\theta^e = \theta^m$, and $\phi > 0$. Then, the dictatorship of the middle class generates higher income per capita when $A^m > A^e$ and the dictatorship of the elite generates higher income per capita when $A^e > A^m$.*

Proof. See Exercise 23.3. □

This proposition therefore gives a simple example of a situation whereby which political institutions will lead to better economic performance (in terms of income per capita) depends on whether the group that is more productive also holds political power. When political power and economic power are decoupled, there is greater inefficiency. An immediate implication of this result is that it is difficult to think of "efficient political institutions" without considering the self-interested objectives of those who hold and wield political power and without fully analyzing how their productivity and their economic activities compare to those of others. Naturally, one can dream of political institutions that will outperform both the elite dominated politics of the previous chapter and the middle class dominated politics of this chapter. For example, we can think of a set of political institutions that constitutionally force all taxes to be equal to zero—so that in the context of the simple model we are focusing on here, there are no distortionary policies. In this environment, this alternative arrangement will outperform both elite dominated and middle class dominated polities. However, such political institutions are not realistic. First, there are numerous reasons why societies need to raise taxes, for example, they need to finance productivity-enhancing public goods as in the model of Section 22.8 in the previous chapter, and they also need to engage in some amount of redistribution to ensure a safety net to their citizens. Once we allow for positive taxes, then the social groups or the politicians that are in power can also misuse these tax revenues and the associated fiscal instruments. Second, constitutional limits on taxes are difficult to enforce. Once a particular group is in power and has the capability to dictate policies, there is no easy way of preventing them to rewrite the constitution as has been the practice in many countries over the last two centuries. This discussion indicates that we can think of "ideal political institutions" that may prevent the distortions of simpler institutions that vest power with a particular group of individuals, but such political institutions are difficult to create, implement and maintain. Most likely, they are not even feasible. And this implies that the choice of political institutions in practice will be between arrangements that will create different types of distortions and different winners and losers.

23.2.2. Democracy or Dictatorship of the Workers? The previous subsection contrasted the dictatorship of the middle class to the dictatorship of the elite. A third possibility is to have a more democratic political system in which the majority decides policies. Since in realistic scenarios, the workers will outnumber both the elite and the middle class, this means the choice of policies that favor the economic interests of the workers (who have so far been passive in this model, simply supplying their labor at the equilibrium wage rate) will be implemented. While such a system does resemble democracy in some ways, it can also be viewed as a dictatorship of the workers, since it will now be the workers who will dictate policies, in the same way that the elite or the middle class dictated policies under their own dictatorship.[2] This emphasizes once more that different political institutions will create different winners and losers depending on which group has more political power.

[2]What the difference between the dictatorship of workers or poor segments of the society and a true "democracy" may be is an important question, of interest to philosophers, political scientists and economists. However, since there is as yet no satisfactory answer to this important question and it falls beyond the scope of my focus here, I will not discuss it further.

The analysis is again straightforward, though the nature of the political equilibrium does depend even more strongly on whether or not Condition 22.1 holds (i.e., whether there is excess supply or not). The following proposition summarizes the equilibrium choices of policies by the workers when they monopolize political power.

Proposition 23.3. *Consider the environment of subsection 22.2.4 with Cobb-Douglas technology and suppose that workers hold political power.*

(1) *Suppose that Condition 22.1 fails to hold (so that there is excess labor supply), then the unique Markov Perfect Political Economy Equilibrium features* $\tau^m(t) = \tau^e(t) = \tau^{RE} \equiv \min\{1-\alpha, \bar{\tau}\}$.
(2) *Suppose that Condition 22.1 holds (so that there is no excess labor supply) and that* $\theta^e = \theta^m = \theta$. *Then unique Markov Perfect Political Economy Equilibrium is as follows:*
   (a) *if* $A^m > A^e$, *then* $\tau^e(t) = 0$, *and* $\tau^m(t) = \tau^{Dm}$ *where* $(1-\tau^{Dm})^{1/(1-\alpha)}A^m = A^e$, *or* $\tau^{Dm} = 1-\alpha$ *and* $\alpha^{1/(1-\alpha)}A^m \geq A^e$*;*
   (b) *if* $A^m < A^e$, *then* $\tau^m(t) = 0$, *and* $\tau^e(t) = \tau^{De}$ *where* $(1-\tau^{De})^{1/(1-\alpha)}A^e = A^m$, *or* $\tau^{De} = 1-\alpha$ *and* $\alpha^{1/(1-\alpha)}A^e \geq A^m$.

The most interesting implication of this proposition comes from the comparison of the cases with and without excess supply. When Condition 22.1 fails to hold, there is excess labor supply and taxes have no effect on wages. Anticipating this, workers favor taxes on both groups of producers to raise revenues to be redistributed to themselves. The dictatorship of the workers ("democracy") will then generate this outcome as the political equilibrium. Clearly, this is more distortionary than either the dictatorship of the elite or the middle class, because in these political scenarios at least one of the producer groups was not taxed (but the resulting allocation is once again Pareto optimal for the same reasons as stressed above). The situation is very different when Condition 22.1 holds. In that case, recall that both the dictatorships of the elite and of the middle class generated significant distortions owing to the factor price manipulation effect—in particular, they imposed taxes on competing producers precisely to keep wages low. In contrast, workers now dislike taxes precisely because of their effect on wages. Consequently, in this case, workers have more moderate preferences regarding taxation, and democracy generates lower taxes than both the dictatorship of the elite and the dictatorship of the middle class. This proposition therefore again highlights that which set of political institutions will generate a greater level of income per capita (or higher economic growth) depends on investment opportunities and market structure. When workers (or a subgroup that is influential in democracy) can tax entrepreneurs without suffering the consequences, democracy will generate high levels of redistributive taxation and can lead to a lower income per capita than elite or middle class dominated politics. However, when workers recognize the impact of taxes on their own wages, democracy will generate more moderate political outcomes.

The simple analysis in this section therefore already gives us some clues about why there are no clear-cut relationships between political regimes and economic growth. If the forces highlighted here are important, we would expect democracy to generate higher growth under certain circumstances, for example, when the equivalent of Condition 22.1 holds. In contrast, democracy will lead to worse economic performance by pursuing populist policies and imposing high taxes when the equivalent of Condition 22.1 fails to hold. Naturally, the model presented here is very simple in many ways, and Condition 22.1 or its close cousins may not be the right ones for evaluating whether democracy or other regimes are more growth-enhancing. Nevertheless, this analysis emphasizes that democratic regimes, like the dictatorships of the elite and of the middle class, will look after the interests of the groups that have political power and the resulting allocations will often involve different types of distortions. Whether these distortions are more or less severe than those generated by alternative political regimes will depend on technology, factor endowments, and the types of policies available to the political system. In light of our analysis so far this result is not surprising, but its implications are nonetheless important to emphasize. In particular, it highlights that there are no a priori theoretical reasons to expect that there should be a simple empirical relationship between democracy and growth. On balance, we may believe that the distortions created by democracy should be less than those created by dictatorships (nondemocracies), but this will be a conclusion to be reached with more detailed theoretical and empirical analysis. Moreover, in Section 23.5 I will present another set of reasons, which I find more compelling than those implied by the simple models here, for why democracies may not generate more growth than dictatorships.

23.3. Dynamic Tradeoffs

The previous section contrasted economic allocations under different political regimes (in particular, the dictatorship of the elite, the dictatorship of the middle class and democracy, which here amounts to the dictatorship of the workers). Although the underlying economic environment was a simplified version of the infinite-horizon neoclassical growth model, the tradeoffs among the regimes were static. In this section, I will study an environment, which also involves entry into entrepreneurship, social mobility and a simple form of creative destruction. Using this environment, I will contrast the implications of democracy to oligarchy for economic performance. The emphasis will be on the dynamic tradeoffs between the two regimes.

23.3.1. The Baseline Model. The model economy is similar to that analyzed in Section 22.2 and more specifically, to the Cobb-Douglas economy in subsection 22.2.4. The economy is populated by a continuum 1 of infinitely-lived agents, each with preferences given by (22.1) as in Section 22.2. In addition, for reasons that will become clear soon, I assume that each each individual dies with a small probability ε in every period, and a mass ε of new individuals are born (with the convention that after death there is zero utility and β is the discount factor inclusive of the probability of death). I will consider the limit of this economy with $\varepsilon \to 0$.

There are two occupations in this economy, production workers and entrepreneurs. The key difference between the models in the previous chapter (and that in the previous section) and the one here is the possibility of social mobility, i.e., the fact that individuals may choose their occupations. In particular, each agent can either be employed as a worker or set up a firm to become an entrepreneur. I assume that all agents have the same productivity as workers, but their productivity in entrepreneurship differs. In particular, agent i at time t has entrepreneurial talent/skills $a_i(t) \in \{A^L, A^H\}$ with $A^L < A^H$. To become an entrepreneur, an agent needs to set up a firm, if he does not have an active firm already. Setting up a new firm may be costly because of entry barriers created by existing entrepreneurs.

Each agent therefore starts period t with skill level $a_i(t) \in \{A^H, A^L\}$ and some amount of capital $k_i(t)$ invested from the previous date (recall that, as in the models we have seen so far, capital investments are made one period in advance), and another state variable denoting whether he already possesses a firm. We will denote this by $e_i(t) \in \{0, 1\}$, with $e_i(t) = 1$ corresponding to the individual having chosen entrepreneurship at date $t-1$ (for date t). More importantly, if the individual is already an "incumbent" entrepreneur at t, i.e., $e_i(t) = 1$, this may make it cheaper for him to become an entrepreneur at date $t+1$, i.e., to choose $e_i(t+1) = 1$, because of potential entry barriers into entrepreneurship for non-incumbents. I will refer to an agent would $e_i(t) = 1$ as a member of the "elite" at time t, both because he will avoid the entry costs and also because in oligarchy he will be a member of the political elite making the policy choices.

In summary, at each period t, each agent makes the following decisions: an occupation choice $e_i(t+1) \in \{0, 1\}$, and in addition if $e_i(t+1) = 1$, i.e., if he becomes an entrepreneur, he also makes an investment decision for next period $k_i(t+1) \in \mathbb{R}_+$. In addition, those who are currently entrepreneurs, i.e., those with $e_i(t) = 1$, decide how much labor $l_i(t) \in \mathbb{R}_+$ to hire.

Agents also make the policy choices in this society. How the preferences of various agents map into policies differs depending on the political regime, which will be discussed below. There are three policy choices. Two of those are similar to the policies we have seen so far; a tax rate $\tau(t) \in [0, \hat{\tau}]$ on output and a lump-sum transfer distributed to all agents denoted by $T(t) \in [0, \infty)$. Notice that I have already imposed an upper bound on taxes $\hat{\tau} < 1$. This upper bound can be derived from the ability of individuals to hide their output in the informal sector or because of the standard distortionary effects of taxation. It simplifies the analysis to take it as given here. This parameter will have important implications on what type of political regime will lead to greater income per capita. The new policy instrument is a cost $B(t) \in [0, \infty)$ imposed on new entrepreneurs setting up a firm. I assume that the entry barrier $B(t)$ is pure waste, for example corresponding to the bureaucratic procedures that individuals have to go through to open a new business. This implies that lump-sum transfers are financed only from taxes.

An entrepreneur with skill level $a_i(t)$ and capital level $k_i(t)$ produces

$$y_i(t) = \frac{1}{\alpha} k_i(t)^{\alpha} (a_i(t) l_i(t))^{1-\alpha} \tag{23.3}$$

units of the final good, when he hires $l_i(t) \in \mathbb{R}_+$ units of labor. Notice that entrepreneurial skill enters the production function as a labor-augmenting productivity term. As in subsection 22.2.4, I assume that there is full depreciation of capital at the end of the period, so $k_i(t)$ is also the level of investment of entrepreneur i at time $t-1$, which is in terms of the unique final good of the economy.

I will further simplify the analysis by assuming that all firms have to operate at the same size, $\bar{L}$, so $l_i(t) = \bar{L}$ (see Exercise 23.6 for the implications of relaxing this assumption). Finally, I adopt the convention that the entrepreneur himself can work in his firm as one of the workers, which implies that the opportunity cost of becoming an entrepreneur is 0.

The most important assumption here is that each entrepreneur has to run the firm himself, so it is his productivity, $a_i(t)$, that matters for output. An alternative would be to allow costly delegation of managerial positions to other, more productive agents. In this case, low-productivity entrepreneurs may prefer to hire more productive managers. If delegation to managers can be done costlessly, entry barriers would create no distortions. Throughout I assume that delegation is prohibitively costly.

To simplify the expressions below, I define $b(t) \equiv \beta^{-1} B(t) / \bar{L}$, which corresponds to discounted per worker entry cost (and will be the relevant object when we look at the profitability of different occupational choices and thus simplify the expressions). Profits (the returns to entrepreneur i gross of the cost of entry barriers) at time t are then equal to $\pi_i(t) = (1 - \tau(t)) y_i(t) - w(t) l_i(t) - \beta^{-1} k_i(t)$, which takes into account that investment $k_i(t)$ has to be made in the previous period, thus the opportunity cost of investment (which is forgone consumption) is multiplied by the inverse of the discount factor. This expression for profits takes into account that the entrepreneur produces an output of $y_i(t)$, pays a fraction $\tau(t)$ of this in taxes, and also pays a total wage bill of $w(t) l_i(t)$. Given a tax rate $\tau(t)$ and a wage rate $w(t) \geq 0$ and using the fact that $l_i(t) = \bar{L}$, the net profits of an entrepreneur with talent $a_i(t)$ at time t are:

$$\pi\left(k_i(t) \mid a_i(t), w(t), \tau(t)\right) = (1-\tau(t))\,\alpha^{-1} k_i(t)^{\alpha}\left(a_i(t)\,\bar{L}\right)^{1-\alpha} - w(t)\,\bar{L} - \beta^{-1} k_i(t), \tag{23.4}$$

where recall that the tax rate is in the range $0 \leq \tau(t) \leq \hat{\tau}$ and the cost of capital is multiplied by β^{-1} because it is incurred in the previous period. Given this expression, the (instantaneous) gain from entrepreneurship for an agent of talent $z \in \{L, H\}$ at time t as a function of the tax rate, $\tau(t)$, and the wage rate, $w(t)$, is:

$$\Pi^z(\tau(t), w(t)) = \max_{k_i(t)} \pi\left(k_i(t) \mid a_i(t) = A^z, w(t), \tau(t)\right). \tag{23.5}$$

Note that this is the *net gain* to entrepreneurship since the agent receives the wage rate $w(t)$ irrespective (either working for another entrepreneur when he is a worker, or working for himself—thus having to hire one less worker—when he is an entrepreneur). More importantly, the gain to becoming an entrepreneur for an agent with $e_i(t-1) = 0$ and ability $a_i(t) = A^z$ is $\Pi^z(\tau(t), w(t)) - \beta^{-1}B(t) = \Pi^z(\tau(t), w(t)) - b(t)\,\bar{L}$, since this agent will have to pay the additional cost imposed by the entry barriers, which, like the costs of investment, is incurred in the previous period and is thus multiplied by β^{-1}.

Labor market clearing requires the total demand for labor not to exceed the supply. Since entrepreneurs also work as production workers, the supply is equal to 1, so:

$$\int_0^1 e_i(t)\, l_i(t)\, di = \int_{i \in S_t^E} \bar{L} di \leq 1, \tag{23.6}$$

where S_t^E is the set of entrepreneurs at time t.

Finally, I specify the law of motion of entrepreneurial talent, $a_i(t)$, I assume that there is imperfect correlation between the entrepreneurial skill over time with the following Markov structure:

$$a_i(t+1) = \begin{cases} A^H & \text{with probability } \sigma^H & \text{if } a_i(t) = A^H \\ A^H & \text{with probability } \sigma^L & \text{if } a_i(t) = A^L \\ A^L & \text{with probability } 1-\sigma^H & \text{if } a_i(t) = A^H \\ A^L & \text{with probability } 1-\sigma^L & \text{if } a_i(t) = A^L \end{cases}, \tag{23.7}$$

where σ^H, $\sigma^L \in (0,1)$. Here σ^H is the probability that an agent has high skill in entrepreneurship conditional on being high skill in the previous period, and σ^L is the probability transitioning from low skill to high skill. It is natural to suppose that $\sigma^H \geq \sigma^L > 0$, so that skills are persistent and low skill is *not* an absorbing state. What is essential for the results is imperfect correlation of entrepreneurial talent over time, i.e., $\sigma^H < 1$, so that the identities of the entrepreneurs necessary to achieve productive efficiency change over time and thus necessitate a type of creative destruction.

The imperfect over-time correlation in $a_i(t)$ can be interpreted in three alternative and complementary ways. First, we can suppose that the productivity of an individual is not constant over time, and changes in comparative advantage necessitate changes in the identity of entrepreneurs. Second, we can think of the infinitely-lived agents as representing dynasties, and the imperfect over-time correlation in $a_i(t)$ may represent imperfect correlation between the skills of parents and children. Thrid and perhaps most interestingly, it may be that each individual has a fixed competence across different activities, and comparative advantage in entrepreneurship changes as the importance of different activities evolves over time. For example, some individuals may be better in industrial entrepreneurship, while some are better in agriculture, and as industrial activities become more profitable than agriculture, individuals who have a comparative advantage in industry should enter into entrepreneurship and those who have a comparative advantage of agriculture should exit. Both of these stories are parsimoniously captured by the Markov chain for talent given in (23.7).

This Markov chain also implies that the fraction of agents with high skill in the stationary distribution is (see Exercise 23.7):

$$M \equiv \frac{\sigma^L}{1-\sigma^H+\sigma^L} \in (0,1)\,. \tag{23.8}$$

Since there is a large number (continuum) of agents, the fraction of agents with high skill at any point is M. Throughout I assume that

$$M\bar{L} > 1,$$

so that, without entry barriers, high-skill entrepreneurs generate more than sufficient demand to employ the entire labor supply. Moreover, I think of M as small and $\bar{L}$ as large; in particular, I assume $\bar{L} > 2$, which ensures that the workers are always in the majority and simplifies the political economy discussion below.

The timing of events within every period can be summarized as follows. At the beginning of time t, $a_i(t)$, $e_i(t)$ and $k_i(t)$ are given for all individuals as a result of their decision from date $t-1$. Then the following sequence of moves takes place.

(1) Entrepreneurs demand labor and the labor market clearing wage rate, $w(t)$, is determined.
(2) The tax rate on entrepreneurs, $\tau(t) \in [0,\hat{\tau}]$, is set.
(3) The skill level of each agent for next period, $a_i(t+1)$, is realized.
(4) The entry barrier for new entrepreneurs $b(t+1)$ is set.
(5) All agents make occupational choices, $e_i(t+1)$, and entrepreneurs make investment decisions, $k_i(t+1)$ for next period.

Entry barriers and taxes will be set by different agents in different political regimes as will be specified below. Notice that taxes are set after the investment decisions. This raises the holdup problems discussed in the previous chapter and acts as an additional source of inefficiency. The fact that $\tau(t) \leq \hat{\tau} < 1$ puts a limit on these holdup problems. It is also important to note that individuals make their occupational choices and investment decisions knowing their ability level, i.e., $a_i(t+1)$ is realized before the decisions on $e_i(t+1)$ and $k_i(t+1)$. Notice also that if an individual does not operate his firm, he loses "the license", so next time he wants to set up a firm, he needs to incur the entry cost (and the assumption that $l_i(t) = \bar{L}$ rules out the possibility of operating the firm at a much smaller scale). Finally, we need to specify the initial conditions: I assume that the distribution of talent in the society is given by the stationary distribution, nobody starts out as an entrepreneur, so that $e_i(-1) = 0$ for all i, and the initial level of capital holdings is not important, since negative consumption is allowed, thus individuals can always increase their capital holdings by choosing a negative level of consumption.

Let us again focus on Markov Perfect Political Economy Equilibrium, where strategies are only a function of the payoff relevant states. For individual i the payoff relevant state at time t includes his own state $(e_i(t), a_i(t), k_i(t), a_i(t+1))$,[3] and potentially the fraction of entrepreneurs that are high skill, denoted by $\mu(t)$, and defined as

$$\mu(t) = \Pr\left(a_i(t) = A^H \mid e_i(t) = 1\right) = \Pr\left(a_i(t) = A^H \mid i \in S_t^E\right).$$

The equilibrium can be characterized by writing the net present discounted values of different agents recursively and then characterizing the optimal strategies within each time period by backward induction. I start with the "economic equilibrium," which is the equilibrium of the economy described above given a policy sequence $\{b(t), \tau(t)\}_{t=0,1,...}$. Let $x_i(t) = (e_i(t+1), k_i(t+1))$ be the vector of choices of agent i at time t (for entrepreneurship and capital investment at time $t+1$) and let $x(t) = [x_i(t)]_{i\in[0,1]}$ denote the choices for all agents, and $p(t) = (\tau(t), b(t+1))$ denote the vector of policies at time t. Moreover, let $p^t = \{p(n)\}_{n=t}^{\infty}$ denote the infinite sequence of policies from time t onwards, and similarly w^t and x^t denote the sequences of wages and choices from t onwards. Then $\hat{x}^t$ and a sequence of wage rates $\hat{w}^t$ constitute an economic equilibrium given a policy sequence p^t if, given $\hat{w}^t$ and p^t and his state $(e_i(t-1), a_i(t))$, $x_i(t)$ maximizes the utility of agent i, and $\hat{w}_t$ clears the labor market at time t, i.e., equation (23.6) holds. Each agent's type in the next period, $(e_i(t+1), a_i(t+1))$, is then given by his decision at time t as to whether to become an entrepreneur and by the law of motion in (23.7).

I now characterize this equilibrium. Since $l_i(t) = \bar{L}$ for all $i \in S_t^E$ (where, recall that, S_t^E is the set of entrepreneurs at time t), profit-maximizing investments are given by:

$$k_i(t) = (\beta(1-\tau(t)))^{1/(1-\alpha)} a_i(t)\bar{L}, \tag{23.9}$$

where is $\tau(t)$ is the equilibrium tax rate that entrepreneurs anticipate correctly along the equilibrium path. This equation implies that the level of investment is increasing in the skill

[3]Here $e_i(t)$, $k_i(t)$ and $a_i(t)$ are part of the individual's state at time t, because they influence an entrepreneur's labor demand. In addition, $a_i(t+1)$ is revealed at time t and influences his occupational choice and investment decisions $e_i(t+1)$ and $k_i(t+1)$ for $t+1$ and is also part of his state.

level of the entrepreneur, $a_i(t)$, and the level of employment, $\bar{L}$, and decreasing in the tax rate, $\tau(t)$.

Now using (23.9), the net current gain to entrepreneurship for an agent of type $z \in \{L, H\}$ (i.e., of skill level A^L or A^H) can be obtained as:

$$\Pi^z(\tau(t), w(t)) = (1-\alpha)\alpha^{-1}(\beta(1-\tau(t)))^{1/(1-\alpha)} A^z \bar{L} - w(t)\bar{L}. \tag{23.10}$$

Moreover, the labor market clearing condition (23.6) implies that the total mass of entrepreneurs at any time is $\int_{i\in S_t^E} di = 1/\bar{L}$. Tax revenues at time t and the per capita lump-sum transfers are then given as:

$$T(t) = \int_{i\in S_t^E} \tau(t) y_i(t)\, di = \alpha^{-1}\tau(t)(\beta(1-\tau(t)))^{\alpha/(1-\alpha)}\bar{L}\int_{i\in S_t^E} a_i(t)\, di. \tag{23.11}$$

To economize on notation, let us now denote the sequence of future policies and equilibrium wages by $q^t \equiv (p^t, w^t)$. Then the time t value of an agent with skill level $z \in \{L, H\}$ if he chooses production work (for time t) is

$$W^z(q^t) = w(t) + T(t) + \beta CW^z(q^{t+1}), \tag{23.12}$$

where it is explicitly conditioned on future policies and wages, q^t, since these influence continuation values, and $CW^z(q^{t+1})$ is the relevant continuation value for a worker of type z from time $t+1$ onwards, given by

$$\begin{aligned} CW^z(q^{t+1}) \;=\;& \sigma^z \max\left\{W^H(q^{t+1}); V^H(q^{t+1}) - b(t+1)\bar{L}\right\} \\ &+ (1-\sigma^z)\max\left\{W^L(q^{t+1}); V^L(q^{t+1}) - b(t+1)\bar{L}\right\}, \end{aligned} \tag{23.13}$$

where $V^z(q^t)$ is defined similarly to $W^z(q^t)$ and is the time t value of an agent of skill z when he is an entrepreneur. The expressions for both (23.12) and (23.13) are intuitive. A worker of type $z \in \{L, H\}$ receives a wage income of $w(t)$ (independent of his skill), a transfer of $T(t)$, and the continuation value $CW^z(q^{t+1})$. This continuation value encodes the major dynamic tradeoffs facing individuals in this model. A worker of type $z \in \{L, H\}$ today—that is, an individual i with $e_i(t) = 0$—will be high skill in the next period with probability σ^z, and in this case, he can either choose to remain a worker, receiving value W^H, or decide to become an entrepreneur, $e_i(t+1) = 1$, by incurring the entry cost $b(t+1)\bar{L}$, receiving the value of a high-skill entrepreneur, V^H. The reason why this individual has to pay the cost $b(t+1)\bar{L}$ when he chooses $e_i(t+1) = 1$ is that he is not currently an entrepreneur, i.e., $e_i(t) = 0$, thus he has to pay the costs associated with the entry barriers. The max operator makes sure that the individual chooses whichever option gives higher value. With probability $1-\sigma^z$, he will be low skill, and receives the corresponding values.

Similarly, the value functions for entrepreneurs are given by:

$$V^z(q^t) = w(t) + T(t) + \Pi^z(\tau(t), w(t)) + \beta CV^z(q^{t+1}), \tag{23.14}$$

where Π^z is given by (23.10) and now crucially depends on the skill level of the agent, and $CV^z\left(q^{t+1}\right)$ is the continuation value for an entrepreneur of type z:

(23.15)

$$CV^z\left(q^{t+1}\right)=\sigma^z\max\left\{W^H\left(q^{t+1}\right);V^H\left(q^{t+1}\right)\right\}+\left(1-\sigma^z\right)\max\left\{W^L\left(q^{t+1}\right);V^L\left(q^{t+1}\right)\right\}.$$

An entrepreneur of ability A^z also receives the wage $w\left(t\right)$ (working for his own firm) and the transfer $T\left(t\right)$, and in addition makes profits equal to $\Pi^z\left(\tau\left(t\right),w\left(t\right)\right)$. The following period, this entrepreneur has high skill with probability σ^z and low skill with probability $1-\sigma^z$, and conditional on the realization of this event, he decides whether to remain an entrepreneur or become a worker. Two points are noteworthy here. First, in (23.15), in contrast to the expression in (23.13), there is no additional cost of becoming an entrepreneur since this individual already owns a firm. Second, if an entrepreneur decides to become a worker, he obtains the value as given by the expressions in (23.13) so that the next time he wishes to operate a firm, he has to incur the cost of doing so.

Inspection of (23.13) and (23.15) immediately reveals that the occupational choices of individuals for time t will depend on the *net value* of entrepreneurship conditional on their current occupational status, $e_i\left(t-1\right)=\mathbf{e}$. We write this is

$$NV\left(q^t\mid a_i\left(t\right)=A^z,e_i\left(t-1\right)=\mathbf{e}\right)=V^z\left(q^t\right)-W^z\left(q^t\right)-\left(1-\mathbf{e}\right)b\left(t\right)\bar{L},$$

which is defined as a function of an individual's skill a and current entrepreneurship status, $\mathbf{e}$. The last term is the entry cost incurred by agents with $\mathbf{e}=0$. The max operators in (23.13) and (23.15) imply that if $NV>0$ for an agent, then he prefers to become an entrepreneur.

Who will become an entrepreneur in this economy? The answer depends on the NV's. Standard dynamic programming arguments from Chapters 6 and 16, combined with the fact that instantaneous payoffs are strictly monotone, imply that $V^z\left(q^t\right)$ is strictly monotonic in $w\left(t\right)$, $T\left(t\right)$ and $\Pi^z\left(\tau\left(t\right),w\left(t\right)\right)$, so that $V^H\left(q^t\right)>V^L\left(q^t\right)$ (see Exercise 23.5). By the same arguments, $NV\left(q^t\mid a_i\left(t\right)=A^z,e_i\left(t-1\right)=\mathbf{e}\right)$ is also increasing in $\Pi^z\left(\tau\left(t\right),w\left(t\right)\right)$. This in turn implies that for all a and $\mathbf{e}$,

$$\begin{aligned}NV\left(q^t\mid a_i\left(t\right)=A^H,e_i\left(t-1\right)=1\right)&\geq NV\left(q^t\mid a_i\left(t\right)=a,e_i\left(t-1\right)=\mathbf{e}\right)\\&\geq NV\left(q^t\mid a_i\left(t\right)=A^L,e_i\left(t-1\right)=0\right).\end{aligned}$$

In other words, the net value of entrepreneurship is highest for high-skill existing entrepreneurs, and lowest for low-skill workers. However, it is unclear ex ante whether $NV\left(q^t\mid a_i\left(t\right)=A^H,e_i\left(t-1\right)=0\right)$ or $NV\left(q^t\mid a_i\left(t\right)=A^L,e_i\left(t-1\right)=1\right)$ is greater, that is, whether entrepreneurship is more profitable for incumbents with low skill or for outsiders with high skill, who will have to pay the entry cost.

We can then define two different types of equilibria:

(1) *Entry equilibrium* where all entrepreneurs have $a_i\left(t\right)=A^H$.

(2) *Sclerotic equilibrium* where agents with $e_i(t-1) = 1$ remain entrepreneurs irrespective of their productivity.

An entry equilibrium requires the net value of entrepreneurship to be greater for a non-elite high skill agent than for a low-skill elite. Let us define $w^H(t)$ as the threshold wage rate such that high-skill non-elite agents are indifferent between entering and not entering entrepreneurship. That is, $w^H(t)$ has to be such that $NV\left(q^t \mid a_i(t) = A^H, e_i(t-1) = 0\right) = 0$. Using (23.12) and (23.14), we obtain this threshold as:

(23.16)
$$w^H(t) \equiv \max\left\{(1-\alpha)\alpha^{-1}(\beta(1-\tau(t)))^{1/(1-\alpha)}A^H - b(t) + \frac{\beta\left(CV^H\left(q^{t+1}\right) - CW^H\left(q^{t+1}\right)\right)}{\bar{L}}; 0\right\}.$$

Similarly, define $w^L(t)$ as the wage such that low-skill incumbent producers are indifferent between existing entrepreneurship or not, i.e., $w^L(t)$ is such that defined by $NV\left(q^t \mid a_i(t) = A^L, e_i(t-1) = 1\right) = 0$:

(23.17)
$$w^L(t) \equiv \max\left\{(1-\alpha)\alpha^{-1}(\beta(1-\tau(t)))^{1/(1-\alpha)}A^L + \frac{\beta\left(CV^L\left(q^{t+1}\right) - CW^L\left(q^{t+1}\right)\right)}{\bar{L}}; 0\right\}.$$

Both expressions are intuitive. For example, in (23.16), the term $(1-\alpha)\alpha^{-1}(\beta(1-\tau(t)))^{1/(1-\alpha)}A^H$ is the per worker profits that a high-skill entrepreneur will make before labor costs. $b(t)$ is the per worker entry cost (discounted total costs, $\beta^{-1}B(t)$, divided by $\bar{L}$). Finally, the term $\beta\left(CV^H\left(q^{t+1}\right) - CW^H\left(q^{t+1}\right)\right)$ is the indirect (dynamic) benefit, the additional gain from changing status from a worker to a member of the elite for a high-skill agent. Naturally, this benefit will depend on the sequence of policies, for example, it will be larger when there are greater entry barriers in the future. Consequently, if $w^L(t) < w^H(t)$, the total benefit of becoming an entrepreneur for a non-elite high-skill agent exceeds the cost. Equation (23.17) is explained similarly. Evidently, a wage rate lower than both $w^L(t)$ and $w^H(t)$ would lead to excess demand for labor and could not be an equilibrium. Consequently, the condition for an entry equilibrium to exist at time t can simply be written as a comparison of the two thresholds determined above:

$$w^H(t) \geq w^L(t). \tag{23.18}$$

A sclerotic equilibrium emerges, on the other hand, when the converse of (23.18) holds.

Moreover, in an entry equilibrium, i.e., when (23.18) holds, we must have that $NV\left(q^t \mid a_i(t) = A^H, e_i(t-1) = 0\right) = 0$. If it were strictly positive, or in other words, if the wage were less than $w(t)$, all agents with high skill would strictly prefer to become entrepreneurs, which is not possible since, by assumption, $M\bar{L} > 1$. This argument also shows that the total number (measure) of entrepreneurs in the economy will be $1/\bar{L}$. Then, from (23.10), (23.12) and (23.14), the equilibrium wage, which will be denoted $w^E(t)$, is

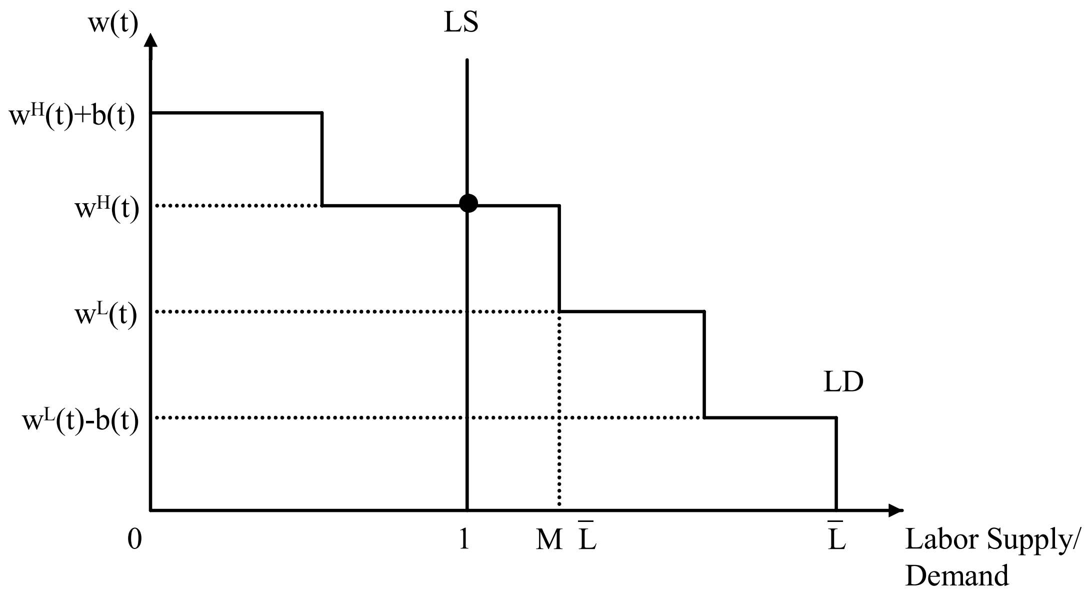

Figure 23.1. Labor market equilibrium when (23.18) holds.

equal to

$$w^E(t) = w^H(t)\,. \tag{23.19}$$

Note also that when (23.18) holds, naturally $NV\left(q^t \mid a_i(t) = A^L, e_i(t-1) = 1\right) \leq 0$, so low-skill incumbents would be worse off if they remained as entrepreneurs at the equilibrium wage rate $w^E(t)$.

Figure 23.1 illustrates the entry equilibrium diagrammatically by plotting labor demand and supply in this economy. Labor supply is constant at 1, while labor demand is decreasing as a function of the wage rate. This figure is drawn for the case where condition (23.18) holds, so that there exists an entry equilibrium. The first portion of the curve shows the willingness to pay of high-skill incumbents, i.e., those who start with $e_i(t-1) = 1$ but have high entrepreneurial skills $a_i(t) = A^H$. This marginal willingness is $w^H(t) + b(t)$ (since entrepreneurship is as profitable for them as for high-skill potential entrants and they do not have pay the entry cost). The second portion is for high-skill potential entrants—those with $e_i(t-1) = 0$ and $a_i(t) = A^H$—and is equal to $w^H(t)$. These two groups together demand $M\bar{L} > 1$ workers, ensuring that labor demand intersects labor supply at the wage given in (23.19).

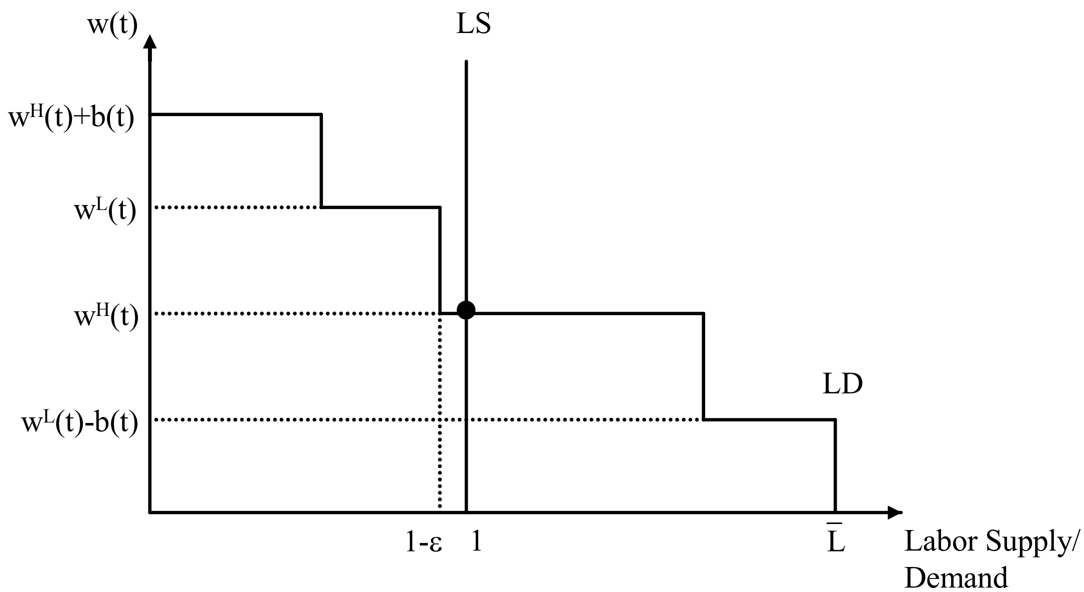

Figure 23.2. Labor market equilibrium when (23.18) does not hold.

In a sclerotic equilibrium, on the other hand, $w^H(t) < w^L(t)$, and low-skill incumbents remain in entrepreneurship, i.e., $e_i(t) = e_i(t-1)$. If there were no deaths so that $\varepsilon = 0$, the total number of entrepreneurs would be $1/\bar{L}$ and for any $w \in \left[w^H(t), w^L(t)\right]$, labor demand would exactly equal labor supply (i.e., $1/\bar{L}$ agents demanding exactly $\bar{L}$ workers each, and a total supply of 1). Hence, there would be multiple equilibrium wages. In contrast, when $\varepsilon > 0$, the total number of entrepreneurs who could pay a wage of $w^L(t)$ will be less than $1/\bar{L}$ for all $t > 0$, thus there would be excess supply of labor at this wage, or at any wage above the lower support of the above range. This implies that the equilibrium wage must be equal to this lower support, $w^H(t)$, which is identical to (23.19). Since at this wage agents with $e_i(t-1) = 0$ and $a_i(t) = A^H$ are indifferent between entrepreneurship and production work, in equilibrium a sufficient number of them enter entrepreneurship, so that total labor demand is equal to 1. In the remainder, I focus on the limiting case of this economy where $\varepsilon \to 0$, which picks $w^E(t) = w^H(t)$ as the equilibrium wage even when labor supply coincides with labor demand for a range of wages.[4]

Figure 23.2 illustrates this case diagrammatically. Because (23.18) does not hold in this case, the second flat portion of the labor demand curve is for low-skill incumbents

[4]In other words, the wage $w^H(t)$ at $\varepsilon = 0$ is the only point in the equilibrium set where the equilibrium correspondence is (lower-hemi) continuous in ε (recall the definition of lower hemi-continuity in Definitions A.24 and A.25 in Appendix Chapter A). In fact, the feature that there will be multiple equilibrium wage levels in dynamic models with entry barriers is not a feature of the setup here, which involves two types of entrepreneurs. This is demonstrated in Exercise 23.13.

($e_i(t-1) = 1$ and $a_i(t) = A^L$) who, given the entry barriers, have a higher marginal product of labor than high-skill potential entrants.

The equilibrium law of motion of the fraction of high-skill entrepreneurs, $\mu(t)$, is:

$$\mu(t) = \begin{cases} \sigma^H \mu(t-1) + \sigma^L (1 - \mu(t-1)) & \text{if (23.18) does not hold} \\ 1 & \text{if (23.18) holds} \end{cases}, \tag{23.20}$$

starting with some $\mu(0)$. The exact value of $\mu(0)$ will play an important role below. Recall that we have assumed above that $e_i(-1) = 0$ for all i. Under this assumption, any $b(0)$ would apply equally to all potential entrants and as long as it is not so high as to shut down the economy, the equilibrium would involve $\mu(0) = 1$. I consider $\mu(0) = 1$ to be the baseline case in the analysis below. While an economy with $e_i(-1) = 0$ is a plausible benchmark, I will also discuss below how the results differ if $e_i(-1) = 1$ for some i, so that the economy already starts with some privileged "elites" at the initial period. This will also open the way for a discussion of the potential adverse consequences of other selection mechanisms into entrepreneurship in the initial period.

To obtain a full political equilibrium, we need to determine the policy sequence p^t. I consider two extreme cases: (1) *Democracy*: the policies $b(t)$ and $\tau(t)$ are determined by majoritarian voting, with each agent having one vote. (2) Elite control (*Oligarchy*): the policies $b(t)$ and $\tau(t)$ are determined by majoritarian voting among the elite—the current entrepreneurs—at time t.

23.3.2. Democracy. A democratic equilibrium is a MPE where $b(t)$ and $\tau(t)$ are determined by majoritarian voting at time t. The timing of events implies that the tax rate at time t, $\tau(t)$, is decided after investment decisions, whereas the entry barriers are decided before. The assumption $\bar{L} > 2$ above ensures that workers (non-elite agents) are always in the majority.

At the time taxes are set, investments are sunk, agents have already made their occupation choices, and workers are in the majority. Therefore, taxes will be chosen to maximize per capita transfers given by

$$\alpha^{-1} \tau(t) k(t)^{\alpha} \bar{L} \sum_{i \in S_t^E} a_i(t),$$

which takes into account that $k(t)$ is already given from the investment in the previous period. Since this expression is increasing in $\tau(t)$ and $\tau(t) \leq \hat{\tau}$, the optimal tax for a worker is $\tau(t) = \hat{\tau}$ for all t. In view of this, total tax revenues are

$$T^E(t) = \alpha^{-1} \hat{\tau} (\beta(1 - \hat{\tau}))^{\alpha/(1-\alpha)} \bar{L} \sum_{i \in S_t^E} a_i(t). \tag{23.21}$$

The entry barrier, $b(t)$, is then set at the end of period $t-1$ (before occupational choices) to maximize this expression. Low-productivity workers (with $e_i(t-1) = 0$ and $a_i(t) = A^L$) know that they will remain workers, and in MPE, the policy choice at time t has no influence on strategies in the future except through its impact on payoff-relevant state variables. Therefore, given $\tau(t) = \hat{\tau}$, the utility of agent i with $e_i(t-1) = 0$ and $a_i(t) = A^L$ depends on $b(t)$ only through the equilibrium wage, $w^E(t)$, and the transfer, $T^E(t)$. High-productivity workers (those with $e_i(t-1) = 0$ and $a_i(t) = A^H$) may become entrepreneurs, but as the above analysis shows, in this case, $NV\left(q^t \mid a_i(t) = A^H, e_i(t-1) = 0\right) = 0$, we have $W^H = W^L$, so their utility is also identical to those of low-skill workers. Consequently, all workers prefer a level of $b(t)$ that maximizes $w^E(t) + T^E(t)$. Since the preferences of all workers are the same and they are in the majority, the democratic equilibrium will maximize these preferences.

A democratic equilibrium is therefore given by policy, wage and economic decision sequences $\hat{p}^t$, $\hat{w}^t$, and $\hat{x}^t$ such that $\hat{w}^t$ and $\hat{x}^t$ constitute an economic equilibrium given $\hat{p}^t$, and $\hat{p}^t = (\hat{\tau}, b(t+1))$ is such that:

$$b(t+1) \in \arg\max_{b(t+1)\geq 0} \left\{w^E(t+1) + T^E(t+1)\right\}.$$

Inspection of (23.19) and (23.21) immediately shows that wages and tax revenue are both maximized when $b(t+1) = 0$ for all t, so the democratic equilibrium will not impose any entry barriers. This is intuitive; workers have nothing to gain by protecting incumbents, and a lot to lose, since such protection reduces labor demand and wages. Since there are no entry barriers, only high-skill agents will become entrepreneurs, or in other words $e_i(t) = 1$ only if $a_i(t) = A^H$ at all t. Given this stationary sequence of MPE policies, we can use the value functions (23.12) and (23.14) to obtain

$$V^H = W^H = W^L = W = \frac{w^D + T^D}{1-\beta}, \tag{23.22}$$

where w^D is the equilibrium wage in democracy, and T^D is the level of transfers, given by $\hat{\tau}Y^D$. Since there are no entry barriers now or in the future and $\tau(t) = \hat{\tau}$, equation (23.16) then implies that $w^D = (1-\alpha)\alpha^{-1}(\beta(1-\hat{\tau}))^{\alpha/(1-\alpha)} A^H$. The following proposition therefore follows immediately:

Proposition 23.4. *There exists a unique democratic equilibrium, which features $\tau(t) = \hat{\tau}$ and $b(t) = 0$. Moreover, we have $e_i(t) = 1$ if and only if $a_i(t) = A^H$, so $\mu(t) = 1$. The equilibrium wage rate is given by*

$$w(t) = w^D \equiv (1-\alpha)\alpha^{-1}(\beta(1-\hat{\tau}))^{\alpha/(1-\alpha)} A^H, \tag{23.23}$$

and the aggregate output is

$$Y^D(t) = Y^D \equiv \alpha^{-1}(\beta(1-\hat{\tau}))^{\alpha/(1-\alpha)} A^H. \tag{23.24}$$

An important feature of the democratic equilibrium is that aggregate output is constant over time. This will contrast with the oligarchic equilibrium, where the skill composition of entrepreneurs and the level of output will change over time. Another noteworthy feature is that there is perfect equality because the excess supply of high-skill entrepreneurs ensures that they receive no rents.

It is useful to note that Y^D corresponds to the level of output inclusive of consumption and investment. "Net output" and consumption can be obtained by subtracting investment costs from Y^D, and in this case, they will be given by $Y_n^D \equiv \left(\alpha^{-1} - \beta\left(1-\hat{\tau}\right)\right)\left(\beta(1-\hat{\tau})\right)^{\alpha/(1-\alpha)} A^H$. All the results stated for output here also hold for net output. I focus on output only because the expressions are slightly simpler.

23.3.3. Oligarchic Equilibrium. In oligarchy, policies are determined by majoritarian voting among the elite.[5] At the time of voting over the entry barriers, $b\left(t\right)$, the elite consist of those with $e_i\left(t-1\right)=1$, and at the time of voting over the taxes, $\tau\left(t\right)$, the elite are those with $e_i\left(t\right)=1$.

Let us start with the taxation decision among those with $e_i\left(t\right)=1$ and also impose the following condition:

Condition 23.1.

$$\bar{L} \geq \frac{1}{2}\frac{A^H}{A^L} + \frac{1}{2}.$$

When this condition is satisfied, both high-skill and low-skill entrepreneurs prefer zero taxes, i.e., $\tau\left(t\right)=0$. I simplify the analysis here by assuming that this condition holds. Exercise 23.10 discusses the case when this condition is relaxed. Intuitively, Condition 23.1 requires the productivity gap between low and high-skill elites not to be so large that low-skill elites wish to tax profits in order to indirectly transfer resources from high-skill entrepreneurs to themselves.

When Condition 23.1 holds, the oligarchy will always choose $\tau\left(t\right)=0$. Then, at the stage of deciding the entry barriers, high-skill entrepreneurs would like to choose $b\left(t\right)$ to maximize V^H, and low-skill entrepreneurs would like to maximize V^L (both groups anticipating that $\tau\left(t\right)=0$). Both of these expressions are maximized by setting a level of the entry barrier that ensures the minimum level of equilibrium wages. Recall from (23.19) that equilibrium wages in this case are still given by $w^E\left(t\right)=w^H\left(t\right)$, so they will be minimized by ensuring that $w\left(t\right)=0$, i.e., by choosing any

$$b\left(t\right) \geq b^E\left(t\right) \equiv \left(1-\alpha\right)\alpha^{-1}\beta^{1/(1-\alpha)}A^H + \beta\left(\frac{CV^H\left(q^{t+1}\right) - CW^H\left(q^{t+1}\right)}{\bar{L}}\right). \tag{23.25}$$

[5]Notice that this assumption means political power rests with current entrepreneurs. As discussed in the previous chapter, there may often be a decoupling between economic and political power, so that key decisions are not made by current entrepreneurs, but by those who are politically powerful for historical or other reasons. The analysis in the previous chapter and also in Section 23.2 in this chapter illustrated the distortionary policies that would arise from such decoupling. The model here goes to the other extreme and places all political power in the hands of the current entrepreneurs and highlights a different set of inefficiencies that this will cause.

Without loss of any generality, let us assume that they will set the entry barrier as $b(t) = b^E(t)$ in this case.

An oligarchic equilibrium then can be defined as a policy sequence $\hat{p}^t$, wage sequence $\hat{w}^t$, and economic decisions $\hat{x}^t$ such that $\hat{w}^t$ and $\hat{x}^t$ constitute an economic equilibrium given $\hat{p}^t$, and $\hat{p}^t$ is such $\tau(t+n) = 0$ and $b(t+n) = b^E(t+n)$ for all $n \geq 0$. In the oligarchic equilibrium, there is no redistributive taxation and entry barriers are sufficiently high to ensure a sclerotic equilibrium with zero wages.

Imposing $w^E(t+n) = 0$ for all $n \geq 0$, we can solve for the equilibrium values of high- and low-skill entrepreneurs from the value functions (23.14) as

$$\tilde{V}^L = \frac{1}{1-\beta}\left[\frac{\left(1-\beta\sigma^H\right)A^L + \beta\sigma^L A^H}{\left(1-\beta\left(\sigma^H-\sigma^L\right)\right)}(1-\alpha)\,\alpha^{-1}\beta^{\alpha/(1-\alpha)}\bar{L}\right], \tag{23.26}$$

and

$$\tilde{V}^H = \frac{1}{1-\beta}\left[\frac{\left(1-\beta\left(1-\sigma^L\right)\right)A^H + \beta\left(1-\sigma^H\right)A^L}{\left(1-\beta\left(\sigma^H-\sigma^L\right)\right)}(1-\alpha)\,\alpha^{-1}\beta^{\alpha/(1-\alpha)}\bar{L}\right]. \tag{23.27}$$

These expressions are intuitive. First, consider $\tilde{V}^L$ and the case where $\beta \to 1$; then, starting in the state $e(t-1) = L$, an entrepreneur will spend a fraction $\sigma^L/\left(1-\sigma^H+\sigma^L\right)$ of his future with high skill A^H and a fraction $\left(1-\sigma^H\right)/\left(1-\sigma^H+\sigma^L\right)$ with low skill A^L. The fact that $\beta < 1$ implies discounting and the low-skill states which occur sooner are weighed more heavily (since the agent starts out as low skill). The intuition for $\tilde{V}^H$ is identical.

Since there will be zero equilibrium wages and no transfers, it is clear that $W = 0$ for all workers. Therefore, for a high-skill worker, $NV = \tilde{V}^H - b$, implying that

$$b(t) = b^E \equiv \frac{1}{1-\beta}\left[\frac{\left(1-\beta\left(1-\sigma^L\right)\right)A^H + \beta\left(1-\sigma^H\right)A^L}{\left(1-\beta\left(\sigma^H-\sigma^L\right)\right)}(1-\alpha)\,\alpha^{-1}\beta^{\alpha/(1-\alpha)}\bar{L}\right] \tag{23.28}$$

is sufficient to ensure zero equilibrium wages.

In this oligarchic equilibrium, aggregate output is:

$$Y^E(t) = \alpha^{-1}\beta^{\alpha/(1-\alpha)}\left[\mu(t)\,A^H + (1-\mu(t))A^L\right], \tag{23.29}$$

where $\mu(t) = \sigma^H\mu(t-1) + \sigma^L(1-\mu(t-1))$ as given by (23.20), starting with some $\mu(0)$.

As noted above, if, as in our benchmark assumption, all individuals start with $e_i(-1) = 0$, then the equilibrium will feature $\mu(0) = 1$. In this case, and in fact, for any $\mu(0) > M$, $\mu(t)$ will be a decreasing sequence converging to M and aggregate output $Y^E(t)$ will also be decreasing over time with:

$$\lim_{t\to\infty} Y^E(t) = Y^E_\infty \equiv \alpha^{-1}\beta^{\alpha/(1-\alpha)}\left[A^L + M(A^H - A^L)\right]. \tag{23.30}$$

Intuitively, the comparative advantage of the members of the elite in entrepreneurship gradually disappears because of the imperfect correlation between ability over time.

Nevertheless, it is also possible to imagine societies in which $\mu(0) < M$, because there is some other process of selection into the oligarchy in the initial period that is *negatively* correlated with skills in entrepreneurship. In this case, somewhat paradoxically, $\mu(t)$ and thus $Y^E(t)$ would be increasing over time. While interesting in theory, this case appears less relevant in practice, where we would expect at least some positive selection in the initial period, so that high-skill agents are more likely to become entrepreneurs at time $t = 0$ and $\mu(0) > M$.

Another important feature of the oligarchic equilibrium is that there is a high degree of (income) inequality. Wages are equal to 0, while entrepreneurs earn positive profits—in fact, each entrepreneur earns $y_i(t)\bar{L}$ (gross of investment expenses), where $y_i(t)$ depends on the current skill level of the entrepreneur. Since wages are equal to 0, total entrepreneurial earnings are equal to aggregate output. This contrasts with relative equality in democracy.

Proposition 23.5. *Suppose that Condition 23.1 holds. Then there exists a unique oligarchic equilibrium, with $\tau(t) = 0$ and $b(t) = b^E$ as given by (23.28). The equilibrium is sclerotic, with equilibrium wages $w^E(t) = 0$, and the fraction of high-skill entrepreneurs given by $\mu(t) = \sigma^H \mu(t-1) + \sigma^L(1 - \mu(t-1))$ starting with $\mu(0)$. Aggregate output is given by (23.29) and satisfies $\lim_{t\to\infty} Y^E(t) = Y^E_\infty$ as in (23.30). Moreover, as long as $\mu(0) > M$, aggregate output is decreasing over time.*

Proof. See Exercise 23.8. □

23.3.4. Comparison Between Democracy and Oligarchy. The first important result in the comparison between democracy and oligarchy is that if initial selection into entrepreneurship is on the basis of entrepreneurial skills (e.g., because $e_i(-1) = 0$ for all i) so that $\mu(0) = 1$, then aggregate output in the initial period of the oligarchic equilibrium, $Y^E(0)$, is greater than the constant level of output in the democratic equilibrium, Y^D. In other words,

$$Y^D = \alpha^{-1}\left(\beta(1-\hat{\tau})\right)^{\alpha/(1-\alpha)} A^H < Y^E(0) = \alpha^{-1}\beta^{\alpha/(1-\alpha)} A^H.$$

Therefore, oligarchy initially generates greater output than democracy, because it is protecting the property rights of entrepreneurs (whereas democracy is imposing distortionary taxes on entrepreneurs). However, the analysis also shows that, in this case, $Y^E(t)$ declines over time, while Y^D is constant. Consequently, the oligarchic economy may subsequently fall behind the democratic society. Whether it does so or not depends on whether Y^D is greater than Y^E_∞ as given by (23.30). This will be the case if $(1-\hat{\tau})^{\alpha/(1-\alpha)} A^H > A^L + M(A^H - A^L)$, or if

Condition 23.2.

$$(1-\hat{\tau})^{\alpha/(1-\alpha)} > \frac{A^L}{A^H} + M\left(1 - \frac{A^L}{A^H}\right).$$

If Condition 23.2 holds, then at some point the democratic society will overtake ("leapfrog") the oligarchic society.

As noted above, it is possible to imagine societies in which even in the initial period, there are "elites" that are not selected into entrepreneurship on the basis of their skills. In this case, we will typically have $\mu(0) < 1$. In the extreme case where there is negative selection into entrepreneurship in the initial period, we have $\mu(0) < M$. To analyze these cases, let us define

$$\bar{\mu}(0) \equiv \frac{(1-\hat{\tau})^{\alpha/(1-\alpha)} - A^L/A^H}{1 - A^L/A^H}. \tag{23.31}$$

It can be verified that as long as $\mu(0) > \bar{\mu}(0)$, oligarchy will generate greater output than democracy in the initial period. Notice also that $\bar{\mu}(0) > M$ if and only if Condition 23.2 holds.

This discussion and inspection of Condition 23.2 establish the following result (proof in the text):

Proposition 23.6. *Suppose that Condition 23.1 holds.*

(1) *Suppose also that $\mu(0) = 1$. Then at $t = 0$, aggregate output is higher in an oligarchic society than in a democratic society, i.e., $Y^E(0) > Y^D$. If Condition 23.2 does not hold, then aggregate output in oligarchy is always higher than in democracy, i.e., $Y^E(t) > Y^D$ for all t. If Condition 23.2 holds, then there exists t' such that for $t \leq t'$, $Y^E(t) \geq Y^D$ and for $t > t'$, $Y^E(t) < Y^D$, so that the democratic society leapfrogs the oligarchic society. Leapfrogging is more likely when $\hat{\tau}$, A^L/A^H and M are low.*

(2) *Suppose next that $\mu(0) < 1$. If $\mu(0) > \max\{M, \bar{\mu}(0)\}$, then the results from part 1 apply. If Condition 23.2 holds and $\mu(0) < \bar{\mu}(0)$, then aggregate output in oligarchy, $Y^E(t)$, is always lower than that in democracy, Y^D. If Condition 23.2 does not hold and $\mu_0 < M$, then aggregate output in oligarchy, $Y^E(t)$, is always higher than that in democracy, Y^D.*

Proof. See Exercise 23.9. □

This proposition implies that when $\mu(0)$ is not excessively low (i.e., when there is no *negative* correlation between initial entry into entrepreneurship and skills), an oligarchic society will start out as more productive than a democratic society, but will decline over time.

This proposition shows that oligarchy is more likely to be relatively inefficient in the long run:

(1) when $\hat{\tau}$ is low, meaning that democracy is unable to pursue highly populist policies with a high degree of redistribution away from entrepreneurs/capitalists. The

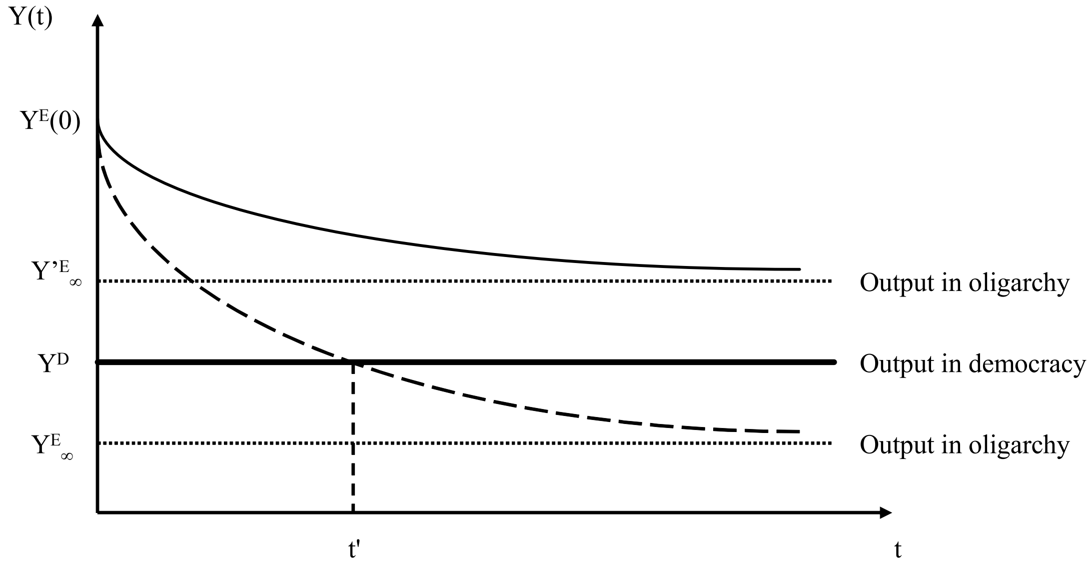

Figure 23.3. Dynamic comparison of output in oligarchy and democracy. The dashed line represents output in oligarchy when Condition 23.2 holds, and the solid line represents output in oligarchy when this condition does not hold.

parameter $\hat{\tau}$ may correspond both to certain institutional impediments limiting redistribution, or more interestingly, to the specificity of assets in the economy; with greater specificity, taxes will be limited, and redistributive distortions may be less important.

(2) when A^H is high relative to A^L, so that the creative destruction process—the selection of high-skill agents for entrepreneurship—is important for the efficient allocation of resources.

(3) M is low, so that a random selection of agents contains a small fraction of high-skill agents, making oligarchic sclerosis highly distortionary. Alternatively, M is low when σ^H is low, so oligarchies are more likely to lead to low output in the long run when the efficient allocation of resources requires a high degree of "churning" in the ranks of entrepreneurs, which is another measure of the importance of creative destruction.

On the other hand, if the extent of taxation in democracy is high and the failure to allocate the right agents to entrepreneurship only has limited costs, then an oligarchic society will generate greater output than a democracy in the long run.

These comparative static results may be useful in interpreting why, as discussed in Section 23.1, the Northeastern United States so conclusively outperformed the Caribbean plantation economies during the 19th century. First, the American democracy was not highly redistributive, corresponding to low $\hat{\tau}$ in terms of the model here. More important, the 19th century was the age of industry and commerce, where the allocation of high-skill agents to entrepreneurship appears to have been probably quite important, and potentially only a small fraction of the population were really talented as inventors and entrepreneurs. This can be thought of as a low value of A^L/A^H and M.

Figure 23.3 illustrates the case with $\mu(0) = 1$ (or $\mu(0) > \max\{M, \bar{\mu}(0)\}$), and depicts both the situation in which Condition 23.2 holds and the converse. The thick flat line shows the level of aggregate output in democracy, Y^D. The other two curves depict the level of output in oligarchy, $Y^E(t)$, as a function of time for the case where Condition 23.2 holds and for the case where it does not. Both of these curves asymptote to some limit, either Y^E_∞ or Y'^E_∞, which may lie below or above Y^D. The dashed curve shows the case where Condition 23.2 holds, so after date t', oligarchy generates less aggregate output than democracy. When Condition 23.2 does not hold, the solid curve applies, and aggregate output in oligarchy asymptotes to a level higher than Y^D.

Naturally, both of these major results, the greater short-term efficiency and the dynamic costs of oligarchy, are derived from the underlying assumptions of the model. In addition to $\mu(0)$ being sufficiently large, the first result is a consequence of the assumption that the only source of distortion in oligarchy is the entry barriers. In practice, an oligarchic society could pursue other distortionary policies to reduce wages and increase profits, in which case it might generate lower output than a democratic society even at time $t = 0$. I will emphasize how these costs arise and affect equilibrium dynamics in Section 23.5 below. The dynamic costs of oligarchy are also stark in this model, since output and distortions in democracy are constant, whereas the allocation of talent deteriorates in oligarchy because of the entry barriers. In more general models, democracy may also create intertemporal distortions. For example, if democracy is expected to tax capital incomes in the future and and there is less than full depreciation, this will create dynamic distortions by affecting the whole sequence of investment levels, though in this case, it is also reasonable to think that oligarchy may tax human capital more, creating similar distortions. Which set of distortions dominate is an empirical question. Nevertheless, the dynamic distortions of oligarchy emphasized in this paper are new and potentially important, and thus need to be considered in evaluating the allocative costs of these regimes.

The second part of the proposition also highlights the role of selection of individuals into entrepreneurship (and oligarchy) in the initial period. It shows that the results highlighted so far hold even if $\mu(0)$ is less than one, as long as it is not very small. On the other hand, if $\mu(0)$ is very small to start with, oligarchy may always generate less output than democracy, and in fact, if $\mu(0)$ starts out less than M, oligarchy may even have increasing level of output. A very low level of $\mu(0)$ may emerge if the oligarchy is founded by individuals that are talented in non-economic activities (e.g., by elites specialized in fighting in pre-modern times) and these non-economic talents are negatively correlated with entrepreneurial skills. Nevertheless, as noted above, a significant amount of positive selection on the basis of skills even in the initial period seems to be the more reasonable case.

On the basis of this analysis, the current model not only adds to the arguments we have made already, that there is no unambiguous theoretical result on whether democracy or non-democracy will generate greater growth, but it also highlights a different dimension of the tradeoff between different regimes—that related to the dynamics they imply. While democracy may create short-run distortions, it can lead to better long-run performance because it avoids *political sclerosis*—that is, incumbents becoming politically powerful and erecting entry barriers against new and better entrepreneurs. This model therefore suggests precisely the type of patterns we already discussed in Section 23.1; lack of a clear relationship between democracy and growth over the past 50 years combined with the examples of democracies that have been able to achieve industrialization during critical periods in the 19th century. In fact, a simple extension of the framework here provides additional insights that are useful in thinking about why democracies may be successful in preventing political sclerosis; the forces highlighted here also suggest that democracies are more "*flexible*" than oligarchies. In particular, Exercise 23.12 considers a simple extension of the framework here and demonstrates that democracies will typically be better able to adapt to the arrival of new technologies, because there are no incumbents with rents to protect, who can successfully block or slow down the introduction of new technology. This type of flexibility might, ultimately, be one of the more important advantages of democratic regimes.

Even though the model presented in this section provides a range of ideas and comparative static results that are useful for understanding the comparative development experiences of democratic and nondemocratic regimes, like the model discussed in the previous section, it focuses on the costs of democracy resulting from its more redistributive nature—in particular, it emphasizes that democratic regimes redistribute income away from the rich and the entrepreneurs towards the poorer segments of the society and this leads to distortions reducing income per capita. An alternative source of distortions in democracy, which will complement the mechanisms discussed here, will be the resistance of the elite against democratic redistributive policies, which will often lead to additional inefficiencies. This will be discussed in Section 23.5.

Before doing this, however, it is useful to return to the induced preferences over different regimes. As a first step in this direction, let us look at income inequality and the preferences of different groups over regimes. First, it is straightforward to see that oligarchy always generates more (consumption) inequality relative to democracy, since the latter has perfect equality—the net incomes and consumption of all agents are equalized in democracy because of the excess supply of high-skill entrepreneurs.

Moreover, non-elites are always better off in democracy than in oligarchy, where they receive zero income. In contrast, and more interestingly, it is possible for low-skill elites to be better off in democracy than in oligarchy (though high-skill elites are always better off in oligarchy). This point will play a role in our first result on regime change in subsection 23.3.5, so it is useful to understand the intuition. Recall that the utility of low-skill elites in oligarchy is given by (23.26) above, whereas combining (23.22), (23.23) and (23.24), these low-skill agents in democracy would receive:

$$W^L = \frac{1}{1-\beta}\left[\left(\left(\alpha^{-1}(1-\alpha)(1-\hat{\tau}) + \alpha^{-1}\hat{\tau}\right)(\beta(1-\hat{\tau}))^{\alpha/(1-\alpha)}\right)A^H\right].$$

Comparing this expression to (23.26) makes it clear that if $\hat{\tau}$, A^L/A^H, σ^L and/or $\bar{L}$ are sufficiently low, these low-skill elites would be better off in democracy than in oligarchy. More specifically, we have:

Proposition 23.7. *Suppose that Condition 23.1 holds. Then, provided that*

$$\frac{\left(1-\beta\sigma^H\right)A^L/A^H + \beta\sigma^L}{1-\beta\left(\sigma^H-\sigma^L\right)}\bar{L} < \left((1-\hat{\tau}) + \frac{\hat{\tau}}{1-\alpha}\right)(\beta(1-\hat{\tau}))^{\alpha/(1-\alpha)}, \tag{23.32}$$

low-skill elites would be better off in democracy.

Proof. See Exercise 23.11 □

Note, however, that so far even when (23.32) holds, low-skill elites prefer to remain in entrepreneurship. This is because, given the structure of the political game, if the low-skill incumbent elites give up entrepreneurship, the new entrepreneurs will make the political choices, and they will naturally choose high entry barriers and no redistribution. Therefore, by quitting entrepreneurship, low-skill elites would be giving up their political power. Consequently, they are choosing between being elites and being workers in oligarchy, and clearly, the former is preferred.

23.3.5. First Thoughts on Regime Changes. I have so far characterized the political equilibrium under two different scenarios; democracy and oligarchy. Which political system prevails in a given society was treated as exogenous. Why are certain societies democratic, while others are oligarchic? One possibility is to appeal to historical accident, while another is to construct a "behind-the-veil" argument, whereby whichever political system leads to greater efficiency or ex ante utility would prevail. This approach is not satisfactory, however. Political regimes matter precisely because they regulate the conflict of interest between different groups (in this context, between workers and entrepreneurs). The behind-the-veil approach is unsatisfactory, since it recognizes and models this conflict to determine the equilibrium within a particular regime, but then ignores it when there is a choice of regime.

Moreover, this approach does not provide a framework for the analysis of regime changes in the context of a dynamic equilibrium.

A more satisfactory approach would be to use the induced preferences of the agents over political regimes and analyze a game over the determination of political institutions, where agents have these induced preferences. Broadly speaking two types of processes can lead to political change in this case. The first is some type of voluntary agreement within the society (or within a decisive subset of the society) involving an equilibrium change in political institutions. The second results from conflict over political institutions. In this subsection, I will start with the first alternative, which is simpler and fits well with the model presented here. However, most political changes in practice are better approximated by the second approach, and this will be the topic of the next two sections. For now, let us return to our basic model in this section and assume that $\mu(0) = 1$. Let us also make one modification to the baseline framework: the current elite can now vote to disband oligarchy, upon which a permanent democracy is established (previously, for example, in Proposition 23.7, such a vote was not allowed). I denote this choice by $d(t) \in \{0, 1\}$, with 0 standing for continuation with the oligarchic regime. To describe the law of motion of the political regime, let us denote oligarchy by $D(t) = 0$ and democracy by $D(t) = 1$. Since transition to democracy is permanent, we have

$$D(t) = \begin{cases} 0 & \text{if } d(t-n) = 0 \text{ for all } n \geq 0 \\ 1 & \text{if } d(t-n) = 1 \text{ for some } n \geq 0 \end{cases}.$$

Voting over $d(t)$ in oligarchy is at the same time as voting over $b(t)$ (there are no votes over $d(t)$ in democracy, since a transition to democracy is permanent), so agents with $e_i(t) = 1$ get to vote over these choices. I assume that after the vote for $d(t) = 1$, there is immediate democratization and all agents participate in the vote over taxes starting in period t.

First, imagine a situation where condition (23.32) does not hold so that even low-skill elites are better off in oligarchy. Then all elites will always vote for $d(t) = 0$, and also choose $b(t) = b^E$ and $\tau(t) = 0$ (as in Proposition 23.5). Hence, in this case, the equilibrium remains oligarchic throughout.

What happens when (23.32) holds? Current low-skill elites, i.e., those with $e_i(t) = 1$ and $a_i(t+1) = A^L$, would be better off in democracy (recall Proposition 23.7). If they vote for $d(t) = 0$, they stay in oligarchy, which gives them a lower payoff. If, instead, they vote for $d(t) = 1$ and $b(t) = 0$, then this will lead to democracy. Consequently, following this vote, high-skill agents enter entrepreneurship and there are redistributive taxes at the rate $\tau(t) = \hat{\tau}$ as in Proposition 23.4. Now, when they are in the majority, low-skill elites can induce a transition to a permanent democracy by voting for $d(t) = 1$. Since $\mu(0) = 1$, however, they are initially in the minority, and the oligarchic equilibrium persists for a while. Nevertheless, the fraction of high-skill elites will decrease over time. Provided that $M < 1/2$ and that entry barriers are kept throughout, low-skill agents will eventually become the majority and succeed in disbanding the oligarchic regime. One complication is that as $\mu(t)$ approaches 1/2, high-skill elites may prefer to temporarily reduce the entry barrier and include new entrepreneurs in order to prevent the disbanding of the regime. Nevertheless, this strategy will not be attractive when the future is discounted heavily. Consequently, we can establish the following proposition:

Proposition 23.8. *Suppose that Condition 23.1 holds, $M < 1/2$ and the society starts as oligarchic.*

If (23.32) does not hold, then for all t the society remains oligarchic with $d(t) = 0$; the equilibrium involves no redistribution, $\tau(t) = 0$ and high entry barriers, $b(t) = b^E$ as given by (23.25), and the fraction of high-skill entrepreneurs is given by $\mu(t) = \sigma^H \mu(t-1) + \sigma^L(1 - \mu(t-1))$ starting with $\mu(0) = 1$.

If (23.32) holds, then there exists $\bar{\beta} \in (0,1)$ such that for all $\beta \leq \bar{\beta}$, the society remains oligarchic, $d(t) = 0$, with no redistribution, $\tau(t) = 0$ and high entry barriers, $b(t) = b^E$ as given by (23.25) until date $t = \tilde{t}$ where $\tilde{t} = \min\{t' \in \mathbb{N}\}$ such that $\mu(t') \leq 1/2$ (whereby $\mu(t) = \sigma^H \mu(t-1) + \sigma^L(1 - \mu(t-1))$ for $t < \tilde{t}$ starting with $\mu(0) = 1$). At $\tilde{t}$, the society transitions to democracy with $d(t) = 1$, and for $t \geq \tilde{t}$, we have $\tau(t) = \hat{\tau}$, $b(t) = 0$ and $\mu(t) = 1$.

Proof. See Exercise 23.14. □

Intuitively, when (23.32) holds, low-skill entrepreneurs are better off transitioning to democracy than remaining in the oligarchic society, while high-skill entrepreneurs are always better off in oligarchy. Because they discount the future heavily, high-skill entrepreneurs are not willing to reduce entry barriers and sacrifice current profits. As a result, the society remains oligarchic as long as high-skill entrepreneurs are in the majority, i.e., as long as $t < \tilde{t}$, and the first period in which low-skill entrepreneurs become majority within the oligarchy, i.e., at $\tilde{t}$ such that $\mu(t) < 1/2$ for the first time, the oligarchy disbands itself transitioning to a democratic regime (and at that point $\mu(t)$ jumps up to 1).

This configuration is especially interesting when Condition 23.2 holds such that oligarchy ultimately would have led to lower output than democracy. In this case, as long as (23.32) holds, oligarchy transitions to democracy avoiding the long-run adverse efficiency consequences of oligarchy (though when this condition does not hold, oligarchy survives forever with negative consequences for efficiency and output). This extension therefore provides a simple framework for thinking about how a society can transition from oligarchy to a more democratic system, before the oligarchic regime becomes excessively costly. Interestingly, however, the reason for the transition from oligarchy to democracy is *not* increased inefficiency in the oligarchy, but conflict between high and low-skill agents *within* the oligarchy; the transition takes place when the low-skill elites become the majority.

The result presented in Proposition 23.8 only leads to equilibrium regime change when (23.32) holds. When it fails to hold, this proposition states that society will remain permanently as an oligarchy (provided that it starts as an oligarchy). However, the induced preferences of different agents are radically opposed to each other. Those who are the incumbent entrepreneurs—the elite—are happy in oligarchy, whereas those left out of the elite have the worst possible allocation, with zero wages and zero consumption. They are also the majority. Can we expect the oligarchic system that serves the interests of a small minority of the population to last forever? To answer this question, we need to introduce additional ideas about how political change can happen when there is social conflict over the set of political institutions in the society. This is what I turn to next.

23.4. Understanding Endogenous Political Change

23.4.1. General Insights. The analysis so far has focused on the implications of different political institutions on economic growth and how their economic consequences shape the preferences of different agents over these political institutions. The only example of institutional change we have seen was one of voluntary transition from oligarchy to democracy in the previous section. In practice, however, most institutional change does not happen voluntarily, but is a result of social conflict. Consider, for example, the democratization of most Western European nations during the 19th and early 20th centuries or the democratization experience in Latin America during the 20th century. In both cases, democracy was not voluntarily granted by the existing elites, but resulted from the process of social conflict, in which those previously disenfranchised demanded political rights and in some cases were able to secure them. But how does this happen? A nondemocratic regime, by its nature, vests political power with a narrow group. Those who are excluded from this group, the non-elites, do not have the right to vote or nor do they have any voice in collective decisions. So how can they influence, the course of the political equilibrium and induce equilibrium political change? The answer to this question lies in drawing a distinction between *de jure* (formal) and *de facto* political power. De jure political power refers to power that originates from the political institutions in society, and has been the form of political power that has been our focus so far. One may view it as the more "legitimate" type of political power. Political institutions determine who gets to vote, how representatives make choices, and the general rules of collective decision-making in society. In Max Weber's famous description, they also reserve "the legitimate use of violence" to the state (and to the actors that control the state). These various different types of political power are all of the de jure kind. However, there is another, equally important type of political power that features importantly in equilibrium political changes—de facto political power. The political power of protesters that marched against the existing regime before the First Reform Act in Britain in 1832 was not of the de jure kind. The law of the land did not empower them to influence the political course of actions—in fact, they were quite explicitly disenfranchised. But they had a different kind of power, emanating from their ability to solve the collective action problem and organize protests. This power was also supported by the fact that they were the majority in the society. This type of political power, which lives outside the political institutions, is de facto political power.

De facto political power is ever present around us. Civil wars, revolutions, and social unrests are manifestations of the use of de facto political power by various groups. Military excursions are another example. More interesting for our purposes are the types of de facto political power that coexist with de jure political power in orderly (or semi-orderly societies). For example, in many Latin American countries governments are elected via democratic means, as it should be according to the political institutions that have specified the distribution of de jure power in society, but at the same time there is ample fraud, vote buying, and use of violence via paramilitaries and other organizations to influence the outcomes of elections. All of these fall within the category of the exercise of de facto political power. Then there are grey areas. For example, the ability of the rich and well-organized groups to use money for campaign contributions or for lobbying, and thus influencing the policy choices and platforms of politicians can be viewed as an example of the exercise of de facto power, though it can also be viewed as part of the regular functioning of political institutions, since in many societies, like the United States, lobbies are legal.

De facto political power is important for political change, since de jure political power itself will act as a source of *persistence*—not of change. For example, consider the model of the previous section when (23.32) does not hold, so that the oligarchs are happy to maintain the oligarchic regime. If de jure power is the only source of power, the elite will be the only one with the decision-making powers in the society, and they will never change the political regime away from oligarchy towards democracy. However, if the non-elites had some source of power—which, by its nature, has to be de facto power—then, political change becomes a possibility. Perhaps in some periods, the non-elites will be able to solve their collective action problem and thus exercise enough pressure on the system to force some changes. In the extreme, they can induce the elites to disband oligarchy and transition to democracy, or they can themselves topple the oligarchic regime.

I will argue that the interaction between de jure and de facto political power is the most promising way to approach the analysis of equilibrium political change. Moreover, this interaction becomes particularly interesting when studied in a dynamic framework. This is for at least two reasons. First, most of the issues we are discussing are dynamic in nature—they refer to political *change*. Second, whether the distribution of de facto political power is permanent or changing stochastically over time has major consequences for the structure of political equilibrium. When a particular (disenfranchised) group has permanent (and unchanging) amount of de facto political power, it can use this at each date to demand concessions from those holding de jure political power. Such a situation may lead to an equilibrium without political change (though the equilibrium will have a very different distribution of resources because of the concessions induced by the de facto power of the disenfranchised group. Next consider a situation in which the de facto political power of the disenfranchised group is highly *transient*—in the sense that, they have been able to solve their collective action problem and exercise de facto political power today, but it is unlikely that they will have the same type of power tomorrow. Then, the disenfranchised group cannot rely on the use of their de facto political power in the future to receive concessions. If they want concessions and redistribution of resources towards themselves in the future, they have to use their current power in order to secure such a change. This generally involves a change in political institutions as a way of changing the future distribution of de jure power. More explicitly, consider a situation in which a particular group of individuals know that today they have the power to change institutions and create a playing field favoring themselves in the future, but they also understand that this de facto political power will be gone tomorrow. Thus any limited transfer of resources or other concessions made to them today will be either reversed or will be insufficient relative to the benefits from changing the playing field in their favor. It will therefore be precisely the transient nature of their de facto political power that will encourage them to take actions to change political institutions in order to cement their power more firmly (so that they can change their transients de facto political power into more durable de jure political power). This informal discussion therefore suggests a particular channel via which the interaction between de facto and de jure political power can lead to equilibrium changes in political institutions. I next give a historical example to illustrate this point further.

23.4.2. An Example. As a brief example, consider the development of property rights in Europe during the Middle Ages. There is broad agreement in the literature that lack of property rights for non-elite landowners, merchants, and early industrialists was detrimental to economic growth during this epoch. Since political institutions at the time placed political power in the hands of kings and various types of hereditary monarchies, such rights were largely decided by these monarchs. The monarchs often used their powers to expropriate producers, impose arbitrary taxation, renege on their debts, and allocate the productive resources of society to their allies in return for economic benefits or political support. Consequently, economic institutions during the Middle Ages provided little incentive to invest in land, physical or human capital, or technology, and failed to foster economic growth. These economic institutions also ensured that the monarchs and their allies controlled a large fraction of the economic resources in society, solidifying their political power and ensuring the continuation of the political regime.

The 17th century witnessed major changes in the economic and political institutions that paved the way for the development of property rights and limits on monarchs' power, especially in England after the Civil War of 1642 and the Glorious Revolution of 1688, and in the Netherlands after the Dutch Revolt against the Hapsburgs. How did these major institutional changes take place? In England until the 16th century the king also possessed a substantial amount of de facto political power, and leaving aside civil wars related to royal succession, no other social group could amass sufficient de facto political power to challenge the king. But changes in the English land market and the expansion of Atlantic trade in the 16th and 17th centuries gradually increased the economic fortunes, and consequently the de facto power of landowners and merchants opposed to the absolutist tendencies of the Kings.

By the 17th century, the growing prosperity of the merchants and the gentry, based both on internal and overseas, especially Atlantic, trade, enabled them to field military forces capable of defeating the king. This de facto power overcame the Stuart monarchs in the Civil War and Glorious Revolution, and led to a change in political institutions that stripped the king of much of his previous power over policy. These changes in the distribution of political power led to major changes in economic institutions, strengthening the property rights of both land and capital owners and spurring a process of financial and commercial expansion. The consequence was rapid economic growth, culminating in the Industrial Revolution, and a very different distribution of economic resources from that in the Middle Ages.

This discussion poses, and also gives clues about the answers to, a crucial question: why do groups with political power want to change political institutions in their favor? In the context of the example above, why did the gentry and merchants use their de facto political power to change political institutions rather than simply implement the policies they wanted? The answer lies with the *transient* nature of de facto political power and the lack of *commitment* to future policies.

As already discussed in the previous chapter, the commitment problem arises because groups with political power cannot commit to not using their power to change the distribution of resources in their favor. For example, economic institutions that increased the security of property rights for land and capital owners during the Middle Ages would not have been credible as long as the monarch monopolized political power. He could promise to respect others' property rights, but then at some point, renege on his promise, as exemplified by the numerous financial defaults by medieval kings. Credible secure property rights necessitated a reduction in the political power of the monarch. Although these more secure property rights would foster economic growth, they were not appealing to the monarchs who would lose their rents from predation and expropriation as well as various other privileges associated with their monopoly of political power. This is why the institutional changes in England as a result of the Glorious Revolution were not simply conceded by the Stuart kings. James II had to be deposed for the changes to take place.

The reason why de facto political power is often used to change political institutions is closely related to the transient nature of this power and to commitment problems. Individuals care not only about economic outcomes today but also in the future. In the example above, we presume that the gentry and merchants were interested in their profits and therefore in the security of their property rights, not only in the present but also in the future. Therefore, they would have liked to use their (de facto) political power to secure benefits in the future as well as the present. However, commitment problems make this difficult. If the gentry and merchants would have been sure to maintain their de facto political power, this would not have been a problem. However, de facto political power is often *transient*, for example because the collective action problems that are solved to amass this power are likely to resurface in the future, or other groups, especially those controlling de jure power, can become stronger. The commitment problems, in turn, imply that promises made today cannot be trusted and any change in policies and economic institutions that relies purely on de facto political power is likely to be reversed in the future. A plausible story for the nature of political changes in early modern Europe is therefore that when they had the transient political power, the English gentry, merchants, and industrialists strove not just to change economic institutions in their favor following their victories against the Stuart monarchy, but they also sought, and managed, to alter political institutions and the future allocation of de jure power in their favor. Using political power to change political institutions then emerges as a useful strategy to make gains more durable. Consequently, political institutions are an important as way in which future political power can be manipulated, and thus indirectly shaping future, as well as present, economic institutions and outcomes.

23.4.3. Modeling. The discussion so far illustrated how we can use the interaction between de facto and de jure political power in order to study equilibrium political changes, and their implications for economic growth. While the discussion has given some clues about what the incentives of different parties with and without de jure political power will be in a dynamic game, it is so far unclear how one would construct models to analyzed these forces and generate useful comparative statics. In this subsection, I suggest a general framework that is useful for thinking about the dynamic interactions between de facto and de jure political power. In the next subsection and the next section, I will illustrate this framework further.

Imagine a dynamic model in which there are two *state variables*, political institutions and the distribution of resources. For example, $P(t) \in \mathcal{P}$ denotes a specific set of political institutions in place at time t. This can be democracy or nondemocracy, parliamentary versus presidential system, different types of oligarchic institutions, etc. The set $\mathcal{P}$ denotes the entire set of feasible political institutions relevant for the situation we are studying. Similarly, let $W(t) \in \mathcal{W}$ denote a variable encoding the distribution of resources at time t. For example, in a society consisting of two groups, the rich and the poor, this could be the relative incomes of the two groups. In a society with many individuals, it could be the density function of income or wealth. Again, $\mathcal{W}$ is the set of all possible distributions of resources in society. It is useful to think of both $P(t)$ and $W(t)$ as state variables for three reasons. First, they are relatively slow-changing, thus corresponding to the loose notion of a state variable. Second, they will typically be the payoff-relevant variables when we set up the problem is a dynamic game, thus they will be the "Markovian states". Third and perhaps most important, these two variables will determine the two sources of political power essential for understanding equilibrium political change. The variable $P(t)$ will determine the distribution of de jure political power, which I denote by $J(t) \in \mathcal{J}$, in particular, it determines who has the right to vote and which politicians are subject to what types of constraints and what decisions they can take. The distribution of resources is not the only variable that affects de facto political power but it is one of its main determinants. In particular, as already hinted in the discussion above, de facto political power is typically the result of the ability of certain groups to solve their collective action problem, or emerges when certain groups have the resources to hire their own armies, paramilitaries, and supporters, or simply use the money for lobbying and bribing. Let the distribution of de facto political power in the society at time t be $F(t) \in \mathcal{F}$. As in the beginning of Part 8, let us also denote economic institutions by $R(t) \in \mathcal{R}$, and let $Y(t) \in \mathcal{Y}$ be a measure of economic performance, such as income per capita or growth (though it could also include other performance-related variables, such as the level of poverty, health, human capital etc.).

A dynamic framework that is useful for thinking about political change and its implications for economic growth would consist of a mapping $\varphi : \mathcal{P} \times \mathcal{Z} \to \mathcal{J}$, which determines the distribution of de jure power at time t as a function of political institutions at time t, $P(t) \in \mathcal{P}$, as well as some potential stochastic elements, captured by $z(t) \in \mathcal{Z}$. It will also consist of a mapping determining the distribution of de facto power in a similar manner, $\phi : \mathcal{W} \times \mathcal{Z} \to \mathcal{F}$, where the types of stochastic elements influencing whether a particular group has de facto political power will be different than those affecting the distribution of de jure power, but we can summarize both of them with the variable $z(t) \in \mathcal{Z}$. Then given the realization of $J(t) \in \mathcal{J}$ and $F(t) \in \mathcal{F}$, another mapping $\iota : \mathcal{J} \times \mathcal{F} \to \mathcal{R} \times \mathcal{P}$ determines both economic institutions today and also one of the future state variables, the political institutions tomorrow, $P(t+1) \in \mathcal{P}$. Intuitively speaking, the distribution of de facto and de jure political power regulates what types of economic institutions will be in place today and also leaves the door open for potential political change, for example a switch from nondemocracy to democracy if such a change is necessary because of the balance of de facto and de jure power today. Finally, an economic equilibrium mapping $\rho : \mathcal{R} \to \mathcal{Y} \times \mathcal{W}$ determines both the economic performance variables and the distribution of economic resources. For example, if economic institutions are competitive markets, they may lead to high wages and high output, and if they are repressive labor markets, they will lead to low wages, high profits, but perhaps lower output because of greater distortions they may be creating because of monopsony distortions or the induced misallocation of workers to tasks.

The next chart summarizes this discussion diagrammatically.

$$
\begin{array}{ccccccc}
\begin{array}{c}\text{political}\\ \text{institutions}\,(t)\end{array} & \Longrightarrow & \left.\begin{array}{c}\text{de jure}\\ \text{political}\\ \text{power}\,(t)\\ \&\\ \text{de facto}\\ \text{political}\\ \text{power}\,(t)\end{array}\right\} & \begin{array}{c}\Longrightarrow\\ \\ \\ \Longrightarrow\end{array} & \begin{array}{c}\text{economic}\\ \text{institutions}\,(t)\\ \\ \\ \text{political}\\ \text{institutions}\,(t+1)\end{array} & \Longrightarrow & \left\{\begin{array}{c}\text{economic}\\ \text{performance}\,(t)\\ \&\\ \text{distribution}\\ \text{of resources}\,(t+1)\end{array}\right. \\
\begin{array}{c}\text{distribution}\\ \text{of resources}\,(t)\end{array} & \Longrightarrow & & & & &
\end{array}
$$

This framework and the associated diagram emphasize both the effects of economic institutions on economic performance and the distribution of resources—what we have tried to understand so far. But they have also introduced the dynamics of political power and political institutions. Of course, at this level of generality, such a dynamic framework is somewhat vacuous. It would only be useful and meaningful if we can put more content into the set of political institutions and the distribution of resources that need to be considered, derive the mappings φ, ϕ, ι and ε from economic interactions with sound microfoundations and then conduct useful comparative statics. This is a tall order, and a full dynamic model of the sort, that is able to deliver on all of these counts, does not currently exist. Nevertheless, a number of models that are useful for the analysis of political change and the interaction between politics and economics can be viewed through the lenses of this framework. This suggests that an abstract framework like the one presented here might be useful in emphasizing what the important frontiers for research are and what types of models we may want to think about for furthering our understanding of political change and the relationship between political institutions and economic growth.

In the next subsection, I will provide an informal discussion of one application of this framework, and then in the next section, I will present another application in greater detail, because it will not only illustrate the basic ideas highlighted here, but also shed light on the economic tradeoffs between different political regimes.

23.4.4. Another Example: The Emergence of Democracy. The framework presented above is largely inspired by the the models of the emergence of democracy developed in Acemoglu and Robinson (2000a, 2006a). Acemoglu and Robinson construct a model of the emergence of democracy based on social conflict between the elite, who originally hold de jure political power, and the masses, that are initially without de jure political power, but can sometimes solve their collective action problems and gather significant de facto political power. A historical case that illustrates the main issues emphasized by Acemoglu and Robinson is the emergence of democracy in 19th century Europe. Many European nations during the 19th century were run by small elites. Most had elected legislatures, often descendents of medieval parliaments, but the franchise was highly restricted to males with relatively large amounts of assets, incomes or wealth. However, as the century and the Industrial Revolution progressed, this political monopoly was challenged by the disenfranchised who engaged in collective action to force political change.

In response to these developments, the elites responded in three ways. First, the elites could use repression to prevent social unrest, as they have done in much of Europe during the revolutionary waves of 1848. Second, as in Germany under Bismarck, they may try to use economic concessions to buy off opposition. Finally, if neither repression nor concessions were attractive or effective, elites expanded the franchise and gave political power to the previously disenfranchised—they created the precedents of modern democracy.

The first important move towards democracy in Europe came in Britain came with the First Reform Act of 1832, which we have already discussed above. This act removed many of the worst inequities under the old electoral system, in particular the "rotten boroughs" where several members of parliament were elected by very few voters. The 1832 reform also established the right to vote based uniformly on the basis of property and income. The reform was passed in the context of rising popular discontent at the existing political status quo in Britain.

By the 1820s the Industrial Revolution was well under way and the decade prior to 1832 saw continual rioting and popular unrest. Notable were the Luddite Riots from 1811-1816, the Spa Fields Riots of 1816, the Peterloo Massacre in 1819, and the Swing Riots of 1830. Another catalyst for the reforms was the July revolution of 1830 in Paris. Much of this was led and orchestrated by the new middle-class groups who were being created by the spread of industry and the rapid expansion of the British economy. For example, under the pre-1832 system neither Manchester nor Sheffield had any members of the House of Commons.

There is little dissent amongst historians that the motive for the 1832 Reform was to avoid social disturbances. The 1832 Reform Act increased the total electorate from 492,700 to 806,000, which represented about 14.5% of the adult male population. Yet, the majority of British people could not vote, and the elite still had considerable scope for patronage, since 123 constituencies still contained less than 1,000 voters. There is also evidence of continued corruption and intimidation of voters until the Ballot Act of 1872 and the Corrupt and Illegal Practices Act of 1883. The Reform Act therefore did not create mass democracy, but rather was designed as a strategic concession. As a result, parliamentary reform was still very much on the agenda in the middle of the century, especially with the greater demands of the Chartist movement. Despite its various achievements and fame, the Chartist movement could not secure a major democratic reform, in part, because the de facto political power of the disenfranchised groups was not strong enough to force reform. This, however, changed in the latter half of the 19th century, partly because of the sharp business cycle downturn that caused significant economic hardship and the increased threat of violence. Also significant was the founding of the National Reform Union in 1864 and the Reform League in 1865, and the Hyde Park riots of July 1866 provided the most immediate catalyst. Following these events, major electoral reform got underway with the Second Reform Act in 1867, which increased the total electorate from 1.36 million to 2.48 million, and made working class voters the majority in all urban constituencies. The electorate was doubled again by the Third Reform Act of 1884, which extended the same voting regulations that already existed in the boroughs (urban constituencies) to the counties (rural constituencies). The Redistribution Act of 1885 removed many remaining inequalities in the distribution of seats and from this point on Britain only had single member electoral constituencies (previously many constituencies had elected two members—the two candidates who gained the most votes). After 1884 about 60% of adult males were enfranchised. Once again social disorder appears to have been an important factor behind the 1884 act.

In Britain, the Reform Acts of 1867-1884 were a turning point in the history of the British state. Economic institutions also began to change. In 1871 Gladstone reformed the civil service, opening it to public examination, making it meritocratic. Liberal and Conservative governments introduced a considerable amount of labor market legislation, fundamentally changing the nature of industrial relations in favor of workers. During 1906-1914, the Liberal Party, under the leadership of Asquith and Lloyd George, introduced the modern redistributive state into Britain, including health and unemployment insurance, government financed pensions, minimum wages, and a commitment to redistributive taxation. As a result of the fiscal changes, taxes as a proportion of National Product more than doubled in the 30 years following 1870, and then doubled again. In the meantime, the progressivity of the tax system also increased. Finally, there is also a consensus amongst economic historians that inequality in Britain fell after the 1870s. At the same time as the fiscal reforms were taking place, there were also major educational reforms changing the distribution of resources and distribution of opportunities in the British society in a major way. The Education Act of 1870 committed the government to the systematic provision of universal education for the first time, and this was made free in 1891. The school leaving age was set at 11 in 1893. In 1899, it was further increased to 12 and special provisions for the children of needy families were introduced. As a result of these changes, the proportion of 10-year olds enrolled in school that stood at 40 percent in 1870 increased to 100 percent in 1900. Finally, a further act in 1902 led to a large expansion in the resources for schools and introduced the grammar schools which subsequently became the foundation of secondary education in Britain.

Overall, the picture that emerges from British political history is clear. Beginning in 1832, when Britain was governed by the relatively rich, primarily rural aristocracy, a series of strategic concessions were made over an 86 year period. These concessions were aimed at incorporating the previously disenfranchised into politics since the alternative was seen to be social unrest, chaos, and possibly revolution. The concessions were gradual because in 1832, social peace could be purchased by buying off the middle classes. Moreover, the effect of the concessions was diluted by the specific details of political institutions, particularly the continuing unrepresentative nature of the House of Lords. Although challenged during the 1832 reforms, the House of Lords provided an important bulwark for the wealthy against the potential of radical reforms emanating from a democratized House of Commons. Later, as the working classes reorganized through the Chartist movement and through trade unions, further concessions had to be made. The Great War and the fallout from it sealed the final offer of full democracy.

Faced with the threat of revolt and social chaos, political elites may also attempt to avoid giving away their political power by making concessions, such as income redistribution or other policies that favor non-elites and the disenfranchised. However, because the promise of concessions is typically non-credible when threats are transient, such promises are typically insufficient to defuse social unrest. Democratization can then be viewed as a credible commitment to the disenfranchised. In particular, democratization is a credible commitment to future redistribution because it redistributes de jure political power away from the elites to the masses. In democracy, the poorer segments of the society would be more powerful, and could vote and use their de jure political power to implement economic institutions and policies consistent with their interests. Therefore, democratization was a way of transforming the transitory de facto power of the disenfranchised poor into more durable de jure political power.

The above account of events makes it quite clear that democracy in many Western societies, and particularly in Britain, did not emerge from the voluntary acts of an enlightened elite. Democracy was, in many ways, forced on the elite, because of the threat of revolution. Nevertheless, democratization was not the only potential outcome in the face of pressure from the disenfranchised, or even in the face of the threat of revolution. Many other countries faced the same pressures and political elites decided to repress the disenfranchised rather than make concessions to them. This happened with regularity in Europe in the 19th century, though by the turn of the 20th century most West European nations had accepted that democracy was inevitable. Repression lasted much longer as the favorite response of elites in Latin America, and it is still the preferred option for current political elites in China or Burma. And yet, repression is costly not only for the repressed, but also for the elites. Therefore, faced with demands for democracy political elites face a tradeoff. If they grant democracy, then they lose power over policy and face the prospect of, possibly radical, redistribution. On the other hand, repression risks destroying assets and wealth. In the urbanized environment of 19th century Europe (Britain was 70% urbanized at the time of the Second Reform Act), the disenfranchised masses were relatively well organized and therefore difficult to repress. Moreover, industrialization had led to an economy based on physical, and increasing human, capital. Such assets are easily destroyed by repression and conflict, making repression an increasingly costly option for elites. In contrast, in predominantly agrarian societies like many parts of Latin America earlier in the century or current-day Burma, physical and human capital are relatively unimportant and repression is easier and cheaper. Moreover, not only is repression cheaper in such environments, democracy is potentially much worse for the elites because of the prospect of radical land reform. Since physical capital is much harder to redistribute, elites in Western Europe found the prospect of democracy much less threatening.

So far I have offered a verbal account how one might develop a theoretical model of the democratization process in line with the abstract framework of the previous subsection. Once the main ideas are understood, a formal framework is not difficult to construct. In this bare-bone form, Acemoglu and Robinson (2006a) consider the following framework (see Exercise 23.15 for a simplified version of this theory, which also highlights some additional implications): the society consists of two groups, the elite and the masses. Political power is initially in the hands of the elite, but the masses are more numerous. Thus if there is ever democratization, the masses will become politically more powerful and dictate the policies. All individuals are infinitely lived and the elite are richer than the masses. Because the society starts as a nondemocracy, de jure power is in the hands of the elite. Let us suppose that the only policy choice is a redistributive tax, τ, the proceeds of which are distributed lump sum. The elite prefer zero taxation, $\tau = 0$, since they are richer and any taxation will redistribute income away from them to the poorer masses.

Let us imagine that while de jure power in nondemocracy lies with the elite, the poor may have de facto political power. In particular, suppose that with probability q in each period, the masses are able to solve their collective action problem and can threaten to undertake a revolution. A revolution is very costly for the elite, but generates only limited gains for the masses. These limited gains may nonetheless be better than living under elite control and the inequitable distribution of resources that this involves. So when they are able to solve their collective action problem (with probability q), the *revolution constraint* of the masses becomes binding. In this case, the rich need to take some action and make concessions to avoid a revolution.

As in the historical account I provided in the previous subsection, the elites in the theory also have have three options to defuse the revolutionary threat. The first is to make concessions through redistributive policies today. This will work if q is high. In the limit, where $q = 1$, the masses can undertake a revolution in each date, thus the rich can credibly commit to making redistribution towards them at each date, because if they fail to do so, the masses can immediately undertake a revolution, which is costly for the elite. However, the same strategy does not work when q is small. Consider the polar case where $q \to 0$. In this case, the masses essentially expect never to have the same type of de facto political power in the future. Presuming that the amount to redistribution that the elite can give to the masses during a particular period is limited, they will not be satisfied by temporary concessions. If temporary concessions are not sufficient, the elite may want to use repression. Repression will be successful if the revolutionary threat is not very well-organized and it will be profitable for the rich elite if they have a lot to lose from democratization. Thus repression will be the action of choice for elites that fear major redistribution, in the form of fiscal taxation or land reform, after democratization, such as the land-based elites in Central America and Burma. But in a highly urbanized and industrialized society like Britain, where the costs repression will be significant and the elite would have less to fear from democratization, the third option, which is enfranchisement, becomes an attractive choice. This third option involves the elite changing the political system and manufacturing a transition to democracy. As long as this change in institutions is credible, the distribution of de jure power has changed and transferred at least some of the decision-making power to the masses. With their newly-gained decision-making power, the masses know that they can choose policies in the future that will create a more equitable distribution of resources for themselves and will typically be happy to accept democratic institutions instead of a revolution that is costly for themselves as well. This shows how one can build a dynamic model of endogenous changes in political institutions.

Compared to the abstract framework in the previous subsection, the model described here is stripped down (and to save space, I have not even provided the equations to establish the main claims). First, the distribution of resources is no longer a state variable (it is constant and does not affect transitions or the distribution of political power). In addition, de jure political power is simply a non-stochastic outcome of political institutions; in nondemocracy, the elite make the decisions; and in democracy, there is one person one vote, and the masses, thanks to their majority, become the decisive voters. Finally, there are very limited economic decisions. The only relevant decision is one of taxation. Thus, in its current form, this is not a satisfactory framework for analyzing the impact of political institutions on economic institutions or the relationship between political regimes and economic growth. Acemoglu and Robinson (2006a) consider extensions of the framework, which go some way towards a framework for the analysis of economic institutions and economic growth. Instead of discussing these extensions, however, in the next section, I will present a related model, in which there is more explicit interaction between economic and political institutions, and it will again be the sum total of de facto and de jure powers that will determine the evolution of equilibrium institutions.

23.5. Dynamics of Political and Economic Institutions: A First Look

23.5.1. Baseline Model. In this section, I will discuss a model based on Acemoglu and Robinson (2007), which will feature both the interaction of de jure and de facto political power and also illustrate how democracy can be captured and lead to poor economic outcomes for this reason. Interestingly, while the model of democratization described at the end of the previous section emphasized how the de facto political power of the citizens can affect equilibrium dynamics, here the emphasis will be on the de facto political power of the elite in democracy and how they can use this to capture democratic politics.

Consider the following infinite-horizon economy in discrete time, with the unique final good. The society is populated by a finite number L of citizens/workers and M elites. Let me assume to simplify the analysis here that citizens are significantly more numerous than the elites, loosely written as $L >> M$. What the exact relative sizes of the two groups need to be for the main results to apply is discussed in greater detail in Exercise 23.16 below.

Let us use $h \in \{E, C\}$ to denote whether an individual is from the elite or a citizen, and $\mathcal{E}$ and $\mathcal{C}$ to denote the set of elites and citizens, respectively. All agents have the usual risk-neutral preferences given by

$$\sum_{t=0}^{\infty} \beta^t c_i^h(t), \tag{23.33}$$

where $c_i^h(t)$ denotes consumption of agent i from group $h \in \{E, C\}$ at time t in terms of the unique final good and $\beta \in (0, 1)$ is the common discount factor.

Each citizen owns one unit of labor, which they supply inelastically. Each member of the elite $i \in \mathcal{E}$ has access to a linear production function to produce the unique private good with constant marginal productivity of A. Let us consider two different *reduced-form economic* *institutions*. In the first, labor markets are *competitive* and we index these institutions by the subscript c (indicating "pro-citizen" or "competitive"). Let $I(t) \in \{e, c\}$ denote the institutional choice in period t. Given the production technology each elite will make zero returns and each citizen will receive their marginal product of labor, A. When there are competitive labor markets, $I(t) = c$, the wage rate (and the wage earnings of each citizen) is:

$$w_c \equiv A. \tag{23.34}$$

The return to a member of the elite with competitive markets is similarly

$$R_c \equiv 0. \tag{23.35}$$

The alternative set of economic institutions favor the elite and are *labor repressive* $(I(t) = e)$ and allow the elite to use their political power to reduce wages below competitive levels. We parameterize the distribution of resources under labor repression as follows: $\lambda < 1$ denotes the share of national income accruing to citizens and $\delta \in [0, 1)$ is the fraction of potential national income, AL, that is lost because of the inefficiency of labor repression. For instance, $\delta > 0$ may result from standard monopsony distortions in the labor market. Note, however, that none of the results presented in this paper depend on the value of δ. The case where $\delta = 0$ would correspond to a situation in which there is no distortion from labor repression and the choice of economic institutions is purely redistributive. Alternatively, one could also consider the case in which $\delta < 0$, so that economic institutions favored by the elite are more "efficient" than those preferred by the citizens. However, given the emphasis on "labor repressive" institutions and the focus on whether democracy may fail to generate greater income per capita even when it is more efficient, the case of $\delta > 0$ is more relevant. The straightforward implication of this assumption is that, when economic institutions are labor repressive, there will be lower income per capita and thus worse economic performance.

Another reason for introducing the parameters δ and λ is that the model will have interesting comparative statics with respect to these parameters. For now, it suffices to start with the levels of factor prices under different economic institutions as functions of δ and λ. In particular, we have that factor prices as functions of economic institutions are

$$w_e \equiv \lambda (1 - \delta) A, \tag{23.36}$$

and

$$R_e \equiv (1 - \lambda)(1 - \delta) \frac{AL}{M}. \tag{23.37}$$

Factor prices can then be written as a function of economic institutions as $w(I(t) = e) = w_e$, $R(I(t) = e) = R_e$, $w(I(t) = c) = w_c$ and $R(I(t) = c) = R_c$. For future reference, let us also define

$$\Delta R \equiv R_e - R_c$$
$$= (1-\lambda)(1-\delta)\frac{AL}{M} > 0, \quad (23.38)$$

and

$$\Delta w \equiv w_c - w_e$$
$$= (1-\lambda(1-\delta))A > 0 \quad (23.39)$$

as the gains to the elite and the citizens from their more preferred economic institutions. Since the citizens are significantly more numerous, i.e., $L >> M$, (23.38) and (23.39) imply that $\Delta R >> \Delta w$.

There are two possible political regimes, democracy and nondemocracy, denoted respectively by D and N. The distribution of de jure political power will vary between these two regimes. At time t, the (payoff-relevant) "state" of this society will be represented by $s(t) \in \{D, N\}$, which designates the political regime that applies at that date. Irrespective of the political regime (state), the identities of the elites and the citizens do not change.

Overall, in line with the discussion the previous section, political power is determined by the interaction of de facto and de jure political power. Both groups can invest to garner further de facto political power. In particular, suppose that elite $i \in \mathcal{E}$ spends an amount $\theta_i(t) \geq 0$ as a contribution to activities increasing their group's de facto power. Then total elite spending on such activities will be $\sum_{i\in\mathcal{E}} \theta_{i,s}(t)$ when the political state is s, and let us assume that their de facto political power is

$$P_s^E(t) = \phi_s^E \sum_{i\in\mathcal{E}} \theta_{i,s}(t), \quad (23.40)$$

where $\phi_s^E > 0$ and dependence on the state $s \in \{D, N\}$ is made explicit to emphasize that investments in de facto power by the elite may be less effective in democracy. The superscript E distinguishes it from the corresponding parameter for the citizens.

Citizens' power comes from three distinct sources. First, they can also invest in their de facto political power. Second, because citizens are more numerous, they may sometimes solve their collective action problem and exercise additional de facto political power. We assume that this second source of de facto of political power is stochastic and fluctuates over time. The reasoning underlying this assumption is similar to that given in the previous section for why de facto political power resulting from solving the collective action problem is often transient. These fluctuations will cause equilibrium changes in political institutions. Finally, again because they are more numerous, citizens will have greater power in democracy than in nondemocracy. Overall, the power of the citizens when citizen $i \in \mathcal{C}$ spends an amount

$\theta_{i,s}(t) \geq 0$ is

$$P_s^C(t) = \phi_s^C \sum_{i \in \mathcal{C}} \theta_{i,s}(t) + \omega(t) + \eta I(s(t) = D), \tag{23.41}$$

where $\phi_s^C > 0$, $\omega(t)$ is a random variable drawn independently and identically over time from a given distribution $F[\cdot]$, $I(s = D) \in \{0, 1\}$ is an indicator function for $s = D$, and η is a strictly positive parameter measuring citizens' de jure power in democracy. Equation (23.41) implies that in democracy the political power of the citizens shifts to the right in the sense of first-order stochastic dominance. To simplify the discussion, let me impose the following assumptions on F: F is defined over $(\underline{\omega}, \infty)$ for some $\underline{\omega} < 0$, is everywhere strictly increasing and twice continuously differentiable (so that its density f and the derivative of the density, f', exist everywhere). Moreover, $f[\omega]$ is single peaked (in the sense that there exists ω^* such that $f'[\omega] > 0$ for all $\omega < \omega^*$ and $f'[\omega] < 0$ for all $\omega > \omega^*$) and satisfies $\lim_{\omega \to \infty} f[\omega] = 0$.

Let us also introduce the variable $\pi(t) \in \{e, c\}$ to denote whether the elite have more (total) political power at time t. In particular, when $P_s^E(t) \geq P_s^C(t)$, we have $\pi(t) = e$ and the elite have more political power and will make the key decisions. In contrast, whenever $P_s^E(t) < P_s^C(t)$, $\pi(t) = c$ and citizens have more political power, and they will make the key decisions.

Finally, suppose that the group with greater political power will decide both economic institutions at time t, $I(t)$, and what the state variable tomorrow, i.e., political regime will be in the following period, $s(t+1)$. Moreover, let us assume that when the elite have more political power, a representative elite agent makes the key decisions, and when citizens have more political power, a representative citizen does so. Since the political preferences of all elites and all citizens are the same, these representative agents will always make the decisions favored by their group.

Summarizing the timing of events, we have that at each date t, the society starts with the state variable $s(t) \in \{D, N\}$. Then:

(1) Each elite agent $i \in \mathcal{E}$ and each citizen $i \in \mathcal{C}$ simultaneously chooses how much to spend to acquire de facto political power for their group, $\theta_i(t) \geq 0$, and $P^E(t)$ is determined according to (23.40).
(2) The random variable $\omega(t)$ is drawn from the distribution F, and $P^C(t)$ is determined according to (23.41).
(3) If $P^E(t) \geq P^C(t)$ (i.e., $\pi(t) = e$), a representative elite agent chooses $(I(t), s(t+1))$, and if $P^E(t) < P^C(t)$ (i.e., $\pi(t) = c$), a representative citizen chooses $(I(t), s(t+1))$.
(4) Given $I(t)$, $R(t)$ and $w(t)$ are determined and paid to elites and citizens respectively, and consumption takes place.

Let us first focus on the symmetric Markov Perfect (Political Economy) Equilibrium (MPE) of this game (the results with non-symmetric equilibria and subgame perfect equilibria are discussed in Exercise 23.17). As usual, a MPE imposes the restriction that equilibrium strategies are mappings from payoff-relevant states, which here only include $s \in \{D, N\}$, and since we formulate the model recursively we drop time subscripts from now on. In a MPE strategies are not conditioned on the past history of the game over and above the influence of this past history on the payoff-relevant state s. A MPE consists of contribution functions $\{\theta_{i,s}(t)\}_{i\in\mathcal{E}}$ for each elite agent as a function of the political state, a corresponding vector of functions $\{\theta_{i,s}(t)\}_{i\in\mathcal{C}}$ for the citizens, and decision variables, $I(\pi)$ and $s'(\pi)$ as a function of the state s and $\pi \in \{e, c\}$, and equilibrium factor prices as given by (23.34)-(23.37). Here the function $I(\pi)$ determines the equilibrium decision about labor repression conditional on who has power and the function $s'(\pi) \in \{D, N\}$ determines the political state at the start of the next period. Symmetric MPE in addition imposes the requirement that contribution functions take the form θ_s^E and θ_s^C, i.e., do not depend on the identity of the individual elite or citizen, $i \in \mathcal{E} \cup \mathcal{C}$.

As usual, MPE can be characterized by backward induction within the stage game at some arbitrary date t, given the state $s \in \{D, N\}$, and taking future plays (as functions of future states) as given. Clearly, whenever $\pi = e$ so that the elite have political power, they will choose economic institutions that favor them ($I(e) = e$) and a political system that gives them more power in the future ($s'(e) = N$). In contrast, whenever citizens have political power, $\pi = c$, they will choose $I(c) = c$ and $s'(c) = D$. This implies that choices over economic institutions and political states are straightforward. Moreover the determination of market prices under different economic institutions has already been specified above by equations (23.34)-(23.37). The only remaining decisions are the contributions of each agent to their de facto power, $\theta_{i,s}(t)$ for $i \in \mathcal{E} \cup \mathcal{C}$ and $s \in \{D, N\}$. A symmetric MPE can thus be summarized by two pairs of contribution vectors $\theta^E = \left(\theta_D^E, \theta_N^E\right)$ and $\theta^C = \left(\theta_D^C, \theta_N^C\right)$. The MPE can be characterized by writing the payoff to agents recursively, and for this reason, we denote the equilibrium value of an elite agent in state $s \in \{D, N\}$ by V_s^E (i.e., V_D^E for democracy and V_N^E for nondemocracy).

Since we are focusing on symmetric MPE, suppose that all other elite agents, except $i \in \mathcal{E}$, have chosen a level of contribution to de facto power equal to θ_s^E and all citizens have chosen a contribution level θ_s^C. Consequently, when agent $i \in \mathcal{E}$ chooses $\theta_i(t)$, the total power of the elite will be

$$P^E\left(\theta_i, \theta_s^E, \theta_s^C \mid s\right) = \phi_s^E\left((M-1)\,\theta_s^E + \theta_i\right).$$

The elite will have political power if

$$P^E\left(\theta_i, \theta_s^E, \theta_s^C \mid s\right) \geq \phi_s^C L \theta_s^C + \eta I(s = D) + \omega(t). \tag{23.42}$$

Expressed differently, the probability that the elite have political power in state $s \in \{N, D\}$ is

$$p\left(\theta_i, \theta_s^E, \theta_s^C \mid s\right) = F\left[\phi_s^E\left((M-1)\,\theta_s^E + \theta_i\right) - \phi_s^C L \theta_s^C - \eta I\left(s = D\right)\right]. \tag{23.43}$$

As noted above, backward induction within the stage game implies that $I(e) = e$, $I(c) = c$, $s'(e) = N$ and $s'(c) = D$. Thus returns to the citizens and the elite will be w_e and R_e as given by (23.36) and (23.37) when $\pi = e$, and w_c and R_c as in (23.34) and (23.35) when $\pi = c$. Incorporating these best responses and using the one-step-ahead deviation principle (see Appendix Chapter C), we can write the payoff of an elite agent i recursively as follows:

$$\begin{aligned} V^E\left(N \mid \theta^E, \theta^C\right) = \max_{\theta_i \geq 0.} \Big\{ & -\theta_i + p\left(\theta_i, \theta_N^E, \theta_N^C \mid N\right)\left(R_e + \beta V^E\left(N \mid \theta^E, \theta^C\right)\right) \\ & \left(1 - p\left(\theta_i, \theta_N^E, \theta_N^C \mid N\right)\right)\left(R_c + \beta V^E\left(D \mid \theta^E, \theta^C\right)\right)\Big\}. \end{aligned} \tag{23.44}$$

This equation incorporates the fact that with probability $p\left(\theta_i, \theta_N^E, \theta_N^C \mid N\right)$ the elite will remain in power and choose $I = e$ and $s' = N$, and with the complementary probability, the citizens will come to power and choose $I = c$ and $s' = D$. Finally, this expression also makes use of the one-step-ahead deviation principle in writing the continuation values as $V^E\left(N \mid \theta^E, \theta^C\right)$ and $V^E\left(D \mid \theta^E, \theta^C\right)$, that is, it restricts attention to symmetric MPE after the current period, where all citizens and elites choose the contribution levels given by the vectors θ^C and θ^E.

Since F is continuously differentiable, $p\left(\theta_i, \theta_N^E, \theta_N^C \mid N\right)$ is also differentiable. Moreover, the continuation values $V^E\left(D \mid \theta^E, \theta^C\right)$ and $V^E\left(N \mid \theta^E, \theta^C\right)$ are taken as given, so the first-order necessary condition for the optimal choice of θ_i by elite agent i can be written as

$$\phi_N^E f\left[\phi_N^E\left((M-1)\,\theta_N^E + \theta_i\right) - \phi_N^C L \theta_N^C\right]\left[\Delta R + \beta \Delta V^E\right] \leq 1, \tag{23.45}$$

and $\theta_i \geq 0$, with complementary slackness, where recall that ΔR is defined in (23.38), f is the density function of the distribution function F, and

$$\Delta V^E \equiv V^E\left(N \mid \theta^E, \theta^C\right) - V^E\left(D \mid \theta^E, \theta^C\right)$$

is the difference in value between nondemocracy and democracy for an elite agent in the symmetric MPE. Intuitively, (23.45) requires the cost of one more unit of investment in de facto political power to be no less than the benefit. The benefit is given by the increased probability that the elite will control politics induced by this investment, $\phi^E(N)$, times the density of the F function evaluated at the equilibrium investments, multiplied by the benefit from controlling politics, which is the current benefit ΔR plus the discounted increase in continuation value, $\beta \Delta V^E$. In addition, the second-order sufficient condition is

$f'\left[\phi_N^E\left((M-1)\,\theta_N^E+\theta_i\right)-\phi_N^C L\theta_N^C\right]<0.$[6] For future reference, let us also introduce the notation that $\theta_i \in \Gamma^E\left[\theta^E,\theta^C \mid N\right]$ if θ_i is a solution to (23.45) that satisfies the second-order condition.

Similarly, the value function for a citizen when the initial political state is $s=N$ is

$$V^C\left(N \mid \theta^E,\theta^C\right)=\max_{\theta_i\geq 0.}\left\{-\theta_i+p_0\left(\theta_i,\theta_N^E,\theta_N^C \mid N\right)\left(w_e+\beta V^C\left(N \mid \theta^E,\theta^C\right)\right)\right.$$
$$\left.\left(1-p_0\left(\theta_i,\theta_N^E,\theta_N^C \mid N\right)\right)\left(w_c+\beta V^C\left(D \mid \theta^E,\theta^C\right)\right)\right\}, \tag{23.46}$$

which is very similar to (23.44) except that the labor market rewards are now given by w_e and w_c instead of R_e and R_c, and the probability that $\pi=e$ is now given by the function

$$p_0\left(\theta_i,\theta_s^E,\theta_s^C \mid s\right)=F\left[\phi_s^E M\theta_s^E-\phi_s^C\left((L-1)\,\theta_s^C+\theta_i\right)-\eta I\left(s=D\right)\right], \tag{23.47}$$

which is the probability that the elite have more power than the citizens in state $s\in\{D,N\}$, when all elite agents choose investment in de facto power, θ_s^E, all citizens except i choose θ_s^C, and individual i chooses θ_i. The first-order necessary condition is similar to (23.45) and can be written as

$$\phi_N^C f\left[\phi_N^E M\theta_N^E-\phi_N^C\left((L-1)\theta_N^C+\theta_i\right)\right]\left[\Delta w+\beta\Delta V^C\right]\leq 1 \tag{23.48}$$

and $\theta_i\geq 0$ with complementary slackness, and

$$\Delta V^C\equiv V^C\left(D \mid \theta^E,\theta^C\right)-V^C\left(E \mid \theta^E,\theta^C\right).$$

The interpretation of this condition is the same as that of (23.45). The second-order sufficient condition is $f'\left[\phi_N^E M\theta_N^E-\phi_N^C\left((L-1)\theta_N^C+\theta_i\right)\right]>0$. If θ_i is a solution to (23.48), we denote this by $\theta_i\in\Gamma^C\left[\theta^E,\theta^C \mid N\right]$.

By analogy, the value function for the elite in democracy is given by:

$$V^E\left(D \mid \theta^E,\theta^C\right)=\max_{\theta_i\geq 0}\left\{-\theta_i+p\left(\theta_i,\theta_D^E,\theta_D^C \mid D\right)\left(R_e+\beta V^E\left(N \mid \theta^E,\theta^C\right)\right)\right. \tag{23.49}$$
$$\left.+\left(1-p\left(\theta_i,\theta_D^E,\theta_D^C \mid D\right)\right)\left(R_c+\beta V^E\left(D \mid \theta^E,\theta^C\right)\right)\right\},$$

where $p\left(\theta_i,\theta_D^E,\theta_D^C \mid D\right)$ is again given by (23.43). The first-order necessary condition for the investment of an elite agent in democracy then becomes:

$$\phi_D^E f\left[\phi_D^E\left((M-1)\,\theta_D^E+\theta_i\right)-\phi_D^C L\theta_D^C-\eta\right]\left[\Delta R+\beta\Delta V^E\right]\leq 1, \tag{23.50}$$

and $\theta_i\geq 0$, again with complementary slackness and with the second-order condition $f'\left[\phi_D^E\left((M-1)\,\theta_D^E+\theta_i\right)-\phi_D^C L\theta_D^C-\eta\right]<0$. We write $\theta_i\in\Gamma^E\left[\theta^E,\theta^C \mid D\right]$ if θ_i solves (23.50) and satisfies the second-order condition. Finally, for the citizens in democracy, we

[6] This condition with strict inequality is sufficient, while with a weak inequality, it would be necessary but not sufficient. I impose the sufficient condition throughout to simplify the discussion.

have

$$V^C\left(D \mid \theta^E, \theta^C\right) = \max_{\theta_i \geq 0} \left\{ -\theta_i + p_0\left(\theta_i, \theta_D^E, \theta_D^C \mid D\right)\left(w_e + \beta V^C\left(N \mid \theta^E, \theta^C\right)\right) \right.$$

$$\left. + \left(1 - p_0\left(\theta_i, \theta_D^E, \theta_D^C \mid D\right)\right)\left(w_c + \beta V^C\left(D \mid \theta^E, \theta^C\right)\right)\right\}, \tag{23.51}$$

where $p_0\left(\theta_i, \theta_D^E, \theta_D^C \mid D\right)$ is given by (23.47). The first-order necessary condition is now

$$\phi_D^C f\left[\phi_D^E M \theta_D^E - \phi_D^C\left((L-1)\theta_D^C + \theta_i\right) - \eta\right]\left[\Delta w + \beta \Delta V^C\right] \leq 1, \tag{23.52}$$

and $\theta_i \geq 0$, with complementary slackness and the second-order condition $f'\left[\phi_D^E M \theta_D^E - \phi_D^C\left((L-1)\theta_D^C + \theta_i\right) - \eta\right] > 0$. We denote solutions to this problem by $\theta_i \in \Gamma^C\left[\theta^E, \theta^C \mid D\right]$.

With these definitions, we can define a symmetric Markov Perfect Political Economy Equilibrium as contribution vectors $\theta^E = \left(\theta_N^E, \theta_D^E\right)$ and $\theta^C = \left(\theta_N^C, \theta_D^C\right)$ such that $\theta_N^E \in \Gamma^E\left[\theta^E, \theta^C \mid N\right]$, $\theta_D^E \in \Gamma^E\left[\theta^E, \theta^C \mid D\right]$, $\theta_N^C \in \Gamma^C\left[\theta^E, \theta^C \mid N\right]$ and $\theta_D^C \in \Gamma^C\left[\theta^E, \theta^C \mid D\right]$. In addition, policy, economic and political decisions $I(\pi)$ and $s'(\pi)$ must be such that, $I(e) = e$, $s'(e) = N$, $I(c) = c$ and $s'(c) = D$, and factor prices must be given by (23.34)-(23.37) as a function of $I \in \{e, c\}$.

The comparison of (23.45) and (23.48) immediately implies that these first-order conditions cannot generally hold as equalities both for the elite and the citizens. The comparison of (23.50) and (23.52) also leads to the same conclusion. In particular, "generically" only one of the two groups will invest to increase their de facto political power. Which group will be the one to invest in their political power? Loosely speaking, the answer is: whichever group has higher gains from doing so. Here the difference in numbers becomes important. In particular, recall that $L >> M$ implies $\Delta R >> \Delta w$. Consequently, it will be the elite that have more to gain from controlling politics and that will invest to increase their de facto power. This leads to the following proposition, with the proof left as an exercise, in which the exact threshold for the number of citizens relative to elites necessary for this result to hold has to be determined.

Proposition 23.9. *Suppose that $L >> M$. Then, any symmetric MPE involves $\theta_D^C = \theta_N^C = 0$.*

Proof. See Exercise 23.16. □

This proposition simplifies the characterization of equilibrium, which is now reduced to the characterization of two investment levels, θ_N^E and θ_D^E, such that $\theta_N^E \in \Gamma^E\left[\theta^E, 0 \mid N\right]$ and $\theta_D^E \in \Gamma^E\left[\theta^E, 0 \mid D\right]$. Given Proposition 23.9, we can also write the equilibrium probabilities that the elite will have more political power as:

$$p_N \equiv F\left[\phi_N^E M \theta_N^E\right] \text{ and } p_D \equiv F\left[\phi_D^E M \theta_D^E - \eta\right]. \tag{23.53}$$

Next, incorporating symmetry and the fact that $\theta_D^C = \theta_N^C = 0$ into the first-order conditions (23.45) and (23.50), and assuming the existence of an interior solution (with $\theta_N^E > 0$ and $\theta_D^E > 0$), we obtain the following two equations that characterize interior equilibria:

$$\phi_N^E f\left[\phi_N^E M \theta_N^E\right]\left[\Delta R + \beta \Delta V^E\right] = 1, \tag{23.54}$$

and

$$\phi_D^E f\left[\phi_D^E M \theta_D^E - \eta\right]\left[\Delta R + \beta \Delta V^E\right] = 1. \tag{23.55}$$

The question is whether there exists such an interior solution. The following assumption imposes that the additional rents that the elite will gain from labor repressive institutions are sufficiently large and ensures that this is the case.

CONDITION 23.3.

$$\min\left\{\phi_N^E f[0]\,\Delta R, \phi_D^E f[-\eta]\,\Delta R\right\} > 1.$$

PROPOSITION 23.10. *Suppose that $L >> M$ and that Condition 23.3 holds. Suppose also that $\phi_N^E = \phi_D^E$. Then there exists a unique symmetric MPE. This equilibrium involves $p_D = p_N \in (0,1)$, so that the probability distribution over economic institutions is non-degenerate and independent of whether the society is democratic or nondemocratic.*

PROOF. Using Proposition 23.9, we have that $\theta_D^C = 0$ and $\theta_N^C = 0$. Then, Condition 23.3 implies that $\theta_D^E = 0$ and $\theta_N^E = 0$ cannot be part of an equilibrium. Since we have also assumed that $f[\omega]$ is continuous and that $\lim_{\omega\to\infty} f[\omega] = 0$, both conditions (23.54) and (23.55) must hold as equalities for some interior values of θ_D^E and θ_N^E establishing existence of an equilibrium. The result that $p_D = p_N > 0$ then follows immediately from the comparison of these two equalities, which establishes (23.56). The fact that $p_D = p_N < 1$ follows from the assumption on F, which implies that it is strictly increasing throughout its support, so for any interior θ_D^E and θ_N^E, $F\left[\phi^E M \theta_D^E - \eta\right] = F\left[\phi^E M \theta_N^E\right] < 1$. In addition, again from the assumption on F, that $f[\omega]$ is single peaked, so only a unique pair of θ_D^E and θ_N^E could satisfy (23.54) and (23.55) with $f'\left[\phi^E M \theta_N^E\right] < 0$ and $f'\left[\phi^E M \theta_D^E - \eta\right] < 0$ for given ΔV^E. The fact that $\Delta V^E = \theta_D^E - \theta_N^E = \eta / \left(\phi^E M\right)$ is uniquely determined (from equation (23.56) below) then establishes the uniqueness of the symmetric MPE. □

This proposition presents the most striking result of the model in this section; when $\phi_N^E = \phi_D^E$, the effects of changes in political institutions are totally offset by changes in investments in de facto power. Consequently, the stochastic distribution for economic institutions is identical starting in democracy or in nondemocracy. The intuition for this result is straightforward and can be obtained by comparing (23.54) and (23.55) in the special case where $\phi_N^E = \phi_D^E = \phi^E$. These two conditions can hold as equality only if

$$f\left[\phi^E M \theta_N^E\right] = f\left[\phi^E M \theta_D^E - \eta\right]. \tag{23.56}$$

The fact that f is single peaked (which has been assumed above) combined with the second-order conditions implies that $\phi^E M\theta_N^E = \phi^E M\theta_D^E - \eta$, or in other words,

$$\theta_D^E = \theta_N^E + \frac{\eta}{\phi^E M}. \tag{23.57}$$

(23.53) then implies that $p_D = p_N$, which is the *invariance* result discussed in the Introduction.

Intuitively, in democracy the elite invest sufficiently more to increase their de facto political power so that they entirely offset the democratic (de jure power) advantage of the citizens. A more technical intuition for this result is that the optimal contribution conditions for the elite both in nondemocracy and democracy equate the marginal cost of contribution, which is always equal to 1, to the marginal benefit. Since the marginal costs are equal, equilibrium benefits in the two regimes also have to be equal. The marginal benefits consist of the immediate gain of economic rents, ΔR, plus the gain in continuation value, which is also independent of current regime. Consequently, marginal costs and benefits can only be equated if $p_D = p_N$. This result illustrates how institutional change and persistence can coexist—while political institutions change frequently, the equilibrium process for economic institutions remains unchanged.

The most important implication of Proposition 23.10 relates to the potential inefficiency of democracy relative to nondemocracy and thus to the discussion in Section 23.1. Recall that the relatively poor performances of democracies in the postwar era is a potential puzzle, especially viewed in light of the presence of some disastrous, kleptocratic nondemocracies. In the current model, an unconstrained democracy would choose competitive labor market institutions, which increase wages and serve the majority of the population, and these institutions would also lead to higher income per capita in the economy, because $\delta > 0$. However, because the elite can invest in de facto political power in democracy to offset the de jure power advantage of the masses, the equilibrium looks very different. In fact, an allocation starting from nondemocracy *weakly Pareto dominates* one that starts in democracy, even though labor repression is socially costly (i.e., $\delta > 0$). This is because citizens are equally well off in the two allocations, while starting in democracy the elite receive the same economic payoff but invest more in de facto power and thus are worse off. This analysis therefore suggests that the high levels of investment in de facto political power by the elite in democracy, which are socially costly, may be one of the reasons why many democratic societies have disappointing economic performances. This source of inefficiency in democracy complements the distortionary effects of democracies emphasized so far (which resulted from distortionary redistributive taxes imposed by democratic regimes). Here democracies are inefficient because the actions by the elites to prevent democracies from choosing their preferred economic institutions and turn the democratic regime into a *dysfunctional democracy.*

Despite its importance, especially in the context of the debate on the relationship between political regimes and growth, Proposition 23.10 may be viewed as a special case, because it depends on the assumption that the technology for de facto political power for the elite is the same in democracy and nondemocracy, i.e., $\phi_N^E = \phi_D^E$. One may reasonably suspect that the elite may may be less effective in using its resources to garner de facto political power in democratic regimes, which may successfully place constraints on their behavior. In this case, we may want to assume that $\phi_N^E > \phi_D^E$ instead of $\phi_N^E = \phi_D^E$. The next proposition presents the relevant results under this assumption.

Proposition 23.11. *Suppose that $L >> M$ and that Condition 23.3 holds. Then, any symmetric MPE leads to a Markov regime switching structure where the society fluctuates between democracy with associated competitive economic institutions ($I = c$) and nondemocracy with associated labor repressive economic institutions ($I = e$), with switching probabilities $1 - p_N \in (0, 1)$ and $1 - p_D \in (0, 1)$. Moreover, provided that $\phi_N^E > \phi_D^E$, $p_D < p_N$.*

Proof. The proof of this proposition builds on the proof of Proposition 23.10. In particular, suppose that $\phi_N^E > \phi_D^E$. Proposition 23.9 and the same argument as in the proof of Proposition 23.10 again imply that $\theta_D^C = 0$ and $\theta_N^C = 0$, and also $\theta_D^E > 0$ and $\theta_N^E > 0$. Again by assumption we have that $f[\omega]$ is continuous with $\lim_{\omega\to\infty} f[\omega] = 0$, so that both conditions (23.54) and (23.55) must hold as equalities for some interior values of θ_D^E and θ_N^E establishing existence. $p_D > 0$ and $p_N > 0$ follows from the fact that $\theta_D^E > 0$ and $\theta_N^E > 0$, and $p_D < 1$ and $p_N < 1$ follows by the assumptions on F. To complete the proof, we need to establish that when $\phi_N^E > \phi_D^E$, $p_D < p_N$. Suppose, to obtain a contradiction, that $p_D \geq p_N$. Then the assumption on F combined with the second-order conditions implies that $\phi_D^E M\theta_D^E - \eta \geq \phi_N^E M\theta_N^E$ and $f\left[\phi_D^E M\theta_D^E - \eta\right] \leq f\left[\phi_N^E M\theta_N^E\right]$. But combined with the hypothesis that $\phi_N^E > \phi_D^E$, this implies that (23.54) and (23.55) cannot both hold, thus leads to a contradiction and establishes that $p_D < p_N$. □

This proposition also has a number additional implications relative to Proposition 23.10. First, the equilibrium now involves endogenous switches between different political regimes. Second, there is "state dependence" or persistence, in the sense that democracy is more likely to follow democracy than it is to follow nondemocracy (i.e., $p_D < p_N$). Third, the effects of the changes in the distribution of de jure power induced by political regime change are partially offset by changes in investments in de facto power (though not fully offset as in Proposition 23.10). This offset is due to the elite's investments in their de facto political power.

Both Propositions 23.10 and 23.11 rely on Condition 23.3, which ensures that investment in de facto power is always profitable for the elite. When this is not the case, democracy can become an absorbing state and changes in political institutions will have more important effects. This is stated in the next proposition.

Proposition 23.12. *Suppose that $L >> M$ and that there exists $\bar{\theta}_N^E > 0$ such that*

$$\phi_N^E f\left[\phi_N^E M \bar{\theta}_N^E\right]\left[\frac{\Delta R - \beta \bar{\theta}_N^E}{1-\beta F\left[\phi_N^E M \bar{\theta}_N^E\right]}\right] = 1; \tag{23.58}$$

and

$$\eta > -\underline{\omega}. \tag{23.59}$$

Then there exists a symmetric MPE in which $p_N \in (0,1)$ and $p_D = 0$.

Proof. See Exercise 23.18. □

Therefore, if we relax part of Condition 23.3, symmetric MPEs with democracy as an absorbing state may arise. Clearly, Condition (23.59), which leads to this outcome, is more likely to hold when η is high. This implies that if democracy creates a substantial advantage in favor of the citizens, it may destroy the incentives of the elite to engage in activities that increase their de facto power. This will then change the future distribution of political regimes and economic institutions.

It is also interesting to note that even when (23.59) holds, the equilibrium with $p_D, p_N > 0$ characterized in Propositions 23.10 and 23.11 may still exist, leading to a symmetric MPE with $p_D = p_N$. Consequently, whether democracy becomes an absorbing state (i.e., whether it becomes fully consolidated) may depend on expectations.

The analysis so far has established how the interplay between de facto and de jure political power leads to the coexistence of persistence in economic institutions and change in political regimes. Equally important, however, is how the likelihood of different institutional outcomes are related to the underlying parameters. I now present a number of comparative static results shedding light on this question. To simplify the analysis, let us focus on the case Proposition 23.10, where $\phi_N^E = \phi_D^E = \phi^E$. The generalization of these results generalize to the case where $\phi_N^E > \phi_D^E$ as discussed in Exercise 23.20. When $\phi_N^E = \phi_D^E$, comparative statics are straightforward since equations (23.44), (23.49) and (23.57) immediately imply that

$$\Delta V^E = \frac{\eta}{\phi^E M} > 0. \tag{23.60}$$

This equation is intuitive. Proposition 23.10 implies that from the viewpoint of the elite, there is only one difference between democracy and nondemocracy; in democracy the elite have to spend more in contributions in order to retain the same political power. In particular, the per elite additional spending is equal to $\eta/\phi^E M$, which is increasing in the de jure political power advantage that democracy creates for the citizens (since, in equilibrium, the elite totally offset this advantage).

Using (23.54) and (23.60) and denoting the equilibrium level of θ_N^E by θ_N^*, we obtain:

$$\phi^E f\left[\phi^E M\theta_N^*\right]\left[\Delta R+\beta\frac{\eta}{\phi^E M}\right]=1. \tag{23.61}$$

Similarly, denoting the equilibrium level of θ_D^E by θ_D^*, we also have

$$\phi^E f\left[\phi^E M\theta_D^*-\eta\right]\left[\Delta R+\beta\frac{\eta}{\phi^E M}\right]=1. \tag{23.62}$$

Finally, let us denote the probability that the elite will have political power by $p^*=p_D=p_N$. This probability corresponds both to the probability that the elite will control political power, and also to the probability that the society will be nondemocratic and economic institutions will be labor repressive rather than competitive. Thus this probability summarizes most of the economic implications of the model.

Proposition 23.13. *Suppose that $L>>M$ and Condition 23.3 holds. Suppose also that $\phi_N^E=\phi_D^E=\phi^E$. Then, θ_N^*, θ_D^* and p^* are strictly increasing in ΔR, β and η, and strictly decreasing in M. Moreover, p^* is strictly increasing in ϕ^E.*

Proof. See Exercise 23.19. □

Many of the comparative statics in Proposition 23.13 are intuitive and do not require much elaboration. For example, the effect of the number of elite agents, M, on investments in de facto power and the equilibrium probability of nondemocracy and the effect of ϕ^E on the equilibrium probability of nondemocracy are straightforward to understand. Observe that M also has an indirect effect on the equilibrium, which goes in the same direction; greater M reduces ΔR (cf. equation (23.38)) and further discourages investments in de facto power via this channel.

The fact that an increase in ΔR increases the probability that the elite control political power is also natural, since ΔR is a measure of how much they have to gain by controlling political power. But this latter result also has interesting economic implications. Since ΔR will be high when λ or δ are low, we also have $\partial p^*/\partial\lambda<0$, and $\partial p^*/\partial\delta<0$, so that political and economic institutions favoring the elite are more likely to arise when the elite will be able to use labor repressive institutions effectively or when the costs of repression are relatively low. A major reason why λ and δ may vary across societies is because of differences in economic structure, economic institutions and factor endowments. For example, we may expect both parameters to be higher in societies where agriculture is more important and physical or human capital-intensive sectors are less important, since labor repression may be more effective in reducing wages and may also create less distortion in such societies than in those with more complex production relations. This interpretation is consistent with the greater prevalence of labor repressive practices in predominantly agricultural societies.

The fact that a higher β also increases the likelihood of labor repressive institutions is somewhat more surprising. In many models, a higher discount factor leads to better allocations. Here, in contrast, a higher discount factor leads to more wasteful activities by the elite and to labor repressive economic institutions. The reason is that the main pivotal agents in this model are the elite, which, by virtue of their smaller numbers, are the ones investing in their de facto political power (recall Proposition 23.9) and thus take the effect of their contributions on equilibrium allocations into account. Contributing to de facto political power is a form of investment and some of the returns accrue to the elite in the future (when they secure nondemocracy instead of democracy). Therefore a higher level of β encourages them to invest more in their political power and makes nondemocracy and labor repressive economic institutions more likely.

The most surprising and interesting comparative static result concerns the effects of η. Since a higher η corresponds to a greater de jure power advantage for the citizens in democracy, one might have expected a greater η to lead to better outcomes for the citizens. In contrast, we find that higher η makes nondemocracy and labor repressive economic institutions more likely (as long as Condition 23.3 still holds). This is because a higher η makes democracy more costly for the elite, inducing each elite agent to invest more in the group's political power in order to avoid democracy. This effect is strong enough to increase the probability that they will maintain political power. However, the overall impact of η on the likelihood of democracy is non-monotonic: if η increases so much that Condition 23.3 no longer holds, then Proposition 23.12 applies and democracy becomes fully consolidated (i.e., an absorbing state).

Some of the comparative static results in Proposition 23.13 are the outcome of two competing forces. The fact that the cost of investing in de facto political power is linear and the assumption that $\phi_N^E = \phi_D^E$ are important for these results. In particular, in the case where $\phi_N^E > \phi_D^E$, the comparative statics with respect to ΔR, β and M still hold. But those with respect to η become ambiguous; a greater democratic advantage for citizens helps them gain power in democracy, but also induces the elite to invest more in their de facto political power. Which effect dominates cannot be determined without imposing further structure.

23.5.2. Captured Democracy. The model presented in the previous two subsections provides a clear reason why democracies may not feature markedly different policies and output levels than nondemocracies. In particular, the analysis demonstrated how the exercise of de facto political power by the elite in democracy can undo the influence of the masses in democratic politics. These patterns can be interpreted as a form of democratic capture by the elite. In this subsection, I will illustrate a stronger form of capture. To do this, I will relax the assumption that the group that has power can immediately change both economic institutions and the political system. Instead, consistent with the discussion in the previous section, I will now assume that political institutions are more durable than economic institutions and policies, thus more difficult to change. I will then show how this leads to a more extreme form of captured democracy, where, in equilibrium, democratic political institutions may emerge and survive for extended periods of time, but the elite are still able to impose their favorite economic institutions.

To capture the greater durability of political institutions, let me assume that overthrowing a democratic regime is more difficult than influencing economic institutions. In particular, the elite require greater political power to force a switch from democracy to nondemocracy than simply influencing economic institutions in democracy. For example, when $s = D$ and $P_D^C(t) + \xi > P_D^E(t) \geq P_D^C(t)$, where $\xi > 0$, the elite can choose economic institutions at time t, but cannot change the political system. If, on the other hand, $P_D^E(t) \geq P_D^C(t) + \xi$, the elite can choose both economic institutions and the future political system. Symmetrically when $s = N$ and $P_N^E(t) + \xi > P_N^C(t) \geq P_N^E(t)$, the citizens can choose economic institutions, but cannot change the political system. This formulation builds in the assumption that changing political institutions is more difficult than influencing economic institutions. Moreover, to simplify the analysis we focus on the case where $\phi_N^E = \phi_D^E = \phi^E$ and let us now assume that F is everywhere strictly increasing and twice continuously differentiable (so that its density f and the derivative of the density, f', exist everywhere), and moreover we have $f'[\omega] < 0$ for all ω and $\lim_{\omega \to \infty} f[\omega] = 0$. This assumption will simplify the analysis and enable us to obtain sharper results. In addition, I make one more important assumption, that citizens prefer to live in democracy, for example, because democracy provides some other benefits such as public goods and amenities, even if the probability of labor repressive economic institutions are higher under democracy.

Given these assumptions, the structure of the model is similar to before and a symmetric MPE is also defined similarly. The value functions are more complicated, but have similar intuition to those in the previous subsection. To simplify the exposition, let us impose the result of Proposition 23.9 in writing the various probabilities. In particular, suppose that citizens choose zero contribution to their de facto political power, all elite agents except $i \in \mathcal{E}$ choose an investment level of θ_D^E and i chooses θ_i. Then, let the probability that the elite have sufficient power to change democracy to nondemocracy be

$$\hat{p}\left(\theta_i, \theta_D^E, \theta_D^C \mid D\right) = F\left[\phi^E\left((M-1)\,\theta_D^E + \theta_i\right) - \eta - \xi\right], \tag{23.63}$$

while the probability that they only have power to choose economic institutions is, as before,

$$p\left(\theta_i, \theta_D^E, \theta_D^C \mid D\right) = F\left[\phi^E\left((M-1)\,\theta_D^E + \theta_i\right) - \eta\right].$$

Correspondingly, the value function for the elite in democracy can be written as:

$$
\begin{aligned}
V^E\left(D \mid \theta^E, \theta^C\right) &= \max_{\theta_i \geq 0} \left\{-\theta_i + p\left(\theta_i, \theta_D^E, \theta_D^C \mid D\right) R_e + \right. \\
&\quad \left(1 - p\left(\theta_i, \theta_D^E, \theta_D^C \mid D\right)\right) R_c + \hat{p}\left(\theta_i, \theta_D^E, \theta_D^C \mid D\right) \beta V^E\left(N \mid \theta^E, \theta^C\right) \\
&\quad \left. + \left(1 - \hat{p}\left(\theta_i, \theta_D^E, \theta_D^C \mid D\right)\right) \beta V^E\left(D \mid \theta^E, \theta^C\right)\right\}, \qquad (23.64)
\end{aligned}
$$

where I have already imposed that when the citizens have sufficient power they will choose democracy. With similar arguments to before, the maximization in (23.64) implies the following first-order condition for an interior equilibrium:

$$
\phi^E f\left[\phi^E M \theta_D^E - \eta\right] \Delta R + \beta \phi^E f\left[\phi^E M \theta_D^E - \eta - \xi\right] \Delta V^E = 1, \qquad (23.65)
$$

which is now sufficient since the stronger assumptions on F now ensure that the second-order condition is always satisfied.

The main difference of this first-order condition from the one before is that the probability with which the elite gain the economic rent ΔR is different from the probability with which they secure a change in the political system. For this reason, two different densities appear in (23.65).

Similarly for nondemocracy, let us define

$$
\hat{p}\left(\theta_i, \theta_N^E, \theta_N^C \mid N\right) = F\left[\phi^E\left((M-1)\theta_N^E + \theta_i\right) + \xi\right], \qquad (23.66)
$$

and

$$
p\left(\theta_i, \theta_N^E, \theta_N^C \mid N\right) = F\left[\phi^E\left((M-1)\theta_N^E + \theta_i\right)\right],
$$

which leads to a modification of the value function for nondemocracy as

$$
\begin{aligned}
V^E\left(N \mid \theta^E, \theta^C, \theta_i\right) &= \max_{\theta_i \geq 0} \left\{-\theta_i + p\left(\theta_i, \theta_N^E, \theta_N^C \mid N\right) R_e + \right. \\
&\quad \left(1 - p\left(\theta_i, \theta_N^E, \theta_N^C \mid N\right)\right) R_c + \hat{p}\left(\theta_i, \theta_N^E, \theta_N^C \mid N\right) \beta V^E\left(N \mid \theta^E, \theta^C\right) \\
&\quad \left. + \left(1 - \hat{p}\left(\theta_i, \theta_N^E, \theta_N^C \mid N\right)\right) \beta V^E\left(D \mid \theta^E, \theta^C\right)\right\}, \qquad (23.67)
\end{aligned}
$$

which again has a similar structure to the value function in democracy except for the presence of the utility benefit from being able to provide the public good which the elite value. Consequently, the first-order (necessary and sufficient) condition for optimal contribution by an elite agent in an interior equilibrium is also similar:

$$
\phi^E f\left[\phi^E M \theta_N^E\right] \Delta R + \beta \phi^E f\left[\phi^E M \theta_N^E + \xi\right] \Delta V^E = 1. \qquad (23.68)
$$

To characterize the equilibrium, let us introduce the additional notation such that $\pi = (e, e)$ denotes the elite keeping total power in nondemocracy or gaining total power in democracy (i.e., $P_N^E(t) \geq P_N^C(t)$ or $P_D^E(t) \geq P_D^C(t) + \xi$); $\pi = (e, c)$ corresponding to the elite keeping control of de jure power but losing control of economic institutions in nondemocracy (i.e., $P_N^E(t) + \xi \geq P_N^C(t) > P_N^E(t)$); $\pi = (c, c)$ means the elite lose power in nondemocracy or fail to gain any power in democracy (i.e., $P_N^C(t) > P_N^E(t) + \xi$ or $P_D^C(t) > P_D^E(t)$); and finally, $\pi = (c, e)$ corresponds to the citizens maintaining de jure power in democracy but losing control over economic institutions (i.e., $P_D^C(t) + \xi > P_D^E(t) \geq P_D^C(t)$).

The interesting result in this case is that once the society becomes democratic, it may remain so potentially for a long time (i.e., $\hat{p}_D$ can be small), but the elite will still be able to control the economic institutions (i.e., p_D could be quite large). This is stated and proved in the next proposition.

Proposition 23.14. ***C****onsider the modified model with durable political institutions. Suppose that $L >> M$ and that Condition 23.3 holds. Then, we have a Markov regime-switching process with state dependence and $1 > \hat{p}_N > \hat{p}_D > 0$. Moreover, democracy is captured in the sense that $0 < p_N < p_D < 1$, i.e., democracy will survive but choose economic institutions in line with the elite's interests with even a higher probability than does nondemocracy.*

Proof. The following actions are clearly best responses: $I(\pi = e) = e$, $I(\pi = c) = c$, $s'(\pi = e) = N$. Suppose also that $s'(\pi = c) = D$. Then Proposition 23.9 implies that the probability of labor repressive economic institutions under democracy and nondemocracy are given by

$$p_D = F\left[\phi^E M \theta_D^E - \eta\right],$$

and

$$p_N = F\left[\phi^E M \theta_N^E\right].$$

Suppose, to obtain a contradiction, that $p_D \leq p_N$. This is equivalent to

$$\phi^E M \theta_D^E - \eta \leq \phi^E M \theta_N^E. \tag{23.69}$$

Since, by assumption, f is decreasing everywhere, this implies

$$f\left[\phi^E M \theta_D^E - \eta\right] \geq f\left[\phi^E M \theta_N^E\right].$$

This equation combined with (23.65) and (23.68) implies that

$$f\left[\phi^E M \theta_D^E - \eta - \xi\right] \leq f\left[\phi^E M \theta_N^E + \xi\right].$$

Using once more that f is decreasing, this is equivalent to

$$\phi^E M \theta_D^E - \eta - \xi \geq \phi^E M \theta_N^E + \xi,$$

which, given $\xi > 0$, contradicts (23.69), establishing that $p_D > p_N$. But, by the same reasoning, $p_D > p_N$ implies $f\left[\phi^E M \theta_D^E - \eta - \xi\right] > f\left[\phi^E M \theta_N^E + \xi\right]$, thus $\phi^E M \theta_D^E - \eta - \xi < \phi^E M \theta_N^E + \xi$. Since F is strictly monotonic, this implies $\hat{p}_N > \hat{p}_D$, establishing the Markov regime-switching structure. □

The equilibrium predictions in this proposition are considerably richer than those in the previous subsection and illustrates the potential rich dynamics of equilibrium economic and political institutions. The equilibrium still takes a Markov regime-switching structure with fluctuations between democracy and nondemocracy. But in democracy, there is no guarantee that economic institutions will be those favored by the citizens. While in the baseline model the elite were able to impose both their political and economic wishes at the same time, here we have an equilibrium pattern whereby democracy persists, but the elite may be able to impose their favorite economic institutions. In fact, the proposition shows that the elite may be able to impose labor repressive economic institutions with a *higher* probability under democracy than in nondemocracy.

The intuition for this (somewhat paradoxical) result is that in democracy there is an additional benefit for the elite to invest in de facto political power, which is to induce a switch from democracy to nondemocracy. Consequently, the elite invest in their de facto power sufficiently more in democracy that they are able to obtain their favorite economic institutions with a greater probability. Nevertheless, the elite are happier in nondemocracy, because the cost of investing in their de facto political power in democracy is significantly higher. In fact, it is precisely because they prefer nondemocracy to democracy that they are willing to invest more in their de facto political power in democracy and obtain the labor repressive economic institutions with a high probability. What about citizens? If there were no additional benefit of democracy, then citizens would be worse off in democracy than in nondemocracy, because they would only care about economic institutions and economic institutions are more likely to be labor repressive in democracy than in nondemocracy. However, if the other benefits to citizens from democracy are sufficiently high, as we have assumed, then citizens are willing to choose a democratic regime, even though economic institutions in democracy will to be no better for them than those in nondemocracy.

The most important implication of this extended model is that a dynamic equilibrium resulting from the interaction between de jure and de facto political power may involve the emergence of democratic political institutions, but also allow for the possibility that these institutions are captured by the elite, so that they choose pro-elite policies. In this equilibrium, democracies lead to two different, and novel, inefficiencies. First, they may in fact choose inefficient pro-elite economic institutions with even a greater likelihood than nondemocratic regimes. Second, this outcome is a democratic equilibrium precisely because the elite engage in a large amount of wasteful investment in de facto power. This type of equilibrium configuration is more likely to emerge when the parameter ξ is large, which implies a high degree of durability in political institutions.

23.6. Taking Stock

This chapter provided a brief overview of some of the issues related to the effects of political institutions on economic growth. How societies can choose economic institutions and policies that are not conducive to economic growth was the focus of the previous chapter.

A natural conjecture based on the analysis there is to relate differences in economic institutions to political institutions. For example, if political power is in the hands of an elite that is opposed to growth, growth-enhancing policies are less likely to emerge. Our analysis in the previous chapter hinted that such considerations could be important. The empirical evidence in Chapter 4 also provided support for such a view, whereby the cluster of economic institutions that provide secure property rights to a broad cross-section of society together with political institutions that place constraints on elites and politicians, and support such economic institutions are conducive to economic growth. Nevertheless, the relationship between political regimes and growth is more complicated for a number of reasons. First, the empirical evidence is less clear-cut than we may have originally presumed—while there are historical examples of the positive effects of democratic institutions on economic growth, the postwar evidence does not provide strong support for the view that democracies and political institutions that constrain rulers and politicians are more conducive to economic growth. Second, political institutions themselves are not given, but are endogenous and change dynamically. These two factors imply that we need to understand how political institutions affect economic outcomes more carefully and should also consider the modeling of equilibrium political institutions. Both of these areas are at the forefront of research in political economy and are likely to play a more important role in the research on economic growth in the coming years. They are also active research areas, very much in their infancy. Thus there is no established framework and no general consensus on what types of models are most useful in thinking about the issues raised in this chapter (nor any consensus on what the facts are when it comes to the effect of political regimes on economic growth).

In this light, I presented a number of models to highlight the ideas about political equilibria and the relationship between political institutions and economic growth that I view to be most important and most promising for future work. I emphasized how preferences over political institutions need to be derived from the implications of these political institutions for economic allocations. I then highlighted how ideal (or perfect) political institutions are unlikely to exist or be feasible in practice, thus different political institutions, by creating different sets of winners and losers, also create tradeoffs. Oligarchies favor the already rich, which brings a range of distortions. Democracies, on the other hand, will typically involve higher taxes on the rich to generate income to redistribute to the less well-off and in the process create distortions on investment and other economic choices. In general, it is impossible to unambiguously conclude whether democracies or oligarchies (or yet other political systems favoring other groups) will be more growth-enhancing. However, certain ideas seem both plausible and consistent with the data. One aspect I tried to emphasize is that the dynamic tradeoffs between democracies and other regimes may be different than the static tradeoffs. While democracies may create static distortions because of their greater redistributive tendencies, they are likely to outperform oligarchies in the long run because they avoid political sclerosis, whereby the incumbents are able to dominate the political system and erect entry barriers to protect their businesses, even when efficiency dictates that other individuals should enter and form new businesses to replace theirs. Thus democracy may be more conducive to the process of creative destruction that is part of modern capitalist growth than other political regimes. A related idea, which also receives support from casual empiricism and the model presented in Section 23.3, is that democracies might be more flexible and adaptable to the arrival of new technologies.

Another important idea I tried to emphasize is that democracies may lead to inefficient outcomes because they may sometimes be *dysfunctional*. The main functional characteristic of democracies is that they create political equality, providing voice to the masses, free entry of parties, and free and fair elections. However, as already emphasized, political equality often comes with a tendency to choose redistributive policies that are costly for the elites (especially, for elites that have most of their assets in land or in other easily taxable forms). This creates an incentive for the elite to invest to increase their de facto political power in order to capture democratic politics. Captured democracies are particular example of dysfunctional democracies and may lead to a range of inefficiencies not because the regime is democratic, but precisely because its true democratic nature remains unfulfilled. The model in Section 23.5 provides a stark example of this tendency, whereby democracy may lead to a Pareto dominated allocation, even though without the intervention from the elite, democratic politics would lead to higher income per capita and much less distorted allocations. Whether, in practice, the majority of distortions in democracies are related to their redistributive tendencies or to the fact that they are captured by the elites is an empirical question that has not been addressed yet. Future empirical work might shed light on this important set of questions. Another type of dysfunctional democracy, discussed at the beginning of this chapter, is exemplified by the populist regimes such as those of Perón in Argentina and Chavez in Venezuela. While these regimes engage in some amount of redistribution, they also pursue highly distortionary policies. Why such regimes arise and sometimes even receive support from the electorate is another important part of the puzzle of the relatively disappointing performances of some democracies, but has not received much attention in the economics literature.

Finally, I also gave a very brief overview of some of the issues that arise when we wish to model the dynamics of political institutions themselves. Section 23.4 provided both an abstract discussion of the types of models that would be useful for such an analysis and examples of how these models can be developed and applied to various situations. Once again, this is an area of active current research and the material presented here is no more than the tip of the iceberg, and it is meant to entice the reader to think more about these issues and to introduce the bare minimum that is necessary for a more coherent discussion of the relationship between political institutions and economic growth.

23.7. References and Literature

This chapter relates to a large literature in political economy and political science. Because of space constraints, I will not provide a comprehensive literature review. Instead, I will simply refer to the relevant books and papers on which the material I presented draws on. The literature on the relationship between political regimes and economic growth, discussed in Section 23.1, is relatively large, but the key references are discussed in that section, so it is not necessary to repeat them here.

Section 23.2 built on the models presented in the previous chapter, which themselves were based on Acemoglu (2006). Section 23.3 is directly based on Acemoglu (2007b). Other models that discuss the functioning of oligarchic societies include Leamer (1998), Bourguinon and Verdier (2000), Robinson and Nugent (2001), Galor, Moav and Vollrath (2005), and Sonin (2003). Section 23.4 provided an abstract discussion of the issues related to the modeling of political change based on Acemoglu and Robinson (2006a) and Acemoglu, Johnson and Robinson (2005a). The distinction between de jure and de facto political power is introduced in Acemoglu and Robinson (2006a) and is also discussed in Acemoglu, Johnson and Robinson (2005a). There are more details on the historical examples discussed in this section in both of these references. Interesting examples of the use of de facto political power by elites in the context of Latin America are provided by Paige (1997) for Central America, by Smith (1979) for Mexico, by Klein (1999) and Mazzuca and Robinson (2004) for Colombia, and by Key (1949), Woodward (1955), Wright (1986), and Ransom and Sutch (2001) for the US South after the Civil War.

Key references on changes in political and economic institutions in medieval Europe include Tawney (1941), Brenner (1976, 1982, 1993), Brewer (1988), Hilton (1981), Ertman (1997), and North and Weingast (1989). The role of Atlantic trade in changing the economic and political landscape of many European nations is emphasized in Davis (1983) and Acemoglu, Johnson on Robinson (2005b). The literature on democratization in Europe and Latin America is summarized in Acemoglu and Robinson (2006a). Important modern reference include Evans (1983), Lee (1994), Lang (1999) and Collier (2000). The fiscal reforms following democratization are documented and discussed in Lindert (2000, 2004), and the educational reforms are discussed in Ringer (1979) and Mitch (1983).

Engerman (1981), Coatsworth (1993), Eltis (1995), Engerman and Sokoloff (1997), and Acemoglu, Johnson and Robinson (2002) provide information on the prosperity the United States in the 17th and 18th centuries relative to the Caribbean and South America. The contrast of industrialization in Britain and France against the experiences of Russia and Austria-Hungary draws on Acemoglu and Robinson (2006b), which includes references to the original literature. Mosse (1992) and Gross (1973) provide an excellent introduction to the policies of Russian and Austria-Hungarian monarchies concerning industrialization and economic development. The model sketched at the end of Section 23.4 builds on Acemoglu and Robinson (2000a, 2006a). Finally, the model presented in Section 23.5 is based on Acemoglu and Robinson (2007).

23.8. Exercises

Exercise 23.1. Consider the following infinite horizon economy populated by two groups of equal size, denoted 1 and 2. All agents in both groups maximize the expected present discounted value of income, with discount factor β. In any period one of the groups is in power while the other group is out of power. When either group is in power, it loses power with probability $q < 1/2$ in every period. Income is generated in the following way: group j has an asset stock of $A_j(t)$ at time t. Using these assets, it can produce income $A_j(t) f(I_j(t))$ if it invests $I_j(t)$ and this costs $I_j(t)$ in terms of utility. Investments are made before nature realizes whether the group in power will lose its position. Initially, the asset stock of both groups is the same $A_1(0) = A_2(0) = A$. Assume that, as long as the assets are not expropriated, income can be hidden in a non-taxable sector that generates a net return of $(1-\tau) A_j(t) f(I_j(t)) - I_j(t)$.

Suppose that the net return to the group is

$$(1 - e_j(t))(1 - \tau_j(t)) A_j(t) f(I_j(t)) + G_j(t) - I_j(t),$$

where $\tau_j(t)$ is a tax rate faced by this group, $e_j(t) \in [0,1]$ denotes the proportion of group j's assets that are expropriated in period t, and $G_j(t)$ is a transfer to group j in period t. The law of motion of assets, as a function of expropriation of assets, is given by:

$$\begin{aligned} A_1(t) &= A_1(t-1) - e_1(t) A_1(t-1) + e_2(t) A_2(t-1) \\ A_2(t) &= A_2(t-1) - e_2(t) A_2(t-1) + e_1(t) A_1(t-1). \end{aligned}$$

(1) First suppose that asset expropriation is not allowed, so $e_j(t) = 0$, and the only decision each group takes is the tax rate it sets when in power. Characterize the pure strategy Markov Perfect Equilibria (MPE) of this repeated game. Show that the output level is less than first-best and is constant over time.
(2) Next suppose that the group in power can expropriate the assets of the other group (so the two decisions now are taxes and expropriation). Characterize the MPE, and show that output can actually be higher in this economy than the economy without asset expropriation. Explain why. Show also that now output is no longer constant, but fluctuates over time.

(3) Next consider a model endogenizing q. In particular, imagine that the group out of power can choose to take power in any period but to do so must pay a non-pecuniary cost c. This cost c is drawn each period from the cumulative distribution $G(c)$. First consider the case without asset expropriation. Show that there will exist a level of c^* such that when $c \leq c^*$, the group out of power will take power [Hint: write the value functions of the members of the two groups in terms of c^*—or the probability of regime change in the future—and obtain a fixed-point recursion for c^*].

(4) Next consider the case with asset expropriation (where the group that comes to power can cost is the expropriate all the assets of the other group). Show that there will exist a level of c^{**} such that when $c \leq c^{**}$, the group out of power will take power, and show that $c^{**} > c^*$. Show also that this economy with endogenous power switches has higher volatility than the corresponding economy with exogenous power switches.

(5) Discuss whether the two theoretical channels, highlighted by the model, linking security of property rights to economic instability are plausible. Feel free to give real world examples.

Exercise 23.2. Prove Proposition 23.1.

Exercise 23.3. (1) Prove Proposition 23.2.

(2) Generalize the result in Proposition 23.2 to the case where $\theta^e \neq \theta^m$. In particular, derive an inequality that determines when the dictatorship of the elite will generate greater output per capita than the dictatorship of the middle class.

Exercise 23.4. Prove Proposition 23.3. [Hint: to prove the second part of this proposition, first note that equilibrium wage will be given by whichever group has lower net (after tax) productivity. Then write the utility of workers under two scenarios, first, when the elite have lower net productivity, and second when the middle class have lower net productivity. In writing these expressions, recall that the group with the lower productivity will employ $1 - \theta\bar{L}$ workers, since Condition 22.1 holds. Derive the optimal tax policies for workers in these two scenarios and then compare the utility at these optimal policies].

Exercise 23.5. In the model of Section 23.3, prove that $V^z\left(q^t\right)$ given in (23.14) is strictly monotonic in $w\left(t\right)$, $T\left(t\right)$ and $\Pi^z\left(\tau\left(t\right), w\left(t\right)\right)$, and therefore that $V^H\left(q^t\right) > V^L\left(q^t\right)$.

Exercise 23.6. In the model of Section 23.3, suppose that $l_i\left(t\right)$ is unbounded above. What problems would this create? Next suppose that $l_i\left(t\right)$ could be arbitrarily small. What problems will this raise for the equilibrium in this section? Could you generalize the results in this section to an environment in which $l_i\left(t\right) \in \left[\underline{L}, \bar{L}\right]$, where $\underline{L} > 0$ and $\bar{L} < \infty$?

Exercise 23.7. Derive equation (23.8).

Exercise 23.8. Prove Proposition 23.5.

Exercise 23.9. Prove Proposition 23.6.

EXERCISE 23.10. Suppose that Condition 23.1 does not hold. Generalize the results in Propositions 23.5 and 23.6.

EXERCISE 23.11. (1) Prove Proposition 23.7.

(2) Show that Condition 23.1 and (23.32) can be jointly satisfied.
(3) Prove Proposition 23.8.

EXERCISE 23.12. Consider the model in Section 23.3, starting with $\mu(0) = 1$ and an oligarchic regime. Suppose that at some time $t' < \infty$ a new technology arises, which is $\psi > 1$ as productive as the old technology. However, entrepreneurial skills with this new technology are uncorrelated with entrepreneurial skills relevant for the old technology. In particular, suppose that entrepreneurial skills for new technology are given by

$$\hat{a}_i(t+1) = \begin{cases} A^H & \text{with probability} \quad \hat{\sigma}^H & \text{if } \hat{a}_i(t) = A^H \\ A^H & \text{with probability} \quad \hat{\sigma}^L & \text{if } \hat{a}_i(t) = A^L \\ A^L & \text{with probability } 1-\hat{\sigma}^H & \text{if } \hat{a}_i(t) = A^H \\ A^L & \text{with probability } 1-\hat{\sigma}^L & \text{if } \hat{a}_i(t) = A^L \end{cases}.$$

(1) Show that there exists $\bar{\psi}$ such that if $\psi > \bar{\psi}$, all existing entrepreneurs will increase entry barriers and switch to the new technology.
(2) Show that if $\psi < \bar{\psi}$, then again entry barriers will be increased and now only entrepreneurs who have low skills with the old technology will switch to the new technology.
(3) Now analyze the response of democracy to the arrival of the same technology.
(4) Compare output per capita in democracy and oligarchy after the arrival of new technology, and explain why democracy is more "flexible" in dealing with the arrival of new technologies.

EXERCISE 23.13. This exercise shows that entry barriers typically lead to multiple equilibrium wages in dynamic models. Consider the following two-period model. The production function is given by (23.3) and the distribution of entrepreneurial talent is given by a continuous cumulative density function $G(a)$. There is an entry cost into entrepreneurship equal to b at each date and each entrepreneur hires one worker (does not work as a worker himself). Total population is equal to 1.

(1) First, ignore the second period and characterize the equilibrium wage and determine which individuals will become entrepreneurs. Show that the equilibrium is unique.
(2) Now consider the second period and suppose that all agents discount the future at the rate β. Show that there are multiple equilibrium wages in the second period and as a result, multiple equilibrium wages in the initial period.
(3) Now suppose that a fraction ε of all agents die in the second period and are replaced by new agents. New agents have to pay the entry cost into entrepreneurship if they

want to become entrepreneurs. Suppose that their talent distribution is also given by $G(a)$. Characterize the equilibrium in this case and show that it is unique.

(4) Now consider the limiting equilibrium in part 3 with $\varepsilon \to 0$. Explain why this limit leads to a unique equilibrium while there are multiple equilibria at $\varepsilon = 0$.

EXERCISE 23.14. Prove Proposition 23.8.

EXERCISE 23.15. Consider an economy populated by λ rich agents who initially hold power, and $1 - \lambda$ poor agents who are excluded from power, with $\lambda < 1/2$. All agents are infinitely lived and discount the future at the rate β. Each rich agent has income θ/λ while each poor agent has income $(1-\theta)/(1-\lambda)$ where $\theta > \lambda$. The political system determines a linear tax rate, τ, the proceeds of which are redistributed lump-sum. Each agent can hide their money in an alternative non-taxable production technology, and in the process they lose a fraction ϕ of their income. There are no other costs of taxation. The poor can undertake a revolution, and if they do so, in all future periods, they obtain a fraction $\mu(t)$ of the total income of the society (i.e., an income of $\mu(t)/(1-\lambda)$ per poor agent). The poor can not revolt against a democracy. The rich lose everything and receive zero payoff after a revolution. At the beginning of every period, the rich can also decide to extend the franchise to the poor, and this is irreversible. If the franchise is extended, the poor decide the tax rate in all future periods.

(1) Define Markov perfect equilibria (MPE) of this game.
(2) First suppose that $\mu(t) = \mu^l$ at all times. Also assume that $0 < \mu^l < 1 - \theta$. Show that in the MPE, there will be no taxation when the rich are in power, and tax rate will be $\tau = \phi$ when the poor are in power. Show that in the MPE, there is no extension of the franchise and no taxation.
(3) Suppose that $\mu^l \in (1-\theta, (1-\phi)(1-\theta) + \phi(1-\lambda))$. Characterize the MPE in this case. Why is the restriction $\mu^l < (1-\phi)(1-\theta) + \phi(1-\lambda)$ necessary?
(4) Now consider the non-Markovian equilibria of this game when $\mu^l > 1-\theta$. Construct an equilibrium where there is extension of the franchise along the equilibrium path. [Hint: first, to simplify, take $\beta \to 1$, and then consider a strategy profile where the rich are always expected to set $\tau = 0$ in the future; show that in this case the poor would undertake a revolution; for bonus points, also explain why the continuation strategy of $\tau = 0$ by the rich in all future periods could be part of a subgame perfect equilibrium path]. Why is there extension of the franchise now? Can you construct a similar non-Markovian equilibrium when $\mu^l < 1 - \theta$?
(5) Explain why the MPE led to different predictions than the non-Markovian equilibria. Which one is more satisfactory?

(6) Now suppose that $\mu(t) = \mu^l$ with probability $1-q$, and $\mu(t) = \mu^h$ with probability q, where $\mu^h > 1-\theta > \mu^l$. Construct a MPE where the rich extend the franchise, and from there on, a poor agent sets that tax rate. Be specific on the parameter values that are necessary for such an equilibrium to exist. Explain why extension of the franchise is useful for rich agents?

(7) Now consider non-Markovian equilibria again. Suppose that the unique MPE has franchise extension. Can you construct a subgame perfect equilibrium, as $\beta \to 1$, where there is no franchise extension?

(8) Contrast the role of restricting strategies to be Markovian in the two cases above [Hint: why is this restriction ruling out franchise extension in the first case, while ensuring that franchise extension is the unique equilibrium in the second?].

EXERCISE 23.16. (1) Show that if $L \geq \bar{L}$, then

$$\bar{L} \equiv \max\left\{\frac{\phi_D^C}{\phi_D^E}, \frac{\phi_N^C}{\phi_N^E}\right\} \frac{(1-\lambda(1-\delta))}{(1-\beta)(1-\lambda)(1-\delta)} M \in (0,\infty). \tag{23.70}$$

Then any MPE involves $\theta_D^C = \theta_N^C = 0$.

(2) Given the result in part 1, prove Proposition 23.9.

EXERCISE 23.17. *

(1) Generalize Proposition 23.10 to non-symmetric Markovian strategies.

(2) Now consider any subgame perfect equilibria. Show that for β sufficiently high, subgame perfect equilibria that are on the Pareto frontier for the elite (meaning that it is impossible to make one elite agent better off without making another one worse off), have the same qualitative features as the equilibrium in Proposition 23.10.

EXERCISE 23.18. Prove Proposition 23.12.

EXERCISE 23.19. Prove Proposition 23.13.

EXERCISE 23.20. Generalize the results on the effects of ΔR, β and M on equilibrium objects in Proposition 23.13 to the case where $\phi_N^E > \phi_D^E$.

CHAPTER 24

Epilogue: Mechanics and Causes of Economic Growth

This chapter contains concluding remarks. Instead of summarizing the models and ideas presented so far, I will end with a brief discussion of what we have learned from the models and analyses presented in this book and how they offer a useful perspective on world growth and cross-country income differences. I will then provide a very quick overview of some of the many remaining questions, which are important to emphasize both as a measure of our ignorance and as potential topics for future research.

24.1. What Have We Learned?

It may be useful to first summarize the most important aspects and takeaway lessons of our analysis. My summary would be as follows.

(1) *Growth as the source of current income differences*: at an empirical level, the investigation of economic growth is important not only for understanding the growth process, but also because the analysis of the sources of cross-country income differences today requires us to understand why some countries have grown rapidly over the past 200 years while others have not (Chapter 1).

(2) *The role of physical capital, human capital, and technology*: cross-country differences in economic performance and growth over time are related to physical capital, human capital and technology. Part of our analysis has focused on theoretical and empirical investigation of the contributions of these factors to production and growth (Chapters 2 and 3), and the remaining, larger part, has focused on understanding physical capital accumulation, human capital accumulation, and technology creation and adoption decisions (Chapters 8-15 and 18-19). One conclusion that has emerged concerns the importance of technology in understanding both cross-country and over-time differences in economic performance. Here, technology refers both to advances in techniques of production, thus to the accumulation of knowledge and blueprints for more efficient machinery, and also to the general efficiency of the organization of production, which will be affected by the incentives that a society (and its government) provides to firms and workers, by its contracting institutions, and by the types of market failures that prevent the development of more productive economic relationships (Section 18.5 in Chapter 18 and Chapter 21).

(3) *Endogenous investment decisions*: while we can make considerable progress by understanding the role of physical capital and human capital, and using cross-country data on differences in investments in machinery and in education to account for the process of economic growth and development, we also need to endogenize these investment decisions. Investments in physical and human capital are forward-looking and depend on the rewards that individuals expect from their investments. Understanding differences in these investments is therefore intimately linked to understanding how "reward structures"—that is, the pecuniary and nonpecuniary rewards and incentives for different activities—differ across societies and how individuals respond to differences in reward structures (Chapters 8 and 10).

(4) *Endogenous technology:* likewise, technology should be thought of as endogenous, not as mana from heaven. There are good empirical and theoretical reasons for thinking that new technologies are created by profit-seeking individuals and firms, via research, development and tinkering. In addition, decisions to adopt new technologies are likely to be highly responsive to profit incentives. Since technology appears to be a prime driver of economic growth over time and a major factor in cross-country differences in economic performance, we must understand how technology responds to factor endowments, market structures, and rewards. Developing a conceptual framework that emphasizes the endogeneity of technology has been one of the major objectives of this book. The modeling of endogenous technology necessitates ideas and tools that are somewhat different from those involved in the modeling of physical and human capital investments. Three factors are particularly important. First, fixed costs of creating new technologies combined with the nonrival nature of technology necessitates the use of models where innovators have ex post (after innovation) monopoly power. The same might apply, though perhaps to a lesser degree, to firms that adopt new technologies. The presence of monopoly power changes the welfare properties of decentralized equilibria and creates a range of new interactions and externalities (Chapters 12 and 13 and Section 21.5 in Chapter 21). Second, the process of innovation is implicitly one of competition and creative destruction. The modeling of endogenous technology necessitates more detailed models of the industrial organization of innovation and R&D, and how this impacts competition among firms and how new firms may or may not be able to replace incumbents. These models shed light on the impact of market structure, competition, regulation and intellectual property rights protection on innovation and technology adoption (Chapters 12 and 14). Third, endogenous technology implies that not only the aggregate rate of technological change but also the types of

technologies that are developed will be responsive to rewards. Key factors influencing the types of technologies that societies develop are again reward structures and factor endowments. For example, changes in relative supplies of different factors are likely to effect which types of technologies will be developed and adopted (Chapter 15).

(5) *Linkages across societies and balanced growth at the world level*: while endogenous technology and endogenous growth are major ingredients in our thinking about the process of economic growth in general and the history of world economic growth in particular, it is also important to recognize that most economies do not invent their own technologies, but adopt them from the world technology frontier or adapt them from existing technologies (Chapter 18). In fact, the process of technology transfer across nations might be one of the reasons why after the initial phase of industrialization, countries that have been part of the global economy have grown at broadly similar rates (Chapter 1). Therefore, the modeling of cross-country income differences and the process of economic growth for a large part of the world necessitates a detailed analysis of technology adoption, technology diffusion and technology transfer. Two topics deserve special attention in thinking about technological linkages across countries and technology adoption decisions. The first is the contracting institutions supporting contracts between upstream and downstream firms, between firms and workers, and between firms and financial institutions. These institutional arrangements will affect the amount of investment, the selection of entrepreneurs and firms, and also the efficiency with which different tasks are allocated across firms and workers. There are marked differences in contracting institutions across societies and these differences appear to be a major factor influencing technology adoption and diffusion in the world economy. Contracting institutions not only have a direct effect on technology and prosperity, but also shape the internal organization of firms, which contributes to the efficiency of production and influences how innovative firms will be (Section 18.5 in Chapter 18). The second is international trading relationships. International trade not only generates static gains familiar to economists, but also influences the innovation and growth process. The international division of labor and the product cycle are examples of how international trading relationships help the process of technology diffusion and enable a more productive specialization of production (Chapter 19).

(6) *Takeoffs and failures*: the last 200 years of world economic growth stand in stark contrast to the thousands of years before. Despite intermittent growth in certain parts of the world during certain epochs, the world economy was largely stagnant until the end of the 18th century. This stagnation had multiple aspects. These

included low productivity, high volatility in aggregate and individual outcomes, a Malthusian-type configuration where increases in output were often accompanied by increases in population does only having a limited effect on per capita income, and a largely rural and agricultural economy. Another major aspect of stagnation has been the failed growth attempts; many societies grew for certain periods of time and then lapsed back into depressions and stagnation. This changed in the 18th century. We owe our prosperity today to the takeoff in economic activity, closely related to industrialization, that started in Britain and Western Europe, and spread to certain other parts of the world, most notably to West European offshoots, such as the United States and Canada. Therefore, the nations that are rich today are precisely those where this process of takeoff originated or else those that were able to rapidly adopt and build upon the technologies underlying this takeoff (Chapter 1). Understanding current income differences across countries therefore necessitates understanding why some countries failed to take advantage of the new technologies and production opportunities.

(7) *Structural changes and transformations:* modern economic growth and development are accompanied by a set of sweeping structural changes and transformations. These include changes in the composition of production and consumption (the shift from agriculture to industry and from industry to services), urbanization, financial development, changes in inequality of income and inequality of opportunity, the transformation of social and living arrangements, changes in the internal organization of firms, and the demographic transition. While the process of economic development is multifaceted, much of its essence lies in the structural transformation of the economy and society at large (Section 17.6 in Chapter 17, Chapters 20 and 21). Many of these transformations are interesting to study for their own sake. Moreover, they are also important ingredients for sustained growth. Lack of structural transformation is not only a symptom of stagnation, but often also one of its causes. Societies may fail to takeoff and benefit from the available technology and investment opportunities in the world partly because they have not managed to undergo the requisite structural transformations, so they do not have the right type of financial relations, the appropriate skills, or the types of firms that would be necessary as conduits of new technologies.

(8) *Policy, institutions and political economy*: the reward structures faced by firms and individuals play a central role in shaping whether they undertake the investments in new technology and in human capital necessary for takeoff, industrialization, and economic growth. These reward structures are determined by policies and institutions. Policies and institutions also directly affect whether a society can embark

upon modern economic growth for a variety of interrelated reasons (Chapter 4). First, they directly determine the society's reward structure, thus shaping whether investments in physical and human capital and technological innovations are profitable. Second, they determine whether the infrastructure investments and contracting arrangements necessary for modern economic relations are present. For example, modern economic growth would be impossible in the absence of some degree of contract enforcement, the maintenance of law and order, and at least a minimum amount of investment in public infrastructure. Third, they influence and regulate the market structure, thus determining whether the forces of creative destruction will be operational and whether new and more efficient firms can replace less efficient incumbents. Finally, institutions and policies may sometimes (or perhaps often) go in the opposite direction and block the adoption and use of new technologies as a way of protecting politically powerful incumbent producers or to stabilize the established political regime. In this light, to understand why takeoff into sustained growth started 200 years ago and not before, why it started in some countries and not others, and why it was followed by some countries and not others, we need to understand the policy and institutional choices that societies make. This means we need to investigate the political economy of growth, paying special attention to which individuals and groups will be the winners from economic growth and which will be the losers. When losers cannot be compensated and have sufficient political power, we may expect the political economy equilibrium to lead to policies and institutions that are not growth enhancing. Our basic analysis of political economy generates insights about what types of distortionary policies may block growth, when these distortionary policies will be adopted, and how technology, market structure and factor endowments interact with the incentives of the social groups in power to encourage or discourage economic growth (Chapter 22).

(9) *Endogenous political institutions:* policies and institutions are central to understanding the growth process over time and cross-country differences in economic performance, but these social choices are in turn determined within society's political institutions. Democracies and dictatorships are likely to make different policy choices and create different types of reward structures. But political institutions themselves are not exogenous in the long-run equilibrium of a society. They are both determined in equilibrium and change along the equilibrium path as a result of their own dynamics and as a result of stimuli coming from changes in technology, trading opportunities, and factor endowments (Chapter 23). We have already seen some simple models that provide various useful insights, but much remains to be done. Towards a more complete understanding of world economic growth and the income

differences today, we therefore need to study the following: (1) how political institutions affect policies and economic institutions, thus shaping incentives for firms and workers; (2) how political institutions themselves change, especially interacting with economic outcomes and technology; (3) why political institutions and the associated economic institutions did not lead to sustained economic growth throughout history, why they enabled economic takeoff 200 years ago, and why in some countries, they blocked the adoption and use of superior technologies and derailed the process of economic growth.

In this brief summary, I focused on the ideas most relevant for thinking about the process of world economic growth and cross-country income differences we observe today. The reader will recall that the focus in the book has been not only on ideas, but also on careful mathematical modeling of these ideas and mechanisms so that coherent and rigorous theoretical approaches to these core issues can be developed. Nevertheless, to avoid repetition I will not mention the theoretical foundations of these ideas, which range from basic consumer, producer and general equilibrium theory to dynamic models of accumulation, models of monopolistic competition, and dynamic models of political economy. But I wish to emphasize again that a thorough study of the theoretical foundations of these ideas is necessary both to develop a satisfactory understanding of the main issues and also to find the best way of making them empirically operational.

24.2. A Possible Perspective on Growth and Stagnation over the Past 200 Years

The previous section gave a brief summary of the most important ideas highlighted in this book. I now discuss how some of these ideas might be useful in shedding light on the process of world economic growth and cross-country divergence that have motivated our investigation starting in Chapter 1. The central questions are these:

(1) Why did the world economy not experience sustained growth before 1800?
(2) Why did economic takeoff start in 1800 and in Western Europe?
(3) Why did some societies manage to benefit from the new technologies and organizational forms that emerged starting in 1800, while others steadfastly refused or failed to do so?

I will now offer a narrative that provides some tentative answers to these three questions. While certain parts of the mechanisms I will propose have been investigated econometrically and certain other parts receive support from historical evidence, the reader should view these as a first attempt at providing a coherent answers to these central questions. Two aspects of these answers are noteworthy. First, they build on the theoretical insights that the models in this book generate. Second, in the spirit of the discussion in Chapter 4, they link the proximate causes of economic phenomena to fundamental causes, and in particular to institutions. And here, I take a shortcut. Although I emphasized in Chapter 23 that there are no perfect political institutions and that each set of different political arrangements is likely to favor some groups at the expense of others, I will simplify the discussion here by making a core distinction between two sets of institutional arrangements. The first, which I will refer to as *authoritarian political systems*, encompasses absolutist monarchies, dictatorships, autocracies and various types of oligarchies that concentrate power in the hands of a small minority and pursue economic policies that are favorable to the interests of this minority. Authoritarian systems often rely on some amount of repression because they seek to maintain an unequal distribution of political power and economic benefits. They also adopt economic institutions and policies that protect incumbents and create rents for those who hold political power. The second set of institutions are *participatory regimes*. These regimes place constraints on rulers and politicians, thus preventing the most absolutist tendencies in political systems, and give voice to new economic interests, so that a strict decoupling between political and economic power is avoided. Such regimes include constitutional monarchies, where broader sections of the society take part in economic and political decision-making, and democracies where political participation is greater than in nondemocratic regimes. The distinguishing feature of participatory regimes is that they provide voice and (economic and political) security to a broader cross-section of society than authoritarian regimes. As a result, they are more open to entry by new businesses and provide a more level playing field and better security of property rights to a relatively broad section of the society. Thus in some ways, the contrast between authoritarian political systems and participatory regimes is related to the contrast between the growth-promoting cluster of institutions and the growth-blocking, extractive institutions emphasized and illustrate in Chapter 4. The reader should note that many different terms could have been used instead of “authoritarian” and “participatory,” and some details of the distinction may be arbitrary. More importantly, it should be borne in mind that even the most participatory regime will involve an unequal distribution of political power and those that have more political power can use the fiscal and political instruments of the state for their own benefits and for the detriment of the society at large (Chapter 23). Why this type of behavior is curtailed or limited sometimes is a question at the forefront of current research and I will not dwell on it here.

Armed with the economic models we have seen so far in the distinction between authoritarian and participatory regimes, we are now ready to discuss some tentative answers to the central questions outlined above.

Why did the world not experience sustained growth before 1800? While sustained growth is a recent phenomenon, growth and improvements in living standards certainly did occur many times in the past. The human history is also full of major technological breakthroughs. Even before the Neolithic Revolution, many technological innovations increased the productivity of hunter gatherers. The transition to farming after about 9000BC is perhaps the most major technological revolution of all times; it led to increased agricultural productivity and to the development of socially and politically more complex societies. Archaeologists have also documented various instances of economic growth in pre-modern periods. Historians estimate that consumption per-capita doubled during the great flowering of Ancient Greek society from 800BC to 50BC (Morris, 2005). Similar improvements in living standards were experienced by the Roman Republic and Empire after 400BC (Hopkins, 1980), and appear to have been experienced by pre-Columbian civilizations in South America, especially by the Olmec, the Maya, and even perhaps the Inca (Mann, 2004, Webster, 2003). Although data on these ancient growth experiences are limited, the available evidence suggests that the basic models we have seen, where growth relies on physical capital accumulation and some technological change, provide a good first description of the developments in these ancient economies (see, for example, Morris, 2005, on capital accumulation and limited technological change in Ancient Greece).

Importantly, however, these growth experiences were qualitatively different from those that the world experienced after its takeoff starting in the late 18th century. Four factors appear to have been particularly important and set these growth episodes apart from modern economic growth. The first is that they were relatively short-lived or took place at relatively slow pace.[1] In most cases, the initial spurt of growth soon crumbled for one reason or another, somehow reminiscent of the failed takeoffs in the model of Section 17.6 in Chapter 17. Secondly and relatedly, growth was never based on continues technological innovations, thus never resembled the technology-based growth emphasized in Chapters 13-15. Thirdly, in most cases economic institutions that would be necessary to support sustained growth did not develop. Financial relations were generally primitive, contracting institutions remained informal, markets were heavily regulated with various internal tariffs, and incomes and savings did not reach levels necessary for the mass market and for simultaneous investments in a range of activities. Put differently, the structural transformations accompanying development discussed in Chapter 21 did not take place. The final factor is arguably more important and is possibly the cause of the first three; all these episodes took place within the context of authoritarian political regimes. They were not broad-based growth experiences. Instead, this was elite-driven growth for the benefit of the elite and exploiting existing comparative

[1]For example, in Ancient Greece Morris (2005) estimates that income per capita doubled or at most tripled in the 500 years between 800 B.C. and 300 B.C., and this was largely caused by "catch-up" type growth starting from possibly unusually low levels in 800 B.C.

advantages. Thus it is not surprising that the improvements in living standards did not affect the entire society, but only a minority.

Why did these growth episodes not turn into a process of takeoff, ultimately leading to sustained growth? My main answer is related to that offered in Section 23.3 in Chapter 23. Growth under authoritarian regimes is possible. Entrepreneurs and workers can become better at what they do, achieve better division of labor, and improve the technologies they work with by tinkering and learning-by-doing. Moreover, those with political power and their allies do have the necessary security of property rights to undertake investments. And some technological breakthroughs can always happen by chance. Nevertheless, a distinguishing feature of growth under authoritarian institutions is that it will look after the interests of the current elite. This means that, in the final analysis, growth must always rely on existing techniques and production relationships. It will never unleash the process of creative destruction and the entry of new talent in new businesses necessary to carry a nation to the state of sustained growth. In addition, technological constraints may have also played a role. Although the progress of technological knowledge is not monotone (and useful production techniques are sometimes forgotten), the technological know-how available to potential entrepreneurs at the end of the 18th century was undoubtedly greater than that available to potential entrepreneurs in Rome or Ancient Greece. It may also be the case that sustained growth with technologically innovativeness and creative destruction necessitates some critical level of technological know-how may be necessary. For example, one may argue that the relatively rapid growth in the 19th century required skilled workers and it would have been prohibitively costly for a critical mass of workers to acquire the necessary human capital before the printing press was invented.

Let me focus on the political economy aspect and provide a few examples may be useful to illustrate the main limits to growth under authoritarian regimes. The Chinese Empire has been technologically innovative during many distinct phases of its history. Productivity in the Chinese economy, especially in the Yangtze Delta and other fertile lands, has been high enough to support a high density of population. But the Chinese economy never came close to sustained growth. Authoritarian political institutions regulated economic activity tightly for most of Chinese history. The society has typically been hierarchical, with a clear distinction between the elite and the masses. This system did not allow free entry into business by new entrepreneurs that would adopt and exploit new technologies and unleash the powers of creative destruction. When prospects for economic growth conflicted with political stability, the elite always opted for maintaining stability, even if this came at the expense of potential economic growth. Thus China tightly controlled overseas and internal trade, did not develop the property rights and contracting institutions necessary for modern economic growth, and did not allow an autonomous middle-class to emerge as an economic and political force (Elvin, 1973, Mokyr, 1991, Wong, 1997).

The Ancient Greek and Roman civilizations are often viewed as the first democratic societies. One might therefore be tempted to count them as participatory regimes that should have achieved sustained economic growth. But this is not necessarily the case. First, participatory regimes do not guarantee sustained economic growth when other preconditions have not been met. But more importantly, these societies were democratic only in comparison to others at the time. Both societies were representative only for a small fraction of the population. Production relied on slavery and coercion. Moreover, despite certain democratic practices, there was a clear distinction between a small elite, which monopolized economic and political power, and the masses, which consisted of both free plebs and slaves. Economic growth in both Greece and Rome did not rely on continuous innovation. Both societies managed to achieve high levels of productivity in agriculture, but without changing the organization of production in a radical manner. Both societies benefited from their military superiority for a while, and challenges to their military power were also important factors in their decline.

The Ottoman Empire provides another example of a society that was successful for an extended period of time but never transitioned into sustained growth. The Ottoman Empire, especially during the 14th, 15th and 16th centuries, achieved relative prosperity and great military strength. Agricultural productivity was high in many parts of the empire and military tribute contributed to state coffers and generated revenues to be distributed to parts of the population. But the state elite, controlling decision-making within the empire, never encouraged broad-based economic growth. There was no private property in land, trade was encouraged as long as it was consistent with the state's objectives, but always tightly controlled, and any new technology that could destabilize the power of the state was steadfastly blocked. Like China, Greece and Rome, the Ottoman growth first tapered off and then turned into decline (Pamuk, 2007).

The final example I will mention is the Spanish monarchy. By the beginning of the 16th century, the Spanish crown had achieved both political control of its own lands under Ferdinand and Isabella and control of the large overseas empire through its colonial enterprises. Many parts of greater Spain, including the South that was recently reconquered from the Moors and the lands of Aragon, were already prosperous in the 15th century. The whole of Spain became much wealthier with the transfer of gold, silver and other resources from the colonies in the 16th century. But this wealth did not translate into sustained growth. The colonial experiment was managed under a highly authoritarian regime set up by Ferdinand and Isabella, and the most lucrative businesses were allocated to the allies of the crown. The greater revenues generated from the colonies only helped to tighten the grip of the crown on the rest of the society and the economy. Instead of abating, absolutism increased. Trade and industry remained highly regulated, and groups not directly allied to the crown were viewed suspiciously and discriminated against. The most extreme example of this, the persecution of Jews that had started under the Inquisition, continued and spilled over to other independent merchants. Spain enjoyed the transfer of wealth from the colonies, but then experience a very lengthy of stagnation, with economic and political decline (Elliott, 1963).

It is also remarkable that in none of these cases did complementary economic institutions develop. Financial institutions were always rudimentary. The Roman Republic developed a precursor to the modern corporation and allowed some contracts between free citizens, but by and large, economic prosperity was built on traditional economic activities that did not necessitate complex relationships among producers and between firms and workers. Consequently, the structural transformations that accompany economic growth never took place in these societies. Life was largely rural, social relations were dominated by the state and community enforcement, and financial markets were rudimentary or nonexistent. Perhaps more important, there was little investment in human capital, except for the elite for whom education was often not a means towards higher productivity.

I suspect that these patterns were not coincidences. Economic life under authoritarian political regimes must have many of these features. Growth relying on practices that increase the productivity of the elite in traditional activities can secure growth for a while. But it will never engender creative destruction. Growth will go hand-in-hand with the political domination of the elite and thus with entry barriers protecting the status and the power of the elite. Therefore, the answer to the question of why not before 1800 is twofold. First, no society before 1800 invested in human capital, allowed new firms to bring new technology, and generally unleashed the powers of creative destruction. This might have been partly due to the difficulty of undertaking broad-based human capital investments in societies without the printing press and with only limited communication technologies. But it was also related to the reward structures for workers and firms. An important consequence of this pattern of growth is that no society experienced the sweeping structural transformations that are an essential part of modern economic growth (recall Chapter 21). Second, no society achieved these crucial steps toward sustained growth because all these societies lived under authoritarian political regimes.

Why did economic takeoff start in 1800 and in Western Europe? Before developing an answer to this question, let me take a slight digression. Faced with the patterns documented so far, one can adopt one of two distinct approaches. Either we can think that stagnation is the usual state of mankind and that something quite unusual, perhaps unlikely, needs to happen in order to break the cycle and lead to economic takeoff (Brenner, 1966).

If this perspective is correct, there is no reason to expect that a similar takeoff would happen in other societies unless they were subject to similar, and similarly unusual, shocks or some other process of intervention or change that induced them to undergo similar changes. Alternatively, one might suppose that the impetus for growth is ever present and is kept in check by certain non-growth-enhancing institutions or market failures (Jones, 1988). Once the growth process starts, it is likely to continue and spread. Then the question is pinpointing what the growth-blocking institutions and market failures are. My perspective is a mixture of these two views.

The division of labor emphasized by Adam Smith and capital accumulation always present growth opportunities to societies. Furthermore, human ingenuity is strong enough to create room for major technological breakthroughs in almost any environment. Thus, consistent with the second perspective, there is always a growth impetus in human societies. Nevertheless, this growth impetus may only be *latent* because it lives within the context of a set of political (economic) institutions. When these institutions are not encouraging growth—when they do not provide the right kind of reward structure, punish rather than reward innovators and investors—we do not expect the growth impetus to lead to sustained growth. Even in such environments, economic growth is possible and this is why China, Greece, Rome and other empires experienced growth for part of their history. But this prosperity did not exploit the full growth impetus. In fact, such prosperity relied on political regimes that, by their nature, had to control the growth impetus, because the growth impetus would ultimately bring these same regimes down. Therefore, growth under authoritarian political institutions must be growth despite the reward structures, not because of them.

West European growth starting in the late 18th century was different because Western Europe underwent three important structural transformations starting in the late Middle Ages. These structural transformations created an environment in which the latent growth impetus could turn into an engine of sustained growth.

The first was the collapse of one of the pillars of the ancient regime, with the decline of feudal relations in Western Europe. Starting in the 13th century and especially after the Black Death during the mid-14th century, the feudal economic relations crumbled in many parts of Western Europe. Serfs were freed from their feudal dues either by default (because the relationship collapsed) or by fleeing to the already expanding city centers (Postan, 1966). This heralded the beginning of an important social transformation; urbanization and changes in social relations. But perhaps more importantly, it created a labor force ready to work at cheap wages in industrial and commercial activities. It also removed one of the greatest sources of conflict between existing elites and new entrepreneurs—competition in the labor market (recall from the analysis in Chapter 22 how competition in factor markets can be the source of a range of distortionary policies). The decline of the feudal order also weakened the power base of the European authoritarian regimes that were largely unchallenged until the end of the Middle Ages (Pirenne, 1937).

The second structural transformation was related. With the decline in population, real incomes increased in much of Europe, and many cities created sufficiently large markets for merchants to seek new imports and for industrialists to seek new products (recall the impact of a decline in population on income per capita in the Solow or the neoclassical growth model, Chapters 2 and 8, or in the Malthusian model of Section 21.2 in Chapter 21, and the evidence in Chapter 4; recall also the importance of aggregate demand in jumpstarting industrialization emphasized in Section 21.5 in Chapter 21). During the Middle Ages, a range of important technologies in metallurgy, armaments, agriculture and basic industry, such as textiles, were already perfected (White, 1978, Mokyr, 1991, 2002). Thus the European economy had reached a certain level of technological maturity and perhaps created a platform for entrepreneurial activity in a range of areas and sufficient income levels to generate investment in physical capital and technology necessary for new production relations.

The more important change, however, was the political one. The late Middle Ages also witnessed the start of a political process that inexorably led to the collapse of absolutist monarchies and to the rise of constitutional regimes. The constitutional regimes that emerged in the 16th and 17th centuries in Western Europe were the first examples of participatory regimes, because they shifted political power to a large group of individuals that were previously "outsiders" to political power, including the gentry, small merchants, proto-industrialists as well as overseas traders and financiers (Section 23.3 in Chapter 23). These regimes then provided secure property rights and growth-enhancing institutions for a broad cross-section of society. These institutional changes created the environment necessary for new investments and technological changes and the beginning of sustained growth, which would culminate in the Commercial Revolution in the Netherlands and Britain during the 17th century and in the British Industrial Revolution at the end of the 18th century. By 19th century, industry and commerce had spread to much of Western Europe (North and Thomas, 1973, Chapter 4).

It is noteworthy to emphasize that constitutional monarchies were not democracies as we understand them today. There was no one-person one-vote and the distinction between the rich and the poor was quite palpable. Nevertheless, these regimes where responses to the demands by the merchants, industrialists, and those with the resources who wished to participate in economic activity. More importantly, these constitutional regimes not only reformed the political institutions of Western Europe, but undertook a series of economic reforms facilitating modern capitalist growth. Internal tariffs and regulations were lifted. Entry into domestic businesses and foreign trade was greatly facilitated. With the founding of the Bank of England and other financial reforms, the process of financial development got underway.

These constitutional regimes that emerged, first in Britain and the Netherlands, then in France and other parts of Western Europe, paved the way for sustained economic growth based on property rights for a broad cross-section of society, investment in contract enforcement, law and order, and infrastructure, and free entry into existing and new business lines. According to the theoretical perspective developed in earlier chapters, these improved conditions should have led to greater investments in physical capital, human capital and technology (Chapters 4 and 22). This is indeed what happened and the process of modern economic growth, unprecedented for the world economy, started. Economic relations now relied on new businesses investing in industry and commerce and on the formation of complex organizational form and production relations. Growth did not immediately accelerate. Economic growth was present but modest during the 17th and 18th centuries (Maddison, 2001). But these institutional changes laid the foundations of the more rapid growth that was soon to come. Financial institutions developed, the urban areas expanded further, new technologies were invented, and markets became the primary arena for transactions and competition (North and Thomas, 1973). By 1800, the process of technological change and investment had progressed enough that many historians now viewed this as the beginning of the Industrial Revolution (Ashton, 1968, Mokyr, 1989). The first phase of the Industrial Revolution was followed by the production of yet newer technologies, more complex organizations, greater reliance on skills and human capital in the production process and increasing globalization of the world economy. By the second half of the 19th century, Western Europe had reached growth levels that were unprecedented.

Naturally, a complete answer to the question above requires an explanation for why the constitutional regimes that were so important for modern economic growth emerged in Western Europe starting in the late 16th and 17th centuries. These institutions had their roots in the late medieval aristocratic parliaments in Europe, but more importantly, they were the outcome of radical reform resulting from the change in the political balance of power in Europe starting in the 16th century (Ertman, 1997). The 16th century witnessed a major economic transformation of Europe following the increase in international trade due to the discovery of the New World and the rounding of the Cape of Good Hope (Davis, 1973, Acemoglu, Johnson and Robinson, 2005a). Together with increased overseas trade came greater commercial activity within Europe. These changes led to a modest increase in living standards and more importantly, increased economic and political power of a new group of merchants, traders and industrialists. These new men were not the traditional allies of the European monarchies. They therefore demanded, and often were powerful enough to obtain, changes in political institutions that provided them with greater security of property rights and state action to help them in their economic endeavors. By this time, with the collapse of the feudal order, the foundations of the authoritarian regimes that were in place in the the Middle Ages were already weak. Nevertheless, the changes leading to the constitutional regimes did not come easy. The Dutch had to fight the Hapsburg monarchy to gain their independence as a republic. Britain had to endure the Civil War and the Glorious Revolution. France had to go through the Revolution of 1789 (Acemoglu, Johnson and Robinson, 2005a). But in all cases, the ancien régime gave way to more representative institutions, with greater checks on absolute power and greater participation by merchants, industrialists and entrepreneurs. It was important that the social changes led to a new set of political institutions, not simply to concessions. This is related to the theoretical ideas emphasized in Chapter 23; the nascent groups demanded long-term guarantees for the protection of their property rights and their participation in economic life, and this was most easily delivered by changes in political institutions, not by short-term concessions.

These changes created the set of political institutions that would then enable the creation of the economic institutions mentioned above. The collapse of the authoritarian political regimes and the rise of the first participatory regimes then opened the way for modern economic growth.

Why did some societies manage to benefit from the new technologies while others failed? The economic takeoff started in Western Europe, but quickly spread to certain other parts of the world. The chief importer of economic institutions and economic growth was the United States. The United States already had participatory political institutions, founded by settler colonists, who had just defeated the British crown to gain their independence and set up a smallholder society. This was a society built by the people who would live in it, and they were particularly keen on creating checks and balances to prevent a strong political or economic elite. This environment turned out to be a perfect conduit for modern economic growth. The lack of a strong political and economic elite meant that a broad cross-section of society could take part in economic activity, import technologies from Western Europe and then build their own technologies to quickly become the major industrial power in the world (Galenson, 1996, Engerman and Sokoloff, 1997, Keyssar, 2000, Acemoglu, Johnson and Robinson, 2002). In the context of this example, the importance of technology adoption from the world technology frontier is in line with the emphasis in Chapter 18, while the growth-promoting effects of a lack of elite creating entry barriers is consistent with the approach in Section 23.3 in Chapter 23.

Similar processes took place in other West European offshoots, for example, in Canada. Yet in other parts of the world, adoption of new technologies and the process of economic growth came as part of a movement towards defensive modernization. Japan started its economic and political modernization with the Meiji restoration (or perhaps even before) and a central element of this modernization effort was the importation of new technologies.

But these attitudes to new technologies were by no means universal. New technologies were not adopted, they were in fact resisted, in many parts of the world. This included most of Eastern Europe, for example Russia and Austria-Hungary, where the existing land-based elites saw the technologies as a threat both to their economic interests, because they would lead to the end of the feudal relations that still continued in this part of Europe, and to their political interests, which relied on limiting the power of new merchants and slowing down the process of peasants migrating to cities to become the new working class (see Freudenberger, 1967, and Mosse, 1992, for evidence and Chapter 22 for a theoretical perspective). Similarly, the previously-prosperous plantation economies in the Caribbean had no interest in introducing new technologies and allowing free entry by entrepreneurs. These societies continued to rely on their agricultural staples; industrialization, competition in free labor markets and workers investing in their human capital were seen as potential threats to the economic and political powers of the elite. The newly independent nations in Latin America were also dominated by a political elite, which continued the tradition of the colonial elites and showed little interest in industrialization. Much of South East Asia, the Indian subcontinent, and almost all of sub-Saharan Africa were still West European colonies, and were run under authoritarian and repressive regimes (often as producers of raw materials for the rapidly industrializing West European nations or as sources of tribute income). Free labor markets, factor mobility, entrepreneurial spirit, creative destruction and new technologies did not feature in the colonial political trajectories of these countries (Chapter 4).

Thus the 19th century, the age of industrialization, was only to see the industrialization of a few select places. Modern economic growth did not start in much of the world until early 20th century. By the 20th century, however, more and more nations that started importing some of the technologies that had been developed and used in Western Europe. The process of technology transfer pulling all of the countries integrated in the global economy towards greater income levels, as emphasized and studied in Chapter 19, thus started by the 20th century, but still not for all nations. Many more had to wait for their independence from their colonial masters, and even then, the end of colonialism led to a period of instability and infighting among would-be elites. Once some degree of political stability was achieved and economic institutions that encourage growth were put in place, growth started. For example, growth in Australia and New Zealand was followed by Hong Kong, then by South Korea, then by the rest of South Asia and finally by India. In each of these cases, as emphasized in Chapters 20 and 21, growth went hand-in-hand with structural transformations. Once the structural transformations were under way, they facilitated further growth. Consistent with the picture in Chapter 19, societies integrated into the global economy started importing technologies and achieved average growth rates in line with the growth of the world technology frontier (and often exceeding those during their initial phase of catch-up). In most cases, this meant growth for the new members of the global economy, but not necessarily the disappearance of the income gap between these new members and the earlier industrializers.

In the meantime, many parts of the world continued to suffer political instability discouraging investment in capital and new technology, or even exhibited overt hostility to new technologies. These included parts of sub-Saharan Africa and until recently much of Central America. Returning to some of the examples discussed in Chapter 1, Nigeria and Guatemala, for example, failed to create incentives for its entrepreneurs or workers both during its colonial period and after independence. Both of these countries also experienced significant political instability and economically disastrous civil wars in the postwar era (recall the discussion of the implications of political instability in Chapter 23). Brazil managed to achieve some degree of growth, but this was mostly based on investment by large, heavily protected corporations and not on a sustained process of technological change and creative destruction (thus more similar to the oligarchic growth in terms of the model of Section 23.3 in Chapter 23). In these cases and others, policies that failed to provide secure property rights to new entrepreneurs and those that blocked the adoption of new technologies, as well as political instability and infighting among the elites, seem to have played an important role in the failure to join the world economy and its growth process. Overall, these areas fell behind the world average in the 19th century and continued to do so for most of the 20th century. Many nations in sub-Saharan Africa, such as Congo, Liberia, Sudan and Zimbabwe, are still amidst political turmoil and fail to offer even the most basic security of property rights to their entrepreneurs. Consequently, many are still falling further behind the world average.

24.3. Many Remaining Questions

The previous section provided a narrative emphasizing how technological changes transformed the world economy starting in the 18th century and how certain societies took the advantage of these changes while others failed to do so. Parts of the story receive support from the data. The importance of industrialization in the initial takeoff is now well documented. There is broad consensus that economic institutions protecting property rights and allowing for free entry and introduction of new technologies were important in the 19th century and continue to be important today in securing economic growth (see Chapter 4). There is also general consensus that political instability, weak property rights, and lack of infrastructure are major impediments to growth in sub-Saharan Africa. Nevertheless, the narrative above is speculative. These factors might be important, but they do not need to be the main ones explaining the evolution of the world income distribution over the past 200 years. Moreover, as yet there is no consensus on the role of political institutions in this process.

Thus what I have presented above should be taken for what it is; a suggested answer that needs to be investigated more. But this is only one of the many remaining questions. The political economy of growth is important because it enables us to ask and answer questions about the fundamental causes of economic growth. But many other aspects of the process of growth require further investigation. In some sense, the field of economic growth is one of the more mature areas in economics, and certainly within macroeconomics it is the area where there is broadest agreement on what types of models are useful for the study of economic dynamics and for empirical analysis. And yet, there is so much that we still do not know.

I will now end by mentioning a few of the areas where the potential for more theoretical and empirical research is clear. First, while in this chapter I have largely focused on factors facilitating or preventing the adoption of technologies in less-developed nations, there is still much to be done to understand the pace at which technological progress happens at the frontier economies. Our models of endogenous technological change give us the basic framework for thinking about how profit incentives shape investments in new technologies. But much needs to be done to understand how market structure affects innovation. Chapter 12 highlighted how different market structures will create different incentives for technological change. We saw in Chapter 14 how competition among firms within an industry can be important for the growth process. But most of our understanding of these issues is at a qualitative level. For example, we lack a framework similar to that used for the analysis of the effects of capital and labor income taxes and indirect taxes in public finance, which we could used to analyze the effects of various regulations, of intellectual property right policies and of anticompetitive laws on innovation and economic growth. Since the pace at which the world economic frontier progresses has a direct effect on the growth of many nations, even small improvements in the environment for innovation in advanced economies could have important dividends for the rest of the world.

In addition to the industrial organization of innovation, the contractual structure of innovation needs further study. We live in a complex society, in which most firms are linked to others as suppliers or downstream customers, and most firms are connected to the rest of the economy indirectly through their relationship with the financial markets. All of these relationships are mediated by various explicit and implicit contracts. Similarly, the employment relationship that underlies the productivity of most firms relies on contractual relations between employers and employees. We know that moral hazard and holdup problems occur in all these contractual relationships. But how important are they for the process of economic growth? Can improvements in contracting institutions improve innovation and technological upgrading in frontier economies? Can they also facilitate technology transfer? These are basic, but as yet, unanswered questions. The contractual foundations of economic growth are still in their infancy and require much work.

The previous section emphasized how many economies started the growth process by importing technologies and thus integrating into the global economy. Today we live in an increasingly globalized and globalizing economy. But there is still much to understand about how technology is transferred from some firms to others, and from advanced economies to less-developed ones. The models I presented in Chapter 19 emphasized the importance of human capital, barriers to technology adoption, issues of appropriate technology, and contracting problems. Nevertheless, most of the models are still at the qualitative level and we lack a framework that can make quantitative predictions about the pace of technology diffusion. We have also not yet incorporated many important notions related to technology transfer into our basic frameworks. These include ideas related to tacit knowledge, the workings of the international division of labor, the role of trade secrecy, and the system of international intellectual property rights protection.

The reader will have also noticed that the material presented in Chapter 21 is much less unified and perhaps more speculative than the rest of the book. Although some of this reflects the fact that I had to simplify a variety of models to be able to present them in a limited space, much of it is because we are far from a unified framework for understanding the process of economic development and the structural transformations that it involves. We know that economic growth is accompanied by structural change. Some of the structural change can be viewed as a simple byproduct of economic growth, such as the increase in services relative to agriculture. But other structural transformations, including developments in financial markets, changes in contract enforcement regimes, urbanization, and the amount and composition of human capital investments are not simple byproducts of economic growth. They are intimately linked to the process of economic development. Moreover, it may be the case that lack of certain structural transformation might delay or prevent economic growth. To understand these questions, we require models with stronger theoretical foundations, a unified approach to these related issues, and a greater effort to link the models of economic development to the wealth of empirical evidence that the profession has now accumulated on economic behavior in less-developed economies.

Last but not least, given the narrative in the last section and the discussion in Chapters 4, 22 and 23, the reader will not be surprised that I think many important insights about economic growth lie in political economy. But understanding politics is in many ways harder than understanding economics, because political relations are even more multifaceted. Although I believe that the political economy and growth literatures have made important advances in this area over the past decade or so, much remains to be done. The political economy of growth is in its infancy and as we further investigate why societies make different choices, we will gain a better understanding of the process of economic growth.

Part 9

Mathematical Appendices

CHAPTER A

Odds and Ends in Real Analysis and Applications to Optimization

This chapter is included as a review of some basic material from real analysis. Its main purpose is to make the book self-contained and also include explicit statements of some of the theorems that are appealed to in the text. The material here is not meant to be a comprehensive treatment of real analysis. Space restrictions preclude me from attempting to do justice to any of the topics here, so my purpose is only a brief review. Accordingly, many results will be given without proof and other important results will be omitted as long as they are not referred to in the text or do not play an important role in the development or the proof of some of the other results presented here. I will state some useful results as *Fact* (often leaving their proof as an exercise). These results are typically used or referred to in the text, or are inputs into proving the more important results in this appendix. These more important results will be stated as *Theorem.*

It should be emphasized that the material here is not a substitute for a basic "*Mathematics for Economists*" type review or textbook. An excellent book of this sort is Simon and Blume (1994), and I will presume that the reader is familiar with most of the material in this or a similar book. In particular, I assume that the reader is comfortable with linear algebra, functions, relations, set theoretic language, calculus of multiple variables, and basic proof techniques.

To gain a deeper understanding and appreciation of the material here, the reader is encouraged to consult one of many excellent books on real analysis, functional analysis, and general topology. Some of the material here is simply a review of introductory real analysis more or less at the level of the classic books by Rudin (1976) or Apostol (1974). Some of the material, particularly those concerning topology and infinite-dimensional analysis, are more advanced and can be found in Conway (1990), Kelley (1955), Kolmogorov and Fomin (1970), Royden (1994), and Aliprantis and Border (1999). Excellent references for applications of these ideas to optimization problems include Luenberger (1969) and Berge (1963). A recent treatment of some of these topics with economic applications is presented in Ok (2007).

A.1. Distances and Metric Spaces

Throughout, X denotes a set and $x \in X$ is a generic element of the set X. A set X can be viewed as a space or as a subset of a larger set (space) Y. I denote a subset Y of X as $Y \subset X$ (which includes the case where $Y = X$). For any $Y \subset X$, $X \backslash Y$ stands for the *complement* of Y in X, i.e., $X \backslash Y = \{x : x \in X \text{ and } x \notin Y\}$. Whenever I use the expression $X \backslash Y$ it is implicit that Y is a subset of X.[1]

Of special importance for our purposes here are two types of spaces: (1) finite-dimensional Euclidean spaces, which I will denote by $X \subset \mathbb{R}^K$, where K is an integer; (2) infinite-dimensional spaces, such as spaces of sequences or spaces of functions, which feature in discrete-time and continuous-time dynamic optimization problems. For our purposes, the most useful sets are those that are equipped with a *metric*, so that they can be treated as a *metric space*. Metric spaces play a major role in the analysis of dynamic programming problems in Chapters 6 and 16.

Definition A.1. *Let X be a nonempty set. A function $d : X \times X \to \mathbb{R}_+$ is a* ***metric*** *(distance function) if, for any x, y, and z in X, it satisfies the following three conditions:*

1. ***(Properness)*** *$d(x, y) = 0$ if and only if $x = y$.*
2. ***(Symmetry)*** *$d(x, y) = d(y, x)$.*
3. ***(Triangle Inequality)*** *$d(x, y) \leq d(x, z) + d(z, y)$.*

A nonempty set X equipped with a metric d constitutes a ***metric space*** *(X, d).*

The same set can be equipped with different metrics. This will sometimes facilitate the analysis, but in many cases, different metrics give equivalent results. In this case, we say that two metrics are *equivalent*, and the definition for this is given below in Definition A.4 (but I am mentioning this here, since I will refer to equivalent metrics in the next example).

Example A.1. The following are examples of metric spaces. In each case, properness and symmetry are easy to verify, but verifying that the triangle inequality holds requires some work (see Exercise A.2).

1. For any $X \subset \mathbb{R}^K$, let x_i be the ith component of $x \in X$. Then the usual *Euclidean distance* $d(x, y) = \left(\sum_{i=1}^K |x_i - y_i|^2\right)^{1/2}$ is a metric and thus the Euclidean space with its usual distance constitutes a metric space. It is typically referred to as the *K-dimensional Euclidean space*. Moreover, one can construct alternative metrics for Euclidean spaces that are *equivalent*, in the sense that which of these metrics one uses has no bearing on the topological properties or on any of the other issues we focus on here. These metrics include the family $d_p(x, y) = \left(\sum_{i=1}^K |x_i - y_i|^p\right)^{1/p}$ for

[1] Throughout this chapter, I will simplified the notation and use "=" for definitions instead of "≡".

$1 \leq p < \infty$. An extreme element of this family, which also defines an equivalent metric on finite dimensional Euclidean spaces, is $d_\infty(x,y) = \sup_i |x_i - y_i|$.

(2) For any nonempty set X, one can construct the *discrete metric* defined as $d(x,y) = 1$ if $x \neq y$ and $d(x,y) = 0$ if $x = y$. In this case (X,d) is a *discrete space*.

(3) Let $X \subset \mathbb{R}^K$ and consider the set of continuous real-valued functions $f : X \to \mathbb{R}$ denoted by $\mathbf{C}(X)$. A natural metric for $\mathbf{C}(X)$ is the *sup metric* $d_\infty(f,g) = \sup_{x \in X} |f(x) - g(x)|$. Thus $(\mathbf{C}(X), d_\infty)$ is a metric space. The same metric can be used for the set of bounded continuous functions, $\mathbf{B}(X)$, leading to the metric space $(\mathbf{B}(X), d_\infty)$ or for the set of bounded (but not necessarily continuous) functions, $\mathbf{B}^0(X)$, with the associated metric space $\left(\mathbf{B}^0(X), d_\infty\right)$.

(4) Let $\ell \subset \mathbb{R}^\infty$ be a set consisting of infinite sequences of real numbers. For example, $x = (x_0, x_1, x_2, ...)$ would be a typical element of ℓ provided that $x_i \in \mathbb{R}$ for each $i = 0, 1, 2, ...$ A family of metrics for this set is given by $d_p(x,y) = (\sum_{i=1}^{\infty} |x_i - y_i|^p)^{1/p}$ for $1 \leq p < \infty$ or by $d_\infty(x,y) = \sup_i |x_i - y_i|$. For any $p \in [1, \infty]$, (ℓ, d_p) is a metric space, sometimes denoted by ℓ_p.

Metric spaces are particularly useful because they enable us to define neighborhoods and open sets, which are the building blocks of mathematical analysis and essential elements for our investigation of optimization problems. Below, I will define notions of neighborhood and openness somewhat more generally, but it is useful to start from the following simpler definition.

Definition A.2. *Let (X,d) be a metric space and $\varepsilon > 0$ be scalar. Then for any $x \in X$,*

$$\mathcal{N}_\varepsilon(x) = \{y \in X \colon d(x,y) < \varepsilon\}$$

*is the ε-**neighborhood** of x.*

Example A.2. In the simplest case where $X \subset \mathbb{R}$ and $d(x,y) = |x - y|$, $\mathcal{N}_\varepsilon(x) = (x - \varepsilon, x + \varepsilon)$.

Definition A.3. *Let (X,d) be a metric space. Then $Y \subset X$ is **open** in X if for each $y \in Y$, there exists $\varepsilon > 0$ such that $\mathcal{N}_\varepsilon(y) \subset Y$. $Z \subset X$ is **closed** in X if $X \backslash Z$ is open in X.*

The *closure* of a set Y in X is $\overline{Y} = \{y \in X : \text{for each } \varepsilon > 0, \mathcal{N}_\varepsilon(y) \cap Y \neq \varnothing\}$, that is, every neighborhood of each point in $\overline{Y}$ contains at least one point of Y. Clearly, $Y \subset \overline{Y}$. Moreover, if Y is closed, then $Y = \overline{Y}$. The *interior* of a set Y in X can then be defined as $\text{Int}Y = Y \backslash \overline{(X \backslash Y)}$. Clearly if Y is open in Y, then $\overline{(X \backslash Y)} = X \backslash Y \cap Y = \varnothing$, and therefore $\text{Int}Y = Y$.

Example A.3. Again in the simplest case where $X = [0,1]$ and $d(x,y) = |x - y|$, for any $x \in (0,1)$ and $\varepsilon > 0$ sufficiently small, $(x - \varepsilon, x + \varepsilon)$ is open in X, whereas

$[0,1]\setminus(x-\varepsilon,x+\varepsilon) = [0,x-\varepsilon]\cup[x+\varepsilon,1]$ is closed in X. Also, $\text{Int}(x-\varepsilon,x+\varepsilon) = (x-\varepsilon,x+\varepsilon)$, $\text{Int}([0,1]\setminus(x-\varepsilon,x+\varepsilon)) = (0,x-\varepsilon)\cup(x+\varepsilon,1)$, $\overline{(x-\varepsilon,x+\varepsilon)} = [x-\varepsilon,x+\varepsilon]$, and $\overline{[0,1]\setminus(x-\varepsilon,x+\varepsilon)} = [0,x-\varepsilon]\cup[x+\varepsilon,1]$.

Fact A.1. *Let (X,d) be a metric space. X and $\varnothing$ are both open and closed sets.*

The importance of the following theorem will become clear once we turn to the somewhat more abstract topological characterization of closed and open sets. Let A be a set of real numbers (for example, $\mathbb{N}$). Then $\{X_\alpha\}_{\alpha\in A}$ is a *collection of sets*. If A is countable [finite], then $\{X_\alpha\}_{\alpha\in A}$ is a countable [finite] collection of sets, but it can also be an arbitrary collection of sets. In referring to collections of sets, I will take it as implicit that A is a set of real numbers. Let us also use X_α^c to denote the complement of X_α in X, i.e., $X_\alpha^c = X\backslash X_\alpha$.

Theorem A.1. ***(Properties of Open and Closed Sets)*** *Let (X,d) be a metric space and $\{X_\alpha\}_{\alpha\in A}$ be a collection of sets with $X_\alpha\subset X$ for all $\alpha\in A$.*

(1) *If each X_α is open in X, then $\bigcup_{\alpha\in A} X_\alpha$ is open.*
(2) *If each X_α is open in X and $\{X_\alpha\}_{\alpha\in A}$ is a finite collection of sets, then $\bigcap_{\alpha\in A} X_\alpha$ is open.*
(3) *If each X_α is closed in X, then $\bigcap_{\alpha\in A} X_\alpha$ is closed.*
(4) *If each X_α is closed in X and $\{X_\alpha\}_{\alpha\in A}$ is a finite collection of sets, then $\bigcup_{\alpha\in A} X_\alpha$ is closed.*

Proof. **(Part 1)** Let $\{X_\alpha\}_{\alpha\in A}$ be an arbitrary collection of open sets in X. If $\bigcup_{\alpha\in A} X_\alpha$ is empty, then it is open by Fact A.1. If it is nonempty, then for each $x\in\bigcup_{\alpha\in A} X_\alpha$, it must be the case that $x\in X_{\alpha'}$ for some $\alpha'\in A$. Since $X_{\alpha'}$ is open, there exists $\varepsilon>0$ such that $\mathcal{N}_\varepsilon(x)\subset X_{\alpha'}\subset\bigcup_{\alpha\in A} X_\alpha$, establishing that for each $x\in\bigcup_{\alpha\in A} X_\alpha$ there exists an ε-neighborhood of x in $\bigcup_{\alpha\in A} X_\alpha$ so that $\bigcup_{\alpha\in A} X_\alpha$ is open.

(Part 2) Let $\{X_\alpha\}_{\alpha\in A}$ be a finite collection of open sets in X (enumerated by $\alpha = 1,2,...,N$). Once again, if $\bigcap_{\alpha\in A} X_\alpha$ is empty, it is open by Fact A.1. If it is nonempty, then for each $x\in\bigcap_{\alpha\in A} X_\alpha$, we have that $x\in X_\alpha$ for $\alpha = 1,2,...,N$. Since X_α is open, then by definition there exists $\varepsilon_\alpha>0$ such that $\mathcal{N}_{\varepsilon_\alpha}(x)\subset X_\alpha$ for each $\alpha = 1,2,...,N$. Let $\varepsilon = \min\{\varepsilon_1,\varepsilon_2,...,\varepsilon_N\}$. Clearly, $\varepsilon>0$. Moreover, by construction, $\mathcal{N}_\varepsilon(x)\subset\mathcal{N}_{\varepsilon_\alpha}(x)\subset X_\alpha$ for each $\alpha = 1,2,...,N$. Therefore, $\mathcal{N}_\varepsilon(x)\subset\bigcap_{\alpha\in A} X_\alpha$, proving the claim.

(Parts 3 and 4) These follow immediately from De Morgan's Law that

$$\left(\bigcup_{\alpha\in A} X_\alpha\right)^c = \bigcap_{\alpha\in A} X_\alpha^c.$$

□

The restriction to finite collections is important in Part 2 of Theorem A.1. Consider, the following example.

Example A.4. $X_\alpha = \left(0, 1+\alpha^{-1}\right)$ and consider the infinite intersection $\bigcap_{\alpha\in\mathbb{N}} X_\alpha$. It can be verified that $\bigcap_{\alpha\in\mathbb{N}} X_\alpha = (0,1]$, which is not an open set.

Definition A.4. *Two metrics d and d' defined on sX are* ***equivalent*** *if they both generate the same collection of open sets in X. Alternatively, let $\mathcal{N}_\varepsilon$ and $\mathcal{N}'_\varepsilon$ refer to neighborhoods defined by these metrics. The two metrics are equivalent if for each $x \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ and $\delta' > 0$ such that $\mathcal{N}'_\varepsilon(x) \subset \mathcal{N}_\delta(x)$ and $\mathcal{N}_\varepsilon(x) \subset \mathcal{N}'_{\delta'}(x)$.*

Exercise A.4 verifies that the two parts of Definition A.4 imply each other.

Definition A.5. *Let (X,d) be a metric space. Then $Y \subset X$ is* ***bounded*** *if there exists $x \in X$ and $\delta \in (0,+\infty)$ such that $Y \subset \mathcal{N}_\delta(x)$. If $Y \subset X$ is not bounded, then it is* ***unbounded****.*

Example A.5. Let $X = \mathbb{R}$ and $d(x,y) = |x-y|$. The subsets $(0,1)$ and $[0,1]$ of $\mathbb{R}$ are bounded, while the subset $\mathbb{R}_+ = [0,\infty)$ of $\mathbb{R}$ is unbounded.

A.2. Mappings, Functions, Sequences, and Continuity

I refer to a "relation" ϕ that assigns an element of some set Y for an element of some set X as a *mapping* from X to Y, and denote this by $\phi : X \to Y$. Throughout $\phi : X \to Y$ implies that $\phi(x)$ is defined for each $x \in X$. I have also adopted the convention that ϕ assigns a single element of the set Y to $x \in X$ and thus write $\phi(x)$ as an element of Y (i.e., $\phi(x) \in Y$). This is without any loss of generality, since the space Y is quite general. For example, for a set Z, we could specify $Y = \mathcal{P}(Z)$ (where, recall that, $\mathcal{P}(Z)$ denotes the set of all subsets of Z). In this case, an element of Y would be a subset of Z. Thus, one can alternatively write that for $x \in X$, $\phi(x) \in Y$ or $\phi(x) \subset Z$. I will also use the notation $\phi(X')$ for some $X' \subset X$ to designate the *image* of the set X', defined as

$$\phi\left(X'\right) = \left\{y \in Y : \exists x \in X' \text{ with } \phi(x) = y\right\}.$$

For a mapping $\phi : X \to Y$, X is sometimes referred to as the *domain* of ϕ, while Y is its *range*. One might want to reserve the term range to $Y' \subset Y$ such that $Y' = \phi(X)$. For our purposes here, this distinction is not important.

The notation ϕ^{-1} is standard to denote the inverse of the mapping ϕ. Notice that ϕ^{-1} will not be single-valued even if ϕ a single valued, since more than one x in X can have the same image in Y. For $Y' \subset Y$, let

$$\phi^{-1}\left(Y'\right) = \left\{x \in X : \exists y \in Y' \text{ with } \phi(x) = y\right\}.$$

By a *function* f I typically refer to a *real-valued* mapping, i.e., $f : X \to \mathbb{R}$ for some arbitrary set X. I will use lowercase letters to refer to functions. I will use the term *correspondence* to refer to a set-valued mapping, i.e., $F : X \to \mathcal{P}(Z)$ for some set Z. This means that the mapping F assigns a subset of Z to each element of x. I will use uppercase letters to refer to correspondences. Since these will play an important role below, the following common notation will be used for correspondences: $F : X \rightrightarrows Z$. When the range of the correspondence is the real numbers, then we naturally have $F : X \rightrightarrows \mathbb{R}$.

Definition A.6. *Let (X, d) be a metric space. A* ***sequence,*** *denoted by $\{x_n\}_{n=1}^{\infty}$, is a mapping ϕ with domain given by the natural numbers, $\mathbb{N}$, and range given by X.*

Many sequences we will encounter will have ϕ replaced by a real-valued function, so that each $x_n \in \mathbb{R}$. The important point is that in all cases the domain is $\mathbb{N}$, so that $\{x_n\}_{n=1}^{\infty}$ is a countable (infinite) sequence. One can easily generalize the notion of a sequence to that of *nets*. We say that A is an *ordered set*, if there exists a transitive relation "$>$" such that for any distinct α, α' in A, we have either $\alpha > \alpha'$ or $\alpha' > \alpha$, but not both. The simplest example would be $A \subset \mathbb{R}$.

Definition A.7. *A* ***net****, denoted by $\{x_\alpha\}_{\alpha \in A}$ for some ordered A, is a real-valued function with domain given by A.*

Whenever sequences and nets have real numbers as elements, the underlying metric space relevant for the convergence is $(\mathbb{R}, d)$, with d referring to the usual Euclidean metric $d(x, y) = |x - y|$.

Example A.6. $\{x_n\}_{n=1}^{\infty}$ such that $x_n = 1/n$ for each $n \in \mathbb{N}$ is a sequence, while $\{x_\alpha\}_{\alpha \in A}$ $x_\alpha = 1/\alpha$ for each $\alpha \in (0, 1]$ is a net.

Definition A.8. *Consider the sequence $\{n_k\}_{k=1}^{\infty}$ of positive increasing integers (such that $n_k > n_{k'}$ whenever $k > k'$) and let N_K be a subset of these integers. Then for a given sequence $\{x_n\}_{n=1}^{\infty}$, $\{x_{n_k}\}_{n_k \in N_K}$ is a* ***subsequence*** *of $\{x_n\}_{n=1}^{\infty}$.*

A subnet can be defined in a similar manner.

Definition A.9. *Let (X, d) be a metric space. A sequence $\{x_n\}_{n=1}^{\infty}$ in X is* ***convergent*** *with* ***limit point*** *$x_\infty \in X$ if for every $\varepsilon > 0$, there exists an integer $N(\varepsilon)$ such that $n \geq N(\varepsilon)$ implies $d(x_n, x_\infty) < \varepsilon$. We write this as $\lim_{n\to\infty} x_n = \lim x_n = x_\infty$ or simply as $\{x_n\}_{n=1}^{\infty} \to x_\infty$.*

Definition A.10. *Let $\{x_\alpha\}_{\alpha \in A}$ be a net in a metric space (X, d). Then $\{x_\alpha\}_{\alpha \in A}$ is* ***convergent*** *with* ***limit point*** *x_∞ if for each neighborhood $\mathcal{N}_\varepsilon(x_\infty)$, there exists $\bar{\alpha}$ such that for all $\alpha \geq \bar{\alpha}$, $x_\alpha \in \mathcal{N}_\varepsilon(x_\infty)$.*

FACT A.2. *If a sequence* $\{x_n\}_{n=1}^{\infty}$ *or a net* $\{x_\alpha\}_{\alpha\in A}$ *in* X *is convergent, then it has a unique limit point* $x_\infty \in X$.

PROOF. See Exercise A.7. □

FACT A.3. $\{x_n\}_{n=1}^{\infty}$ *in* X *is convergent if and only if every subsequence of* $\{x_n\}_{n=1}^{\infty}$ *in* X *is convergent.*

PROOF. See Exercise A.8. □

EXAMPLE A.7. Note, however, that convergence of subsequences (or subnets) does not guarantee convergence of the original sequence. Consider the sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n = (-1)^n$. Clearly, this sequence is not convergent. But picking $\{n_k\}_{n=1}^{\infty}$ as the even integers, we construct a convergent subsequence $\{x_{n_k}\}_{n_k\in N_K}$ with limit point 1.

Let $\bar{\mathbb{R}}$ denote the *extended real numbers,* that is, $\bar{\mathbb{R}} = \mathbb{R}\cup\{-\infty\}\cup\{\infty\}$. It is straightforward to verify that $(\bar{\mathbb{R}}, \bar{d})$ is a metric space, where $\bar{d}(x,y) = d(x,y)/(1+d(x,y))$, with $d(x,y)$ denoting the standard Euclidean metric now allowed to take infinite values.

FACT A.4. *Let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence in* $\bar{\mathbb{R}}$ *(equipped with its usual metric). If* $\{x_n\}_{n=1}^{\infty}$ *is monotone (nondecreasing or nonincreasing), then it is convergent.*

PROOF. See Exercise A.9. □

DEFINITION A.11. *Let* $X \subset \mathbb{R}$. *Then the* ***supremum*** *of* X, *denoted by* $\sup X$, *is the smallest* $\bar{x} \in \bar{\mathbb{R}}$ *such that* $\bar{x} \geq x$ *for all* $x \in X$. *If there does not exist* $\bar{x} \in \mathbb{R}$ *for which this is true, then clearly* $\bar{x} = \infty$. *Similarly, the* ***infimum*** *of* X, *denoted by* $\inf X$, *is the greatest* $\underline{x}$ *such that* $\underline{x} \leq x$ *for all* $x \in X$, *where again* $\underline{x} = -\infty$ *is allowed. If* $\bar{x} = \sup X \in X$, *then we refer to* $\bar{x}$ *as the maximum of* X *and denote it by* $\bar{x} = \max X$. *Similarly, if* $\underline{x} = \inf X \in X$, *then* $\underline{x}$ *is the minimum of* X *and is denoted by* $\underline{x} = \min X$.

Since X here itself can be taken to be a sequence of numbers, supremum and infimum can be defined for sequences. In particular, for $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ construct the sequences $\{x'_n\}_{n=1}^{\infty}$ and $\{x''_n\}_{n=1}^{\infty}$ such that $x'_n = \sup_{k\geq n}\{x_k\}$ and $x''_n = \inf_{k\geq n}\{x_k\}$. Clearly, $\{x'_n\}_{n=1}^{\infty}$ is monotone (nonincreasing) and $\{x''_n\}_{n=1}^{\infty}$ is monotone (nondecreasing). Therefore, by Fact A.4 $\lim_{n\to\infty} x'_n$ exists and is denoted by $\limsup x_n$, and also $\lim_{n\to\infty} x''_n$ exists and is denoted by $\liminf x_n$. The following results are then immediate.

FACT A.5. *Let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence in* $\bar{\mathbb{R}}$. *Then:*

(1)

$$\inf_n x_n \leq \liminf x_n \leq \limsup x_n \leq \sup_n x_n.$$

(2) $\{x_n\}_{n=1}^{\infty}$ *is convergent if and only if*

$$\liminf x_n = \limsup x_n,$$

and in this case, we denote both of these as $\lim x_n = x_\infty$.

(3) *Let* $\{y_n\}_{n=1}^{\infty}$ *be another sequence in* $\bar{\mathbb{R}}$ *with* $x_n \leq y_n$ *for all* n. *Then*

$$\liminf x_n \leq \liminf y_n \text{ and } \limsup x_n \leq \limsup y_n,$$

and moreover, if the limits exist,

$$\lim x_n \leq \lim y_n.$$

PROOF. See Exercise A.11. □

DEFINITION A.12. *Let* (X, d) *be a metric space. A sequence* $\{x_n\}_{n=1}^{\infty}$ *in* X *is a* ***Cauchy sequence*** *if for each* $\varepsilon > 0$, *there exists an integer* $M(\varepsilon)$ *such that for any* $n, m \geq M(\varepsilon)$, $d(x_n, x_m) < \varepsilon$.

LEMMA A.1. *Let* (X, d) *be a metric space and* $\{x_n\}_{n=0}^{\infty}$ *be a convergent sequence in* X. *Then it is a Cauchy sequence.*

PROOF. Fix $\varepsilon > 0$. Since $\{x_n\}_{n=0}^{\infty}$ is convergent, $\lim_{n\to\infty} x_n = x_\infty$ exists. Then by the triangle inequality, for any x_n, x_m,

$$d(x_n, x_m) \leq d(x_n, x_\infty) + d(x_m, x_\infty).$$

Since $\lim_{n\to\infty} x_n = x_\infty$, by Definition A.9 there exists $M(\varepsilon)$ such that for any $n \geq M(\varepsilon)$, $d(x_n, x_\infty) < \varepsilon/2$. Combining this with the previous inequality implies that $d(x_n, x_m) < \varepsilon$, establishing the desired result. □

The converse of this lemma is not true, as illustrated by the following example.

EXAMPLE A.8. Let $X = (0, 1]$ and $d(x, y) = |x - y|$. Consider the sequence $x_n = 1/n$. This is clearly Cauchy, but does not converge to any point in X, and is thus not convergent.

DEFINITION A.13. *A metric space* (X, d) *is* ***complete*** *if every Cauchy sequence in* (X, d) *is convergent.*

The importance of complete metric spaces is illustrated by the Contraction Mapping Theorem, Theorem 6.7, which was presented in Section 6.3.

I now briefly discuss continuity of mappings and functions in metric spaces.

DEFINITION A.14. *Let* (X, d_X) *and* (Y, d_Y) *be metric spaces and consider the mapping* $\phi : X \to Y$. ϕ *is* ***continuous at*** $x \in X$ *if and only if for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that whenever* $d_X(x, x') < \delta$, *then* $d_Y(\phi(x), \phi(x')) < \varepsilon$. ϕ *is* ***continuous*** *on* X *if it is continuous at each* $x \in X$.

FACT A.6. *Equivalently, ϕ is continuous at x if for all $\{x_n\}_{n=0}^{\infty} \to x$, $\{\phi(x_n)\}_{n=0}^{\infty} \to \phi(x)$.*

FACT A.7. *Let (X, d_X), (Y, d_Y) and (Z, d_Z) be metric spaces and consider the mappings $\phi : X \to Y$ and $\gamma : Y \to Z$. If ϕ is continuous at x' and γ is continuous at $\phi(x')$, then $\gamma(\phi(x)) = \gamma \circ \phi$ is continuous at x'.*

PROOF. See Exercise A.12. □

Similarly, sums of continuous functions are continuous and ratios of real-valued continuous functions are continuous as long as the denominator is not equal to zero.

The following is an important theorem in its own right and also will motivate the somewhat more general treatment of continuity in the next section.

THEOREM A.2. ***(Open Sets and Continuity I)*** *Let (X, d_X) and (Y, d_Y) be metric spaces and consider the mapping $\phi : X \to Y$. ϕ is continuous if and only if for every $Y' \subset Y$ that is open in Y, $\phi^{-1}(Y')$ is open in X.*

PROOF. ($\Longrightarrow$) Suppose that ϕ is continuous and Y' is open in Y. Then consider $x \in X$ such that $\phi(x) \in Y'$. Since Y' is open, there exists $\varepsilon > 0$ such that $d_Y(\phi(x), y) < \varepsilon$ implies $y \in Y'$. Since ϕ is continuous at x, for the same $\varepsilon > 0$ there exists $\delta > 0$ such that for all x' with $d_X(x, x') < \delta$, $d_Y(\phi(x), \phi(x')) < \varepsilon$. This establishes that $x' \in \phi^{-1}(Y')$ and thus $\phi^{-1}(Y')$ is open in X.

($\Longleftarrow$) Suppose that $\phi^{-1}(Y')$ is open in X for every open Y' in Y. For given $\varepsilon > 0$ and $x \in X$, let $Y' = \{y \in Y : d_Y(\phi(x), y) < \varepsilon\}$, which is clearly an open set and thus $\phi^{-1}(Y')$ is open in X. Therefore, there exists $\delta > 0$ such that $x' \in \phi^{-1}(Y')$ whenever $d_X(x, x') < \delta$. Next $x' \in \phi^{-1}(Y')$ implies that $\phi(x') \in Y'$, so that $d_Y(\phi(x), \phi(x')) < \varepsilon$, completing the proof. □

Before moving to a more abstract treatment of continuity, it is also useful to state a simple but useful theorem.

THEOREM A.3. ***(The Intermediate Value Theorem)*** *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Suppose that $f(a) \neq f(b)$. Then for c intermediate between $f(a)$ and $f(b)$ (e.g., $c \in (f(a), f(b))$ if $f(a) < f(b)$), there exists $x^* \in (a, b)$ such that $f(x^*) = c$.*

PROOF. The simplest way to prove this result as follows: the image of the interval $[a, b]$ under the function f, $f([a, b])$, must be connected, in the sense that the set $f([a, b])$ cannot be the union of two disjoint sets W, W', that is, $f([a, b]) = W \cup W'$ and $\overline{W} \cap \overline{W'} = \varnothing$ (where recall that $\overline{W}$ and $\overline{W'}$ are the closures of W and W'). Suppose not. Then there would exist two disjoint open sets V and V' such that $f([a, b]) \subset V \cup V'$. But from Theorem A.2, this implies that $f^{-1}(V)$ and $f^{-1}(V')$ are open in $[a, b]$, and by the fact that $f([a, b]) \subset V \cup V'$, we have $[a,b] \subset f^{-1}(V) \cup f^{-1}(V')$, which implies that $[a,b]$ is not connected, which is clearly incorrect and yields a contradiction. Theorem A.3 then follows immediately since $f([a,b])$ is connected and thus includes any value between $f(a)$ and $f(b)$. $\square$

The Intermediate Value Theorem is, in many ways, the simplest "fixed point theorem" that economists can use in some applications (see Theorems A.16 and A.17 below for more general fixed point theorems). Fixed point theorems require a mapping $\phi : X \to X$ to have a point $x^* \in X$ such that $x^* = \phi(x^*)$. The usefulness of this construction stems from the fact that many equilibrium problems can be formulated as fixed point problems. It is also clear that a fixed point is nothing but a "zero" of a slightly different mapping. In particular, define $\tilde{\phi}(x) = \phi(x) - x$. Then a fixed point of ϕ corresponds to a zero of $\tilde{\phi}$. Perhaps the most useful application of the Intermediate Value Theorem is for the case in which $f(a) < 0$ and $f(b) > 0$ (or $f(a) > 0$ and $f(b) < 0$). In this case, the theorem states that the continuous function f has a "zero" over the interval $[a,b]$, that is, some value $x^* \in (a,b)$ such that $f(x^*) = 0$. This motivates my description of the Intermediate Value Theorem as the "simplest fixed point theorem".

A.3. A Minimal Amount of Topology: Continuity and Compactness

Theorem A.2 implies that only the structure of open sets are relevant for thinking about continuity of mappings. This motivates our brief introduction to topology. Topology is the study of open sets and their properties. Our main interest in introducing notions from topology is to be able to talk about *compactness*. While compactness can be discussed just using ideas from metric spaces, for some of the results on infinite-dimensional (dynamic) optimization, a slightly more general treatment of compactness is necessary. I first define a topology.

Definition A.15. *A **topology** $\tau = \{V_\alpha\}_{\alpha \in A}$ on a nonempty set X is a collection of subsets $\{V_\alpha\}_{\alpha \in A}$ of X, such that*

1. $\varnothing \in \tau$ *and* $X \in \tau$.
2. *Each* V_α *(for* $\alpha \in A$*) is in* τ.
3. *For any* $A' \subset A$, $\bigcup_{\alpha \in A'} V_\alpha$ *is in* τ.
4. *For any finite* $A' \subset A$, $\bigcap_{\alpha \in A'} V_\alpha$ *is in* τ.

*Given a topology τ on X, V is an **open** set in X if $V \in \tau$ and it is a **closed** set in X if $X \backslash V \in \tau$.*

The pair (X, τ) is a *topological space*.

The parallel between this definition and the properties of unions and intersections of open sets given in Theorem A.1 is obvious. Sometimes it is convenient to describe a topology not by all of the open sets, but in a more economical fashion. Two convenient ways of doing this are as follows. First, a topological space can be derived from a metric space. In particular, since a topological space (X, τ) is defined by a collection of open sets and a metric space (X, d) defines the collection of open sets in the space X, it also immediately defines a topological space with the topology *induced* by the metric d. Second, a topological space can be described by a smaller collection of sets (instead of the collection of open sets). This leads us to the concept of a *base* for a topology.

Definition A.16. *Given a topological space (X, τ), $\{W_\alpha\}_{\alpha \in A'}$ is a* ***base*** *for (X, τ) if for every $V \in \tau$, there exist $A'' \subset A'$ such that $V = \bigcup_{\alpha \in A''} W_\alpha$.*

If $\{W_\alpha\}_{\alpha \in A'}$ is a base for (X, τ), we can also say that τ is *generated* by $\{W_\alpha\}_{\alpha \in A'}$.

The following are some examples of topological spaces. The parallel to the metric spaces in Example A.1 is clear.

Example A.9.

(1) For any $X \subset \mathbb{R}^K$, define a collection of open sets (in the sense of Definition A.3) according to the family of metrics $d_p(x, y) = \left(\sum_{i=1}^{K} |x_i - y_i|^p\right)^{1/p}$ for $1 \leq p < \infty$ and $d_\infty(x, y) = \max_{i=1,\dots,K} |x_i - y_i|$ denoted by τ_p for $p \in [1, \infty]$. Then (X, τ_p) is a topological space. (X, τ_2) is sometimes referred to as the *Euclidean topology*, though since the other metrics are also equivalent (Exercise A.9), it would not be wrong to refer to any (X, τ_p) as the Euclidean topology.

(2) For any nonempty set X, the *discrete topology* is defined equivalently either by the discrete metric introduced in Example A.1 or by declaring all subsets of X as open sets.

(3) The *indiscrete topology* τ' on X only has $\varnothing$ and X as open sets.

(4) Consider the $(\mathbf{C}(X), d_\infty)$ metric space of all continuous functions with the sup metric. Define the collection of open sets on $\mathbf{C}(X)$ according to d_∞ by τ_∞, then $(\mathbf{C}(X), \tau_\infty)$ is a topological space.

(5) Consider the set of infinite sequences of real numbers $\ell \subset \mathbb{R}^\infty$ and the family of metrics for this set given by $d_p(x, y) = \left(\sum_{i=0}^{\infty} |x_i - y_i|^p\right)^{1/p}$ for $1 \leq p < \infty$ and by $d_\infty(x, y) = \sup_i |x_i - y_i|$. For any $p \in [1, \infty]$, d_p defines a topology τ_p and (ℓ, τ_p) is a topological space, which is sometimes denoted by the same symbol as the corresponding metric space, ℓ_p.

As suggested by this example, many topological spaces of interest are derived from a metric space. In this case, we say that they are *metrizable* and for all practical purposes, we can treat metrizable topological spaces as metric spaces. In particular, we have:

Definition A.17. *A topological space (X, τ) is* ***metrizable*** *if there exists a metric d on X such that if $V \in \tau$, then V is open in the metric space (X, d) (according to Definition A.3).*

Fact A.8. *If a topological space (X, τ) is metrizable with some metric d, then it has all the topological properties of the metric space (X, d).*

Proof. This follows immediately from the fact that (X, τ) and (X, d) have the same open sets. □

The preceding definition and fact are provided, because metric spaces are easier to work with in practice than topological spaces. Nevertheless, sometimes (as with the product topology introduced in the next section), it may be more convenient to work with more general topological spaces. One disadvantage of general topological spaces is that they do not have all of the nice properties of metric spaces. However, this will not be an issue for the properties of topological spaces that are related to continuity and compactness, which will focus on here. Nevertheless, it is useful to note that a particularly relevant property of general topological spaces is the *Hausdorff* property, which requires that any distinct points x and y of a topological space (X, τ) should be separated, that is, there should exist $V_x, V_y \in \tau$ such that $x \in V_x$, $y \in V_y$ and $V_x \cap V_y = \varnothing$. It is clear that every metric space will have the Hausdorff property (see Exercise A.13). For our purposes, the Hausdorff property will not be necessary.

Returning to general topological spaces, the notions of convergence of sequences, subsequences, nets and subnets can be stated for general topological spaces. Here I will only give the definitions for convergence of sequences and nets (those for subsequences and subnets are defined very similarly).

Definition A.18. *Let (X, τ) be a topological space. A sequence $\{x_n\}_{n=1}^{\infty}$ [a net $\{x_\alpha\}_{\alpha \in A}$] in X is* ***convergent*** *with* ***limit point*** *$x_\infty \in X$ if for $V \in \tau$ with $x_\infty \in V$, there exists an integer N [there exists some $\bar{\alpha} \in A$] such tha $x_n \in V$ for all $n \geq N$ [$x_\alpha \in V$ for all $\alpha \geq \bar{\alpha}$]. We write this as $\lim_{n\to\infty} x_n = \lim x_n = x_\infty$ or as $\{x_n\}_{n=1}^{\infty} \to x_\infty$.*

Continuity is also defined in a similar manner.

Definition A.19. *Let (X, τ_X) and (Y, τ_Y) be topological spaces and consider the mapping $\phi : X \to Y$. ϕ is* ***continuous at*** *$x \in X$ if and only if for every $U \in \tau_Y$ with $\phi(x) \in U$, there exists $V \in \tau_X$ with $x \in V$ such that $\phi(V) \subset U$. ϕ is* ***continuous*** *on X if it is continuous at each $x \in X$.*

The parallel between this definition and the equivalent characterization of continuity in metric spaces in Theorem A.14 is evident. In fact, we have

Theorem A.4. ***(Open Sets and Continuity II)****Let (X, τ_X) and (Y, τ_Y) be topological spaces and consider the mapping $\phi : X \to Y$. ϕ is continuous if and only if for every $Y' \subset Y$ that is open in Y, $\phi^{-1}(Y')$ is open in X.*

The proof of this theorem is identical to that of Theorem A.2 and is thus omitted.

Unfortunately, in general topological spaces, convergence in terms of sequences is not sufficient to characterize continuity. However, convergence in terms of nets is.

Theorem A.5. ***(Continuity and Convergence of Nets)*** *Let (X, τ_X) and (Y, τ_Y) be topological spaces. The mapping $\phi : X \to Y$ is continuous at $x \in X$ if and only if $\{\phi(x_\alpha)\}_{\alpha \in A} \to \phi(x)$ for any net $\{x_\alpha\}_{\alpha \in A} \to x$.*

Proof. ($\Longrightarrow$) Suppose ϕ is continuous at x and consider a net $\{x_\alpha\}_{\alpha \in A} \to x$. Given $U \in \tau_Y$ with $\phi(x) \in U$, $\phi^{-1}(U) \in \tau_X$ and $x \in \phi^{-1}(U)$. Therefore, for some $\bar{\alpha} \in A$, we have that $\alpha \geq \bar{\alpha}$ implies $x_\alpha \in \phi^{-1}(U)$ and thus $\phi(x_\alpha) \in U$ for all $\alpha \geq \bar{\alpha}$, establishing $\{\phi(x_\alpha)\}_{\alpha \in A} \to \phi(x)$.

($\Longleftarrow$) Suppose that ϕ is not continuous at x. Then, there exists $U \in \tau_Y$ with $\phi(x) \in U$ such that $\phi(U) \notin \tau_X$. Let $V \in N(x)$ denote $V \in \tau_X$ with $x \in V$. Then for each $V \in N(x)$ there exists $x_V \in V$ such that $\phi(x_V) \in U$. By construction, $\{x_V\}_{V \in N(x)}$ is a net converging to x, but $\{\phi(x_V)\}_{V \in N(x)} \nrightarrow \phi(x)$, completing the proof. □

Fact A.9. *Consider a function $f : X \to \mathbb{R}$ and suppose that X is endowed with the discrete topology. Then f is continuous.*

Proof. This immediately follows from the fact that any subset X' of X is open in X according to the discrete topology. □

Definition A.20. *Let (X, τ) be a topological space with $\tau = \{V_\alpha\}_{\alpha \in A}$ and $X' \subset X$. A collection of open sets $\{V_\alpha\}_{\alpha \in A'}$ for some $A' \subset A$ is an* ***open cover*** *of X' if $X' \subset \bigcup_{\alpha \in A'} V_\alpha$.*

Fact A.10. *Every $X' \subset X$ has an open cover.*

Proof. By Definition A.15 $X \in \tau$, so that $\{X\}$ is an open cover of X'. □

Definition A.21. *A subset X' of a topological space (X, τ) is* ***compact*** *if every open cover of X' contains a* ***finite subcover****, i.e., for every open cover $\{V_\alpha\}_{\alpha \in A'}$ of X', there exists a finite set $A'' \subset A'$ such that $X' \subset \bigcup_{\alpha \in A''} V_\alpha$*

Compactness is a major property, since compact sets have many nice features and some of these will be used below. Compactness has a particularly simple meaning in Euclidean spaces, which is given by the following famous theorem.

Theorem A.6. ***(Heine-Borel Theorem)*** *Let $X \subset \mathbb{R}^K$ be a Euclidean space (with a Euclidean metric or topology). Then $X' \subset X$ is compact if and only if X' is closed and bounded.*

A proof of this proposition can be found in any real analysis textbook and I will not repeat it here. Its main implication for us is that any K-dimensional interval $\prod_{i=1}^{K}[a_i, b_i]$, with $a_i, b_i \in \mathbb{R}$ and $a_i \leq b_i$, is compact. The assumption that X is a Euclidean space is important for Theorem A.6, as illustrated by the following example.

Example A.10. Consider the topological space (ℓ, τ) where τ is the topology induced by the discrete metric and the subset $\ell' = \left\{x \in \{0,1\}^\infty : \sum_{i=1}^\infty x_i^2 = 1\right\}$. Clearly ℓ' is closed and bounded, but not every open cover of ℓ' has a finite subcover. In particular, note that each point in ℓ' has the form $v_1 = (1,0,0,0,...)$, $v_2 = (0,1,0,0,0,...)$, $v_3 = (0,0,1,0,0,...)$, etc.. Since τ is the discrete topology, $v_n \in \tau$ for each n and moreover the collection $\bigcup_{n\in\mathbb{N}} v_n$ is an open cover of ℓ'. But clearly, this open cover does not a have finite subcover.

Nevertheless, there are important connections between closed sets and compact sets. For example:

Lemma A.2. *Let (X, τ) be a topological space and suppose that $X' \subset X$ is compact. Then*

Fact A.11. (1) *Any $X'' \subset X'$ that is closed is also compact.*
(2) *For any $X'' \subset X$ that is closed, $X'' \cap X'$ is compact.*

Proof. See Exercise A.14. □

One of the important implications of compactness is the following theorem.

Theorem A.7. ***(The Bolzano-Weierstrass Theorem)*** *Let (X, d) be a metric space and let $\{x_n\}_{n=1}^\infty$ be a sequence in X. If X is compact, then $\{x_n\}_{n=1}^\infty$ has a convergent subsequence.*

Proof. Suppose to obtain a contradiction that no such convergent subsequence exists. Then each $x \in X$ must have neighborhood V_x that contains at most one element of the sequence $\{x_n\}_{n=1}^\infty$. Clearly, $\{V_x\}_{x\in X}$ is an open cover of X. But since N_K is an infinite set, $\{V_x\}_{x\in X}$ has no finite subcover, contradicting compactness. □

It is possible to state an equivalent of Theorem A.7 for nets and subnets, but this result is not necessary for our purposes here. The reader may also wonder whether an equivalent of Theorem A.7 applies in a general topological space. Unfortunately, this is not the case (but it is true for topological space that have the Hausdorff property and also have a countable base, see Kelley, 1955).

Theorem A.8. ***(Continuity and Compact Images)*** *Let (X, τ_X) and (Y, τ_Y) be topological spaces, and consider the mapping $\phi : X \to Y$. If ϕ is continuous and $X' \subset X$ is compact, then $\phi(X')$ is compact.*

Proof. Let $\{V_\alpha\}_{\alpha \in A'}$ be an open cover of $\phi(X')$. Since ϕ is continuous, Theorem A.4 implies that $\phi^{-1}(V_\alpha)$ is open for each $\alpha \in A'$. Since X' is compact, every open cover has a finite subcover and therefore there exists $A'' \subset A'$ such that $X' \subset \bigcup_{\alpha \in A''} \phi^{-1}(V_\alpha)$. Since, by definition, $\phi\left(\phi^{-1}(X'')\right) \subset X''$ for any $X'' \subset X$, this implies that

$$\phi\left(X'\right) \subset \bigcup_{\alpha \in A''} (V_\alpha),$$

thus $\{V_\alpha\}_{\alpha \in A''}$ is a finite subcover of $\{V_\alpha\}_{\alpha \in A'}$, completing the proof. □

Despite its simplicity Theorem A.8 has many fundamental implications. The most important is the Weierstrass's Theorem.[2] Recall that for a real-valued function $f : X \to \mathbb{R}$, $\max_{x \in X} f(x)$ and $\min_{x \in X} f(x)$ are the maximum and the minimum of the function over the set X. These may not exist. When they do, we also define the following nonempty sets $\arg\max_{x \in X} f(x) = \{x' \in X : f(x') = \max_{x \in X} f(x)\}$ and $\arg\min_{x \in X} f(x) = \{x' \in X : f(x') = \min_{x \in X} f(x)\}$.

Theorem A.9. ***(Weierstrass's Theorem)*** *Consider the function $f : X \to \mathbb{R}$. If X' is a compact subset of a topological space (X, τ), then $\max_{x \in X'} f(x)$ and $\min_{x \in X'} f(x)$ exist, and $\arg\max_{x \in X'} f(x)$ and $\arg\min_{x \in X'} f(x)$ are nonempty.*

Proof. By Theorem A.8, $f(X')$ is compact. A compact subset of $\mathbb{R}$ contains a minimum and a maximum, thus $\max_{x \in X'} f(x)$ and $\min_{x \in X'} f(x)$ exist. The nonemptiness of $\arg\max_{x \in X'} f(x)$ and $\arg\min_{x \in X'} f(x)$ then follows immediately. □

This theorem implies that if we can formulate a maximization problem as one of maximizing a real-valued function subject to a constraint set that is compact subset of a topological space, then existence of solutions and nonemptiness of the set of maximizers are guaranteed. An immediate corollary is also useful in many applications. A real-valued function $f : X \to \mathbb{R}$ is *bounded on* X if there exists $M < \infty$ such that $|f(x)| < M$ for all $x \in X$.

Corollary A.1. *If $f : X \to \mathbb{R}$ is continuous and (X, τ) is a topological space, then f is bounded on X.*

A.4. The Product Topology

One of the main reasons for introducing topological spaces rather than simply working with metric spaces is to introduce the *product topology.* The product topology is particularly useful when dealing with infinite-dimensional optimization problems, since we can represent the space of sequences, ℓ, as the infinite product of $\mathbb{R}$, i.e., as $\mathbb{R}^\infty$. What are the topological

[2] In fact, there are many theorems that go under the name of "Weierstrass's Theorem," including on uniform continuity of a family of functions and approximation of continuous functions by polynomials. However, since these theorems are not used commonly in economic applications, there should be little confusion in referring to Theorem A.9 as Weierstrass's Theorem.

properties of such product spaces? The answer is provided by the following definition and the famous Tychonoff Theorem, Theorem A.12 below.

Definition A.22. *Let $A \subset \mathbb{R}$ and $\{(X_\alpha, \tau_\alpha)\}_{\alpha \in A}$ be a collection of topological spaces. Let $X = \prod_{\alpha \in A} X_\alpha$ and for each $\alpha \in A$, define the* ***projection map*** *$P_\alpha : X \to X_\alpha$ such that $P(x) = x_\alpha$. Then, the* ***product topology*** *$\tau = \prod_{\alpha \in A} \tau_\alpha$ is the topology such that all sets of the form $\bigcup_{j \in J} V^j$ are open, where $V^j = \prod_{\alpha \in A} V_\alpha^j$ with $V_\alpha^j \in \tau_\alpha$ and $V_\alpha^j = X_\alpha$ for all but finitely many αs.*

A different way of stating this definition is that sets of the form $V^j = \prod_{\alpha \in A} V_\alpha^j$ with $V_\alpha^j \in \tau_\alpha$ and $V_\alpha^j = X_\alpha$ for all but finitely many αs form a *base* for the product topology (recall Definition A.16 above).

The product topology is also referred to as **topology of pointwise convergence** because of the following:

Fact A.12. *A sequence $\{x_n\}_{n=1}^{\infty}$ or a net $\{x_j\}_{j \in J}$ in $X = \prod_{\alpha \in A} X_\alpha$ converges to some $\bar{x}$ if and only if all of the projections of $P_\alpha(x_n)$ or $P_\alpha(x_j)$ converge to $P_\alpha(\bar{x})$.*

This immediately implies that the product topology will be the right tool for analyzing convergence of infinite sequences. An alternative to the product topology would be the *box topology*, which is defined similarly, except that it does not have the last qualifier "$V_\alpha^j = X_\alpha$ for all but finitely many αs". This implies that the box topology has an abundance of open sets and thus compactness is difficult to achieve in the box topology. Exercise A.15 investigates this issue further.

Another major reason for the usefulness of the product topology is related to the fact that it ensures continuity of the projection maps (which seems like a minimal requirement for any reasonable topology), without introducing too many open sets. To formalize this, we need to rank topologies according to how "fine" or "weak" they are.

Definition A.23. *A topology τ defined on some set X is* ***weaker*** *(less fine) than some other topology τ' defined on the same set, if whenever V_α is open in τ it is also open in τ'.*

Theorem A.10. ***(Projection Maps and the Product Topology)*** *The product topology is the weakest topology that makes each projection map P_α continuous.*

Proof. Let τ be the product topology and τ' any other topology in which each projection map is continuous. This implies that for each $\alpha \in A$, whenever $V_\alpha \in X_\alpha$ is open in X_α, then $P_\alpha^{-1}(V_\alpha)$ is open according to τ', i.e., $P_\alpha^{-1}(V_\alpha) \in \tau'$. But this implies that finite intersections of all sets of the form $P_\alpha^{-1}(V_\alpha)$ are members of τ', and therefore all open sets in the product topology τ belong to τ'. Thus τ' must be finer than τ and establishes that the product topology is the weakest topology in which each projection map is continuous. $\square$

An implication of this lemma is that that a mapping $\phi : Y \to \prod_{\alpha \in A} X_\alpha$ is continuous according to the product topology if $P_\alpha \circ \phi : Y \to X_\alpha$ is continuous for each $\alpha \in A$. The product topology is particularly useful in dynamic optimization problems because of the combination of the following two results.

Theorem A.11. ***(Continuity in the Product Topology)***

(1) *Suppose that $f_n : X_n \to \mathbb{R}$ is continuous, X_n is a compact metric space for every $n \in \mathbb{N}$, the collection of functions $\{f_n\}_{n\in\mathbb{N}}$ is uniformly bounded in the sense that there exists $M \in \mathbb{R}$ such that $|f_n(x_n)| \leq M$ for all $x_n \in X_n$ and $n \in \mathbb{N}$, and $\beta < 1$. Then, $f = \sum_{n=1}^{\infty} \beta^n f_n : \prod_{n\in\mathbb{N}} X_n \to \mathbb{R}$ is continuous in the product topology.*

(2) *Suppose that $f_t : X_t \to \mathbb{R}$ is continuous, X_t is a compact metric space for every $t \in [0,\infty]$, the collection of functions $\{f_t\}_{t\in[0,\infty]}$ is uniformly bounded in the sense that there exists $M \in \mathbb{R}$ such that $|f_t(x_t)| \leq M$ for all $x_t \in X_t$ and $t \in [0,\infty]$ and $\rho > 0$. Then, $f = \int_0^\infty \exp(-\rho t) f_t dt : \prod_{t=0}^{\infty} X_t \to \mathbb{R}$ is continuous in the product topology.*

Proof. I will provide a proof for Part 1. The proof for Part 2 is identical and is left to Exercise A.21.

First note that uniform boundedness of the functions $\{f_n\}_{n\in\mathbb{N}}$ ensures that f is well-defined for all $x \in \prod_{n\in\mathbb{N}} X_n$. From Theorem A.5, f is continuous in the product topology if and only if for any $x^\infty \in \prod_{n\in\mathbb{N}} X_n$ we have that $\{f(x_j)\}_{j\in J} \to f(x^\infty)$ for any net $\{x_j\}_{j\in J} \in \prod_{n\in\mathbb{N}} X_n$ with $\{x_\alpha\}_{\alpha\in A} \to x^\infty$ in the product topology. Now take a net $\{x_j\}_{j\in J} \to x^\infty$. By Fact A.12, $\{x^j\}_{j\in J} \to x^\infty$ in the product topology if and only if $\left\{x_n^j\right\}_{j\in J} \to x_n^\infty$ for each $n \in \mathbb{N}$. Then, by continuity of each f_n, $\left\{f_n\left(x_n^j\right)\right\}_{j\in J} \to f(x_n^\infty)$. Fix $\varepsilon > 0$, and let $\overline{n}$ be such that $\frac{\beta^{\overline{n}}}{1-\beta}2M < \varepsilon/2$. Since $\left\{f_n\left(x_n^j\right)\right\}_{j\in J} \to f(x_n^\infty)$ for each $n < \overline{n}$, there exists $\overline{j} \in J$ such that $\left|f_n\left(x_n^j\right) - f(x_n^\infty)\right| \leq \varepsilon(1-\beta)/2$ for each $n < \overline{n}$ and $j \geq \overline{j}$. Therefore, for all $j \in J$ such that $j \geq \overline{j}$, we have

$$\begin{aligned}
\left|\sum_{n=1}^{\infty} \beta^n f_n\left(x_n^j\right) - \sum_{n=1}^{\infty} \beta^n f_n(x_n^\infty)\right| &\leq \sum_{n=1}^{\overline{n}-1} \beta^n \left|f_n\left(x_n^j\right) - f(x_n^\infty)\right| + \sum_{n=\overline{n}}^{\infty} \beta^n 2M \\
&\leq \sum_{n=1}^{\overline{n}-1} \beta^n \frac{\varepsilon(1-\beta)}{2} + \frac{\varepsilon}{2} < \varepsilon,
\end{aligned}$$

where the first line uses the triangle inequality and the fact that $\{f_n\}_{n\in\mathbb{N}}$ is uniformly bounded, and the second line uses the definition of $\bar{j}$. This inequality shows that $\{f(x^j)\}_{j\in J} \to f(x^\infty)$ and establishes the continuity of f. □

Discounting is important in the previous result. The following example shows why.

Example A.11. Suppose that $f_n : X \to \mathbb{R}$ is continuous and X is a compact metric space, and let $f = \sum_{n=1}^{\infty} f_n : X^\infty \to \mathbb{R}$. It can be verified that f is not continuous and tends to infinity for any $\{x^j\}_{j=1}^{\infty} \to x^\infty$ such that $f_n(x_n^\infty) > \varepsilon$ for all n for some $\varepsilon > 0$.

Theorem A.12. ***(Tychonoff's Theorem)*** *Let $A \subset \mathbb{R}$ and consider the family of topological spaces $\{(X_\alpha, \tau_\alpha)\}_{\alpha\in A}$. If each X_α is compact, then $X = \prod_{\alpha\in A} X_\alpha$ is compact in the product topology, i.e., (X,τ) is compact, where $\tau = \prod_{\alpha\in A} \tau_\alpha$.*

The proof of this theorem is somewhat involved, and can be found in Kelley (1955) or Royden (1994).

Combined with Theorem A.11, this theorem implies that problems involving the maximization of discounted utility in standard dynamic economic environments has a continuous objective function in the product topology. We can then appeal to Tychonoff's Theorem to make sure that the relevant constraint set is compact (again in the product topology). This combination then enables us to apply Weierstrass's Theorem, Theorem A.9, to show the existence of solutions. The reader will recall that we have used this technique in Chapters 6, 7, and 16.

A.5. Correspondences and Berge's Maximum Theorem

In this section, I state one of the most important theorems in economic analysis, Berge's Maximum Theorem. This theorem is not only essential for dynamic optimization, but it plays a major role in general equilibrium theory, game theory, little economy,producer theory, public finance and industrial organization. In fact, it is hard to imagine any area of economics where it does not play a major role. Despite its enormous importance, this theorem is left out of most basic "Mathematics for Economists" courses and textbooks. This motivates my somewhat detailed treatment of it here. The first step for this theorem is to have a brief review of correspondences, which were already mentioned above.

Throughout the rest of this Appendix, I focus on metric spaces. Recall that F is a correspondence from a metric space (X, d_X) into (Y, d_Y) if to each $x \in X$ it assigns a subset of Y. We write this as

$$F : X \rightrightarrows Y \text{ or } F : X \to \mathcal{P}(Y) \backslash \varnothing,$$

where $\mathcal{P}(Y)$ is the power set of Y (and I have subtracted the empty set $\varnothing$, so that the correspondence is not empty valued). We are interested in correspondences for three fundamental reasons. First, even when a mapping into real numbers is a well behaved function, $f : X \to \mathbb{R}$, its inverse f^{-1} will typically be set-valued, thus a correspondence. Second, our main interest in most economic problems is with the "arg max" sets defined above, which are the subsets of values in some set X that maximize a function. These will correspond to utility-maximizing consumption, investment or price levels in simple economic problems. Finally, and perhaps most importantly, correspondences will play a key role both as representing economically relevant constraints and as a way of expressing the properties of maximizers in Berge's Maximum Theorem.

As with functions, for a correspondence $F : X \rightrightarrows Y$, I will use the notation $F(X')$ to denote the image of the set X' under the correspondence F:

$$F\left(X'\right) = \left\{y \in Y : \exists x \in X' \text{ with } y \in F\left(x\right)\right\}.$$

Definition A.24. *Let (X, d_X) and (Y, d_Y) be metric spaces and consider the correspondence $F : X \rightrightarrows Y$. Let $\mathcal{N}_\varepsilon(x)$ refer to neighborhoods in (X, d_X). Then*

1. *F is **upper hemi-continuous** at $x \in X$ if for every open subset Y' of Y with $F(x) \subset Y'$, there exists $\varepsilon > 0$ such that $F(\mathcal{N}_\varepsilon(x)) \subset Y'$.*
2. *F is **upper hemi-continuous** on the set X if it is upper hemi-continuous at each $x \in X$.*
3. *F is **lower hemi-continuous** at $x \in X$ if for every open subset Y' of Y with $F(x) \cap Y' \neq \varnothing$, there exists $\varepsilon > 0$ such that $F(x') \cap Y' \neq \varnothing$ for all $x' \in \mathcal{N}_\varepsilon(x)$.*
4. *F is **lower hemi-continuous** on the set X if it is lower hemi-continuous at each $x \in X$.*
5. *F is **continuous** at $x \in X$ if and only if it is both upper- and lower hemi-continuous at $x \in X$.*
6. *F is **continuous** on the set X if and only if it is both upper- and lower hemi-continuous on the set X.*

These notions are slightly easier to understand if we specialize them to Euclidean spaces. First, we say that a correspondence $F : X \rightrightarrows Y$ is *closed-valued* [*compact-valued*] if $F(x)$ is closed [compact] in Y for each x. For Euclidean spaces, the following equivalent definition (see Exercise A.17) is equivalent to Definition A.24 and in general, as Fact A.13 shows, it implies Definition A.24.

Definition A.25. *Let $X \subset \mathbb{R}^{K_X}$ and $Y \subset \mathbb{R}^{K_Y}$, where K_X and K_Y are integers and consider the compact-valued correspondence $F : X \rightrightarrows Y$.*

1. *F is **upper hemi-continuous** at $x \in X$ if for every sequence $\{x_n\}_{n=1}^\infty \to x$ and every sequence $\{y_n\}_{n=1}^\infty$ with $y_n \in F(x_n)$ for each n, there exists a convergent subsequence $\{y_{n_k}\}_{n_k \in N_K}$ of $\{y_n\}_{n=1}^\infty$ such that $\{y_{n_k}\}_{n_k \in N_K} \to y \in F(x)$.*

(2) *F is **lower hemi-continuous** at $x \in X$ if $F(x)$ is nonempty and for every $y \in F(x)$ and every sequence $\{x_n\}_{n=1}^{\infty} \to x$, there exists some integer N and a sequence $\{y_n\}_{n=1}^{\infty}$ with $y_n \in F(x_n)$ for all $n \geq N$, and $\{y_n\}_{n=1}^{\infty} \to y$.*

Upper hemicontinuity and lower hemicontinuity according to Definition A.25 imply the corresponding concepts in Definition A.24 for general metric spaces.

FACT A.13. *Let (X, d_X) and (Y, d_Y) be metric spaces and consider the correspondence $F : X \rightrightarrows Y$. If F is upper hemi-continuous [lower hemi-continuous]at $x \in X$ according to Definition A.25, then it is upper hemi-continuous [lower hemi-continuous] at $x \in X$ according to Definition A.24.*

PROOF. Suppose, to obtain a contradiction, that part 1 of Definition A.25 holds at x, but F is not upper hemi-continuous at x. Then, there exists an open set $Y' \subset Y$ such that $F(x) \subset Y'$ but for any $\varepsilon > 0$, $F(\mathcal{N}_{\varepsilon}(x))$ is not a subset of Y'. Then, for any $\varepsilon > 0$, there exists $x_{\varepsilon} \in \mathcal{N}_{\varepsilon}(x)$ and $y_{\varepsilon} \in F(x_{\varepsilon})$ such that $y_{\varepsilon} \notin Y'$. Construct the sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$ such that each (x_n, y_n) satisfies this property for $\varepsilon = 1/n$. Clearly, $\{x_n\}_{n=1}^{\infty} \to x$. Therefore, by hypothesis, there exists a convergent subsequence $\{y_{n_k}\}_{n_k \in N_K} \to y \in F(x)$. Since Y' is open, $Y \setminus Y'$ is closed, and since $y_{n_k} \in Y \setminus Y'$ for each n_k, the limit point y must also be in the closed set $Y \setminus Y'$. But $y \in Y \setminus Y'$ together with $y \in F(x)$ yields a contradiction in view of the fact that $F(x) \subset Y'$, proving the first part of the Fact.

Suppose, to obtain a contradiction, that part 2 of Definition A.25 holds at x, but F is not lower hemi-continuous at x. Then, there exists an open set $Y' \subset Y$ such that $F(x) \cap Y' \neq \emptyset$, but for any $\varepsilon > 0$, there exists $x_{\varepsilon} \in F(\mathcal{N}_{\varepsilon}(x))$ such that $F(x_{\epsilon}) \cap Y' = \emptyset$. Consider the sequence $\{x_n\}_{n=1}^{\infty}$ such that each x_n satisfies this property for $\varepsilon = 1/n$ and let $y \in F(x) \cap Y'$. Clearly, $x_n \to x$. Therefore, there exists a sequence $\{y_n\}_{n=1}^{\infty}$ and some $N \geq 1$ such that $y_n \in F(x_n)$ for all $n \geq N$ and $\{y_n\}_{n=1}^{\infty} \to y$. However, by the construction of the sequence $\{x_n\}_{n=1}^{\infty}$, we have that $y_n \notin Y'$. Once again, since $Y \setminus Y'$ is closed, it must be the case that the limit point y also lies in the closed set $Y \setminus Y'$. This contradicts $y \in F(x) \cap Y'$ and establishes the second part of the Fact. □

DEFINITION A.26. *Let (X, d_X) and (Y, d_Y) be metric spaces and consider the correspondence $F : X \rightrightarrows Y$. Then F has a **closed graph** (is closed) at $x \in X$ if for every sequence $\{(x_n, y_n)\}_{n=1}^{\infty} \to (x, y)$ such that $y_n \in F(x_n)$ for each n, we also have $y \in F(x)$. F has a closed graph on the set X if it is closed at each $x \in X$.*

The following fact is a simple consequence of Definition A.25.

FACT A.14. *Let Let $X \subset \mathbb{R}^{K_X}$ and $Y \subset \mathbb{R}^{K_Y}$ and consider the correspondence $F : X \rightrightarrows Y$ that is upper hemi-continuous. If $F(x)$ is a closed set in Y (i.e., if F is* closed-valued*) for each $x \in X$, then F has a closed graph.*

Proof. See Exercise A.19. □

For finite dimensional spaces, correspondences with closed graph are also upper hemi-continuous, provided that they satisfy a simple boundedness hypothesis.

Fact A.15. *Let $X \subset \mathbb{R}^{K_X}$ and $Y \subset \mathbb{R}^{K_Y}$ and consider the correspondence $F : X \rightrightarrows Y$. Suppose that F has closed graph at $x \in X$ and that there exists a neighborhood V_x of x such that $F(V_x)$ is bounded. Then F that is upper hemi-continuous at x.*

Proof. Consider sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ such that $y_n \in F(x_n)$ for each n. Suppose that $\{x_n\}_{n=1}^{\infty} \to x$. Then by definition, there exists $N \in \mathbb{N}$ such that $x_n \in V_x$ for all $n \geq N$, where V_x is the neighborhood specified in the statement of the claim, satisfying the property that $F(V_x)$ is bounded. Since Y is a Euclidean space, this implies that the closure of V_x, $\overline{F(V_x)}$, is compact. Then by Theorem A.7, $\{y_n\}_{n=1}^{\infty}$ has a subsequence $\{y_{n_k}\}_{n_k \in N_k}$ converging to some $y \in Y$. This implies that the (sub)sequence $\{(\acute{x}_{n_k}, y_{n_k})\}_{n_k \in N_k} \to (x, y)$. Moreover, since F has closed graph at x, $y \in F(x)$, which establishes that F is upper hemi-continuous at $x \in X$ according to Definition A.25. Then, from Fact A.13, it is upper hemi-continuous at $x \in X$ according to Definition A.24. □

The hypothesis that there exists a neighborhood V_x with $F(V_x)$ bounded can not be dispensed with in this result. This is shown by the following example:

Example A.12. Consider the correspondence $F : [0,1] \to \mathbb{R}$ given by $F(x) = \{0\}$ if $x = 0$ and $F(x) = \{\log x, 0\}$ if $x \in (0,1]$. F has closed graph, but is not upper hemi-continuous at $x = 0$. It can be verified easily that F does not satisfy the hypothesis that there exists a neighborhood V_x with $F(V_x)$ bounded at $x = 0$.

Theorem A.13. ***(Berge's Maximum Theorem)*** *Let (X, d_X) and (Y, d_Y) be metric spaces. Consider the maximization problem*

$$\begin{aligned} &\sup_{y \in Y} f(x, y) \\ &\text{subject to} \\ &y \in G(x), \end{aligned}$$

where $G : X \rightrightarrows Y$ and $f : X \times Y \to \mathbb{R}$. Suppose that f is continuous and G is compact-valued and continuous at x. Then

(1) $M(x) = \max_{y \in Y} \{f(x, y) : y \in G(x)\}$ *exists and is* ***continuous*** *at x.*
(2) $\Pi(x) = \arg\max_{y \in Y} \{f(x, y) : y \in G(x)\}$ *is* ***nonempty, compact-valued, upper hemi-continuous*** *and has* ***closed graph*** *at x.*

Proof. In view of Fact A.13, I will work with Definition A.25.

The fact that $M(x)$ exists and thus $\Pi(x)$ is nonempty for all $x \in X$ follows from Theorem A.9. Consider a sequence$\{y_n\}_{n=1}^{\infty} \to y$ such that $y_n \in \Pi(x_n)$ for each n. Since $G(x)$ is closed, we have $y \in G(x)$. Moreover, by definition, $f(x, y_n) = M(x)$ for each n. Since f is continuous, $f(x, y) = M(x)$ follows. Therefore, $y \in \Pi(x)$ and thus $\Pi(x)$ is closed. Since $\Pi(x)$ is a closed subset of the compact set $G(x)$, we can invoke Lemma A.2 to conclude that $\Pi(x)$ is compact-valued.

Now again take $\{x_n\}_{n=1}^{\infty} \to x$, $\{y_n\}_{n=1}^{\infty}$ with $y_n \in G(x_n)$ for all n, with a convergent subsequence $\{y_{n_k}\}_{n_k \in N_K} \to y$. Since $G(x)$ is upper hemi-continuous, $y \in G(x)$. Take any $z \in G(x)$. Since $G(x)$ is continuous and thus lower-hemi-continuous, there exists $\{z_{n_k}\}_{n_k \in N_K} \to z$ with $z_{n_k} \in G(x_{n_k})$ for all $n_k \in N_K$. By the definition that $y_{n_k} \in \Pi(x_{n_k})$, $M(x_{n_k}) = f(x_{n_k}, y_{n_k}) \geq f(x_{n_k}, z_{n_k})$. Moreover, since f is continuous, by Fact A.5,

$$M(x) = f(x, y) \geq f(x, z).$$

Since this holds for all $z \in G(x)$, we have that $y \in \Pi(x)$ and therefore $\Pi(x)$ is upper hemi-continuous. Applying Fact A.14 once more, we conclude that $\Pi(x)$ also has a closed graph.

To complete the proof, we need to show that $M(x)$ is continuous at x. This follows from the fact that $\Pi(x)$ is upper hemi-continuous. Take $\{x_n\}_{n=1}^{\infty} \to x$, and consider, $\{y_n\}_{n=1}^{\infty}$ such that $y_n \in \Pi(x_n)$ for each n. Since, Π is upper hemi-continuous, Definition A.25 implies that there exists a subsequence $\{y_{n_k}\}_{n_k \in N_K}$ converging to $y \in F(x)$. The continuity of f implies that $M(x_{n_k}) = f(x_{n_k}, y_{n_k}) \to f(x, y) = M(x)$ and establishes that $M(x)$ is continuous at x. □

Note that I wrote the maximization problem as $\sup_{y \in Y}$ instead of $\max_{y \in Y}$. There would have been no loss of generality in using the latter notation, since the theorem establishes that the maximum is attained. Nevertheless, the former might be slightly more appropriate, since when we first consider the problem with do not know whether the maximum is attained or not. Throughout the appendix, I used the "sup" notation, while in the text I typically use the simpler "max" notation.

One difficulty in using Theorem A.13 is that constraint sets do not always define continuous correspondences. This is illustrated in Exercise A.18. However, Fact A.16 shows that in some important cases they do in fact define continuous correspondences.

A.6. Convexity, Concavity, Quasi-Concavity and Fixed Points

Theorem A.13 shows how we can ensure certain desirable properties of the set of maximizers in a variety of problems arising in economic analysis. However, it is not strong enough to assert uniqueness of maximizers or continuity of the set of maximizers (instead, we have upper hemi-continuity, which is weaker than continuity). In this section, I will show how these results can be strengthened when we focus on problems with concave objective functions and convex constraint sets, and then I will provide a brief illustration of how these strengthened results can be used. Throughout the rest of this appendix, let X be a *vector space* (or a linear space) so that if $x, y \in X$ and λ is a scalar, then $x + y \in X$ and $\lambda x \in X$. Properties of vector spaces will be discussed further in Section 10.1 below.

Definition A.27. *A set X is convex, if for any $\lambda \in [0,1]$, and any $x, y \in X$, we have that $\lambda x + (1-\lambda) y \in X$.*

A correspondence $G : X \rightrightarrows Y$ is *convex-valued* at x if $G(x)$ is a convex set (in Y).

Definition A.28. *Let X be convex, $f : X \to \mathbb{R}$ be a real-valued function and $\lambda \in (0,1)$. Suppose that $f(x)$, $f(y)$ and $f(\lambda x + (1-\lambda) y)$ are well defined. Then:*

(1) *f is **concave** if $f(\lambda x + (1-\lambda) y) \geq \lambda f(x) + (1-\lambda) f(y)$ for all $\lambda \in (0,1)$ and all $x, y \in X$.*
(2) *f is **strictly concave** if $f(\lambda x + (1-\lambda) y) > \lambda f(x) + (1-\lambda) f(y)$ for all $\lambda \in (0,1)$ and all $x, y \in X$ with $x \neq y$.*
(3) *f is **convex** if $f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y)$ for all $\lambda \in (0,1)$ and all $x, y \in X$.*
(4) *f is **strictly convex** if $f(\lambda x + (1-\lambda) y) < \lambda f(x) + (1-\lambda) f(y)$ for all $\lambda \in (0,1)$ and all $x, y \in X$ with $x \neq y$.*
(5) *f is **quasi-concave** if $f(\lambda x + (1-\lambda) y) \geq \min\{f(x), f(y)\}$ for all $\lambda \in (0,1)$ and all $x, y \in X$.*
(6) *f is **strictly quasi-concave** if $f(\lambda x + (1-\lambda) y) > \min\{f(x), f(y)\}$ for all $\lambda \in (0,1)$ and all $x, y \in X$ with $x \neq y$.*
(7) *f is **quasi-convex** if $f(\lambda x + (1-\lambda) y) \leq \max\{f(x), f(y)\}$ for all $\lambda \in (0,1)$ and all $x, y \in X$.*
(8) *f is **strictly quasi-convex** if $f(\lambda x + (1-\lambda) y) < \max\{f(x), f(y)\}$ for all $\lambda \in (0,1)$ and all $x, y \in X$ with $x \neq y$.*

Naturally, one can define all of these concepts for a subset X' of the domain X of the function f, since a function could be concave over a certain range, but not everywhere.

We can now state the following generalization of Theorem A.13.

Theorem A.14. ***(Properties of Maximizers)*** *Consider the maximization problem*

$$\sup_{y \in Y} f(x, y)$$

subject to

$$y \in G(x),$$

where $G : X \rightrightarrows Y$ and $f : X \times Y \to \mathbb{R}$. Suppose that f is continuous and G is convex-valued, compact-valued and continuous at x. Then

(1) *If f is quasi-concave, then $\Pi(x) = \arg\max_{y \in Y} \{f(x, y) : y \in G(x)\}$ is nonempty, compact-valued, upper hemi-continuous, has closed graph and is* ***convex-valued*** *at x.*
(2) *If f is strictly quasi-concave in a neighborhood of x, then $\Pi(x)$ is a* ***singleton****.*
(3) *If f satisfies the conditions in part 2 everywhere in X, then $\Pi(x)$ is a* ***continuous*** *(single-valued) function in X.*

Proof. **(Part 1)** Most of the statements here follow from Theorem A.13. We only need to prove that $\Pi(x)$ is convex-valued. Suppose, to obtain a contradiction, that this is not the case. This implies that there exist y and $y' \neq y$ in $\Pi(x)$ such that for some $\lambda \in (0, 1)$, $y'' = \lambda y + (1 - \lambda) y' \notin \Pi(x)$. But since $G(x)$ is convex-valued, $y'' \in G(x)$. Then by quasi-concavity, $f(\lambda y + (1 - \lambda) y') \geq \min\{f(y), f(y')\}$. But since y, $y' \in \Pi(x)$, we have $f(y) = f(y')$, and thus $f(\lambda y + (1 - \lambda) y') \geq f(y) = f(y')$, implying that $y'' = \lambda y + (1 - \lambda) y' \in \Pi(x)$. This yields a contradiction and establishes that $\Pi(x)$ is convex-valued.

(Part 2) Suppose, to obtain a contradiction, that there exist y and $y' \neq y$ in $\Pi(x)$. Since $G(x)$ is convex-valued, $y'' = \lambda y + (1 - \lambda) y' \in G(x)$ for any $\lambda \in (0, 1)$ and moreover, by strict quasi-concavity of f, $f(\lambda y + (1 - \lambda) y') > \lambda f(y) + (1 - \lambda) f(y')$. Again since y, $y' \in \Pi(x)$, $f(y) = f(y')$, and thus $f(\lambda y + (1 - \lambda) y') > f(y) = f(y')$, contradicting that y, $y' \in \Pi(x)$ and establishing the result.

(Part 3) Part 2 implies that $\Pi(x)$ is single-valued everywhere and Part 1 implies that it is upper hemi-continuous. From Definition A.24, this implies that for every sequence $\{x_n\}_{n=1}^{\infty} \to x$ and every sequence $\{y_n\}_{n=1}^{\infty}$ with $y_n = \Pi(x_n)$ for each n, there exists a convergent subsequence $\{y_{n_k}\}_{n_k \in N_K}$ of $\{y_n\}_{n=1}^{\infty}$ such that $\{y_{n_k}\}_{n_k \in N_K} \to y = \Pi(x)$. When $\Pi(x_n)$ is single valued for all x_n, this is the definition of continuity. □

Clearly, all of these results can be generalized to minimization problems. In particular, we have the following theorem, with proof identical to that of Theorems A.13 and A.14.

Theorem A.15. ***(Properties of Minimizers)*** *Consider the minimization problem*

$$\inf_{y \in Y} f(x, y)$$
$$\text{subject to}$$
$$y \in G(x),$$

where $G : X \rightrightarrows Y$ and $f : X \times Y \to \mathbb{R}$. Suppose that f is continuous and G is convex-valued, compact-valued and continuous at x. Then

(1) *If f is quasi-convex, then $\Pi(x) = \arg\min_{y \in Y} \{f(x, y) : y \in G(x)\}$ is nonempty, compact-valued, upper hemi-continuous, has closed graph and is convex-valued at x.*

(2) *If f is strictly quasi-convex, then in addition, $\Pi(x)$ is a singleton.*
(3) *If f satisfies the conditions in part 2 everywhere, then $\Pi(x)$ is a continuous single-valued mapping.*

What makes continuous correspondences and thus Theorems A.13 and A.14 particularly interesting in many applications is the following simple result.

Fact A.16. *Let (X, d_X) and (Y, d_Y) be metric spaces and consider the continuous concave function $g : Y \to \mathbb{R}$. Then the set-valued mapping*

$$G(x) = \{y \in Y : y \leq g(x)\}$$

defines a continuous correspondence $G : X \rightrightarrows Y$.

Proof. See Exercise A.20. □

The following well-known and important theorem shows why convex-valuedness is important.

Theorem A.16. ***(Kakutani's Fixed Point Theorem)*** *Suppose $X \subset \mathbb{R}^K$ is a nonempty, compact, convex set and let*

$$F : X \rightrightarrows X$$

*be a nonempty, convex-valued, and upper hemi-continuous correspondence. Then F has **a fixed point** in X, that is, there exists $x^* \in X$ such that $x^* \in F(x^*)$.*

The proof of this theorem is nontrivial and can be found in Berge (1963), Aliprantis and Border (1999), or Ok (2006). Exercise A.22 shows why convex-valuedness is important and Exercise A.23 presents an application of the results in this section and of Theorem A.16 to the existence of pure strategy Nash equilibria in normal-form games.

While some of the proofs of Kakutani's Fixed Point Theorem start from the slightly simpler Brouwer's Fixed Point Theorem, now that we have stated Theorem A.16, we can also obtain Brouwer's Fixed Point Theorem as a corollary.

Theorem A.17. ***(Brouwer's Fixed Point Theorem)*** *Suppose $X \subset \mathbb{R}^K$ is a nonempty, compact, convex set and let*

$$\phi : X \to X$$

*be a continuous map. Then ϕ has **a fixed point** in X, that is, there exists $x^* \in X$ such that $x^* = \phi(x^*)$.*

Proof. The result follows immediately from Theorem A.16, using Part 3 of Theorem A.14, which shows that a continuous map is a nonempty, convex-valued and upper hemi-continuous correspondence. □

A.7. Differentiation, Taylor Series and the Mean Value Theorem

In this and the next section, I briefly discuss differentiation and some important results related to differentiation that are useful for the analysis in the text. The material in this section should be more familiar, thus I will be somewhat more brief in my treatment. In this section, the focus is on a real-valued function of one variable $f : \mathbb{R} \to \mathbb{R}$. Functions of several variables and vector-valued functions are discussed in the next section.

The reader will recall that the derivative (function) for $f : \mathbb{R} \to \mathbb{R}$ has a simple definition. Take a point x in an open set X' on which the function f is defined. Then, when it exists, the derivative of f at x is defined by the following limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{A.1}$$

Clearly, the term $f(x+h)$ is well defined for h sufficiently small since x is in the open set X'. Moreover, this limit will exist at point x only if f is continuous at $x \in X$. This is a more general property; differentiability implies continuity (see Fact A.17). Using the elementary properties of limits, expression in (A.1) can be rearranged as

$$\lim_{h \to 0} \left| \frac{f(x+h) - f(x) - L(x)\, h}{h} \right| = 0, \tag{A.2}$$

where $L(x) = f'(x)$. This expression emphasizes that we can think of the derivative of the function $f(x)$, $f'(x)$, as a *linear operator*. In fact, one might want to define $f'(x)$ precisely as the *linear operator* $L(x)$ that satisfies equation (A.2). Note that $f'(x)$ is linear in h not in x. It is generally a nonlinear function of x, but it defines a linear function from X' (the open subset of X where f is defined) to $\mathbb{R}$ that assigns the value $f'(x)\, h$ to each h such that $x + h \in X'$. This perspective will be particularly useful in the next section.

DEFINITION A.29. *When $f'(x)$ exists at x, f is **differentiable** at x. If $f'(x)$ exist at all x in some subset $X'' \subset X$, then f is **differentiable** on the entire X''. If, in addition, f' is a continuous function of x on X'', f is **continuously differentiable** and is denoted as a $\mathcal{C}^1$ function.*

When X' is a closed set, then f being differentiable or continuously differentiable on X' is equivalent to f being differentiable or continuously differentiable in the interior of X' and then also having *an extension* (or a continuous extension) of its derivative to the boundary of X'. A slightly stronger requirement, which will also guarantee (continuous) differentiability on X', is that there exists an open set $X'' \supset X'$ such that f is (continuously) differentiable on X''.

Differentiability is a stronger requirement than continuity. In fact, we have

FACT A.17. *Let $X \subset \mathbb{R}$ and $f : X \to \mathbb{R}$ be a real-valued function. If f is differentiable at $x \in X$, then it is also continuous at x.*

PROOF. See Exercise A.24. □

It is also useful to note that differentiability over some set X' does not imply continuous differentiability. The following example illustrates this point.

EXAMPLE A.13. Consider the function f such that $f(x) = x^2 \sin(1/x)$ for all $x \neq 0$ and $f(0) = 0$. It can be verified that f is continuous and differentiable, with derivative $f'(x) = 2x \sin(1/x) - \cos(1/x)$ and $f'(0) = 0$. But clearly, $\lim_{x \downarrow 0} f'(x) \neq 0$.

Higher order derivatives are defined in a similar manner. Again starting with a real-valued function f, suppose that this function has a continuous derivative $f'(x)$. Then, again taking x in some open set X' where $f'(X')$ is well-defined, the second derivative of f, denoted $f''(x)$, is

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}.$$

Higher than the second order derivatives are defined similarly. If a real-valued function f has continuous derivatives up to order n on some set X', then it is said to be $\mathcal{C}^n$. If f is a $\mathcal{C}^2$ function, we also say that it is *twice continuously differentiable.* A $\mathcal{C}^\infty$ has continuous derivatives of any order (which may be constant after some level, for example, as is the case with polynomials). These functions are also referred to as *real analytical functions.*

Before moving to more general mappings, I present three results that are often very useful in applications. The first one is a generalization of the Intermediate Value Theorem, Theorem A.3, to derivatives.

THEOREM A.18. ***(Mean Value Theorems)*** *Suppose that $f : [a,b] \to \mathbb{R}$ is continuously differentiable on $[a,b]$. Then:*

(1) *Suppose that $f'(a) \neq f'(b)$, then for any c intermediate between $f'(a)$ and $f'(b)$, there exists $x^* \in (a,b)$ such that $f'(x^*) = c$.*
(2) *There exists x^* such that*

$$f'(x^*) = \frac{f(b) - f(a)}{b-a}.$$

PROOF. See Exercise A.25. □

A particular difficulty often encountered in evaluating limits of the form $\lim_{x \to x^*} f(x)/g(x)$ (where f and g are continuous real-valued functions) is that we may have both $f(x^*) = 0$ and $g(x^*) = 0$. The following result, known as *L'Hospital's Rule* (or L'Hospital's Theorem) provides one way of evaluating these types of limits.

THEOREM A.19. ***(L'Hospital's Rule)*** *Suppose that $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ are differentiable functions on $[a,b]$, and suppose that for $g'(x) \neq 0$ for $x \in (a,b)$ and let $c \in (a,b)$. If*

$$\lim_{x \to c} \frac{f'(x)}{g'(x)}$$

exists and

$$\text{either } \lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0 \text{ or } \lim_{x\to c} f(x) = \lim_{x\to c} g(x) = \infty,$$

then we have that

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

Proof. See Exercise A.26. □

The final result in this section are the Taylor Theorem and the resulting Taylor Series approximation to differentiable real-valued functions. For this theorem, let the nth derivative of a real-valued function f be denoted by $f^{(n)}$ (e.g., $f' = f^{(1)}$, etc.).

Theorem A.20. ***(Taylor's Theorem I)*** *Suppose that $f : [a, b] \to \mathbb{R}$ is a $\mathcal{C}^{n-1}$ function and moreover its nth derivative, $f^{(n)}(x)$ exists for all $x \in (a, b)$. Then for any x and $y \neq x$ in $[a, b]$, we have that for some z between x and y*

$$f(y) = f(x) + \sum_{k=1}^{n-1} \frac{f^{(k)}(x)}{k!}(y - x)^k + \frac{f^{(n)}(z)}{n!}(y - x)^n.$$

Proof. Suppose, for simplicity, that $y > x$. The proof requires that we show the existence of $z \in (x, y)$ such that

$$f^{(n)}(z) = n!(y - x)^{-n}\left[f(y) - f(x) - \sum_{k=1}^{n-1} \frac{f^{(k)}(x)}{k!}(y - x)^k\right].$$

Let

$$g(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(x)}{k!}(t - x)^k - \frac{(t - x)^n}{(y - x)^n}\left(f(y) - \sum_{k=1}^{n-1} \frac{f^{(k)}(x)}{k!}(y - x)^k\right).$$

Clearly g is n times differentiable. Thus the proof is equivalent to showing that there exists $z \in (x, y)$ such that $g^{(n)}(z) = 0$. It is straightforward to verify that $g^{(k)}(x) = 0$ for $k = 0, 1, ..., n - 1$ and also $g(x) = g(y) = 0$. The Mean Value Theorem, Theorem A.18, then implies that $g^{(1)}(z_1) = 0$ for some $z_1 \in (x, y)$. Next since $g^{(1)}(x) = g^{(1)}(z_1) = 0$, again from Theorem A.18, we have that there exists $z_2 \in (x, z_1)$ such that $g^{(2)}(z_2) = 0$. Continuing inductively for $n - 1$ more steps, establishes the existence of $z \in (x, y)$ such that $g^{(n)}(z) = 0$. □

The following corollary provides both an equivalent form of Theorem A.20 and also an implication of this theorem.

Corollary A.2. (1) *Suppose that $f : [a, b] \to \mathbb{R}$ is a $\mathcal{C}^n$ function. Then*

$$f(y) = f(x) + \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}(y - x)^k + R_n(y - x, x),$$

where $R_n(y - x, x) / |y - x|^n \to 0$ as $y \to x$.

(2) *Suppose that* $f : [a, b] \to \mathbb{R}$ *is a* $\mathcal{C}^{\infty}$ *(real analytical) function. Then*

$$f(y) = f(x) + \lim_{n \to \infty} \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} (y - x)^k .$$

Proof. See Exercise A.27. □

A somewhat more useful corollary, which was used in the text, is

Corollary A.3. *Suppose that* $f : [a, b] \to \mathbb{R}$ *is twice continuously differentiable and concave. Then for any* $x, y \in [a, b]$,

$$f(y) \leq f(x) + f'(x)(y - x) .$$

Proof. By Theorem A.20, $f(y) = f(x) + f'(x)(y - x) + f''(z)(y - x)^2 / 2$ for some z between x and y. Since, by definition, $f''(z) \leq 0$ for a concave function, the conclusion follows. □

A.8. Functions of Several Variables and the Inverse and Implicit Function Theorems

Throughout this section, I limit myself to differentiation in Euclidean spaces, that is, our interests will be with a mapping

$$\phi : X \to Y,$$

where

$$X \subset \mathbb{R}^{K_X} \text{ and } Y \subset \mathbb{R}^{K_Y},$$

for some integers K_X and K_Y. The theory of differentiation and the types of results that I will present below can be developed in more general spaces than Euclidean spaces. For example, Luenberger's (1969) classic treatment of general optimization problems considers X and Y to be Banach spaces (complete normed vector spaces, which allow for a convenient definition of *linear operators*, see Section 10.1 below). Similarly, some of these ideas can be developed in general metric spaces. Nevertheless, for the results presented here, restricting attention to Euclidean spaces is without loss of any generality and enables me to reduce notation and avoid unnecessary complexities.

The case $K_x = K_Y = 1$ was treated in the previous section. Building on the results and the intuitions of that section, let us now move to more general mappings. For $\phi : X \to Y$ (where $X \subset \mathbb{R}^{K_X}$ and $Y \subset \mathbb{R}^{K_Y}$), the equivalent of the derivative is the linear operator $J(x) : X \to Y$, referred to as the *Jacobian* or the *Jacobian Matrix* (in this chapterI do *not* use boldface letters to denote matrices and vectors). With analogy to (A.2), for $x \in X'$, where X' is an open set with $\phi(X') \subset Y$ well-defined, the Jacobian is a matrix $J(x)$ such that

$$\lim_{h \to 0} \left| \frac{\phi(x+h) - \phi(x) - J(x)h}{\|h\|} \right| = 0, \tag{A.3}$$

where $h \in X$ is a vector, and $\|h\|$ is the usual Euclidean norm of the vector h. We say that the mapping ϕ is a differentiable at x if the above limit exists and defines the unique $J(x)$. In this case, the derivative of $\phi(x)$ is denoted by $\phi'(x) = J(x)$. Once again, the derivative is a linear operator; it depends on x, but it assigns the value $J(x)h$ to any vector h such that $x+h$ in X'. The Jacobian matrix is also sometimes denoted by $D\phi(x)$, which is probably a better notation than $J(x)$, since it indicates which function we are referring to. Moreover, using this notation, we can denote the matrix of partial derivatives by $D_{x_1}\phi(x_1, x_2)$ for $x_1 \in \mathbb{R}^{K_1}$ and $x_2 \in \mathbb{R}^{K_2}$.

The expression for the derivative (or the Jacobian) in (A.3) also immediately shows that one could define weaker notions of derivatives for only some types of vectors. Such weaker notions would include *directional* and *Gateaux* derivatives, which are discussed in Luenberger (1969). For our purposes, there is no gain in introducing these weaker notions, though examples of functions that are only differentiable from some direction are easy to construct.

Example A.14. An immediate example of a function that has directional derivatives but is not differentiable according to the stronger notion here would be f defined by $f(x) = x$ for $x \geq 0$ and $f(x) = -x$ for $x < 0$, which has derivatives from the left and the right at 0, but is not differentiable according to (A.1) or (A.3), since a unique $f'(x)$ does not exist.

Let us from now focus on mappings that are differentiable. For such mappings, the convenient feature of the Jacobian matrix is that it has a simple representation in terms of *partial derivatives*. In particular, suppose $X \subset \mathbb{R}^{K_X}$, then $\phi : X \to \mathbb{R}$ is also referred to as a *function of several variables*. Its partial derivatives with respect to each component of X are defined identically to the derivative of a real-valued function of one variable (holding all the other variables constant). Let $x = (x_1, ..., x_{K_X})$ and assume that ϕ is differentiable with respect to its kth component. Then the kth partial derivative of ϕ is

$$\frac{\partial \phi(x_1, ..., x_{K_X})}{\partial x_k} = \phi_k(x),$$

where

$$\phi_k(x) = \lim_{h \to 0} \frac{\phi(x_1, ., x_{k-1}, x_k + h, x_{k+1}, .., x_{K_X}) - \phi(x_1, ., x_{k-1}, x_k, x_{k+1}, .., x_{K_X})}{h}.$$

Now assuming that ϕ has partial derivatives with respect to each x_k for $k = 1, ..., K_X$, the Jacobian in this case is simply a *row vector,*

$$J(x) = \begin{pmatrix} \phi_1(x) & \cdot & \cdot & \cdot & \phi_{K_X}(x) \end{pmatrix}.$$

A general mapping $\phi : X \to Y$, where Y is a subset of $\mathbb{R}^{K_Y}$ can then be thought of as consisting of K_Y real-valued functions of several variables, $\phi^1(x)$,...,$\phi^{K_Y}(x)$. We can define the partial derivatives of each of these functions in a similar fashion and denote them by $\phi_k^j(x)$. The Jacobian can then be written as

$$J(x) = \begin{pmatrix} \phi_1^1(x) & \cdot & \cdot & \cdot & \phi_{K_X}^1(x) \\ \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ \cdot & & & & \cdot \\ \phi_1^{K_Y}(x) & \cdot & \cdot & \cdot & \phi_{K_X}^{K_Y}(x) \end{pmatrix}.$$

Higher-order derivatives can be defined in a similar fashion, though they will now correspond to higher-dimensional objects. When $\phi : X \to X$, $J(x)$ is a $K_X \times K_X$ matrix, and in this case, we can investigate whether it is invertible or not (i.e., whether the inverse $J^{-1}(x)$ at x exists). This will play an important role in the Inverse Function and Implicit Function Theorems below.

Definition A.30. *A mapping ϕ is of the class $\mathcal{C}^n$ on some set X' if it has continuous derivatives up to the nth order.*

Fact A.18. *A mapping $\phi : X \to Y$ with $X \subset \mathbb{R}^{K_X}$, $Y \subset \mathbb{R}^{K_Y}$ and X open is of class $\mathcal{C}^1$ on X if $\phi_k^j(x)$ for $k = 1, ..., K_X$ and $j = 1, ..., K_Y$ exist and are continuous functions of x for each $x \in X$.*

Proof. See Exercise A.28. □

Taylor's Theorem and its corollaries can be generalized to mappings discussed here. Let me simply state the result for $\phi : X \to \mathbb{R}$ with $X \subset \mathbb{R}^{K_X}$. Moreover, let $D\phi$ and $D^2\phi$ denote the row vector of first derivatives and the Jacobian of ϕ, let $\|y - x\|$ be the Euclidean norm of the K_X dimensional vector $y - x$, and let z^T be the transpose of vector z. The following is a simpler version of the equivalent form of Taylor's Theorem in Corollary A.2. Its proof is similar to that of Theorem A.20 but is longer and requires more notation. Since this theorem does not play an important role in the results in this book, I omit the proof.

Theorem A.21. ***(Taylor's Theorem II)*** *Suppose that $\phi : X \to \mathbb{R}$ is a $\mathcal{C}^2$ function and moreover its third derivative $D^3\phi(x)$ exists for all $x \in X$. Then for any x and $y \neq x$ in X, we have that*

$$\phi(y) = \phi(x) + D\phi(x)^T(y - x) + (y - x)^T D^2\phi(x)(y - x) + R_3(y - x, x),$$

where $R_3(y - x, x) / \|y - x\|^3 \to 0$ as $y \to x$.

The following two theorems are the basis of much of the comparative static results in economic models. They are therefore among the most important mathematical results for economic analysis. Consider a mapping $\phi : X \to X$ for $X \subset \mathbb{R}^{K_X}$. One obvious question is whether this mapping will have *an inverse* $\phi^{-1} : X \to X$. If for some subset X' of X, ϕ is single-valued, has an inverse ϕ^{-1}, and also its inverse is also a single valued, then we say that it is *one-to-one*.

Theorem A.22. ***(The Inverse Function Theorem)*** *Consider a $\mathcal{C}^1$ mapping $\phi : X \to X$ for $X \subset \mathbb{R}^{K_X}$. Suppose that the Jacobian of ϕ, $J(x)$ evaluated at some interior point x^* of X is invertible. Then, there exists open sets X' and X'' in X such that $x^* \in X'$, $\phi(x^*) \in X''$, and ϕ is one-to-one on X' with $\phi(X') = X''$. Moreover, $\phi^{-1}(\phi(x)) = x$ for all x in X' and ϕ^{-1} is also a $\mathcal{C}^1$ mapping.*

The proof of this theorem is not difficult, but somewhat long and can be found in any real analysis book, so I will not provide it here. The following theorem is directly used in most comparative static exercises in economics.

Theorem A.23. ***(The Implicit Function Theorem)*** *Consider a $\mathcal{C}^1$ mapping $\phi : X \times Y \to Y$ with $X \subset \mathbb{R}^{K_X}$ and $Y \subset \mathbb{R}^{K_Y}$. Suppose that $(x^*, y^*) \in X \times Y$, $\phi(x^*, y^*) = 0$, all the entries of the Jacobian of ϕ with respect to (x, y), $D_{(x,y)}\phi(x^*, y^*)$, are finite and $D_y\phi(x^*, y^*)$ is invertible Then there exists an open set $X' \ni x^*$ and a unique $\mathcal{C}^1$ mapping $\gamma : X' \to Y$ such that $\gamma(x^*) = y^*$ and*

$$\phi(x, \gamma(x)) = 0 \quad \text{(A.4)}$$

for all $x \in X'$.

This theorem is called the Implicit Function Theorem because the mapping γ is defined implicitly. Exercise 6.5 in Chapter 6 provided the proof of a special case of this theorem. The more general case can also be proved with exactly the same methods as in that exercise. An alternative proof uses the Inverse Function Theorem. Since the former proof has already been discussed and the latter one is contained in most real analysis books, I will not provide the proof of Theorem A.23 here.

It is important to note that the main utility of this theorem comes from the fact that since ϕ and γ are $\mathcal{C}^1$ and (A.4) holds for an open set around x^*, (A.4) can be differentiated with respect to x, to obtain an expression for how the solutions y to the set of equations $\phi(x, y) = 0$ behave as a function of x. If we think of x as representing a set of parameters and of y as the endogenous variables determined by some economic relationship summarized by (A.4), then this procedure can tell us how the endogenous variables change in response to the changes in the environment captured by the parameter x. I made repeated use of this approach throughout the book.

A.9. Separation Theorems

In this section, I will briefly discuss the separation of convex disjoint sets using linear functionals (or hyperplanes). These results form the basis of the Second Welfare Theorem, provided in Theorem 5.7 in Chapter 5. They also provide the basis of many important results in constrained optimization (see Section A.10).

For this section, we take X be a *vector space* (linear space). Recall from Section A.6) that this implies: if $x, y \in X$ and λ is a scalar, then $x + y \in X$ and $\lambda x \in X$. The element of X with the property that $x = \lambda x$ for all scalar λ is denoted by θ. Moreover, we have:

Definition A.31. *The real-valued nonnegative function $\|\cdot\| : X \to \mathbb{R}_+$ is taken to be a* ***norm*** *on X, which implies that for any $x, y \in X$ and any scalar λ, we have*

(1) ***(Properness)*** $\|x\| \geq 0$ *and* $\|x\| = 0$ *if and only if* $x = \theta$.
(2) ***(Linearity)*** $\|\lambda x\| = |\lambda| \|x\|$.
(3) ***(Triangle Inequality)*** $\|x + y\| \leq \|x\| + \|y\|$.

A vector space equipped with a norm is a ***normed vector space****. A complete normed vector space is a* ***Banach space****.*

If a function $p : X \to \mathbb{R}_+$ satisfies properness and triangle inequality, but not necessarily the linearity condition, then it is referred to as a *semi-norm*.

Many of the metric spaces given in Example A.1 are also normed vector spaces with the appropriate norm. In fact, a simple way of obtaining the norm in many cases is to take the distance function d and try the norm $\|x\| = d(x, \theta)$. Notice, however, that this will not always work, since metrics do not need to satisfy the *linearity* condition in Definition A.31.

Example A.15. The first four spaces are normed vector spaces, while the fifth one is not.

(1) For any $X \subset \mathbb{R}^K$, let x_i be the ith component of $x \in X$. Then the *K-dimensional Euclidean space* is a normed vector space with norm given by $\|x\| = \left(\sum_{i=1}^K |x_i|^{1/2}\right)^2$.
(2) Let $X \subset \mathbb{R}^K$ and consider the set of continuous real-valued functions $f : X \to \mathbb{R}$ denoted by $\mathbf{C}(X)$. $\mathbf{C}(X)$ is a normed vector space with $\|f\| = \sup_{x \in X} |f(x)|$.
(3) $\ell \subset \mathbb{R}^\infty$, the set consisting of infinite sequences of real numbers, is a normed vector space with norm $\|x\|_p = (\sum_{i=1}^\infty |x_i|^p)^{1/p}$ for $1 \leq p < \infty$ or by $\|x\|_\infty = \sup_i |x_i|$. For any $p \in [1, \infty]$, the corresponding normed vector space is denoted by ℓ_p.The one of greatest interest for us is ℓ_∞.
(4) Let $\mathbf{c} \subset \mathbb{R}^\infty$, be the set consisting of infinite sequences of real numbers that are equal to zero after point (e.g., $(x_1, ..., x_M, 0, 0, ...)$, where $M \in \mathbb{N}$. Let the sup norm on $\mathbf{c}$ be defined as usual $\|x\|_\infty = \sup_i |x_i|$. Then $\mathbf{c}$ with the sup norm is a normed vector space.

(5) For X nonempty, consider the discrete metric $d(x,y) = 1$ if $x \neq y$ and $d(x,y) = 0$ if $x = y$. The metric space (X,d) is *not* a normed vector space.

When the norm is understood implicitly, we refer to X as a normed vector space.

Definition A.32. *Let X be a normed vector space. Then $\phi : X \to \mathbb{R}$ is a **linear functional** on X if for any $x, y \in X$ and any scalars λ and μ, we have*

$$\phi(\lambda x + \mu y) = \lambda \phi(x) + \mu \phi(y).$$

Linear functionals on normed vector spaces have many nice properties. For example, if $X \subset \mathbb{R}^K$, then any linear functional on X can be expressed as an *inner product* of x with another K-dimensional vector η, i.e., $\phi(x) = \eta \cdot x = \sum_{i=1}^{K} \eta_i x_i$, where $\eta = (\eta_1, ..., \eta_K) \in \mathbb{R}^K$. Therefore, on Euclidean spaces, linear functionals correspond to inner products. In many other spaces, linear functionals will have properties similar to inner products. Some other nice properties of linear functionals are provided in the following result.

Theorem A.24. ***(Continuity of Linear Functionals)*** *Let X be a normed vector space. Then:*

(1) *The linear functional $\phi : X \to \mathbb{R}$ is continuous on X if and only if it is continuous at θ.*
(2) *The linear functional $\phi : X \to \mathbb{R}$ is continuous on X if and only if it is bounded in the sense that there exists M such that $|\phi(x)| \leq M \|x\|$ for all $x \in X$.*

Proof. **(Part 1)** We only need to prove that ϕ is continuous on X if it is continuous at θ. Suppose that it is continuous at θ. Fix an arbitrary $x \in X$ and consider a sequence in X, $\{x_n\}$, converging to x. By the linearity of ϕ,

$$|\phi(x_n) - \phi(x)| = |\phi(x_n - x + \theta) - \phi(\theta)|.$$

Since $x_n \to x$, $x_n - x + \theta \to \theta$ and since ϕ is continuous at θ, $\phi(x_n - x + \theta) \to \phi(\theta)$. Therefore $|\phi(x_n) - \phi(x)| \to 0$, proving that ϕ this continuous at x.

(Part 2) To prove the "if" part, suppose that ϕ is bounded. Consider a sequence $\{x_n\}$ converging to θ. This implies

$$|\phi(x_n)| \leq M \|x_n\|$$

and since $x_n \to \theta$, $|\phi(x_n)| \to 0$, proving that ϕ is continuous at θ. Then by Part 1, ϕ is continuous on X.

To prove the "only if" part, suppose that ϕ is continuous at θ. Fix $\varepsilon > 0$. Then there exists $\delta > 0$ such that for $\|x\| < \delta$, $|\phi(x_n)| < \varepsilon$. Note that $\delta x / \|x\|$ has norm equal to δ.

Therefore,

$$\begin{aligned}|\phi(x)| &= \left|\phi\left(\frac{\delta x}{\|x\|}\right)\right| \cdot \frac{\|x\|}{\delta} \\ &< \varepsilon \cdot \frac{\|x\|}{\delta} \\ &= M\|x\|,\end{aligned}$$

with $M = \varepsilon/\delta$, completing the proof. □

The smallest M that satisfies $|\phi(x)| \leq M\|x\|$ for all $x \in X$ is defined as the norm of the linear functional ϕ and is sometimes denoted by $\|\phi\|$. Theorem A.24 therefore implies that a continuous linear functional has a finite norm.

DEFINITION A.33. *Let X be a normed vector space. The space of all continuous linear functionals on X is the **normed dual** of X and is denoted by X^*.*

Dual spaces have many nice features. For example:

FACT A.19. *If X is a normed vector space, then its dual X^* is a Banach space.*

The following example gives the duals of some common spaces (see Exercise A.30).

EXAMPLE A.16. (1) For any integer K, the dual of $\mathbb{R}^K$ is $\mathbb{R}^K$.

(2) For any $p \in [1, \infty)$, the dual of ℓ_p is ℓ_q where $p^{-1} + q^{-1} = 1$.

A nonobvious fact is the following. Let $\mathbf{c} = \{x = (x_1, x_2, ...) \in \ell_\infty : \lim_{n\to\infty} x_n = 0\}$.

FACT A.20. (1) *The dual of ℓ_∞ is not ℓ_1 (it contains ℓ_1).*

(2) *The dual of $\mathbf{c}$ is ℓ_1.*

Dual spaces are particularly useful in economics, since if X is a commodity space, then X^* corresponds to the space of "price functionals" for X. For example, the dual of $X \subset \mathbb{R}^K$ is $X^* \subset \mathbb{R}^K$ and indeed consists of functionals of the form $\phi(x) = \sum_{i=1}^K \eta_i x_i$ as noted above. Loosely speaking, we can interpret the η_is as "prices" corresponding to the commodity vector x, so that $\phi(x)$ is the "cost" of x at the price vector η. The particular usefulness of this construction for economics stems from the following famous theorem. We refer to a linear functional ϕ defined on X as *nonzero* if it is not identically equal to zero for all $x \in X$.

THEOREM A.25. ***(Geometric Hahn-Banach Theorem)*** *Let X be a normed vector space and let $X^1, X^2 \subset X$. Suppose that X^1 and X^2 are convex, $IntX^1 \neq \varnothing$ and $X^2 \cap IntX^1 = \varnothing$, then there exists a nonzero continuous linear functional ϕ on X such that*

$$\phi(x^1) \leq c \leq \phi(x^2) \text{ for all } x^1 \in X^1,\ x^2 \in X^2 \text{ and some } c \in \mathbb{R}.$$

This theorem is a straightforward consequence of the Hahn-Banach Theorem. The Hahn-Banach Theorem states that if ϕ is a continuous linear functional on a subspace M of X and is dominated by a semi-norm $p(x)$, i.e., $f(x) \leq p(x)$ for all $x \in M$, then there is an extension Φ of ϕ to the entire X such that Φ is a continuous linear functional on X, $\Phi(x) = \phi(x)$ for all $x \in M$ and $\Phi(x) \leq p(x)$ for all $x \in X$. This theorem therefore establishes that normed vector spaces are "abundant" in linear functionals. More important for our purposes, it also implies Theorem A.25. Since its proof is not particularly useful for our purposes here, it is omitted. A proof of this theorem together with further separation theorems can be found in Conway (2000), Kolmogorov and Fomin (1970), and Luenberger (1969).

Notice the non-intuitive requirement that $\text{Int}X^1 \neq \varnothing$, which implies that X^1 should contain an interior point. This is not a stringent requirement when X is a subset of the Euclidean space (and in fact, this condition is not even necessary in that case). However, some common infinite dimensional normed vector spaces, such as ℓ_p for $p < \infty$ do not contain interior points when we restrict attention to their economically relevant subspaces, that is, ℓ_p^+, which restricts all sequences to consist of nonnegative numbers (this is rather nonobvious, but Exercise A.31 illustrates why). This might be a problem if we wished to model the allocations (for example, the sequence of consumption levels or capital stocks) in an infinite-horizon economy as elements of ℓ_p^+. Nevertheless, this is not an issue when we focus on the economically more natural space of sequences of allocations ℓ_∞, because ℓ_∞^+ does contain interior points (see Exercise A.32). The only complication that arises from the use of ℓ_∞ is that not all linear functionals on ℓ_∞ have an inner product representation and thus may not correspond to economically meaningful price systems (recall Fact A.20). This problem can be handled, however, by making somewhat stronger assumptions on preferences and technology to ensure that the relevant linear functionals on ℓ_∞ have the desired inner product representation. This is the reason why the Second Welfare Theorem, Theorem 5.7, impose additional conditions on preferences and technology.

It is also useful to note the following immediate corollary of Theorem A.25.

Theorem A.26. ***(Separating Hyperplane Theorem)*** *Let $X \subset \mathbb{R}^K$ and $X^1, X^2 \subset X$. Suppose that X^1 and X^2 are convex and $X^2 \cap \text{Int}X^1 = \varnothing$, then there exists a hyperplane $H = \left\{x \in X\colon \sum_{i=1}^K \eta_i x_i = c \text{ for } \eta \in \mathbb{R} \text{ and } \eta \neq 0\right\}$ such the H separates X^1 and X^2, or in other words,*

$$\eta \cdot x^1 \leq c \leq \eta \cdot x^2 \text{ for all } x^1 \in X^1,\ x^2 \in X^2 \ ,$$

where recall that $\eta \cdot x = \sum_{i=1}^K \eta_i x_i$.

Note that the statement of this theorem disposes of the hypothesis that $\text{Int}X^1 \neq \varnothing$, which is not necessary when the two sets are subsets of Euclidean spaces. Moreover, the theorem does not add the qualification that the hyperplane H is "nonzero" (in the same way as Theorem A.25 did for linear functionals), since the definition of the hyperplane already incorporates this requirement.

A.10. Constrained Optimization

Many of the problems we encountered in this book are formulated as constrained optimization problems. Chapters 6, 7, and 16 dealt with dynamic (infinite-dimensional) constrained optimization problems. Complementary insights about these problems can be gained by using the separation theorems of the previous section. Let me illustrate this here by focusing on finite-dimensional optimization problems. In particular, suppose that $X \subset \mathbb{R}^K$ and consider the maximization problem

$$\begin{aligned}&\sup_{x\in X} f(x)\\ &\text{subject to}\\ &g(x) \leq 0,\end{aligned} \tag{A.5}$$

where $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}^N$ for some integer N.

The constrained maximization problem (A.5) satisfies the *Slater* condition if there exists $x' \in X$ such that $g(x') < 0$ (meaning that each component of the mapping g takes a negative value). This is equivalent to the set $G = \{x{:}g(x) \leq 0\}$ having an interior point. We say that g is convex, if each component function of g is convex. This implies that the set G is also convex (but the converse is not necessarily true (see Exercise A.33).As is usual, we define the Lagrangian function as

$$\mathcal{L}(x, \lambda) \equiv f(x) - \lambda \cdot g(x)$$

for $\lambda \in \mathbb{R}^N_+$. The vector λ is referred to as the *Lagrange multiplier*. Note also that $\lambda \cdot g(x)$ denotes the inner product of the two N dimensional vectors. A central theorem in constrained maximization is the following.

Theorem A.27. ***(The Saddle Point Theorem)*** *Suppose that in (A.5) f is a quasi-concave function, g is convex and the **Slater condition** is satisfied. Then:*

(1) *If x^* is a solution to (A.5), then there exists $\lambda^* \in \mathbb{R}^N_+$ such that*

$$\mathcal{L}(x, \lambda^*) \leq \mathcal{L}(x^*, \lambda^*) \leq \mathcal{L}(x^*, \lambda) \text{ for all } x \in X \text{ and } \lambda \in \mathbb{R}^N_+ . \tag{A.6}$$

*In this case, (x^*, λ^*) satisfies the **complementary slackness condition***

$$\lambda^* \cdot g(x^*) = 0. \tag{A.7}$$

(2) *If $(x^*, \lambda^*) \in X \times \mathbb{R}^N_+$ satisfies $g(x^*) \leq 0$ and (A.6), then x^* is a solution to (A.5).*

Proof. The proof follows from Theorem A.26.

(Part 1) Consider the space $Y = \mathbb{R}^{N+1}$, with subsets

$$Y^1 = \{(a, b) \in Y{:}a > f(x^*) \text{ and } b < 0\},$$

and

$$Y^2 = \{(a, b) \in Y{:}\ \exists x \in X \text{ with } a \leq f(x) \text{ and } b \geq g(x)\},$$

where $b < 0$ again means that each element of the N-dimensional vector b is negative. Y^1 is clearly convex. Moreover, the quasi-concavity of f and the convexity of g ensure that Y^2 is also convex.

By the hypothesis that x^* is a solution to (A.5), the two sets are disjoint. Then Theorem A.26 implies that there exists a hyperplane separating these two sets. In other words, there exists a nonzero vector $\eta \in \mathbb{R}^{N+1}$ such that

$$\eta \cdot y^1 \leq c \leq \eta \cdot y^2 \text{ for all } y^1 \in Y^1, y^2 \in Y^2.$$

Moreover, the same conclusion holds for all $y^2 \in \overline{Y}^2$. Therefore, let $\eta = (\rho, \lambda)$, so that

$$\rho a^1 + \lambda \cdot b^1 \leq \rho a^2 + \lambda \cdot b^2 \text{ for all } \left(a^1, b^1\right) \in Y^1,\ \left(a^2, b^2\right) \in \overline{Y}^2. \tag{A.8}$$

For $(f(x^*), 0) \in \overline{Y}^2$, we have

$$\rho a^1 + \lambda \cdot b^1 \leq \rho f(x^*)$$

for all $\left(a^1, b^1\right) \in \overline{Y}^1$. Now taking $a^1 = f(x^*)$ and $b^1 < 0$ implies $\lambda \geq 0$ (suppose instead that one component of the vector λ is negative; then take b^1 to have zeros everywhere except for that component, yielding a contradiction to the preceding inequality). Similarly, taking $b^1 = 0$ and $a^1 > f(x^*)$ implies $\rho \leq 0$. Moreover, by the definition of a hyperplane, either ρ or a component of λ must be strictly positive.

Next the optimality of x^* implies that for any $x \in X$, $(f(x), g(x)) \in Y^2$, we have $f(x) \leq f(x^*)$ or x violates the constraint, thus $g(x) \geq 0$, with at least one strict inequality. Therefore, from (A.8),

$$\rho f(x^*) \leq \rho f(x) + \lambda \cdot g(x) \text{ for all } x \in X \tag{A.9}$$

Suppose to obtain a contradiction that $\rho = 0$. Then, by the Slater condition, there exists $x' \in X'$ such that $g(x') < 0$, so that $\lambda \cdot g(x') < 0$ for any nonzero vector λ, violating (A.9). Therefore, we must have $\lambda = 0$. However, this in turn contradicts the fact that the separating hyperplane is nonzero (so that we cannot have both $\rho = 0$ and $\lambda = 0$). Therefore, $\rho < 0$. Now define:

$$\lambda^* = -\frac{\lambda}{\rho} \geq 0.$$

The complementary slackness condition now follows immediately from (A.9). In particular, evaluate the right-hand side at $x^* \in X$, which implies $\lambda \cdot g(x^*) \geq 0$. Since $\lambda \geq 0$ and $g(x^*) \leq 0$, we must have

$$\lambda \cdot g(x^*) = -\rho\left(\lambda^* \cdot g(x^*)\right) = 0.$$

Now using the complementary slackness condition and (A.9) together with the fact that $\rho < 0$, we have

$$\mathcal{L}(x, \lambda^*) = f(x) - \lambda^* \cdot g(x) \leq f(x^*) = \mathcal{L}(x^*, \lambda^*) \text{ for all } x \in X,$$

which establishes the first inequality in (A.6). To establish the second inequality, again use the complement the slackness condition and the fact that $g(x^*) \leq 0$ to obtain

$$\mathcal{L}(x^*, \lambda^*) = f(x^*) \leq f(x^*) - \lambda \cdot g(x^*) = \mathcal{L}(x^*, \lambda) \text{ for all } \lambda \in \mathbb{R}_+^N,$$

which completes the proof of the first part.

(Part 2) Suppose to obtain a contradiction that (A.6) holds, but x^* is not a solution to (A.5). This implies that there exists $x' \in X$ with $g(x') \leq 0$ and $f(x') > f(x^*)$. Then,

$$f\left(x'\right) - \lambda^* \cdot g\left(x'\right) > f(x^*) - \lambda^* \cdot g(x^*),$$

which exploits the fact that $\lambda^* \cdot g(x^*) = 0$ and $\lambda^* \cdot g(x') \leq 0$ (since $\lambda^* \geq 0$ and $g(x') \leq 0$). But this contradicts (A.6) and establishes the desired result. □

Exercise A.34 shows that the Slater condition cannot be dispensed with in this theorem.

An immediate corollary of (A.6) is that if $x^* \in \text{Int}X$ and if f and g are differentiable, then

$$D_x f(x^*) = \lambda^* \cdot D_x g(x^*), \tag{A.10}$$

where, as usual, $D_x f$ and $D_x g$ denote the vector of derivatives of f and the Jacobian of g. (A.10) is the usual first-order necessary condition for interior constrained maximum. In this case, (A.10), together with $g(x^*) \leq 0$, is also sufficient for a maximum (which follows both from Theorems A.27 and A.14).

The next result is the famous Kuhn-Tucker Theorem, which shows that (A.10) is necessary for an interior maximum (provided that f and g are differentiable) even when the quasi-concavity and the convexity assumptions do not hold.

THEOREM A.28. ***(Kuhn-Tucker Theorem)*** *Consider the constrained maximization problem*

$$\sup_{x \in X} f(x)$$
subject to
$$g(x) \leq 0$$
$$h(x) = 0$$

where $f : X \to \mathbb{R}$, $g : X \to \mathbb{R}^N$ and $h : X \to \mathbb{R}^M$ are differentiable and $X \subset \mathbb{R}^K$ (for some integers K, N and M). Let $x^* \in \text{Int}X$ be a solution to this maximization problem and suppose that $N_1 \leq N$ of the inequality constraints are ***active****, in the sense that they hold as equality at x^*. Define $\bar{h} : X \to \mathbb{R}^{M+N_1}$ to be the mapping of these N_1 active constraints stacked with $h(x)$ (so that $\bar{h}(x^*) = 0$). Suppose that the following* ***constraint qualification condition*** *is satisfied: the Jacobian matrix $D_x\left(\bar{h}(x^*)\right)$ has rank N_1+M. Then, the following Kuhn-Tucker condition is satisfied. There exists $\lambda^* \in \mathbb{R}_+^N$ and $\mu^* \in \mathbb{R}^M$ such that*

$$D_x f(x^*) - \lambda^* \cdot D_x g(x^*) - \mu^* \cdot D_x h(x^*) = 0, \tag{A.11}$$

and the ***complementary slackness condition***

$$\lambda^* \cdot g(x^*) = 0$$

holds.

Proof. **(Sketch)** The constraint qualification condition ensures that there exists a $N_1 + M$-dimensional manifold at x^*, defined by the equality and active inequality constraints. Since g and h are differentiable, this manifold is differentiable at x^*. Let $v_\varepsilon(x)$ denote a feasible direction along this manifold for small $\varepsilon \in \mathbb{R}^K$, in particular, such that $x^* \pm \varepsilon v_\varepsilon(x^* + \varepsilon)$ remains along this manifold and thus satisfies $D_x \bar{h}(x^*) \cdot \varepsilon v_\varepsilon(x^* + \varepsilon) = 0$. For ε sufficiently small, the $N - N_1$ nonactive constraints are still satisfied, thus $x^* \pm \varepsilon v_\varepsilon(x^* + \varepsilon)$ is feasible. If $D_x f(x^*) \cdot \varepsilon v_\varepsilon(x^* + \varepsilon) \neq 0$, then $f(x^* + \varepsilon v_\varepsilon(x^* + \varepsilon)) > f(x^*)$ or $f(x^* + \varepsilon v_\varepsilon(x^* + \varepsilon)) > f(x^*)$, implying that x^* cannot be a local (and thus global) maximum. Now consider the $M + N_1 + 1 \times K$ dimensional matrix A, where the first row is $D_x f(x^*)^T$ and the rest is given by $D_x\left(\bar{h}(x^*)\right)$. The preceding argument implies that for all nonzero $\varepsilon \in \mathbb{R}^K$ such that $D_x \bar{h}(x^*) \cdot \varepsilon v_\varepsilon(x^* + \varepsilon) = 0$, we also have $A \cdot (\varepsilon + v_\varepsilon(x^* + \varepsilon)) = 0$. Therefore, both $D_x \bar{h}(x^*)$ and A have the same rank, which by the constraint qualification condition is equal to $M + N_1$. Since A has $M + N_1 + 1$ rows, this implies that the first row of A must be a linear combination of its remaining $M + N_1$ rows, which equivalently implies that there exists an $M + N_1$ vector $\bar{\mu}$ such that $D_x f(x^*) = \bar{\mu} D_x \bar{h}(x^*)$. Assigning zero multipliers to all nonactive constraints, this is equivalent to (A.11). The complementary slackness condition then follows immediately since we have zero multipliers for the nonactive constraints and $g_j(x^*) = 0$ for the active constraints. □

The constraint qualification condition, which required that the active constraints should be linearly independent, plays a similar role to the Slater condition in Theorem A.27. Exercise A.35 shows that this constraint qualification condition cannot be dispensed with (though somewhat weaker conditions can be used instead of the full rank condition used in Theorem A.28).

We have made frequent use of the necessary and sufficient condition (A.10) and the necessary condition (A.11) throughout the text.

A.11. Exercises

Exercise A.1. *

(1) Prove the *Minkowski's inequality* that for any $x = (x_1, x_2, ..., x_K) \in \mathbb{R}^K$, $y = (y_1, y_2, ..., y_K) \in \mathbb{R}^K$ with $K \in \mathbb{N}$ and any $p \in [1, \infty)$,

$$\left(\sum_{k=1}^{K} |x_k + y_k|^p\right)^{1/p} \leq \left(\sum_{k=1}^{K} |x_k|^p\right)^{1/p} + \left(\sum_{k=1}^{K} |y_k|^p\right)^{1/p}.$$

(2) Prove the generalization of this inequality for $K = \infty$.

Exercise A.2. Using Minkowski's inequality show that the metric spaces in Example A.1 part 1 satisfy the triangle inequality.

Exercise A.3. Show that the *sup metric* $d_\infty(f, g) = \sup_{x \in X} |f(x) - g(x)|$ on $\mathbf{C}(X)$ in Example A.1 satisfies the triangle inequality.

Exercise A.4. Using the definition of equivalent metrics in Definition A.4, show that if d and d' are equivalent metrics on X, and a subset X' of X is open according to the collection of neighborhoods generated by metric d, then it is open according to the collection of neighborhoods generated by metric d'.

Exercise A.5. Prove that all of the family of metrics d_p defined on the Euclidean space in Example A.1 are equivalent according to Definition A.4.

Exercise A.6. Prove that $X' \subset X$ is closed if and only if every convergent sequence $\{x_n\}_{n=1}^{\infty}$ in X' has has a limit point $x_\infty \in X'$.

Exercise A.7. Prove Fact A.2.

Exercise A.8. Prove Fact A.3.

Exercise A.9. Prove Fact A.4.

Exercise A.10. Using an argument similar to that in the proof of Theorem A.3 show that if (X, d) is a metric space and $\phi : X \to Y$ a continuous mapping, then $f(X')$ is a connected subset of Y for every connected subset X' of X.

Exercise A.11. Prove Fact A.5.

Exercise A.12. Prove Fact A.7.

Exercise A.13. Show that every metric space will have the Hausdorff property.

Exercise A.14. Prove Lemma A.2.

Exercise A.15.

(1) Show that if $X = \prod_{\alpha=1}^{K} X_\alpha$, i.e., if X is a finite-dimensional product, then the box and the product topologies are equivalent in the sense that they define the same open sets (recall Definition A.4 for equivalence of metrics, which applies to the equivalence of topologies as well).

(2) Show that if X is not finite-dimensional, then the box and the product topologies are not equivalent.

(3) Show that projection maps are always continuous in the box topology.

Exercise A.16. * Suppose that X_α is a metric space for each $\alpha \in A$. Show that the space $X = \prod_{\alpha \in A} X_\alpha$ endowed with the product topology satisfies the Hausdorff property.

Exercise A.17. Prove that the properties of upper and lower hemi-continuity in Definition A.24 imply the properties in Definition A.25 when X and Y are Euclidean.

Exercise A.18.

(1) Show that $G(x) = \{y \in \mathbb{R} : xy \leq 0\}$ is not a continuous correspondence.

(2) Show that if $G_1(x)$ and $G_2(x)$ are continuous, $G_1(x) \cap G_2(x)$ is not necessarily continuous. [Hint: consider $G_1(x) = (-\infty, x]$ and $G_2(x) = \{a, b\}$ for some $a \neq b$].

EXERCISE A.19. Prove Fact A.14.

EXERCISE A.20. Prove Fact A.16.

EXERCISE A.21. (1) Prove Part 2 of Theorem A.11.

(2) Prove that the set

$$\mathcal{F} = \left\{ [y(t)]_{t=0}^{\infty} : \dot{x}(t) = g(x(t), y(t)) \text{ with } x(0) = x_0 \text{ and } \lim_{t\to\infty} x(t) \geq x_1 \right\}$$

is compact in the product topology if it is bounded in the sense that feasible sequences $[y(t)]_{t=0}^{\infty}$ satisfy $\limsup y(t) < \infty$.

EXERCISE A.22. Give an example of an upper hemi-continuous correspondence from $[0, 1]$ into $[0, 1]$ that is not convex-valued and does not have a fixed point.

EXERCISE A.23. Consider a N-person normal-form game. Player i's strategy is denoted by $a_i \in A_i$ and his payoff function is given by the real-valued function $u_i(a_1, ..., a_N)$.

(1) Using Theorems A.13, A.14, and A.16, prove that if each A_i is nonempty, compact and convex and each u_i is continuous in a_j for $j \neq i$ and continuous and quasi-concave in a_i, there exists a strategy profile $(a_1^*, ..., a_N^*)$ that constitutes a pure strategy Nash equilibrium.
(2) Give counterexamples showing why each of the assumptions of (1) compactness of A_i, (2) convexity of A_i, (3) continuity of u_i, and (4) quasi concavity of u_i in own strategy cannot be dispensed with.

EXERCISE A.24. Prove Fact A.17.

EXERCISE A.25. Prove Theorem A.18.

EXERCISE A.26. Prove Theorem A.19. [Hint: use Theorem A.18].

EXERCISE A.27. Proof Corollary A.2.

EXERCISE A.28. Prove Fact A.18.

EXERCISE A.29. Show that the first four spaces given in Example A.15 are normed vector spaces, while the fifth one is not. [Hint: in each case, verify the triangle inequality and the linearity conditions].

EXERCISE A.30. Prove the claims in Example A.16.

EXERCISE A.31. Consider the subspace of ℓ_p, ℓ_p^+, where all elements of the sequence are nonnegative. Suppose $1 \leq p < \infty$. Now consider $x \in \ell_p^+$ and the ε-neighborhood of x, $\mathcal{N}_\varepsilon(x)$. Show that for any $x \in \ell_p^+$ and any $\varepsilon > 0$, $\mathcal{N}_\varepsilon(x) \nsubseteq \ell_p^+$. [Hint: fix $\varepsilon > 0$ and $x = (x_1, x_2, ...) \in \ell_p^+$. Since $x \in \ell_p$, for any $\varepsilon > 0$, there exists an integer N such that for all $n \geq N$, $|x_n| < \varepsilon/2$. Then define z such that $z_n = x_n$ for all $n \neq N$ and $z_N = x_N - \varepsilon/2$. Show that $z \in \mathcal{N}_\varepsilon(x)$ but $z \notin \ell_p^+$].

EXERCISE A.32. Show that $x = (1, 1, 1, ...)$ is an interior point of ℓ_∞^+. [Hint: consider $z_\varepsilon = (1 + \varepsilon, 1 + \varepsilon, ...)$, and show that $z \in \mathcal{N}_\varepsilon(x) \subset \ell_\infty^+$].

EXERCISE A.33. For the mapping $g : X \to \mathbb{R}^N$ for some $X \subset \mathbb{R}^K$, construct the set $G = \{x: g(x) \leq 0\}$. Show that even when each component of g is not a convex function, the set G can be convex.

EXERCISE A.34. Consider the problem of maximizing x subject to the constraint that $x^2 \leq 0$. Show that there exists a unique solution to this program, but there exists no Lagrange multiplier contrary to the claim in Theorem A.27. Show that this is because the Slater condition is not satisfied.

EXERCISE A.35. Consider the constrained maximization problem $\max_{x_1, x_2} -x_1$ subject to $x_1^2 \leq x_2$ and $x_2 = 0$. Show that there exists a unique solution, which is $(x_1, x_2) = (0, 0)$. Show that there does not exist a Lagrange multiplier vector (λ, μ) at which $(0, 0)$ satisfies (A.11). Explain how this is related to the failure of the constraint qualification condition.

CHAPTER B

Review of Ordinary Differential Equations

In this chapter, I give a very brief overview of some basic results on differential equations and also include a few results on difference equations. I limit myself to results that are useful for the material covered in the body of the text. In particular, I provide the background for the major theorems on stability, Theorems 2.2, 2.3, 2.4, 2.5, 7.18, and 7.19, which were presented and then extensively used in the text. I will also provide some basic theorems on existence, uniqueness and continuity of solutions to differential equations. Most of the material here can be found in Simon and Blume (1994) or in basic differential equation textbooks, such as Boyce and DiPrima (1977). Luenberger (1979) is an excellent reference, since it gives a symmetric treatment of differential and difference equations. The results on existence, uniqueness and continuity of solutions can be found in more advanced books, such as Walter (1991). Before presenting the results on differential equations, I also provide a brief overview of eigenvalues and eigenvectors, and some basic results on integrals. Throughout, I continue to assume basic familiarity with matrix algebra and calculus.

B.1. Review of Eigenvalues and Eigenvectors

Let $\mathbf{A}$ be a $n \times n$ (square) real matrix—meaning that all of its entries are real numbers. The square $(n \times n)$ matrix $\mathbf{D}$ is *diagonal* if all of its non-diagonal elements are equal to zero, i.e.,

$$\mathbf{D} = \begin{pmatrix} d_1 & 0 & \cdot & 0 \\ 0 & d_2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & 0 & d_n \end{pmatrix}.$$

The $n \times n$ *identity matrix*, $\mathbf{I}$, is the diagonal matrix with 1's on the diagonal, that is,

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & \cdot & 0 \\ 0 & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & 0 \\ 0 & \cdot & 0 & 1 \end{pmatrix}.$$

Throughout, this chapter I denote matrices and vectors by boldface letters, so $\mathbf{0}$ is the vector or matrix of zeros, whereas 0 is simply the number zero.

Let the real number $\det \mathbf{A}$ denote the determinant of a square matrix $\mathbf{A}$. A matrix $\mathbf{A}$ is *nonsingular* or *invertible* if $\det \mathbf{A} \neq 0$ or alternatively if the only $n \times 1$ column vector $\mathbf{v}$ that is a solution to the equation

$$\mathbf{A}\mathbf{v} = \mathbf{0}$$

is the zero vector $\mathbf{v} = (0, ..., 0)$. If $\mathbf{A}$ is invertible, then there exists $\mathbf{A}^{-1}$ such that

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}.$$

Conversely, if there exists a nonzero solution $\mathbf{v}$ or if $\det \mathbf{A} = 0$, then $\mathbf{A}$ is singular and does not have an inverse.

Let $a, b \in \mathbb{R}$ and define the imaginary number $i = \sqrt{-1}$. Then $\chi = a + bi$ is a *complex number*. A complex number ξ is an *eigenvalue* of $\mathbf{A}$ if

$$\det(\mathbf{A} - \xi\mathbf{I}) = 0,$$

or in other words, if by subtracting ξ times the identity matrix from $\mathbf{A}$, we create a singular matrix. Clearly, if $\mathbf{A}$ is invertible, then none of its eigenvalues are equal to zero. Given an eigenvalue ξ of $\mathbf{A}$, the $n \times 1$ nonzero column vector $\mathbf{v}_\xi$ is an eigenvector of $\mathbf{A}$ if

$$(\mathbf{A} - \xi\mathbf{I})\,\mathbf{v}_\xi = \mathbf{0}.$$

Clearly, if $\mathbf{v}_\xi$ satisfies this equation, so those $\lambda\mathbf{v}_\xi$ for any $\lambda \in \mathbb{R}$. The linear space $\mathbf{V} = \{\lambda\mathbf{v} : (\mathbf{A} - \xi\mathbf{I})\,\mathbf{v} = \mathbf{0} \text{ and } \lambda \in \mathbb{R}\}$ is sometimes referred to as the *eigenspace* of $\mathbf{A}$. One of the major uses of eigenvalues and eigenvectors is in "diagonalizing" a non-diagonal square matrix $\mathbf{A}$. In particular, suppose that the square $n \times n$ matrix $\mathbf{A}$ has n *distinct* real eigenvalues, then a standard result in matrix algebra implies that

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D},$$

where $\mathbf{D}$ is a diagonal matrix with the eigenvalues $\xi_1, ..., \xi_n$ on the diagonal and $\mathbf{P} = \left(\mathbf{v}_{\xi_1}, ..., \mathbf{v}_{\xi_n}\right)$ is the matrix of the eigenvectors corresponding to the eigenvalues. This result will be used in the proof of Theorem B.5 below and is more generally useful in deriving explicit solutions to systems of linear differential and difference equations.

As suggested above, even though all of the entries of $\mathbf{A}$ are real, its eigenvalues can be complex numbers. Moreover, we may expect that a $n \times n$ matrix $\mathbf{A}$ should have n eigenvalues, since $\det(\mathbf{A} - \xi\mathbf{I}) = 0$ is a polynomial of degree n. However, some of these may be repeated eigenvalues (corresponding to repeated roots to the polynomial). Both of these possibilities create a range of difficulties in diagonalizing the matrix $\mathbf{A}$. These difficulties are discussed in most linear algebra, matrix algebra, and differential equations textbooks, and will not be discussed in detail here.

B.2. Some Basic Results on Integrals

Before proceeding to differential equations, it is useful to review some basic results on integrals. Throughout this section, I will focus on *Riemann integrals*. In particular, let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Then the Riemann integral of f between a and $b > a$, denoted by $\int_a^b f(x)\,dx$, can be defined as follows. Divide the interval $[a,b]$ into N equal-sized segments, and let

$$x_j \equiv a + j\frac{b-a}{N} \text{ for } j = 0, 1, ..., N.$$

Then, form the *Riemann sum*

$$\frac{b-a}{N}\sum_{j=0}^{N-1} f(x_j).$$

The Riemann integral is obtained by taking arbitrarily finer partitions of this interval, and is thus equal to

$$\int_a^b f(x)\,dx = \lim_{N\to\infty}\frac{b-a}{N}\sum_{j=0}^{N} f(x_j). \tag{B.1}$$

In fact, there are more general ways of creating partitions. When the Riemann integral is well-defined, how these partitions are created and how the limits are taken is not important, since regardless of the exact way in which they are constructed, as the partitions become finer and finer, the Riemann sum converges to the well-defined Riemann integral $\int_a^b f(x)\,dx$. Thus the construction here is just one way of defining a Riemann integral.

The assumption that f is continuous is not necessary for the Riemann integral to be well-defined (for example, the Riemann integral can be defined for monotone discontinuous functions). But for many functions the Riemann integral is not well-defined. For this reason, it is often more convenient to work with more general integrals, such as the Lebesgue integral. Although I made some references to Lebesgue integrals in the text, here I focus exclusively on Riemann integrals to simplify the discussion. When a function f has a well-defined Riemann integral over the interval $[a,b]$, it is said to be *Riemann integrable* over $[a,b]$.

The following four basic results are useful for our analysis. The proofs can be found in standard real analysis or calculus textbooks, and are not repeated here.

Theorem B.1. ***(Fundamental Theorem of Calculus I)*** *Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable on $[a,b]$. For any $x \in [a,b]$, define*

$$F(x) = \int_a^x f(t)\,dt.$$

Then $F : [a,b] \to \mathbb{R}$ is continuous over $[a,b]$. If f is continuous at some $x_0 \in [a,b]$, then $F(x)$ is differentiable at x_0 with derivative

$$F'(x_0) = f(x_0).$$

Theorem B.2. ***(Fundamental Theorem of Calculus II)*** *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$, then there exists a differentiable function $F : [a,b] \to \mathbb{R}$ on $[a,b]$ such that $F'(x) = f(x)$ for all $x \in [a,b]$ and*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Theorem B.3. ***(Integration by Parts)*** *Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ continuous functions and let $F : [a, b] \to \mathbb{R}$ and $G : [a, b] \to \mathbb{R}$ be differentiable functions such that $F'(x) = f(x)$ and $G'(x) = g(x)$ for all $x \in [a, b]$. Then*

$$\int_a^b F(x) g(x) dx = F(b) G(b) - F(a) G(a) - \int_a^b G(x) f(x) dx.$$

Theorem B.4. ***(Leibniz's Rule)*** *Let $f(x, y)$ be continuous in x on $[a, b]$ and differentiable in y at y_0 and suppose that the functions $a(y)$ and $b(y)$ are differentiable at y_0 with derivatives denoted by a' and b'. Then*

$$\frac{d}{dy}\int_{a(y_0)}^{b(y_0)} f(x, y_0) dx = \int_{a(y_0)}^{b(y_0)} \frac{\partial f(x, y_0)}{\partial y} dx + b'(y_0) f(b(y_0), y_0) - a'(y_0) f(a(y_0), y_0).$$

Riemann integrals as in (B.1), which specify lower and upper limits, are referred to as *definite integrals*. One can also define *indefinite integrals*, $\int f(x) dx$, which simply refer to the function $F(x)$ with the property that $F'(x) = f(x)$. For this reason, the indefinite integral $\int f(x) dx$ is also sometimes referred to as an *anti-derivative*. Definite integrals can also be defined for the cases in which $a = -\infty$ and/or $b = \infty$, provided that the integral is finite.

B.3. Linear Differential Equations

Recall the motivation for considering differential equations in dynamic economic models discussed in Chapter 2. In particular, consider a function $x : T \to \mathbb{R}$, where $T \subset \mathbb{R}$. Suppose that for some $\Delta t > 0$

$$x(t + \Delta t) - x(t) = G(x(t), t, \Delta t),$$

where $G(x(t), t, \Delta t)$ is a real valued function. Now divide both sides of this equation by Δt and consider the limit as $\Delta t \to 0$. Suppose that $\lim_{\Delta t \to 0} G(x(t), t, \Delta t) / \Delta t$ exists and let

$$g(x(t), t) \equiv \lim_{\Delta t \to 0} \frac{G(x(t), t, \Delta t)}{\Delta t}.$$

Using this, we obtain the following simple *differential equation*

$$\frac{dx(t)}{dt} \equiv \dot{x}(t) = g(x(t), t). \tag{B.2}$$

This is an *explicit* first-order differential equation. The term explicit refers to the fact that $\dot{x}(t)$ is separated from the rest of the terms. This contrasts with *implicit* first-order differential equations of the form

$$H(\dot{x}(t), x(t), t) = 0.$$

For our purposes, it is sufficient to deal with explicit equations.

A differential equation is *autonomous*, if it can be written in the form

$$\dot{x}(t) = g(x(t)),$$

or simply as

$$\dot{x} = g(x),$$

meaning that time is not a separate argument. Alternatively, if it cannot be written this way, it is a *nonautonomous* equation. In addition to first-order differential equations, we can consider, second order or nth order equations, for example,

$$\frac{d^2x(t)}{dt^2} = g\left(\frac{dx(t)}{dt}, x(t), t\right),$$

or

$$\frac{d^nx(t)}{dt^n} = g\left(\frac{d^{n-1}x(t)}{dt^{n-1}}, ..., \frac{dx(t)}{dt}, x(t), t\right). \tag{B.3}$$

I will focus on first-order equations, since higher-order equations can always be transformed into a system of first-order equations (see Exercise B.2).

The most common form of differential equation is the so-called *initial value* problem. In this case, a differential equation as in (B.2) is specified together with an initial condition $x(0) = x_0$. We saw many examples of such initial value problems in the text. However, many important problems in economics are not initial value problems, since the boundary conditions are specified by *transversality conditions*, that is, by what the terminal value of the solution $x(t)$ should be at some time $T < \infty$ or at $T = \infty$.

Suppose that a first-order differential equation (B.2) is defined for all $t \in \mathcal{D}$, where $\mathcal{D}$ is an interval in $\mathbb{R}$ and an initial value $x(0) = x_0$ has been specified. A *solution* to this initial value problem is given by a function $x : \mathcal{D} \to \mathbb{R}$ that satisfies (B.2) for all $t \in \mathcal{D}$ with $x(0) = x_0$. Sometimes, a family of functions $\mathcal{X} = \{x : \mathcal{D} \to \mathbb{R}$ such that x satisfies (B.2) for all $t \in \mathcal{D}\}$ is referred to as *general solutions*, while an element of $\mathcal{X}$ that satisfies the boundary condition is called the *particular solution*.

Let us now first look at linear first-order equations. This is a good starting point both because such equations are commonly encountered in economics and they have simple solutions. A *linear first-order differential equation* takes a general form

$$\dot{x}(t) = a(t)x(t) + b(t). \tag{B.4}$$

In addition, if $b(t) = 0$, this is referred to as a *homogeneous equation* and if $a(t) = a$ and $b(t) = b$, then it is an equation with *constant coefficients*.

Let us start with the simplest case, which is a homogeneous linear equation with constant coefficients, i.e.,

$$\dot{x}(t) = ax(t). \tag{B.5}$$

A solution to this equation is straightforward to obtain. One can simply guess the solution and then verify that the solution satisfies the differential equation (B.5). Or one can divide both sides by $\dot{x}(t)$, integrate with respect to t and recall that

$$\int \frac{\dot{x}(t)}{x(t)} dt = \ln x(t) + c_0$$

and

$$\int adt = at + c_1,$$

where c_0 and c_1 are constants of integration. Now taking exponents on both sides, the solution to (B.5) is obtained as

$$x(t) = c\exp(at), \tag{B.6}$$

where c is a constant of integration combining c_0 and c_1 (in fact, $c = \exp(c_1 - c_0)$). Differentiating this equation, one easily obtains (B.5) and verifies that (B.6) is indeed a solution to (B.5). If (B.5) is specified as an initial value problem, then we also have a boundary condition, which, without loss of any generality, can be specified at $t = 0$ as $x(0) = x_0$. This boundary condition pins down the unique value of the constant of integration. In particular, since $\exp(a \times 0) = 1$, $c = x_0$.

Next consider a slightly more general equation, homogeneous but not with constant coefficients, that is,

$$\dot{x}(t) = a(t)x(t), \tag{B.7}$$

defined over $t \geq 0$ with an initial condition $x(0) = x_0$. Once again, dividing both sides by $x(t)$, integrating and then finally taking exponents, we obtain

$$x(t) = c\exp\left(\int_0^t a(s)\,ds\right). \tag{B.8}$$

(This follows since the integral of the right-hand side for a bounded function $a(t)$ is simply $\int_0^t a(s)\,ds + c_1$). Since $\lim_{t\to 0}\int_0^t a(s)\,ds = \int_0^0 a(s)\,ds = 0$, the constant of integration is again pinned down by the initial condition, that is, $c = x_0$. That (B.8) is a solution to (B.7) can be verified by differentiating (B.8) using Leibniz's Rule from the previous section (Theorem B.4).

Next consider an autonomous but nonhomogeneous first-order differential equation,

$$\dot{x}(t) = ax(t) + b. \tag{B.9}$$

With a similar analysis (see Exercise B.3), the solution is now obtained as

$$x(t) = -\frac{b}{a} + c\exp(at).$$

The constant of integration must be $c = x_0 + b/a$, thus we have the particular solution that satisfies the boundary condition as

$$x(t) = -\frac{b}{a} + \left(x_0 + \frac{b}{a}\right)\exp(at). \tag{B.10}$$

This equation also enables us to have a simple discussion of stability. Recall that, as in the text, a *steady state* of (B.9) refers to a situation which $\dot{x}(t) = 0$ for all t. Clearly in this case,

$$x(t) = x^* \equiv -\frac{b}{a},$$

is the unique steady state. Inspection of (B.10) immediately shows that $x(t)$ will approach the steady-state value x^* as t increases if $a < 0$ and it will diverge away from it if $a > 0$. This is naturally what we would expect from Theorem 2.4, which states that the steady state is asymptotically stable if $a < 0$.

Finally, let us consider the most general case of the first-order linear differential equation, that given in (B.4). Once again, with an analysis similar to that in Exercise B.3, we obtain the general solution to (B.4) as

$$x(t) = \left[c + \int_0^t b(s) \exp\left(\int_0^s a(v)\, dv \right)^{-1} ds \right] \exp\left(\int_0^t a(s)\, ds \right). \tag{B.11}$$

Differentiation using Leibniz's Rule verifies that (B.11) provides the solution to (B.4) (see Exercise B.4). A similar analysis to that above, allows us to derive the constant of integration from the initial value $x(0) = x_0$ as $c = x_0$.

Notice, however, that in this case there is no steady-state value of x^* for which $\dot{x}(t) = 0$ in (B.4), since $\dot{x}(t) = 0$ implies $x(t) = -b(t)/a(t)$, which is generally not a constant.

A byproduct of this brief discussion is that we have also established the existence of unique solutions to linear differential equations (since we have provided explicit solutions). Thus linear differential equations (formulated as initial value problems) always have a solution. Moreover, the solution is unique. This is a special case of Theorem B.8 below.

The results on the existence of solutions and explicit characterization of solutions for linear first-order differential equations can be extended to systems of differential equations. The general result here is provided in Theorem B.6. However, before presenting this more general result, it is useful to consider the following simpler system of first-order differential equations with *constant coefficients*:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t), \tag{B.12}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, n is an integer and $\mathbf{A}$ is a $n \times n$ matrix. The boundary condition again takes the form of an initial value $\mathbf{x}(0) = \mathbf{x}_0 \in \mathbb{R}^n$. This system of equation does not include a constant, so that the steady state is $\mathbf{x}^* = \mathbf{0}$. This is simply a normalization, since as we just saw, a nonhomogeneous differential equation with constant coefficients can be transformed into a homogeneous one by a simple change of variables. This system of differential equations always has a unique solution (this follows from Theorem B.6 for from Theorem B.10, see Exercise B.5). However, when $\mathbf{A}$ has distinct real eigenvalues, the solution to (B.12) takes a particularly simple form. This case is presented in the next result.

Theorem B.5. ***(Solution to Systems of Linear Differential Equations with Constant Coefficients)*** *Suppose that* $\mathbf{A}$ *has* n *distinct real eigenvalues* $\xi_1, ..., \xi_n$*, then the unique solution to the system of linear differential equations (B.12) with the initial value* $\mathbf{x}(0) = \mathbf{x}_0$

takes the form

$$\mathbf{x}(t) = \sum_{j=1}^{n} c_j \exp\left(\xi_j t\right) \mathbf{v}_{\xi_j},$$

where $\mathbf{v}_{\xi_1}, ..., \mathbf{v}_{\xi_n}$ *denote the eigenvectors corresponding to the eigenvalues* $\xi_1, ..., \xi_n$ *and* $c_1, ..., c_n$ *denote the constants of integration.*

PROOF. The proof follows by diagonalizing the matrix $\mathbf{A}$. In particular, since $\mathbf{A}$ has n distinct real eigenvalues, recall that

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D},$$

where $\mathbf{D}$ is a diagonal matrix with the eigenvalues $\xi_1, ..., \xi_n$ on the diagonal and $\mathbf{P} = \left(\mathbf{v}_{\xi_1}, ..., \mathbf{v}_{\xi_n}\right)$ is the matrix of the eigenvectors corresponding to the eigenvalues. Let $\mathbf{z}(t) \equiv \mathbf{P}^{-1}\mathbf{x}(t)$, which also implies

$$\begin{aligned} \dot{\mathbf{z}}(t) &= \mathbf{P}^{-1}\dot{\mathbf{x}}(t) \\ &= \mathbf{P}^{-1}\mathbf{A}\mathbf{x}(t) \\ &= \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{z}(t) \\ &= \mathbf{D}\mathbf{z}(t). \end{aligned} \tag{B.13}$$

Since $\mathbf{D}$ is a diagonal matrix, writing $\mathbf{z}(t) = (z_1(t), ..., z_n(t))$, (B.13) implies that $z_1(t) = c_1 \exp(\xi_1 t), ..., z_n(t) = c_n \exp(\xi_n t)$, where $c_1, ..., c_n$ are the constants of integration. Now since $\mathbf{x}(t) = \mathbf{P}\mathbf{z}(t)$, the result follows by multiplying the matrix $\left(\mathbf{v}_{\xi_1}, ..., \mathbf{v}_{\xi_n}\right)$ with the vector of solutions, $\mathbf{z}(t)$. □

When the matrix $\mathbf{A}$ has repeated or complex eigenvalues, explicit solutions can still be derived but are somewhat more complicated. Therefore, I will instead focus on the more general results in Theorem B.6 below. One important set of implications of Theorem B.5 are Theorems 2.4 and 7.18 in the text. In particular, Theorem B.5 implies that the steady-state value, here $\mathbf{x}^* = \mathbf{0}$, will be stable only when all eigenvalues are negative. If, instead, $m < n$ of the eigenvalues are negative, then there will exist a m-dimensional subspace, such that the solution will tend to the steady state only starting with an initial value on this subspace.

Now consider the most general form of a system of linear differential equations:

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t), \tag{B.14}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, n is an integer and $\mathbf{A}(t)$ and $\mathbf{B}(t)$ area $n \times n$ matrices for each t. To simplify the discussion, let us assume that each element of $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are continuous or monotone functions of time, so that they are integrable.

I now characterize the solution to (B.14). I proceed in two steps. First, define the *state-transition matrix* $\mathbf{\Phi}(t, s)$ corresponding to $\mathbf{A}(t)$ as the $n \times n$ matrix function that is differentiable in its first argument and satisfies

$$\frac{d}{dt}\mathbf{\Phi}(t,s)=\mathbf{A}(t)\,\mathbf{\Phi}(t,s)\ \text{ and } \mathbf{\Phi}(t,t)=\mathbf{I} \text{ for all } t \text{ and } s. \tag{B.15}$$

The state-transition matrix is useful because it enables us to express the solutions to homogeneous systems and then derive the solutions to (B.14) from the solutions to the corresponding homogeneous systems. In particular, if $\hat{\mathbf{x}}(t)$ is a solution to the homogeneous system

$$\dot{\mathbf{x}}(t)=\mathbf{A}(t)\,\mathbf{x}(t), \tag{B.16}$$

then it is straightforward to verify that (see Exercise B.6):

$$\hat{\mathbf{x}}(t)=\mathbf{\Phi}(t,s)\,\hat{\mathbf{x}}(s)\ \text{ for any } t \text{ and } s. \tag{B.17}$$

Let us next define the *fundamental set of solutions* to (B.16). The $n\times n$ matrix $\mathbf{X}(t)$ is a fundamental set of solutions to (B.16) if its columns consist of vector-valued functions $\mathbf{x}^1(t),\mathbf{x}^2(t),...,\mathbf{x}^n(t)$ that are linearly independent from each other and are solutions to (B.16). In this case, clearly,

$$\dot{\mathbf{X}}(t)=\mathbf{A}(t)\,\mathbf{X}(t).$$

Then it can be verified that (see Exercise B.7):

$$\mathbf{\Phi}(t,s)=\mathbf{X}(t)\,\mathbf{X}(s)^{-1}. \tag{B.18}$$

We are now in a position to state the form of the unique solution to the general system a linear equations in (B.14).

Theorem B.6. ***(General Solutions to Systems of Linear Differential Equations)*** *The solution to the system of differential equations in (B.14) with initial condition* $\mathbf{x}(0)=\mathbf{x}_0$ *is given by*

$$\hat{\mathbf{x}}(t)=\mathbf{\Phi}(t,0)\,\mathbf{x}_0+\int_0^t\mathbf{\Phi}(t,s)\,\mathbf{B}(s)\,ds, \tag{B.19}$$

where $\mathbf{\Phi}(t,s)$ *is the state transition matrix corresponding to* $\mathbf{A}(t)$*.*

Proof. We only need to verify that $\hat{\mathbf{x}}(t)$ given in (B.19) is a solution to (B.14). Let us simplify notation of time derivatives by writing this as $\mathbf{x}(t)$. Differentiating (B.19) with respect to time and using Leibniz's Rule (Theorem B.4), we obtain

$$\frac{d}{dt}\mathbf{x}(t)=\frac{d}{dt}\mathbf{\Phi}(t,0)\,\mathbf{x}_0+\int_0^t\frac{d}{dt}\mathbf{\Phi}(t,s)\,\mathbf{B}(s)\,ds+\mathbf{\Phi}(t,t)\,\mathbf{B}(t).$$

By the definition of the state transition matrix, (B.15), $\mathbf{\Phi}(t,t)=\mathbf{I}$ and

$$\frac{d}{dt}\mathbf{\Phi}(t,s)=\mathbf{A}(t)\,\mathbf{\Phi}(t,s).$$

Therefore,

$$\begin{aligned}\dot{\mathbf{x}}(t) &\equiv \frac{d}{dt}\mathbf{x}(t) \\ &= \mathbf{A}(t)\,\mathbf{\Phi}(t,0)\,\mathbf{x}_0 + \mathbf{A}(t)\int_0^t \mathbf{\Phi}(t,s)\,\mathbf{B}(s)\,ds + \mathbf{B}(t) \\ &= \mathbf{A}(t)\,\mathbf{x}(t) + \mathbf{B}(t),\end{aligned}$$

completing the verification that (B.19) satisfies (B.14) with initial condition $\mathbf{x}(0) = \mathbf{x}_0$. □

B.4. Stability for Nonlinear Differential Equations

Systems of nonlinear differential equations can be analyzed in the neighborhood of the steady state by using Taylor's Theorem (Theorem A.21). In particular, consider the system of nonlinear autonomous differential equations

(B.20) $$\dot{\mathbf{x}}(t) = \mathbf{G}(\mathbf{x}(t)),$$

where again $\mathbf{x}(t) \in \mathbb{R}^n$, n is an integer and now $\mathbf{G} : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable mapping. Suppose that this system of differential equations has a steady state $\mathbf{x}^* \in \mathbb{R}^n$ and consider $\mathbf{x}(t)$ in the neighborhood of $\mathbf{x}^*$. Then we can consider only the first-order terms in Taylor's Theorem and obtain

$$\dot{\mathbf{x}}(t) = D\mathbf{G}(\mathbf{x}^*)(\mathbf{x}(t) - \mathbf{x}^*) + \mathbf{R}(\mathbf{x}(t) - \mathbf{x}^*),$$

where $\mathbf{R}(\mathbf{x}(t) - \mathbf{x}^*) / \|\mathbf{x}(t) - \mathbf{x}^*\| \to \mathbf{0}$ as $\mathbf{x}(t) \to \mathbf{x}^*$. On the basis of this, as long as $\mathbf{x}^*$ is a *hyperbolic* steady state, that is, as long as the matrix $D\mathbf{G}(\mathbf{x}^*)$ does not have zero eigenvalues (or complex eigenvalues with zero real parts), the behavior of $\mathbf{x}(t)$ in the neighborhood of the steady state $\mathbf{x}^*$ can be approximated by the system of linear differential equations $\dot{\mathbf{x}}(t) = D\mathbf{G}(\mathbf{x}^*)(\mathbf{x}(t) - \mathbf{x}^*)$. This is the basis of Theorems 2.5 and 7.19 in the text. In fact, the following theorem, which is a slightly stronger version of those results, can be proved by using linearization arguments.

Theorem B.7. ***(Grobman-Hartman Theorem)*** *Let $\mathbf{x}^*$ be a steady state of (B.20) and suppose that $\mathbf{G} : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable mapping. If $\mathbf{x}^*$ is hyperbolic, then there exists an open set of trajectories U of (B.20) around $\mathbf{x}^*$ and an open set of trajectories V of the linear system $\dot{\mathbf{x}}(t) = D\mathbf{G}(\mathbf{x}^*)(\mathbf{x}(t) - \mathbf{x}^*)$ around $\mathbf{x}^*$ such that there exists a one-to-one continuous function $h : U \to V$ that preserves the direction of trajectories in U and V.*

Proof. See Walter (1991, Chapter 7.29). □

B.5. Separable and Exact Differential Equations

We cannot obtain explicit solutions to nonlinear differential equations in general (though existence of solutions can be guaranteed under some conditions as shown in the next section). Nevertheless, two important special classes of differential equations, *separable* and *exact differential equations*, often enable us to derive explicit solutions. I start with separable differential equations. A differential equation

$$\dot{x}(t) = g(x(t), t) \tag{B.21}$$

is *separable* if g can be written as

$$g(x,t) \equiv f(x)\, h(t).$$

In that case, the differential equation (B.21) can be expressed as

$$\begin{aligned} \frac{\dot{x}(t)}{f(x(t))} &= h(t). \\ \frac{dx(t)}{f(x(t))} &= h(t)\, dt. \end{aligned}$$

Integrating both sides, we obtain

$$\int \frac{dx}{f(x)} = \int h(t)\, dt.$$

This equation typically allows us to obtain an explicit solution. The following example illustrates a particular application.

Example B.1. The differential equation

$$\dot{x}(t) = \frac{4t^3 + 3t^2 + 2t + 1}{2x(t)}$$

with initial value $x(0) = 1$ at first looks difficult to solve. However, once we note that it is separable, we can write it as

$$2x \cdot dx = \left(4t^3 + 3t^2 + 2t + 1\right) \cdot dt,$$

and integrate

$$2x \cdot dx = \left(4t^3 + 3t^2 + 2t + 1\right) \cdot dt,$$

to obtain

$$x^2 - 1 = t^4 + t^3 + t^2 + 1 + c,$$

where c is a combination of the two constants of integration. To satisfy the initial value, we need $c = 0$. Therefore, the solution to this initial value problem is given by

$$x(t) = \sqrt{t^4 + t^3 + t^2 + t + 1},$$

where the negative root to the quadratic is eliminated because it does not satisfied the initial value $x(0) = 1$.

Another example, which is more relevant for economic applications, is given in Exercise B.8.

Next, consider a differential equation of the form (B.21) again, and suppose that the function g can be written as

$$g(x(t),t) \equiv \frac{G_1(x(t),t)}{G_2(x(t),t)},$$

where

$$G_1(x(t),t) = \frac{\partial F(x(t),t)}{\partial t} \text{ and } G_2(x(t),t) = -\frac{\partial F(x(t),t)}{\partial x},$$

then (B.21) defines an *exact differential equation*. In particular, in this case, we can write

$$\begin{aligned} \dot{x}(t) &= \frac{G_1(x(t),t)}{G_2(x(t),t)} \\ &= -\frac{\partial F(x(t),t)/\partial t}{\partial F(x(t),t)/\partial x}, \end{aligned}$$

or

$$\dot{x}(t)\frac{\partial F(x(t),t)}{\partial x} + \frac{\partial F(x(t),t)}{\partial t} = 0.$$

Let $\hat{x}(t)$ be a solution to this differential equation. Then we equivalently have that

$$\frac{d}{dt}F(\hat{x}(t),t) = 0, \tag{B.22}$$

where d/dt denotes the total derivative of the function F. (B.22) then implies that

$$F(\hat{x}(t),t) = c, \tag{B.23}$$

where c is the constant of integration. Equation (B.23) implicitly defines the solution $\hat{x}(t)$.

Exact differential equations are straightforward to solve once they have been identified. The following provides a simple example.

EXAMPLE B.2. Consider the differential equation

$$\dot{x}(t) = -\frac{2x(t)\ln x(t)}{t},$$

with initial value $x(1) = \exp(1)$. While this differential equation looks difficult to solve at first, once we recognize that it can be written as

$$\begin{aligned} \dot{x}(t) &= -\frac{2t\ln x(t)}{t^2/x(t)} \\ &= -\frac{\partial\left(t^2\ln(x(t))\right)/\partial t}{\partial\left(t^2\ln(x(t))\right)/\partial x}, \end{aligned}$$

it can be seen to be an exact differential equation. Therefore, its solution $\hat{x}(t)$ is given by

$$t^2\ln(\hat{x}(t)) = c,$$

which implies

$$\hat{x}(t) = \exp\left(ct^{-2}\right),$$

as the general solution, and the initial condition pins down the constant of integration as $c = 1$.

B.6. Existence and Uniqueness of Solutions

Initial value problems generally enable us to establish the existence and uniqueness of solutions under relatively weak conditions. In fact, there are many related existence theorems. I will state the most basic existence theorem here, which extends the original theorem by Picard. Consider a first-order differential equation

$$\dot{x}(t) = g(x(t), t) \tag{B.24}$$

defined on some interval $\mathcal{D} \subset \mathbb{R}$, i.e., defined for all $t \in \mathcal{D}$. Throughout, we assume that 0 is in the interior of $\mathcal{D}$. Let us introduce the following Lipschitz condition:

Definition B.1. *The first-order differential equation (B.24) satisfies the* ***Lipschitz condition*** *on the strip $S = \mathcal{X} \times \mathcal{D}$ if there exists a real number $L < \infty$ such that*

$$\left\| g(x,t) - g\left(x',t\right) \right\| \leq L \left\| x - x' \right\|$$

for all $x, x' \in \mathcal{X}$ and for all $t \in \mathcal{D}$.

Naturally, since in this case $x \in \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, this is equivalent to the simpler Lipschitz condition $|g(x,t) - g(x',t)| \leq L|x - x'|$.

Then, we have the following two existence and uniqueness theorems.

Theorem B.8. ***(Picard's Theorem I)*** *Suppose that g is continuous in both of its arguments and satisfies the Lipschitz condition in Definition B.1. Then, there exists $\delta > 0$ such that the initial value problem defined by (B.24) with $x(0) = x_0$ has a unique solution $x(t)$ over the interval $[-\delta, \delta] \subset \mathcal{D}$.*

This theorem guarantees only the existence of unique solution in the neighborhood of the initial value x_0. A stronger version of this theorem holds when $\mathcal{D}$ is compact.

Theorem B.9. ***(Existence and Uniqueness on Compact Sets I)*** *Suppose that g is continuous in both of its arguments and satisfies the Lipschitz condition in Definition B.1, and that $\mathcal{D}$ is compact. Then, the initial value problem defined by (B.24) with $x(0) = x_0$ has a unique solution $x(t)$ over the entire interval $\mathcal{D}$.*

There are various different proofs of these theorems. Example 6.3 and Exercise 6.4 in Chapter 6 provide standard proofs using the Contraction Mapping Theorem, Theorem 6.7.

This theorem can be easily extended to systems of first-order differential equations. Suppose that $\mathbf{x}(t) \in \mathcal{X} \subset \mathbb{R}^n$, where n is an integer. Consider the following system of first-order differential equations:

$$\dot{\mathbf{x}}(t) = \mathbf{G}(\mathbf{x}(t), t), \tag{B.25}$$

where

$$\mathbf{G} : \mathcal{X} \times \mathcal{D} \rightarrow \mathcal{X}$$

and $t \in \mathcal{D} \subset \mathbb{R}$.

DEFINITION B.2. *The system of first-order differential equation (B.24) satisfies the* ***Lipschitz condition*** *over the strip $S = \mathcal{X} \times \mathcal{D}$ if there exists a real number $L < \infty$ such that*

$$\left|\mathbf{G}(\mathbf{x}, t) - \mathbf{G}(\mathbf{x}', t)\right| \leq L \left|\mathbf{x} - \mathbf{x}'\right|$$

for all $\mathbf{x}, \mathbf{x}' \in \mathcal{X}$ and for all $t \in \mathcal{D}$.

THEOREM B.10. ***(Picard's Theorem II)****Suppose that $\mathbf{G}$ is continuous in all of its arguments and satisfies the Lipschitz condition in Definition B.2. Then, there exists $\delta > 0$ such that the initial value problem defined by the system of differential equations in (B.25) with $\mathbf{x}(0) = \mathbf{x}_0$ has a unique solution $\mathbf{x}(t)$ over the interval $[-\delta, \delta] \subset \mathcal{D}$.*

THEOREM B.11. ***(Existence and Uniqueness on Compact Sets II)*** *Suppose that $\mathbf{G}$ is continuous in all of its arguments and satisfies the Lipschitz condition in Definition B.2 and that $\mathcal{D}$ is compact. Then, the initial value problem defined by the system of differential equations in (B.25) with $\mathbf{x}(0) = \mathbf{x}_0$ has a unique solution $\mathbf{x}(t)$ over the entire interval $\mathcal{D}$.*

The proof of these theorems can be found in Walter (1991, Chapter 3.10).

Another useful theorem is the following.

THEOREM B.12. ***(Peano's Theorem)*** *Suppose that $g : \mathcal{X} \times \mathcal{D} \rightarrow \mathbb{R}$ is continuous in both of its arguments over $\mathcal{X} \times \mathcal{D}$. Then, the differential equation (B.24) has at least one solution that goes through each $(x', t') \in \mathcal{X} \times \mathcal{D}$.*

PROOF. See Walter (1991, Chapter 3.10). □

This theorem can also be extended to systems of differential equations.

B.7. Continuity of Solutions

It is often of interest to knwo whether, when some parameter or the initial condition of a differential equation is changed by a small amount, the solution will also change by an appropriately defined small amount. The following theorem answers this question.

THEOREM B.13. ***(Continuity of Solutions to Differential Equations)*** *Suppose that $g : \mathcal{X} \times \mathcal{D} \rightarrow \mathbb{R}$ is continuous in both of its arguments and that $\mathcal{D}$ is compact. Let $x(t)$ be a solution to (B.25) with initial condition $x(0) = x_0$. For every $\varepsilon > 0$, $x_0 \in \mathcal{X}$, and continuous function $\tilde{g} : \mathcal{X} \times \mathcal{D} \rightarrow \mathbb{R}$, there exists $\delta > 0$ such that if*

$$|\tilde{g}(x, t) - g(x, t)| < \delta \text{ and } |\tilde{x}_0 - x_0| < \delta \text{ for all } (x, t) \in \mathcal{X} \times \mathcal{D},$$

then every solution $\tilde{x}(t)$ *to the perturbed initial value problem*

$$\dot{x}(t) = \tilde{g}(x(t), t) \text{ with } x(0) = \tilde{x}_0$$

satisfies

$$|\tilde{x}(t) - x(t)| < \varepsilon \text{ for all } t \in \mathcal{D}.$$

PROOF. See Walter (1991, Chapter 3.12). □

This theorem can also be extended to systems of differential equations.

B.8. Difference Equations

Solutions to difference equations have many features that are common with the solutions to differential equations. For example, the simple first-order difference equation

$$x(t+1) = ax(t) + b,$$

has a solution similar to the first-order linear differential equation with constant coefficients. In particular, if we specify the initial condition $x(0) = x_0$, then successive substitutions yield

$$\begin{aligned} x(1) &= ax_0 + b \\ x(2) &= a^2 x_0 + ab + b, \end{aligned}$$

and so on. Therefore, the general solution to this equation is

$$x(t) = \begin{cases} x_0 + bt & \text{if } a = 1 \\ a^t \left(x_0 - \frac{b}{1-a}\right) + \frac{b}{1-a} & \text{otherwise} \end{cases}.$$

The reader will recognize $x^* \equiv b/(1-a)$ as the steady-state value (when $a \neq 1$), and the solution makes it clear that if $|a| < 1$, then the first-term will tend to zero and $x(t) \to x^*$ (as $t \to \infty$) which is the essence of the stability results presented in the text. In contrast, when $|a| > 1$, the solution will diverge away from x^*.

Next consider the system of first-order linear difference equations

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t), \tag{B.26}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, n is an integer and $\mathbf{A}$ is a $n \times n$ real matrix. When $\mathbf{A}$ has n distinct real eigenvalues, the solution to the system of equations is very similar to that given in Theorem B.5 above. In particular, we have

THEOREM B.14. ***(Solution to Systems of Linear Difference Equations with Constant Coefficients)*** *Suppose that* $\mathbf{A}$ *has* n *distinct real eigenvalues* $\xi_1, ..., \xi_n$*, then the unique solution to the system of linear difference equations (B.26) with the initial value* $\mathbf{x}(0) = \mathbf{x}_0$ *takes the form*

$$\mathbf{x}(t) = \sum_{j=1}^{n} c_j \xi_j^t \mathbf{v}_{\xi_j},$$

where $\mathbf{v}_{\xi_1}, ..., \mathbf{v}_{\xi_n}$ *denote the eigenvectors corresponding to the eigenvalues* $\xi_1, ..., \xi_n$ *and* $c_1, ..., c_n$ *denote constants determined by the initial conditions.*

Proof. The proof again follows by diagonalizing the matrix $\mathbf{A}$. Recall that since $\mathbf{A}$ has n distinct real eigenvalues, we have $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$, where $\mathbf{D}$ is a diagonal matrix with the eigenvalues $\xi_1, ..., \xi_n$ on the diagonal and $\mathbf{P} = \left(\mathbf{v}_{\xi_1}, ..., \mathbf{v}_{\xi_n}\right)$ is the matrix of the eigenvectors corresponding to the eigenvalues. Let $\mathbf{z}(t) \equiv \mathbf{P}^{-1}\mathbf{x}(t)$ and note that

$$\begin{aligned} \mathbf{z}(t+1) &= \mathbf{P}^{-1}\mathbf{x}(t+1) \\ &= \mathbf{P}^{-1}\mathbf{A}\mathbf{x}(t) \\ &= \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{z}(t) \\ &= \mathbf{D}\mathbf{z}(t). \end{aligned} \tag{B.27}$$

Since $\mathbf{D}$ is a diagonal matrix, writing $\mathbf{z}(t) = (z_1(t), ..., z_n(t))$, (B.27) implies that $z_1(t) = c_1\xi_1^t, ..., z_n(t) = c_n\xi_n^t$. Now since $\mathbf{x}(t) = \mathbf{P}\mathbf{z}(t)$, the result follows by multiplying the matrix $\left(\mathbf{v}_{\xi_1}, ..., \mathbf{v}_{\xi_n}\right)$ with the vector of solutions, $\mathbf{z}(t)$. □

Solutions in the more general case where the system of difference equations do not have constant coefficients or eigenvalues may be complex or repeated are given by the analog of Theorem B.6. The matrix of fundamental solutions, $\mathbf{X}(t)$, is defined as the matrix of fundamental solutions to differential equations. In addition, the state transition matrix is again given by

$$\mathbf{\Phi}(t, s) = \mathbf{X}(t)\mathbf{X}(s)^{-1}.$$

Moreover, as before $\mathbf{\Phi}(t, t) = \mathbf{I}$. Now consider the general system of first-order difference equations given by

$$\mathbf{x}(t+1) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t). \tag{B.28}$$

The solution to this set of difference equations is characterized by the next theorem.

Theorem B.15. ***(General Solutions to Systems of Linear Differential Equations)*** *The solution to the system of difference equations in (B.28) with initial condition* $\mathbf{x}(0) = \mathbf{x}_0$ *is given by*

$$\mathbf{x}(t) = \mathbf{\Phi}(t, 0)\mathbf{x}_0 + \sum_{s=0}^{t-1}\mathbf{\Phi}(t, s+1)\mathbf{B}(s).$$

Proof. See Exercise B.9. □

Linearizing systems of nonlinear difference equations then leads to an analog of Theorem B.7. Finally, existence and uniqueness of solutions is somewhat more straightforward for difference equations. In particular, we have the following simple theorem.

THEOREM B.16. ***(Existence and Uniqueness of Solutions to Difference Equations)*** *Consider the system of first-order nonlinear difference equations*

$$\mathbf{x}(t+1) = \mathbf{G}(\mathbf{x}(t)), \tag{B.29}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$*, n is an integer, and* $\mathbf{G}: \mathbb{R}^n \to \mathbb{R}^n$ *is an arbitrary mapping. Suppose that the initial condition is specified as* $\mathbf{x}(0) = \mathbf{x}_0$*. Then (B.29) has a unique solution for all* $t \in \mathbb{N}$*.*

PROOF. Given $\mathbf{x}_0$, $\mathbf{x}(1)$ is uniquely defined as $\mathbf{G}(\mathbf{x}_0)$. Proceeding iteratively, we determine a unique $\mathbf{x}(t)$ that satisfies (B.29) for any $t \in \mathbb{N}$. □

Using the same method of turning higher-order differential equations into a system of first-order differential equations, now applied to difference equations, this theorem also guarantees existence and uniqueness of solutions to higher-order difference equations when the appropriate initial values are specified (see Exercise B.10).

B.9. Exercises

EXERCISE B.1. Use integration by parts as in Theorem B.3 to evaluate $\int_a^b \ln x dx$.

EXERCISE B.2. Show that a nth order differential equation as in (B.3) can be written as a system of n first-order equations. [Hint: let $z_j(t) = d^j x(t)/dt^j$].

EXERCISE B.3. Derive the solution to (B.9) by first defining a new variable $y(t) = x(t) - b/a$. Then rewrite (B.9) in terms of $y(t)$, divide both sides by $y(t)$, integrate, and obtain the general solution. Then transform it back to $x(t)$ to obtain (B.10).

EXERCISE B.4. Show that (B.11) is the general solution to the first-order differential equation (B.4).

EXERCISE B.5. Verify that the system of linear differential equations in (B.12) satisfies the conditions of Theorem B.10.

EXERCISE B.6. Verify (B.17).

EXERCISE B.7. Prove (B.18).

EXERCISE B.8. This exercise asks you to use the techniques to solve separable differential equations to characterize the family of utility functions with a constant relative risk aversion. In particular, recall that the (Arrow-Pratt) measure of relative risk aversion of a twice differentiable utility function u is given by

$$R_u(c) = -\frac{u''(c)\,c}{u'(c)}.$$

Suppose that $R_u(c) = r > 0$ and let $v(c) = u'(c)$, then we obtain

$$\frac{v'(c)}{v(c)} = -\frac{r}{c}.$$

Using this equation characterize the family of utility functions that have a constant relative risk aversion.

Exercise B.9. Prove Theorem B.15.

Exercise B.10. Consider the nth order difference equation

$$x(t+n) = H(x(t+n-1), ..., x(t), t),$$

where $H : \mathbb{R}^n \to \mathbb{R}$. Prove that if the initial values $x(0), x(1), ..., x(n-1)$ are specified, this equation has a unique solution for any t.

CHAPTER C

Brief Review of Dynamic Games

This chapter provides a very brief overview of some basic definitions, results and notation for infinite-horizon dynamic games. The reader is already assumed to be familiar with basic game theory, and the notions of Nash Equilibrium and Subgame Perfect Nash Equilibrium in finite games. A review of these notions as well as much of the material covered here can be found in standard graduate game theory textbooks such as Fudenberg and Tirole (1994), Myerson (1991), Osborne and Rubinstein (1994) as well as Part 2 of Mascolell, Whinston and Green (1995). My focus is throughout on games of complete information (or the so-called games of *perfect monitoring*). These types of games were used in Section 14.4 in Chapter 14, as well as in Chapters 22 and 23. The material I present here is also included in Fudenberg and Tirole (1994).

C.1. Basic Definitions

I will consider the following class of *dynamic infinite-horizon games.* As the name suggests, these games are not finite and they are also somewhat more general than repeated games, since the stage game played at each date is a function of actions taken in the past.

More formally, there is a set of players denoted by $\mathcal{N}$. This set will be either finite, or when it is infinite (especially uncountable), there will be more structure to make the game tractable and thus variants of the theorems here applicable. In particular, in many of the applications, especially in those considered in Chapters 22 and 23, there will be a continuum of players, but these will be in distinct finite groups and the game can be viewed as one between those distinct groups. With this motivation, in this Appendix I focus on the case in which $\mathcal{N}$ is finite, consisting of N players. Each player $i \in \mathcal{N}$ has a strategy set $A_i(k) \subset \mathbb{R}^{n_i}$ at every date, where $k \in K \subset \mathbb{R}^n$ is the state vector, with value at time t denoted by $k(t)$. A generic element of $A_i(k)$ at time t is denoted by $a_i(t)$, and $a(t) = (a_1(t), ..., a_N(t))$ denotes *the vector of actions* (or the "action profile") at time t, i.e.,

$$a(t) \in A(k(t)) \equiv \prod_{i=1}^{N} A_i(k(t)).$$

I use the standard notation $a_{-i}(t) = (a_1(t), .a_{i-1}(t), a_{i+1}(t), .., a_N(t))$ to denote the vector of actions without i's action, thus we can also write $a_i(t) = (a_i(t), a_{-i}(t))$. Notice that, consistent with the types of models analyzed in the text, the action set of each player $A_i(k)$ is only a function of the state variable k and not of calendar time.

Each player has an instantaneous utility function $u_i(a(t), k(t))$ where

$$u_i : A \times K \to \mathbb{R}$$

is assumed to be continuous and bounded. This notation emphasizes that each player's payoff depends on the entire action profile in that period (and not on past actions) and also on a common vector of state variables, denoted by $k(t)$. Past actions will only have an effect on current payoffs through this vector of state variables.

As usual, each player's objective at time t is to maximize their discounted payoff

$$U_{it} = \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s u_i(a(t+s), k(t+s)), \tag{C.1}$$

where $\beta \in (0,1)$ is the discount factor and $\mathbb{E}_t$ is the expectations operator conditional on information available at time t. The games I focus on here have potential uncertainty about the evolution of the state variable in the future and also strategic uncertainty resulting because of mixed strategies. However, they will not feature *asymmetric information*, since I did not use incomplete information or asymmetric information dynamic games in this book. Consequently, the expectations operator $\mathbb{E}_t$ is not indexed by i.

The law of motion of the state vector $k(t)$ is given by the following Markovian transition function

$$q(k(t+1) \mid a(t), k(t)), \tag{C.2}$$

which denotes the probability density that next period's state vector is equal to $k(t+1)$ when the time t action profile of all the agents is $a(t) \in A(k(t))$ and the state vector is $k(t) \in K$. I refer to this transition function Markovian, since it only depends on the current profile of actions and the current state. Naturally, the probability of all possible states tomorrow integrate to 1:

$$\int_{-\infty}^{\infty} q(k \mid a(t), k(t))\, dk = 1 \text{ for all } a(t) \in A(k(t)) \text{ and } k(t) \in K.$$

Next, we need to specify the information structure of the players. We focus on games with perfect observability or perfect monitoring, so that individuals observe realizations of all past actions (in case of mixed strategies, they observe realizations of actions not the strategies). Then, the public history at time t, observed by all agents up to time t, is therefore

$$h^t = (a(0), k(0), ..., a(t), k(t))$$

the history of the game up to and including time t. With mixed strategies, the history would naturally only include the realizations of mixed strategies not the actual strategy. Let the set of all potential histories at time t be denoted by H^t. It should be clear that any element $h^t \in H^t$ for any t corresponds to a subgame of this game.[1]

Let a (pure) strategy for player i at time t be

$$\sigma_i(t) : H^{t-1} \times K \to A_i,$$

i.e., a mapping that determines what to play given the entire past history h^{t-1} and the current value of the state variable $k(t) \in K$. This is the natural specification of a strategy for time t given that h^{t-1} and $k(t)$ entirely determine which subgame we are in.

A mixed strategy for player i at time t is

$$\sigma_i(t) : H^{t-1} \times K \to \Delta(A_i),$$

where $\Delta(A_i)$ is the set of all probability distributions over A_i. We are using the same notation for pure and mixed strategies to economize on notation. Let $\sigma = (\sigma_1, ..., \sigma_N)$ be the strategy profile in the infinite game, and let $\sigma_i(t) = (\sigma_1(t), ..., \sigma_N(t))$ be the continuation strategy profile after time t induced by the strategy profile of the infinite game, σ. Let S_i be the set of all feasible strategies for player i in the infinite game, and $S = \prod_{i=1}^N S_i$ be the set of all feasible strategy profiles. Let me also use the notation $S_i(t)$ for the set of continuation strategies for player i starting at time t. Naturally, $S(t) = \prod_{i=1}^N S_i(t)$ and $S_{-i}(t) = \prod_{j \neq i} S_j(t)$ are defined in the usual manner.

As is standard, define the best response correspondence as

$$BR\left(\sigma_{-i}(t) \mid h^{t-1}, k(t)\right) = \{\sigma_i(t) \in S_i(t) : \sigma_i(t) \text{ maximizes (C.1) given } \sigma_{-i}(t) \in S_{-i}(t)\ \}$$

Definition C.1. *A Subgame Perfect Equilibrium (SPE) is a strategy profile* $\sigma^* = (\sigma_1^*, ..., \sigma_N^*) \in S$ *such that* $\sigma_i^*(t) \in BR\left(\sigma_{-i}^*(t) \mid h^{t-1}, k(t)\right)$ *for all* $\left(h^{t-1}, k(t)\right) \in H^{t-1} \times K$, *for all* $i \in \mathcal{N}$ *and for all* $t = 0, 1, ...$

Therefore, a SPE requires strategies to be best responses to each other given all possible histories, which is a minimal requirement. What is "strong" (or "weak" depending on the perspective) about the SPE is that strategies are mappings from the entire history. As a result, in infinitely repeated games, there are many subgame perfect equilibria. This has prompted game theorists and economists to focus on a subset of equilibria. One possibility would be to look for "stationary" SPEs, motivated by the fact that the underlying game itself is stationary, i.e., payoffs do not depend on calendar time. Another possibility would be to look at the "best SPEs," i.e., those that are on the Pareto frontier, and maximize the utility of one player subject to the utility of the remaining players not being below a certain level.

[1]Sometimes, it may be useful to distinguish calendar time from the nodes within a stage game. In this case, one might want to use the notation h^t to denote the history up to the beginning of time t and then some other variable, say, $j^t \in J^t$ to summarize actions within the stage game at time t. In that case, the proper history at time t would be given by an element of the set $H^t \times J^t$. For our purposes here, this distinction is not necessary.

Perhaps the most popular alternative concept often used in dynamic games is that of Markov Perfect Equilibrium (MPE). The MPE differs from the SPE in only conditioning on the *payoff-relevant* "state". The motivation comes from dynamic programming where, as we have seen, an optimal plan is a mapping from the state vector to the control vector. MPE can be thought of as an extension of this reasoning to game-theoretic situations. The advantage of the MPE relative to the SPE is that most infinite games will have many fewer MPEs tha SPEs in general. The disadvantage, naturally, is that some economically interesting SPEs will be ignored when we focus on MPEs.

We could define payoff relevant history at time t in general as the smallest partition of $\mathcal{P}^t$ of H^t such that any two distinct elements of $\mathcal{P}^t$ necessarily lead to different payoffs or strategy sets for at least one of the players holding the action profile of all other players constant.

In this case, it is clear that given the Markovian transition function above, the payoff relevant state is simply $k(t) \in K$. Then we define a pure Markovian strategy as

$$\hat{\sigma}_i : K \to A_i,$$

and a mixed Markovian strategy as

$$\hat{\sigma}_i : K \to \Delta(A_i).$$

Define the set of Markovian strategies for player i by $\hat{S}_i$ and naturally, $\hat{S} = \prod_{i=1}^N \hat{S}_i$.

Notice that I have dropped the t subscript here. Given the way we have specified the game, time is not part of the payoff relevant state. This is a feature of the infinite-horizon nature of the game. With finite horizons, time would necessarily be part of the payoff-relevant state. Naturally, it is possible to imagine more general infinite-horizon games where the payoff function is $u_i(a_i(t), k(t), t)$, with calendar time being part of the payoff-relevant state.

Note also that $\hat{\sigma}_i$ has a different dimension than σ_i above. In particular, $\hat{\sigma}_i$ assigns an action (or a probability distribution over actions) to each state $k \in K$, while σ_i does so for each subgame, i.e., for all $\left(h^{t-1}, k(t)\right) \in H^{t-1} \times K$ and all t. To compare Markovian and non-Markovian strategies (and to make sure below that we can compare Markovian strategies to deviations that are non-Markovian), it is useful to consider an *extension* of Markovian strategies to the same dimension as σ_i. In particular, let $\hat{\sigma}'_i$ be an extension of $\hat{\sigma}_i$ such that

$$\hat{\sigma}'_i : K \times H^{t-1} \to \Delta(A_i)$$

with $\hat{\sigma}'_i\left(k, h^{t-1}\right) = \hat{\sigma}_i(k)$ for all $h^{t-1} \in H^{t-1}$ and $k(t) \in K$. Define the set of extended Markovian strategies for player i by $\hat{S}'_i$ and naturally, $\hat{S}' = \prod_{i=1}^N \hat{S}'_i$. Moreover, as before, let $\hat{\sigma}'_i(t)$ be the continuation strategy of player i induced by $\hat{\sigma}'_i$ after time t, and $\hat{\sigma}'_{-i}(t)$ be the continuation strategy profile of all players other than i induced by their Markovian strategies $\hat{\sigma}_{-i}$. I will refer both to $\hat{\sigma}_i$ and its extension $\hat{\sigma}'_i$ as "Markovian strategies".

Let us next define:

Definition C.2. *A Markov Perfect Equilibrium (MPE) is a profile of Markovian strategies* $\hat{\sigma}^* = (\hat{\sigma}_1^*, ..., \hat{\sigma}_N^*) \in \hat{S}$ *such that the extension of these strategies satisfy* $\hat{\sigma}_i'^*(t) \in BR\left(\hat{\sigma}_{-i}'^*(t) \mid h^{t-1}, k(t)\right)$ *for all* $\left(h^{t-1}, k(t)\right) \in H^{t-1} \times K$*, for all* $i \in \mathcal{N}$ *and for all* $t = 0, 1, ...$

Therefore, the only difference between MPE and SPE is that in the former attention is restricted to Markovian strategies. It is important to note that, as emphasized by the extension of the Markovian strategies to $\hat{\sigma}_i'^* \in \hat{S}_i'$ and the requirement that $\hat{\sigma}_i'^*(t) \in BR\left(\hat{\sigma}_{-i}'^*(t) \mid h^{t-1}, k(t)\right)$ (which conditions on history h^t), *deviations are not restricted to be Markovian.* In particular, for a MPE, a Markovian strategy $\hat{\sigma}_i^*$ must be a best response to $\hat{\sigma}_{-i}^*$ among all strategies $\sigma_i(t) : H^{t-1} \times K \to \Delta(A_i)$ available at time t.

It should also be clear that a MPE is a SPE, since the extended Markovian strategy satisfies $\hat{\sigma}_i'^*(t) \in BR\left(\hat{\sigma}_{-i}'^*(t) \mid h^{t-1}, k(t)\right)$, ensuring that $\hat{\sigma}_i'^*$ is a best response to $\hat{\sigma}_{-i}'^*$ in all subgames, i.e., for all $\left(h^{t-1}, k(t)\right) \in H^{t-1} \times K$ and for all t.

C.2. Some Basic Results

The following are some standard results and theorems that are useful to bear in mind in the application of dynamic games. First, we start with the eminently useful *one-stage deviation principle*. Recall that $\sigma(t) = (\sigma_1(t), ..., \sigma_N(t))$ denotes the continuation play for player i after date t, and therefore $\sigma_i(t) = (a_i(t), \sigma_i^*(t+1))$ designates the strategy involving action $a_i(t)$ at date t and then the continuation play given by strategy $\sigma_i^*(t+1)$.

Theorem C.1. ***(One-Stage Deviation Principle)*** *Suppose that the instantaneous payoff function of each player is uniformly bounded, i.e., there exists* $B_i < \infty$ *for all* $i \in \mathcal{N}$ *such that* $\sup_{k \in K, a \in A(k)} u_i(a, k) < B_i$. *Then a strategy profile* $\sigma^* = (\sigma_1^*, ..., \sigma_N^*) \in S$ *is a SPE [respectively* $\hat{\sigma}^* = (\hat{\sigma}_1^*, ..., \hat{\sigma}_N^*) \in \hat{S}$ *is a MPE] if and only if for all* $i \in \mathcal{N}$, $\left(h^{t-1}, k(t)\right) \in H^{t-1} \times K$ *and time* t *and for all* $a_i(t) \in A(k(t))$, $\sigma_i^*(t) = (a_i(t), \sigma_i^*(t+1))$ *[resp.* $\hat{\sigma}_i'(t) = \left(a_{it}, \hat{\sigma}_i'^*(t+1)\right)$*] yields no higher payoff to player* i *than* $\sigma_i^*(t)$ *[resp.* $\hat{\sigma}_i'^*(t)$*].*

Proof. **(Sketch)** Fix the strategy profile of other players. Then the problem of individual i is equivalent to a dynamic optimization problem. Then since $\lim_{T \to \infty} \sum_{s=T}^{\infty} \beta^s u_i(a(t+s), k(t+s)) = 0$ for all $\{a(t+s), k(t+s)\}_{s=0}^{T}$ and all t given the uniform boundedness of instantaneous payoffs and $\beta < 1$, we can apply a slight variant of the principle of optimality from dynamic programming, Theorem 16.2 from Chapter 16. In particular, given the uniform boundedness assumption, the same argument as in the proof of this theorem implies that an optimal plan for an individual, for a fixed profile of strategies of all other players, must be optimal for the next stage given his optimal continuation from then on, and moreover, that any non-optimal plan must violate the principle of optimality at some point. □

This theorem basically implies that in dynamic games, we can check whether a strategy is a best response to other players' strategy profile by looking at one-stage deviations, keeping the rest of the strategy profile of the deviating player as given. The uniform boundedness assumption can be weakened to require "continuity at infinity", which essentially means that discounted payoffs converge to zero along any history (and this assumption can also be relaxed further).

Lemma C.1. *Suppose that $\hat{\sigma}'^{*}_{-i}$ is Markovian (i.e., it is an extension of a Markovian strategy $\hat{\sigma}^{*}_{-i}$) and that for $h^{t-1} \in H^{t-1}$ and $k(t) \in K$, $BR\left(\hat{\sigma}'^{*}_{-i} \mid k(t), h^{t-1}\right) \neq \varnothing$. Then, there exists $\hat{\sigma}'^{*}_{i} \in BR\left(\hat{\sigma}'^{*}_{-i} \mid k(t), h^{t-1}\right)$ that is Markovian.*

Proof. Suppose $\hat{\sigma}'^{*}_{-i}$ is Markovian and $BR\left(\hat{\sigma}'^{*}_{-i} \mid k(t), h^{t-1}\right) \neq \varnothing$, thus includes an element σ^{*}_{i}. Suppose, to obtain a contradiction, that σ^{*}_{i} is not Markovian. This implies that there exists some t, $k(t) \in K$ and $h^{t-1}, \tilde{h}^{t-1} \in H^{t-1}$ such that the continuation play following these two histories are not the same, i.e., $\sigma^{*}_{i}[t]\left(k(t), h^{t-1}\right) \in BR\left(\hat{\sigma}'^{*}_{-i} \mid k(t), h^{t-1}\right)$, $\sigma^{*}_{i}[t]\left(k(t), \tilde{h}^{t-1}\right) \in BR\left(\hat{\sigma}'^{*}_{-i} \mid k(t), \tilde{h}^{t-1}\right)$ and $\sigma^{*}_{i}[t]\left(k(t), h^{t-1}\right) \neq \sigma^{*}_{i}[t]\left(k(t), \tilde{h}^{t-1}\right)$, where $\sigma^{*}_{i}[t]\left(k(t), h^{t-1}\right)$ denotes a continuation strategy for player i starting from time t with state vector $k(t)$ and history h^{t-1}. Now, construct the continuation strategy $\hat{\sigma}'^{*}_{i}[t]$ such that $\hat{\sigma}'^{*}_{i}[t]\left(k(t), h^{t-1}\right) = \sigma^{*}_{i}[t]\left(k(t), \tilde{h}^{t-1}\right)$. Since $\hat{\sigma}'^{*}_{-i}$ is Markovian, $\hat{\sigma}'^{*}_{-i,t}$ is independent of $h^{t-1}, \tilde{h}^{t-1} \in H^{t-1}$, and therefore $\hat{\sigma}'^{*}_{i}[t]\left(k(t), h^{t-1}\right) = \hat{\sigma}'^{*}_{i}[t]\left(k(t), \tilde{h}^{t-1}\right) \in BR\left(\hat{\sigma}'^{*}_{-i} \mid k(t), h^{t-1}\right) \cap BR\left(\hat{\sigma}'^{*}_{-i} \mid k(t), \tilde{h}^{t-1}\right)$. Repeating this argument for all instances in which σ^{*}_{i} is not Markovian establishes that a Markovian strategy $\hat{\sigma}'^{*}_{i}$ is also best response to $\hat{\sigma}'^{*}_{-i}$. □

This lemma states that when all other players are playing Markovian strategies, there exists a best response that is Markovian for each player. This does not mean that there are no other best responses, but since there is a Markovian best response, this gives us hope in constructing Markov Perfect Equilibria. Consequently, we have the following theorem:

Theorem C.2. ***(Existence of Markov Perfect Equilibria)****Let K and $A_i(k)$ for all $k \in K$ be finite sets, then there exists a MPE $\hat{\sigma}^{*} = (\hat{\sigma}^{*}_{1}, ..., \hat{\sigma}^{*}_{N})$.*

Proof. **(sketch)** Consider an extended game in which the set of players is an element (i, k) of $\mathcal{N} \times K$, with payoff function given by the original payoff function for player i starting in state k as in (C.1) and strategy set $A_i(k)$. The set $\mathcal{N} \times K$ is finite, and since $A_i(k)$ is also finite, the set of mixed strategies $\Delta(A_i(k))$ for player (i, k) is the simplex over $A_i(k)$. Therefore, the standard proof of existence of Nash equilibrium based on Kakutani's Fixed Point Theorem (Theorem A.16 from Chapter A) applies and leads to the existence of an equilibrium $\left(\hat{\sigma}^{*}_{(i,k)}\right)_{(i,k)\in\mathcal{N}\times K}$ in this extended game. Now going back to the original game, construct the strategy $\hat{\sigma}_i^*$ for each player $i \in \mathcal{N}$ such that $\hat{\sigma}_i^*(k) = \hat{\sigma}_{(i,k)}^*$, i.e., $\hat{\sigma}_i^* : K \to \Delta(A_i)$. This strategy profile $\hat{\sigma}^*$ is Markovian. Consider the extension of $\hat{\sigma}^*$ to $\hat{\sigma}'^*$ as above, i.e., $\hat{\sigma}_i'^*\left(k, h^{t-1}\right) = \hat{\sigma}_i^*(k)$ for all $h^{t-1} \in H^{t-1}$, $k(t) \in K$, $i \in \mathcal{N}$ and t. Then, by construction, given $\hat{\sigma}_{-i}'^*$, it is impossible to improve over $\hat{\sigma}_i'^*$ with a deviation at any $k \in K$, thus Theorem C.1 implies that $\hat{\sigma}_i'^*$ is best response to $\hat{\sigma}_{-i}'^*$ for all $i \in \mathcal{N}$, so is a MPE strategy profile. □

Similar existence results can be proved for countably infinite sets K and $A_i(k)$, and also for uncountable sets, but in this latter instance, some additional requirements are necessary, and these are rather technical in nature. Since they will play no role in what follows, we do not need to elaborate on these.

We also have

Theorem C.3. ***(Existence of Subgame Perfect Equilibria)*** *Let K and $A_i(k)$ for all $k \in K$ be finite sets, then there exists a SPE $\sigma^* = (\sigma_1^*, ..., \sigma_N^*)$.*

Proof. Theorem C.2 shows that a MPE exists and since a MPE is a SPE, the existence of a SPE follows. □

Existence of SPEs when K and $A_i(k)$ are uncountable sets is much easier to guarantee. For example, compactness and convexity of K and $A_i(k)$ is sufficient for existence of pure strategy equilibria when U_{it} is quasi-concave in σ_{it} for all $i \in \mathcal{N}$ (in addition to the continuity assumptions above). In the absence of convexity of K and $A_i(k)$ or quasi-concavity of $U_i(t)$, mixed strategy equilibria can always be guaranteed to exist under some very mild additional assumptions.

For the next result, let $\hat{\Sigma} = \left\{\hat{\sigma}^* \in \hat{S}\text{: } \hat{\sigma}^* \text{ is a MPE}\right\}$ be the set of MPE strategies and $\Sigma^* = \{\sigma \in S\text{: } \sigma^* \text{ is a SPE}\}$ be the set of SPE strategies. Let $\hat{\Sigma}'$ be the extension of $\hat{\Sigma}$ to include conditioning on histories. In particular, as defined before, recall that $\hat{\sigma}_i' : K \times H^{t-1} \to \Delta(A_i)$ is such that $\hat{\sigma}_i'\left(k, h^{t-1}\right) = \hat{\sigma}_i(k)$ for all $h^{t-1} \in H^{t-1}$ and $k(t) \in K$, and let

$$\hat{\Sigma}' = \left\{ \begin{array}{c} \hat{\sigma}' \in S\text{: } \hat{\sigma}_i'\left(k, h^{t-1}\right) = \hat{\sigma}_i(k) \text{ for all } h^{t-1} \in H^{t-1},\, k(t) \in K \\ \text{and } i \in \mathcal{N} \text{ and } \hat{\sigma} \text{ is a MPE} \end{array} \right\}.$$

Then we have:

Theorem C.4. ***(Markov Versus Subgame Perfect Equilibria)*** *$\hat{\Sigma}' \subset \Sigma^*$.*

Proof. This theorem follows immediately by noting that since $\hat{\sigma}^*$ is a MPE strategy profile, the extended strategy profile, $\hat{\sigma}'^*$, is such that $\hat{\sigma}_i'^*$ is a best response to $\hat{\sigma}_{i-1}^*$ for all $h^{t-1} \in H^{t-1}$, $k(t) \in K$ and for all $i \in \mathcal{N}$, thus is subgame perfect. □

This theorem implies that every MPE strategy profile corresponds to a SPE strategy profile and any equilibrium-path play supported by a MPE can be supported by a SPE.

Finally, a well-known theorem for subgame perfect equilibria from repeated games also generalizes to dynamic games. Let $p(a \mid \sigma)$ be the probability distribution over the equilibrium-path actions induced by the strategy profile σ, with the usual understanding that $\int_{a \in A} p(a \mid \sigma)\, da = 1$ for all $\sigma \in S$, where A is a set of admissible action profiles. With a slight abuse of terminology, I will refer to $p(a \mid \sigma)$ as the equilibrium-path action induced by strategy σ. Then, let

$$U_i^M(k) = \min_{\sigma_{-i}} \max_{\sigma_i} \mathbb{E} \sum_{s=0}^{\infty} \beta^s u_i(a(t+s), k(t+s)),$$

starting with $k(t) = k$ and $k(t+s)$ given by (C.2) be the minmax payoff of player i starting with state k. Moreover, let

$$U_i^N(k) = \min_{\sigma \in \Sigma} \mathbb{E} \sum_{s=0}^{\infty} \beta^s u_i(a(t+s), k(t+s)), \tag{C.3}$$

be the minimum (subgame perfect) equilibrium payoff of player i starting in state $k \in K$. In other words, this is player i's payoff in the equilibrium chosen to minimize this payoff (starting in state k). Then we have:

Theorem C.5. ***(Punishment with the Worst Equilibrium)*** *Suppose $\sigma^* \in S$ is a pure strategy SPE with the distribution of equilibrium-path actions given by $p(a \mid \sigma^*)$. Then there exists a SPE $\sigma^{**} \in S$ (possibly equal to σ^*) such that $p(a \mid \sigma^*) = p(a \mid \sigma^{**})$ and σ^{**} involves a continuation payoff of $U_i^N(k)$ to player i, if i is the first to deviate from σ^{**} at date t after some history $h^{t-1} \in H^{t-1}$ and when the resulting state in the next period is $k(t+1) = k$.*

Proof. **(Sketch)** If σ^* is a SPE, then no player wishes to deviate from it. Suppose that i were to deviate from σ^* at date t after some history $h^{t-1} \in H^{t-1}$ and when $k(t) = k$, and denote its continuation payoff starting at at time t, with state vector $k(t)$, and history h^{t-1} by $U_i^d[t]\left(k(t), h^{t-1} \mid \sigma^*\right)$, whereas its equilibrium payoff under σ^* is similarly defined as $U_i^c[t]\left(k(t), h^{t-1} \mid \sigma^*\right)$. σ^* can be a SPE only if

$$U_i^c[t]\left(k(t), h^{t-1} \mid \sigma^*\right) \geq \max_{a_i(t) \in A_i(k)} \left\{ \mathbb{E} u_i\left(a_i(t), a_{-i}\left(\sigma_{-i}^*\right) \mid k(t), h^{t-1}\right) + \beta \mathbb{E} U_i^d[t+1]\left(k(t+1), h^t \mid \sigma_{-i}^*\right) \right\},$$

where $u_i\left(a_i(t), a_{-i}\left(\sigma_{-i}^*\right) \mid k(t), h^{t-1}, \sigma_{-i}^*\right)$ is the instantaneous payoff of individual i when he chooses action $a_i(t)$ in state $k(t)$ following history h^{t-1} and other players are playing the (potentially mixed) action profiled induced by σ_{-i}^*, denoted by $a_{-i}\left(\sigma_{-i}^*\right)$, and $U_i^d[t+1]\left(k(t+1), h^t \mid \sigma_{-i}^*\right)$ is the continuation payoff following this deviation, with $k(t+1)$ following the transition function $q\left(k(t+1) \mid k(t), a_i(t), a_{-i}\left(\sigma_{-i}^*\right)\right)$ and h^t incorporating the actions $a_i(t), a_{-i}\left(\sigma_{-i}^*\right)$. Note that by construction, the continuation play, following the deviation, will correspond to a SPE, since σ_{-i}^* specifies a SPE action for all players other than

i in all subgames, and in response, the best that player i can do is to play an equilibrium strategy.

By definition of a SPE and the minimum equilibrium payoff of player i defined in (C.3), we have

$$U_{it+1}^{d}\left(k\left(t+1\right),h^{t}\mid\sigma_{-i}^{*}\right)\geq U_{i}^{N}\left(k\left(t+1\right)\right).$$

The preceding two inequalities imply

$$U_{i}^{c}\left[t\right]\left(k\left(t\right),h^{t-1}\mid\sigma^{*}\right)\geq\max_{a_{i}\left(t\right)\in A_{i}\left(k\right)}\mathbb{E}u_{i}\left(a_{i}\left(t\right),a_{-i}\left(\sigma_{-i}^{*}\right)\mid k\left(t\right),h^{t-1}\right)+\beta U_{i}^{N}\left(k\left(t\right)\right).$$

Therefore, we can construct σ^{**}, which is identical to σ^{*} except replacing $U_{i}^{d}\left[t+1\right]\left(k\left(t+1\right),h^{t}\mid\sigma^{*}\right)$ with $U_{i}^{N}\left(k\left(t+1\right)\right)$ following the deviation by player i from σ^{*} at date t after some history $h^{t-1}\in H^{t-1}$ and when in the next period, we have $k\left(t+1\right)=k$. Since $U_{i}^{N}\left(k\left(t+1\right)\right)$ is a SPE payoff, σ^{**} will also be a SPE. □

This theorem therefore states that when looking for the set of SPEs, we can limit attention to those involving the most severe equilibrium punishments.

A stronger version of this theorem is the following:

Theorem C.6. ***(Punishment with Minmax Payoffs)*** *Suppose $\sigma^{*}\in S$ is a pure strategy SPE with the distribution of equilibrium-path actions given by $p\left(a\mid\sigma^{*}\right)$. Then there exists $\bar{\beta}\in\left(0,1\right)$ such that for all $\beta\geq\bar{\beta}$, there exists a SPE $\sigma^{**}\in S$ (possibly equal to σ^{*}) with $p\left(a\mid\sigma^{*}\right)=p\left(a\mid\sigma^{**}\right)$ and σ^{**} involves a continuation payoff of $U_{i}^{M}\left(k\right)$ to player i, if i is the first to deviate from σ^{**} at date t after some history $h^{t-1}\in H^{t-1}$ and when $k\left(t\right)=k$.*

Proof. The proof is identical to that of Theorem C.5, except that it uses $U_{i}^{M}\left(k\right)$ instead of $U_{i}^{N}\left(k\right)$. When β is high enough, the minmax payoff for player i, $U_{i}^{M}\left(k\right)$, can be supported as part of a subgame perfect equilibrium. The details of this proof can be found in Abreu (1988) and a further discussion is contained in Fudenberg and Tirole (1994). □

C.3. Application: Repeated Games With Perfect Observability

For repeated games with perfect observability, both SPE and MPE are easy to characterize and their properties are well-known. In particular, suppose that we have the same stage game played an infinite number of times, so that payoffs are given by

$$U_{it}=\mathbb{E}_{t}\sum_{s=0}^{\infty}\beta^{s}u_{i}\left(a\left(t+s\right)\right),\tag{C.4}$$

which is only different from (8.3) because there is no conditioning on the state variable $k\left(t\right)$. Let us refer to the game $\left\{u_{i}\left(a\right),a\in A\right\}$ as the stage game. Define

$$m_{i}=\min_{a_{-i}}\max_{a_{i}}u_{i}\left(a\right),$$

as the minmax payoff in this stage game. Let $V \in \mathbb{R}^N$ be the set of feasible per period payoffs for the N players, with v_i corresponding to the payoff to player i (so that discounted payoffs correspond to $v_i/(1-\beta)$) Then we have:

THEOREM C.7. ***(The Folk Theorem for Repeated Games)*** *Suppose that $\{A_i\}_{i\in\mathcal{N}}$ are compact. Then for any $v \in V$ such that $v_i > m_i$ for all $i \in \mathcal{N}$, there exists $\bar{\beta} \in [0,1)$ such that for all $\beta > \bar{\beta}$, v can be supported as the payoff profile of a SPE.*

PROOF. **(sketch)** Construct following punishment strategies for any deviation such that the first player to deviate, i, is held down to its minmax payoff m_i (which can be supported as a SPE). Then the payoff from any deviation $a \in A_i$ is $D_i(a \mid \beta) \leq d_i + \beta m_i/(1-\beta)$ where d_i is the highest payoff player i can obtain by deviating, which is finite by the fact that u_i is continuous and bounded and A_i is compact. v_i can be supported if

$$\frac{v_i}{1-\beta} \geq d_i + \beta \frac{m_i}{1-\beta}.$$

Since d_i is finite and $v_i > m_i$, there exists $\bar{\beta}_i \in [0,1)$ such that for all $\beta \geq \beta_i$ this inequality is true. Letting $\bar{\beta} = \max_{i\in\mathcal{N}} \bar{\beta}_i$ establishes the desired result. □

We also have:

THEOREM C.8. ***(Unique Markov Perfect Equilibrium in Repeated Games)*** *Suppose that the stage game has a unique equilibrium a^*. Then there exists a unique MPE such that a^* is played at every date.*

PROOF. This follows immediately since $K = \varnothing$ in this case. □

This last theorem is natural, but also very important. In repeated games, there is no state vector, so strategies cannot be conditioned on anything. Consequently, in MPE we can only look at the strategies that are best response in the stage game.

EXAMPLE C.1. **(Prisoner's Dilemma)** Consider the following standard prisoner's dilemma, which, in fact, has many applications in political economy.

	D	C
D	$(0,0)$	$(4,-1)$
C	$(-1,4)$	$(2,2)$

The stage game has a unique equilibrium, which is (D,D). Now imagine this game being repeated an infinite number of times with both agents having discount factor β. The unique MPE is playing (D,D) at every date.

In contrast, when $\beta \geq 1/2$, then (C,C) at every date can be supported as a SPE. To see this, recall that we only need to consider the minmax punishment, which in this case is $(0,0)$. Playing (C,C) leads to a payoff of $2/(1-\beta)$, whereas the best deviation leads to the payoff of 4 now and a continuation payoff of 0. Therefore, $\beta \geq 1/2$ is sufficient to make sure that the following grim strategy profile implements (C,C) at every date: for both players, the strategy is to play C if h^t includes only (C,C) and play D otherwise.

Why this strategy combination is not a MPE is also straightforward to see. The grim strategy ensures cooperation by conditioning on past history, i.e., it conditions on whether somebody has defected at any point in the past. This history is not payoff relevant for the future of the game given the action profile of the other player—fixing the action profile of the other player, whether somebody has cheated in the past or not has no effect on future payoffs.

C.4. Exercises

Exercise C.1. A simple application of the ideas in this appendix are *common pool games*. Consider a society consisting of $N+1<\infty$ players each with payoff function

$$\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \log\left(c_i\left(t+s\right)\right)$$

at time t where $\beta \in (0,1)$ and $c_i(t)$ denotes consumption of individual $i \in \mathcal{N}$ at time t. The society has a common resource, denoted by $K(t)$, which can be thought of as the capital stock at time t. This capital stock follows the non-stochastic law of motion:

$$K(t+1) = AK(t) - \sum_{i \in \mathcal{N}} c_i(t),$$

where $A>0$, $K(0)$ is given and $K(t) \geq 0$ must be satisfied in every period. The stage game is as follows: at every date all players simultaneously announce $\{c_i(t)\}_{i\in\mathcal{N}}$. If $\sum_{i\in\mathcal{N}} c_i(t) \leq AK(t)$, then each individual consumes $c_i(t)$. If $\sum_{i\in\mathcal{N}} c_i(t) > AK(t)$, then $AK(t)$ is equally allocated among the $N+1$ players.

(1) First consider the single-person decision problem corresponding to this game, where $\{c_i(t)\}_{i\in\mathcal{N}}$ is being chosen by a benevolent planner, wishing to maximize the total discounted payoff of all the agents in the society: $\mathbb{E}_t \sum_{i\in\mathcal{N}} \sum_{s=0}^{\infty} \beta^s \log(c_i(t+s))$. Setup this problem as a dynamic programming problem and show that the value function of the planner given capital stock K, $V(K)$, is uniquely defined, is continuous, concave and also differentiable whenever $S \in (0, AK)$. Also show that the saving level is a function of the capital stock given by $\pi(K)$ a single-valued and continuous function. Moreover, show that this "saving function" is given by $\pi(K) = \beta AK$. Also derive an explicit form equation for the value function, $V(K)$.
(2) Consider the MPE. First show that there are some uninteresting MPEs, where all individuals announcing $c_i(0) = AK(0)$ or some consumption level close to this. Explain why these are MPEs, since there are no profitable deviations by any agents.

(3) Next focus on "continuous" and symmetric MPE, where each agent will pursue a strategy of consuming $c^N(K)$ when the capital stock is K. Given symmetry, this implies that when all other agents are pursuing this strategy and agent i chooses consumption c, this will imply a savings level of

$$S = AK - Nc^N(K) - c.$$

Using this observation, write the value function of an individual as

$$V^N(K) = \max_{S \leq AK - Nc^N(K)} \left\{ \log\left(AK - Nc^N(K) - S\right) + \beta V^N(S) \right\}. \tag{C.5}$$

Explain this expression and provide an intuition.

(4) Assuming differentiability, derive the first-order condition of the maximization problem in (C.5) and show that there exists a symmetric equilibrium where the equilibrium aggregate saving level in the economy will be given by

$$\pi(K) = \frac{\beta A}{1 + N - \beta N} K.$$

Compare this expression to that in Part 2. What is the effect of an increase in N? Provide in intuition for this.

(5) Show that if $\beta A > 1 > \beta A / (1 + N - \beta N)$, than the single-person decision problem would involve growth over time, while the MPE would involve the resources shrinking over time.

(6) Next show that in this game there always exists subgame perfect equilibria that implements the single-person solution for any value of $\beta > 0$. Explain this result.

(7) Now suppose that $\beta A = 1$ and again focus on MPE. Suppose that the game starts with capital stock $K(0)$ and consider following discontinuous Markovian strategy profile:

$$c_i(K) = \begin{cases} \frac{\beta AK}{1+N} & \text{if } K \geq K(0) \\ K & \text{if } K < K(0) \end{cases}.$$

Show that when all players other than i' pursue this strategy, it is a best response for player i' to play this strategy as well, and along the equilibrium path, the single-person solution is implemented. Carefully provide an intuition for this result. Show that the same result cannot be obtained when $\beta A \neq 1$. Why not?

CHAPTER D

List of Theorems

In this appendix, I list the theorems presented in various different chapters for reference. Many of these theorems refer to mathematical results used in different parts of the book. Some of them are economic results that are more general and widely applicable than the results I labeled "propositions". To conserve space, I do not list additional mathematical results given in Lemmas, Corollaries and Facts.

Chapter 2

Theorem 2.1: Euler's Theorem.

Theorem 2.2: Stability for Systems of Linear Difference Equations.

Theorem 2.3: Stability for Systems of Nonlinear Difference Equations.

Theorem 2.4: Stability for Systems of Linear Differential Equations.

Theorem 2.5: Stability for Systems of Nonlinear Differential Equations.

Chapter 5

Theorem 5.1: Debreu-Mantel-Sonnenschein Theorem.

Theorem 5.2: Gorman's Aggregation Theorem.

Theorem 5.3: Existence of a Normative Representative Household.

Theorem 5.4: The Representative Firm Theorem.

Theorem 5.5: The First Welfare Theorem for Economies with Finite Number of Commodities.

Theorem 5.6: The First Welfare Theorem for Economies with Infinite Number of Commodities.

Theorem 5.7: The Second Welfare Theorem.

Theorem 5.8: Equivalence of Sequential and Non-Sequential Trading with Arrow Securities.

Chapter 6

Theorem 6.1: Equivalence of Sequential and Recursive Formulations.

Theorem 6.2: Principle of Optimality in Dynamic Programming.

Theorem 6.3: Existence of Solutions in Dynamic Programming.

Theorem 6.4: Concavity of the Value Function.

Theorem 6.5: Monotonicity of the Value Function.

Theorem 6.6: Differentiability of the Value Function.

Theorem 6.7: The Contraction Mapping Theorem.

Theorem 6.8: Applications of the Contraction Mapping Theorem.

Theorem 6.9: Blackwell's Sufficient Conditions for a Contraction.

Theorem 6.10: Sufficiency of Euler Equations and the Transversality Condition.

Chapter 7

Theorem 7.1: Variational Necessary Conditions for an Interior Optimum with Free End Points.

Theorem 7.2: Variational Necessary Conditions for Interior Optimum with Fixed End Points.

Theorem 7.3: Simplified version of Pontryagin's Maximum Principle.

Theorem 7.4: Mangasarian's Sufficient Conditions for an Optimum.

Theorem 7.5: Arrow's Sufficient Conditions for an Optimum.

Theorem 7.6: Pontyagin's Maximum Principle for Multivariate Problems.

Theorem 7.7: Mangasarian's Sufficient Conditions for Multivariate Problems.

Theorem 7.8: Arrow's Sufficient Conditions for Multivariate Problems.

Theorem 7.9: Pontyagin's Maximum Principle for Infinite-Horizon Problems.

Theorem 7.10: Hamilton-Jacobi-Bellman Equations.

Theorem 7.11: Mangasarian's Sufficient Conditions for Infinite-Horizon Problems.

Theorem 7.12: Arrow's Sufficient Conditions for Infinite-Horizon Problems.

Theorem 7.13: General Transversality Condition for Infinite-Horizon Problems.

Theorem 7.14: The Maximum Principle and the Transversality Conditions for Discounted Infinite-Horizon Problems.

Theorem 7.15: Mangasarian's Sufficient Conditions for Discounted Infinite-Horizon Problems.

Theorem 7.16: Arrow's Sufficient Conditions for Discounted Infinite-Horizon Problems.

Theorem 7.17: Existence of Solutions in Optimal Control.

Theorem 7.18: Saddle Path Stability for Systems of Linear Differential Equations.

Theorem 7.19: Saddle Path Stability for Systems of Nonlinear Differential Equations.

Chapter 10

Theorem 10.1: Separation Theorem for Investment in Human Capital.

Chapter 16

Theorem 16.1: Equivalence of Sequential and Recursive Formulations.
Theorem 16.2: Principle of Optimality in Stochastic Dynamic Programming.
Theorem 16.3: Existence of Solutions in Stochastic Dynamic Programming.
Theorem 16.4: Concavity of the Value Function.
Theorem 16.5: Monotonicity of the Value Function in State Variables.
Theorem 16.6: Differentiability of the Value Function.
Theorem 16.7: Monotonicity of the Value Function in Stochastic Variables.
Theorem 16.8: Sufficiency of Euler Equations and the Transversality Condition.
Theorem 16.9: Existence of Solutions with Markov Processes.
Theorem 16.10: Continuity of the Value Function with Markov Processes.
Theorem 16.11: Concavity of the Value Functions with Markov Processes.
Theorem 16.12: Monotonicity of the Value Functions with Markov Processes.
Theorem 16.13: Differentiability of the Value Function with Markov Processes.

Chapter 22

Theorem 22.1: Arrow's Impossibility Theorem.
Theorem 22.2: The Median Voter Theorem.
Theorem 22.3: The Median Voter Theorem with Strategic Voting.
Theorem 22.4: Downs's Policy Convergence Theorem.
Theorem 22.5: The Median Voter Theorem without Single Peaked Preferences.
Theorem 22.6: Downs's Policy Convergence Theorem without Single Peaked References.
Theorem 22.7: The Probabilistic Voting and Preference Aggregation Theorem.

Appendix Chapter A

Theorem A.1: Properties of Open and Closed Sets in Metric Spaces.
Theorem A.2: Continuity and Open Sets in Metric Spaces.
Theorem A.3: The Intermediate Value Theorem.
Theorem A.4: Continuity and Open Sets in Topological Spaces.
Theorem A.5: Convergence of Nets and Continuity in Topological Spaces.
Theorem A.6: The Heine-Borel Theorem.
Theorem A.7 The Bolzano-Weierstrass Theorem.
Theorem A.8: Continuity and Compactness in Topological Spaces.
Theorem A.9: Weierstrass's Theorem.
Theorem A.10: Continuity of Projection Maps and the Product Topology.
Theorem A.11: Continuity of Discounted Utilities in the Product Topology.
Theorem A.12: Tychonoff's Theorem.

Theorem A.13: Berge's Maximum Theorem.

Theorem A.14: Properties of Maximizers under Quasi-Concavity.

Theorem A.15: Properties of Minimizers under Quasi-Convexity.

Theorem A.16: Kakutani's Fixed Point Theorem.

Theorem A.17: Brouwer's Fixed Point Theorem.

Theorem A.18: Mean Value Theorems.

Theorem A.19: L'Hospital's Rule.

Theorem A.20: Taylor's Theorem and Taylor Approximations.

Theorem A.21: Taylor's Theorem for Functions of Several Variables.

Theorem A.22: The Inverse Function Theorem.

Theorem A.23: The Implicit Function Theorem.

Theorem A.24: Continuity of Linear Functionals in Normed Vector Spaces.

Theorem A.25: Geometric Form of the Hahn-Banach Theorem.

Theorem A.26: Separating Hyperplane Theorem.

Theorem A.27: The Saddle −Point Theorem.

Theorem A.28: The Kuhn-Tucker Theorem.

Appendix Chapter B

Theorem B.1: Fundamental Theorem of Calculus I.

Theorem B.2: Fundamental Theorem of Calculus II.

Theorem B.3: Integration by Parts.

Theorem B.4: Leibniz's Rule.

Theorem B.5: Solution to Systems of Linear Differential Equations with Constant Coefficients.

Theorem B.6: Solution to General Systems of Linear Differential Equations.

Theorem B.7: The Grobman-Hartman Theorem on Stability of Nonlinear Systems of Differential Equations.

Theorem B.8: Picard's Theorem on Existence and Uniqueness for Differential Equations.

Theorem B.9: Existence and Uniqueness for Differential Equations on Compact Domain.

Theorem B.10: Picard's Theorem on Existence and Uniqueness for Systems of Differential Equations.

Theorem B.11: Existence and Uniqueness for Systems of Differential Equations on Compact domain.

Theorem B.12: Peano's Theorem of Existence and Uniqueness for Differential Equations.

Theorem B.13: Continuity of Solutions to Differential Equations.

Theorem B.14: Solution to Systems of Linear Difference Equations with Constant Coefficients.

Theorem B.15: Solution to Systems of Linear Difference Equations with Constant Coefficients.

Theorem B.16: Existence and Uniqueness of Solutions to Difference Equations.

Appendix Chapter C

Theorem C.1: One-Stage Deviation Principle.

Theorem C.2: Existence of Markov Perfect Equilibria in Finite Dynamic Games.

Theorem C.3: Existence of Subgame Perfect Equilibria in Finite Dynamic Games.

Theorem C.4: Relationship between Markov and Subgame Perfect Equilibria.

Theorem C.5: Punishment with the Worst Equilibrium.

Theorem C.6: Punishment with the Minmax Continuation Values.

Theorem C.7: The Folk Theorem for Repeated Games.

Theorem C.8: Uniqueness of Markov Perfect Equilibria in Repeated Games.

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